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On nonlinear perturbations of a periodic integrodifferential Kirchhoff equation with critical exponential growth

Eudes Barboza, Yane Araújo and Gilson de Carvalho

Abstract. In this paper, we investigate the existence of solutions for a class of integrodifferential Kirchhoff equations. These equations involve a nonlocal operator with a measurable kernel that satisfies "structural properties" that are more general than the standard kernel of the fractional Laplacian operator. Additionally, the potential can be periodic or asymptotically periodic, and the nonlinear term exhibits critical exponential growth in the sense of Trudinger–Moser inequality. To guarantee the existence of solutions, we employ variational methods, specifically the mountain-pass theorem. In this context, it is important to emphasize that we have additional difficulties due to the lack of compactness in our problem, because we deal with critical growth nonlinearities in unbounded domains. Moreover, the Kirchhoff term adds complexity to the problem, as it requires suitable calculations for control the estimate the minimax level, representing the main challenge in this work. Finally, we consider two different approaches to estimate the minimax level. The first approach is based on a hypothesis proposed by D. M. Cao, while the second one involves a slightly weaker assumption addressed by Adimurthi and Miyagaki.

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1. Introduction

In this paper, we are concerned with the existence of solutions for a class of integrodifferential Kirchhoff equations

$$[-m(\|u\|^2)]\mathcal{L}_K u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \tag{1.1}$$

where V and f are functions that satisfy mild conditions, $m : \mathbb{R}_+ \to \mathbb{R}_+$ is the Kirchhoff function, $\mathbb{R}_+$ denotes $[0,+\infty)$, and $\mathcal{L}_K u$ stands for the integrodifferential operator defined by

$$-\mathcal{L}_K u(x) = 2P.V. \int_{\mathbb{R}} (u(x) - u(y))K(x,y)\,\mathrm{d}y. \tag{1.2}$$

Here $K(x,y) = K(x-y)$ and belongs to a class of singular symmetric kernels, and $P.V.$ means "in the principal value sense".

If $K(x) = C_{N,s}|x|^{-(N+2s)}$, where

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\,\mathrm{d}\zeta \right)^{-1},$$

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this is, when $-\mathcal{L}_K$ is the fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, (see [1]), several papers have studied the existence of solutions for equations of this type,

$$m\left(\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

with $m:\mathbb{R}_+\to\mathbb{R}_+$ the Kirchhoff function, whose prototype, due to Kirchhoff himself, is $m(t) = a + b\gamma t^{\gamma-1}, a, b \geq 0, a + b > 0, \gamma \geq 1$.

Moreover, when $s \to 1^-$ then problem (1.3) formally reduces to the well-known Kirchhoff equation in the literature with $m(t) = a + tb$

$$-\left(a + b\int\limits_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u + V(x)u = f(u)\,\text{in}\,\mathbb{R}^N,$$

which is related to the stationary analogous of the Kirchhoff-type equation

$$-\frac{\partial^2 u}{\partial t^2}\left(a + b\int\limits_{\Omega}|\nabla u|^2\,\mathrm{d}x\right)\Delta u = f(t,x,u),\,\text{in}\,\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, u denotes the displacement, f is the external force, b is the initial tension and a is related to the intrinsic properties of the string. Equations of this type were first proposed by Kirchhoff [2] to describe the transversal oscillations of a stretched string. Besides, we also point out that such nonlocal problems appear in other fields like biological systems, where u describes a process depending on the average of itself. In this direction, we refer readers to Chipot and Lovat [3], Alves and Corrêa [4]. There is extensive literature on this subject, when $a > 0$, that is in the so-called non-degenerate case. We cite e.g. [5–7], as well as the references therein. For the degenerate case, there are a few papers, see [8,9], as well as the references therein. We mention in passing that in [10] variational techniques were used for the first time to handle Kirchhoff elliptic problems.

Fiscella and Valdinoci [6], proposed the following stationary Kirchhoff variational equation with critical growth

$$\begin{cases} m\left(\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set and $2_s^* = \dfrac{2N}{N-2s}$. This equation models nonlocal aspects of the tension arising from measurements of the fractional length of the string. They obtained the existence of nonnegative solutions when m is an increasing and continuous function, there exists $m_0 > 0$ such that $m(t) \geq m_0 = m(0)$ for any $t \in \mathbb{R}_+$ and f is a continuous function with subcritical growth satisfying suitable assumptions. Autuori et al. [8] considered the existence and the asymptotic behavior of nonnegative solutions of (1.4) for the degenerate case.

To the best of our knowledge, there are few papers in the literature on fractional Kirchhoff equations in $\mathbb{R}^N$. Recently, Ambrosio and Isernia [11] considered the fractional Kirchhoff problem

$$\left(a + b\int\limits_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\,\mathrm{d}x\right)(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N,$$

where f is an odd subcritical nonlinearity satisfying the well-known Berestycki and Lions [12] assumptions. By minimax arguments, the authors established a multiplicity result in the radial space $H^{\alpha}_{rad}(\mathbb{R}^N)$ when the parameter $b > 0$ is sufficiently small.

Liu et al. [7] ensure the existence of positive ground state solutions to the following fractional Kirchhoff equation with the Berestycki–Lions type conditions of critical type

$$\left(a + b \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, \mathrm{d}x \right) (-\Delta)^s u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $u \in H^s(\mathbb{R}^N)$, $u > 0$, a, b are positive constants and $N > 2s$.

Motivated by some of these works de Albuquerque et al. [5], studied the existence of bound and ground state solutions for fractional Kirchhoff equations of the form

$$\left(a + b[u]^2_{1/2} \right) (-\Delta)^{1/2} u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}, \tag{1.5}$$

where $a > 0$, $b \geq 0$, $(-\Delta)^{1/2}$ denotes the *square root of the Laplacian* and the term

$$[u]_{1/2} = \left(\int\limits_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, \mathrm{d}x\mathrm{d}y \right)^{1/2}$$

is the so-called *Gagliardo semi-norm* of the function u, V is a bounded potential which may change the sign and the nonlinear term $f(x,u)$ has the critical exponential growth in the sense of Trudinger–Moser, generalizing the results in [13] by de Souza and Araújo, who address the problem of the fractional Schrödinger equations. We emphasize that the results in [13] were also improved by Barboza et al. [14], once, who treated a problem with a more general non-local operator. More specifically, they studied the following integrodifferential Schrödinger equation

$$\mathcal{L}_K u + V(x)u = f(x,u) \, \text{in} \, \mathbb{R}^N, \tag{1.6}$$

when V is a nonnegative and bounded potential, and the nonlinear term $f(x,u)$ has critical exponential growth with respect to the Trudinger–Moser inequality. This problem is a version of (1.5) in case $a = 1$ and $b = 0$ but generalizing the operator.

We highlight that, when $\mathcal{L}_K$ is the fractional Laplacian operator, (1.6) had been studied by other authors under many different assumptions on the potential $V(x)$ and nonlinearity $f(x,u)$. Almost all works and therein references considered nonlinearities involving polynomial growth of subcritical type in terms of the Sobolev embedding when $N > 2s$. In the borderline case $N = 2s$, this is, $N = 1$ and $s = 1/2$, Sobolev embedding states that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, +\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuous embedded in $L^\infty(\mathbb{R})$; for details see [1,15]. In this case, the maximal growth which allows us to treat this problem type variationally in $H^{1/2}(\mathbb{R})$ is motivated by the Trudinger–Moser inequality proved by Ozawa [15] and improved by Iula [16], Kozono et al. [17] and Takahashi [18]. Precisely, by combining some of the results contained in previous studies, it is established

$$\sup_{\substack{u \in H^{1/2}(\mathbb{R}) \\ \|u\|_{1/2} \leq 1}} \int\limits_{\mathbb{R}} (e^{\alpha u^2} - 1) \, \mathrm{d}x < \infty \, ; \alpha \in [0, \pi], \tag{1.7}$$

where

$$\|u\|_{1/2} := \left(\frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, \mathrm{d}x \, \mathrm{d}y + \|u\|_2^2 \right)^{1/2}.$$

Moreover, it holds

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C(\alpha)\|u\|_2^2, \quad \text{for all} \quad 0 < \alpha \leq \pi, \tag{1.8}$$

for details see [15, Theorem 1] and [18, Proposition 1.1].

Therefore, in order to deal with this class of problem in a variational approach in $H^{1/2}(\mathbb{R})$, the maximal growth on the nonlinearity $f(x,u)$ is given by $e^{\pi u^2}$ when $|u| \to +\infty$ (see also the pioneering works [19,20]).

Recently, the borderline case was approached by Miyagaki and Pucci [21]; more specifically, in this paper, they deal with the existence of solutions for a class of nonlinear elliptic equations, involving a nonlocal Kirchhoff term and possibly Trudinger–Moser critical growth nonlinearities in an unbounded domain. In this context, in order to overcome the lack of compactness of the associated energy functional, it is usually assumed a hypothesis under the nonlinearity which helps to estimate the minimax level.

In general, when we consider a nonlinearity $f(t)$ with critical exponential growth, there exist two kinds of assumptions for this purpose. The first one is due to D. M Cao (see [22]) and is widely used in literature. See, for instance, [5,13] and reference in therein. Precisely, it is supposed that there are constants $p > 2$ and $C_p > 0$ such that

$$f(t) \geq C_p t^{p-1}, \quad \text{for all } t \text{ in domain of } f,$$

where C_p is chosen suitably. For this case, it is crucial to show that the embedding constant of solutions space into appropriated Lebesgue space is attained and a suitable version of Lions' Lemma plays a main role to prove it. As far as we know, almost all papers that are concerned with Kirchhoff equations and nonlinearity with exponential critical growth have assumed this type of hypothesis. In the second case, it is considered a heavy dependence on the asymptotic behavior of $h(t) = f(t)t/e^{\alpha_0 t^2}$ at infinity, as in the pioneer works [23,24]. This asymptotic behavior can appear in different ways. For example, in [24] the authors considered an equation involving the Laplacian operator with f satisfying (among other conditions) that $\lim_{t\to\infty} h(t) = C(r)$, where r is the radius of the largest open ball in the domain. In [25], it was assumed that $\lim_{t\to\infty} h(t) = \infty$ for a equation with the 1/2-Laplacian operator. In papers that use this approach, the Moser functions are used to estimate the minimax level of functional associated to a problem with an exponential nonlinearity. For other works with this kind of conditions, see [26,27].

Our purpose is to generalize the results in [5], for this, we study (1.5) when the fractional Laplacian operator is replaced by a more general integrodifferential operator $-\mathcal{L}_K$ where $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ is a measurable kernel which satisfies "structural properties" and the Kirchhoff function is more general. We also improve some results in [14,21], in different perspectives. In these papers, the potential V is assumed nonnegative; in [14], the Kichhorff function is equal to 1, and in [21], it is considered a weight to control nonlinearity behavior at infinity to recover the compactness of energy functional associated to problem. Here in this paper, we will assume the two types of hypotheses and neglect the weight in the nonlinearity, which requires different techniques to estimate the minimax level. This fact and assumptions more general hypotheses under the potential V, the Kichhorff function m and the integrodifferential kernel K improve some results in [5,13,14,21], more specifically, we obtain new versions of the results in these papers.

As expected, our main difficulties are related to unbounded domains and nonlinearities with critical growth. These difficulties become harder due to the presence of the general Kirchhoff term, once to control it, we need to make some suitable calculations in order to recover the lack of compactness.

For easy reference, we record problems, assumptions, and the main results.

1.1. A periodic problem

Here we present the periodic problem for a positive bounded potential and a nonlinearity with critical exponential growth. For this matter, initially, we will study the following problem

$$\begin{cases} [-m(\|u\|^2)]\mathcal{L}_K u + V_0(x)u = f_0(x,u) \quad \text{in} \quad \mathbb{R}, \\ u \in X_0 \quad \text{and} \quad u \geq 0, \end{cases} \tag{1.9}$$

where $-\mathcal{L}_K u$ is given in (1.2) and we assume that $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ is a measurable function with the properties

(K_1) $\gamma K \in L^1(\mathbb{R})$, where $\gamma(x) = min\{1, |x|^2\}$;
(K_2) there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-2}$, for all $x \in \mathbb{R} \setminus \{0\}$;
(K_3) $K(x) = K(-x)$, $\forall x \in \mathbb{R} \setminus \{0\}$.

These hypotheses allow us to obtain a wide range of nonlocal integrodifferential operators of the fractional, that order is different from $s = 1/2$. For example, K given by

$$K(x) = \begin{cases} C_1|x|^{-r} \text{ with } 1 < r \leq 2 \text{ if } |x| \geq 1; \\ C_2|x|^{-q} \text{ with } 2 \leq q < 3 \text{ if } |x| \leq 1, \end{cases}$$

satisfies (K_1)–(K_3). Moreover, we can chose r and q in the intervals $(1, 2]$ and $[2, 3)$, respectively, and if $C_1 \neq C_2$, K is not continuous in $\mathbb{R} \setminus \{0\}$. For more details, see [14].

Moreover, we assume $m : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function that satisfies

(m_1) there exists $\sigma \in [1, +\infty)$ such that $tm(t) \leq \sigma M(t) \, \forall t \in \mathbb{R}_+$ where $M(t) = \int_0^t m(\tau)\,\mathrm{d}\tau$;
(m_2) for all $\tau > 0$ there exists $\eta(\tau) = \eta > 0$ such that $m(t) \geq \eta \; \forall t \geq \tau$;
(m_3) $t \mapsto \sigma M(t) - m(t)t$ is nondecreasing in $\mathbb{R}_+$.

A typical example for m is given by $m(t) = m_0 + bt^{\sigma-1}$ with $m_0, b \geq 0$, $m_0 + b > 0$ and $1 \leq \sigma < +\infty$. Note that (m_1) implies, in particular, that

$$M(t) \geq M(1)t^\sigma \text{ for all } t \in [0, 1], \tag{1.10}$$

and

$$M(t) \leq \frac{M(t_0)}{t_0^\sigma} t^\sigma \text{ for all } t \geq t_0 \quad \text{for all} \quad t_0 > 0. \tag{1.11}$$

Moreover, (m_2) yields that $M(t) > 0$ for all $t > 0$ as in the Kirchhoff model.

We suppose that the function $V_0 : \mathbb{R} \to \mathbb{R}$ is a continuous 1−periodic function satisfying:

$(V_{0,1})$ there exists a positive constant υ_0 such that $V_0(x) \geq -\upsilon_0$ for all $x \in \mathbb{R}$;
$(V_{0,2})$ The infimum

$$\xi_0 := \inf_{\substack{u \in X_0 \\ \|u\|_2 = 1}} \left(\int_{\mathbb{R}^2} [u(x) - u(y)]^2 K(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V_0(x)u^2(x)\,\mathrm{d}x \right)$$

is positive.

Moreover, we consider $f_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous 1−periodic function in x, which has critical exponential growth in s, that is,

$$\lim_{|s| \to +\infty} f_0(x,s)e^{-\alpha s^2} = \begin{cases} 0, & \text{for all } \alpha > \pi, \\ +\infty, & \text{for all } \alpha < \pi, \end{cases}$$

uniformly in $x \in \mathbb{R}$.

We call the attention that this notion of criticality is driven by (1.7) and it has been used in several papers involving exponential growth, see for instance [13,28]. Since we are interested in the existence of nonnegative solutions, we set $f_0(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$.

We also assume that the nonlinearity $f_0(x,u)$ satisfies the conditions

$(f_{0,1})$ $0 \leq \lim_{t\to 0} \dfrac{f_0(x,t)}{t^{2\sigma-1}} < M(1)$ uniformly in $x \in \mathbb{R}$;

$(f_{0,2})$ there exists a constant $\theta > 2\sigma$ such that

$$0 < \theta F_0(x,s) := \theta \int_0^s f_0(x,t)\,\mathrm{d}t \leq s f_0(x,s) \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

$(f_{0,3})$ for each fixed $x \in \mathbb{R}$, the function $f_0(x,s)/s^{2\sigma-1}$ is increasing with respect to $s \in \mathbb{R}$;

$(f_{0,4})$ there are constants $p > 2\sigma$ and $C_p > 0$ (to be shown precisely later) such that

$$f_0(x,s) \geq C_p s^{p-1}, \quad \text{for all} \quad (x,s) \in \mathbb{R} \times [0,+\infty).$$

Here, we define

$$X_0 := \left\{ u \in L^2(\mathbb{R});\ (u(x)-u(y))K(x,y)^{\frac{1}{2}} \in L^2(\mathbb{R}^2) \right\}$$

which is endowed with norm

$$\|u\|_{X_0} = \left([u]^2_{1/2,K} + \int_{\mathbb{R}} V_0(x)u(x)^2\,\mathrm{d}x \right)^{1/2}$$

where

$$[u]^2_{1/2,K} = \int_{\mathbb{R}^2} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

We would like to point out that space X_0 has suitable properties which give to problem (1.9) a variational framework. More specifically, in light of results proved in [14], even with more general hypotheses under the potential, X_0 is uniformly convex Banach space and therefore is a reflexive space. Moreover, $C_0^\infty(\mathbb{R})$ is dense in X_0.

Throughout this paper, we say that $u \in X_0$ is a weak solution for (1.9) if the following equality holds:

$$m(\|u\|^2_{X_0})\langle u,v\rangle_0 = \int_{\mathbb{R}} f_0(x,u)v\,\mathrm{d}x,$$

for all $v \in X_0$ with

$$\langle u,v\rangle_0 := \int_{\mathbb{R}^2} (u(x)-u(y))(v(x)-v(y))K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V_0(x)uv\,\mathrm{d}x.$$

The main results of this subsection are presented in the following. The first result of the paper involves a classical assumption under the nonlinearity (see assumption $(f_{0,4})$ above) which was first introduced by Cao [22].

Theorem 1.1. *Assume that* (m_1)*–*(m_3)*,* (K_1)*–*(K_3)*,* $(V_{0,1})$*,*$(V_{0,2})$ *and* $(f_{0,1})$*–*$(f_{0,4})$ *hold. Then* (1.9) *has a nonnegative and nontrivial solution.*

The second theorem of the paper also deals with the critical growth nonlinearity but involves a little weaker assumption addressed by Adimurthi and Miyagaki [24,29] instead of Cao assumption $(f_{0,4})$. We assume that

$(\tilde{f}_{0,4})$ $\lim_{t\to\infty} \dfrac{f_0(x,t)t}{\exp(\pi t^2)} = +\infty$, uniformly in x.

In order to establish an existence result of a solution to problem (1.9), without the hypothesis $(f_{0,4})$, in addition to the hypothesis $(\tilde{f}_{0,4})$ we need the following hypothesis additional under kernel

(K_4) There are $x_0, r, \lambda_0 \in \mathbb{R}$ such that if $x \in (x_0 - r, x_0 + r)$ then $K(x) \leq \dfrac{1}{\lambda_0}|x|^{-2}$.

The second result says that

Theorem 1.2. *Assume that (m_1)–(m_3), (K_1)-(K_1), $(V_{0,1})$,$(V_{0,2})$, $(f_{0,1})$–$(f_{0,3})$ and $(\tilde{f}_{0,4})$ hold. Then (1.9) has a nonnegative and nontrivial solution.*

It is important to stress that it is possible to obtain different solutions for (1.9) which depend on the kind of hypothesis assumed to estimate the minimax level of functional associated to this problem. For details, see Remark 5.4.

1.2. A nonperiodic problem

The second problem that we will study in this paper is the following,

$$\begin{cases} [-m(\|u\|^2)]\mathcal{L}_{\mathcal{K}} u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \\ u \in X_1 \quad \text{and} \quad u \geq 0. \end{cases} \tag{1.12}$$

Note that the terms $V(x)$ and $f(x,u)$ are not necessarily periodic anymore. Here we deal with the class of asymptotically periodic functions that were introduced by Lins and Silva [30]. Precisely, we introduce the set

$$\mathcal{F} := \{g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) : |\{x \in \mathbb{R} : |g(x)| \geq \varepsilon\}| < \infty, \text{ for all } \varepsilon > 0\},$$

where $|A|$ denotes the Lebesgue measure of a set A. In order to deal with the difficulties imposed by the lack of periodicity, we require assumptions that compare the periodic terms with the asymptotically periodic terms. On the potential $V(x)$, we assume that:

- (v_1) $V_0 - V \in \mathcal{F}$ and $V_0(x) \geq V(x) \geq -v_0$, for all $x \in \mathbb{R}$;
- (v_2) The infimum

$$\xi_1 := \inf_{\substack{u \in X_1 \\ \|u\|_2 = 1}} \left(\int_{\mathbb{R}^2} |u(x) - u(y)|^2 K(x,y)\, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2\, \mathrm{d}x \right)$$

  is positive.

We assume that the nonlinearity $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that have critical exponential growth, $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$ and satisfies the following conditions

- (f_1) $f(x,s) \geq f_0(x,s)$ for all $(x,s) \in \mathbb{R} \times [0,+\infty)$, and for all $\varepsilon > 0$, there exists $\nu > 0$ such that for $s \geq 0$ and $|x| \geq \nu$,

$$|f(x,s) - f_0(x,s)| \leq \varepsilon e^{\alpha_0 s^2};$$

- (f_2) $0 \leq \lim_{t \to 0} \dfrac{f(x,t)}{t^{2\sigma-1}} < M(1)$ uniformly in $x \in \mathbb{R}$;
- (f_3) there exists a constant $\tilde{\theta} \geq \theta > 2\sigma$ such that

$$0 < \tilde{\theta} F(x,s) := \tilde{\theta} \int_0^s f(x,t)\, \mathrm{d}t \leq s f(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

- (f_4) for each fixed $x \in \mathbb{R}$, the function $f(x,s)/s^{2\sigma-1}$ is increasing with respect to $s \in \mathbb{R}$;
- (f_5) at least one of the nonnegative continuous functions $V_0(x) - V(x)$ and $f(x,s) - f_0(x,s)$ is positive on a set of positive measure.

In order to define the weak solution to problem (1.12), as in problem (1.9), we consider

$$X_1 := \left\{ u \in L^2(\mathbb{R});\ (u(x) - u(y))K(x,y)^{\frac{1}{2}} \in L^2(\mathbb{R}^2) \right\}$$

which is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_1 := \int\limits_{\mathbb{R}^2} (u(x) - u(y))(v(x) - v(y))K(x,y)\, \mathrm{d}x\, \mathrm{d}y + \int\limits_{\mathbb{R}} V(x)uv\, \mathrm{d}x$$

and the correspondent induced norm $\|u\|^2_{X_1} = \langle u, u\rangle$.

We would like to point out that space X_1 also has suitable properties which give to problem (1.12) a variational framework.

Throughout this paper, we say that $u \in X_1$ is a weak solution for (1.12) if the following equality holds:

$$m(\|u\|^2_{X_1})\langle u, v\rangle_1 = \int\limits_{\mathbb{R}} f(x, u)v\, \mathrm{d}x,$$

for all $v \in X_1$.

Considering the functions V_0, f_0 and m, K as in Theorems 1.1 and 1.2, the main result of this subsection is the following.

Theorem 1.3. *Assume that (v_1),(v_2) and (f_1)–(f_5) hold. Then (1.12) has a nonnegative and nontrivial solution.*

Remark 1.4. As mentioned earlier, the results of this paper were motivated by the works [5,13,14,21]. Particularly, our Theorems 1.1–1.3 are generalization of Theorems 1.1 and 1.2 of [5], in the sense of the operator, the Kirchhoff term and the nonlinearity. Consequently, we improve the results in [13]. Moreover, Theorems 1.1–1.2 are versions of Theorem 1.1 in [14] for a integrofferential Kirchhoff equation and we consider a nonlinearity with critical growth and two kinds of assumptions. We also improve some results in [21], because we assume more general hypotheses in order overcome the loss of compactness.

The outline of this paper is as follows: Sect. 2 contains some preliminary results necessary to obtain suitable properties for the solutions spaces. In Sects. 3–5, we approach results related to the periodic problem. More specifically, in Sect. 3, we work with its variational formulation. In Sect. 4, we estimate the minimax level of associated functional, and in Sect. 5 deal with the proof of the main results. Lastly, in Sect. 6, we are concerned with the results related to the nonperiodic problem for the proof of the main theorem.

2. Some preliminary results

We recall the definition of the fractional Sobolev space

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int\limits_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2}\, \mathrm{d}x\mathrm{d}y < \infty \right\},$$

which is endowed with the natural norm

$$\|u\|_{1/2} = \left([u]^2_{1/2} + \int\limits_{\mathbb{R}} u^2\, \mathrm{d}x \right)^{1/2}.$$

Lemma 2.1. *Assume the conditions* $(V_{0,1})$–$(V_{0,2})$ *or* (v_1)–(v_2) *and* (K_1)–(K_3)*. The space* X_i *is embedded in* $H^{1/2}(\mathbb{R})$ *and there exists* $C(\lambda, \xi_i) > 0$ *such that*

$$\|u\|_{1/2} \leq \sqrt{C(\lambda, \xi_i)}\|u\|_{X_i}, \ \forall \ u \in X_i \quad \textit{with} \quad i = 0, 1.$$

Proof. By (K_2), given $u \in X_i$ we have

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^2} \,\mathrm{d}x \,\mathrm{d}y \leq \frac{1}{\lambda} \int_{\mathbb{R}^2} (u(x) - u(y))^2 K(x, y) \,\mathrm{d}x \,\mathrm{d}y. \tag{2.1}$$

Let $u \in X_i$ with $u \neq 0$, then by $(V_{0,2})$ or (v_2) we have

$$\xi_i \leq \left\| \frac{u}{\|u\|_2} \right\|_{X_i}^2 = \frac{\|u\|_{X_i}^2}{\|u\|_2^2}.$$

So

$$\|u\|_2^2 \leq \frac{1}{\xi_i} \|u\|_{X_i}^2, \quad \text{for all } u \in X_i. \tag{2.2}$$

From the estimates (2.1) and (2.2) we obtain

$$\|u\|_{1/2}^2 \leq C(\lambda, \xi_i)\|u\|_{X_i}^2, \quad \text{for all } u \in X_i,$$

where $C(\lambda, \xi_i) = \dfrac{1}{\xi_i} + \dfrac{1}{\lambda}$. □

Corollary 2.2. *Let* $q \in [2, +\infty)$*, then the embedding* $X_i \hookrightarrow L^q(\mathbb{R})$ *is continuous with* $i = 0, 1$*. Moreover, if* $q \in [1, 2]$ *the embedding* $X_i \hookrightarrow L^q_{loc}(\mathbb{R})$ *is compact with* $i = 0, 1$*.*

As a consequence of Corollary 2.2, the norms $\| \cdot \|_{X_0}$ and $\| \cdot \|_{X_1}$ are equivalent.

Since we have the above results hold, following the same ideas as in [14, Lemma 2.4], we can obtain that $C_0^\infty(\mathbb{R})$ is dense in X_i for $i = 0, 1$.

Now we show a suitable version of Trudinger–Moser inequality for X_i with $i = 0, 1$.

Lemma 2.3. *Assume* (K_1)–(K_3) *and* $(V_{0,1})$–$(V_{0,2})$ *or* (v_1)–(v_2)*, then there exists* ω *such that if* $0 < \alpha \leq \omega$*, then one has a constant* $C = C(\omega) > 0$*, such that*

$$\sup_{\substack{u \in X_i \\ \|u\|_{X_i} \leq 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) \,\mathrm{d}x \leq C(\omega) \quad \textit{for} \quad i = 0, 1. \tag{2.3}$$

Moreover, for any $\alpha > 0$ *and* $u \in X_i$*, for* $i = 0, 1$*, we have*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) \,\mathrm{d}x < \infty.$$

Proof. First of all, fix $u \in X_i$ with $\|u\|_{X_i} \leq 1$. Now, consider $C(\lambda, \xi_i)$, given in Lemma 2.1, and define

$$v = \frac{u}{\sqrt{C(\lambda, \xi_i)}},$$

consequently, v is in X_i and $H^{1/2}(\mathbb{R})$. So, using Lemma 2.1

$$\|v\|_{1/2} = \frac{\|u\|_{1/2}}{\sqrt{C(\lambda, \xi_i)}} \leq \|u\|_{X_i} \leq 1. \tag{2.4}$$

Applying (2.4) in (1.8)

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1) \,\mathrm{d}x \leq C(\alpha)\|v\|_2^2, \quad \text{for all} \quad 0 < \alpha \leq \pi.$$

Set $0 < \alpha \leq \omega_i = \dfrac{\pi}{C(\lambda, \xi_i)}$ and $\tilde{\alpha} = \alpha C(\lambda, \xi_i)$, and notice that $0 < \tilde{\alpha} \leq \pi$. By Corollary 2.2 and (2.4)

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x = \int_{\mathbb{R}} (e^{\tilde{\alpha} v^2} - 1)\,\mathrm{d}x \leq C(\tilde{\alpha})\|v\|_2^2 \leq C(\alpha, \lambda)\|v\|_{X_i}^2 \leq C(\alpha, \lambda).$$

So we obtain

$$\sup_{\substack{u \in X_i \\ \|u\|_{X_i} \leq 1}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C(\omega_i).$$

Choosing $\omega = \min\{\omega_0, \omega_1\}$, we obtain (2.3).

Now let us take $\alpha > 0, u \in X_i$ and $\varepsilon > 0$. There exists $\phi \in C_0^\infty(\mathbb{R})$ such that $\|u - \phi\|_{X_i} < \varepsilon$. Observe that

$$e^{\alpha |u|^2} - 1 \leq e^{2\alpha |u-\phi|^2} e^{2\alpha |\phi|^2} - 1 \leq \frac{1}{2}\left(e^{4\alpha |u-\phi|^2} - 1\right) + \left(\frac{1}{2} e^{4\alpha |\phi|^2} - 1\right).$$

Then,

$$\int_{\mathbb{R}} (e^{\alpha |u|^2} - 1)\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha \|u-\phi\|_{X_i}^2} - 1)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha |\phi|^2} - 1)\mathrm{d}x. \tag{2.5}$$

Choosing $\varepsilon > 0$ such that $4\alpha\varepsilon^2 < \omega$, we have $4\alpha\|u - \phi\|_{X_i}^2 < \omega$. By (2.5) and (2.3), we conclude that

$$\int_{\mathbb{R}} (e^{\alpha |u|^2} - 1)\mathrm{d}x \leq \frac{C}{2} + \frac{1}{2}\int_{Supp(\phi)} (e^{4\alpha |\phi|^2} - 1)\mathrm{d}x < \infty.$$

□

Lemma 2.4. *If $\alpha > 0$, $q > 2$, $v \in X_i$ and $\|v\|_{X_i} \leq D$ with $\alpha D^2 < \omega$, then there exists $C = C(\alpha, D, q) > 0$, such that*

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x \leq C\|v\|_{X_i}^q \quad \textit{for} \quad i = 0, 1.$$

Proof. Consider $r > 1$ sufficiently close to 1 such that $\alpha r D^2 < \omega$ and $r'q \geq 2$, where $r' = r/(r-1)$. Using Hölder's inequality, we have

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x \leq \left(\int_{\mathbb{R}} (e^{\alpha v^2} - 1)^r \mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q. \tag{2.6}$$

Notice that given $\beta > r$ there exists $C = C(\beta) > 0$ such that for all $s \in \mathbb{R}$,

$$(e^{\alpha s^2} - 1)^r \leq C(e^{\alpha\beta s^2} - 1). \tag{2.7}$$

Hence, from (2.6) and (2.7) we get

$$\begin{aligned}\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x &\leq C\left(\int_{\mathbb{R}} (e^{\alpha\beta v^2} - 1)\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q \\ &\leq C\left(\int_{\mathbb{R}} \left[e^{\alpha\beta D^2\left(\frac{v}{\|v\|_{X_i}}\right)^2} - 1\right] \mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q.\end{aligned}$$

By choosing $\beta > r$ close to r, in such way that $\alpha\beta D^2 < \omega$, it follows from (2.3) and the continuous embedding $X_i \hookrightarrow L^{r'q}(\mathbb{R})$ that

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \, \mathrm{d}x \leq C\|v\|_{X_i}^q,$$

which proves the lemma. □

3. A functional setting for the periodic problem

In order to use a variational framework considering the space X_0, we assume suitable conditions such that weak solutions of (1.9) become critical points of the Euler functional $I_0 : X_0 \to \mathbb{R}$ defined by

$$I_0(u) = \frac{1}{2}M(\|u\|_{X_0}^2) - \int_{\mathbb{R}} F_0(x,u)\,\mathrm{d}x \tag{3.1}$$

where $F_0(x,t) = \int_0^t f_0(x,\tau)\,\mathrm{d}\tau$. Notice that by the condition $(f_{0,1})$ and the fact that $f_0(x,s)$ has critical exponential growth, for each $\alpha > \pi$, $q > 2$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$f_0(x,t) \leq (M(1)-\varepsilon)t^{2\sigma-1} + C_\varepsilon(e^{\alpha t^2}-1)|t|^{q-1}, \quad \text{for all } (x,t) \in \mathbb{R}^2, \tag{3.2}$$

which implies that

$$F_0(x,t) \leq \frac{(M(1)-\varepsilon)}{2}|t|^{2\sigma} + C_\varepsilon(e^{\alpha t^2}-1)|t|^{q}, \quad \text{for all } (x,t) \in \mathbb{R}^2. \tag{3.3}$$

By using the above estimate jointly with the continuous embedding $X_0 \hookrightarrow L^q(\mathbb{R})$, we can conclude that I_0 is well defined. Moreover, using standard arguments we can check that $I_0 \in C^1(X_0,\mathbb{R})$ with the derivative given by

$$I_0'(u)v = m(\|u\|_{X_0}^2)\langle u,v\rangle_0 - \int_{\mathbb{R}} f_0(x,u)v\,\mathrm{d}x, \quad \forall v \in X_0.$$

Thus, critical points of I_0 are weak solutions of problem (1.9) and conversely.

3.1. The geometric condition

Next using the hypotheses $(f_{0,1})$ and $(f_{0,2})$, we prove some facts about the geometric structure of I_0 required by the minimax procedure.

Lemma 3.1. *There exist $\mu > 0$ and $\varrho > 0$ such that $I_0(u) \geq \mu$, provided that $\|u\|_{X_0} = \varrho$.*

Proof. We can use (3.3) to get

$$\int_{\mathbb{R}} F_0(x,u)\,\mathrm{d}x \leq \frac{(M(1)-\varepsilon)}{2}\int_{\mathbb{R}} |u|^{2\sigma}\,\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} (e^{\alpha u^2}-1)|u|^q\,\mathrm{d}x.$$

By continuous embedding $X_0 \hookrightarrow L^q(\mathbb{R})$ for all $q \in [2,+\infty)$ and Hölder inequality, for $\|u\|_{X_0} \leq \rho_0 < 1$ we obtain

$$\int_{\mathbb{R}} F_0(x,u)\,\mathrm{d}x \leq \frac{(M(1)-\varepsilon)}{2}\|u\|_{X_0}^{2\sigma} + C_\varepsilon\|u\|_{X_0}^q \left[\int_{\mathbb{R}} (e^{2\alpha\rho_0^2 u^2/\|u\|_{X_0}^2}-1)\,\mathrm{d}x\right]^{1/2}. \tag{3.4}$$

From (1.10) and (3.4), we have

$$I_0(u) \geq \frac{M(1)}{2}\|u\|_{X_0}^{2\sigma} - \frac{(M(1)-\varepsilon)}{2}\|u\|_{X_0}^{2\sigma} - C_\varepsilon\|u\|_{X_0}^{q}\left[\int_{\mathbb{R}} (e^{2\alpha\rho_0^2 u^2/\|u\|_{X_0}^2} - 1)\,\mathrm{d}x\right]^{1/2}.$$

If $\rho_0 < \min\{1, \sqrt{\omega/2\alpha}\}$, we obtain

$$I_0(u) \geq \frac{\varepsilon}{2}\|u\|_{X_0}^{2\sigma} - C_2\|u\|_{X_0}^{q}.$$

Choosing $q > 2\sigma$ and $\varepsilon > 0$ small enough, we may choose $0 < \varrho < \rho_0$ such that

$$\frac{\varepsilon}{2}\varrho^{2\sigma} - C_2\varrho^q = \mu > 0. \tag{3.5}$$

□

Lemma 3.2. *There exists $e \in X_0$ with $\|e\|_{X_0} > \varrho$ such that $I_0(e) < 0$.*

Proof. Let $u \in C_0^\infty(\mathbb{R})\backslash\{0\}$ with support Ω. By $(f_{0,2})$ there exist $C_1, C_2 > 0$ such that

$$F_0(x,u) \geq C_1|u|^\theta - C_2 \quad \text{for all} \quad x \in \Omega.$$

Consequently, by (1.11), for $t > 0$ such that $\|tu\|_{X_0}^2 \geq 1$, we have the following estimate

$$I_0(tu) \leq \frac{t^2 M(1)}{2}\|u\|_{X_0}^{2\sigma} - C_1 t^\theta \int_\Omega |u|^\theta\, dx + C_2 \int_\Omega dx.$$

Since $\theta > 2\sigma$, we obtain $I_0(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t large enough, the proof is finished.

□

3.2. Palais–Smale sequence

By using the mountain-pass theorem without the (PS) condition (see [31]), there exists a sequence (u_k) in X_0 satisfying

$$I_0(u_k) \to c_0 \quad \text{and} \quad I_0'(u_k) \to 0, \tag{3.6}$$

where

$$c_0 = \inf_{g\in\Gamma}\max_{t\in[0,1]} I_0(g(t))$$

and $\Gamma = \{g \in C([0,1], X_0) : g(0) = 0 \text{ and } g(1) = e\}$.

Lemma 3.3. *Suppose that $(f_{0,1})$ and $(f_{0,2})$ hold. Then, the sequence (u_k) is bounded in X_0.*

Proof. Using well-known arguments, it is not difficult to check that (u_k) is a bounded sequence in X_0. Indeed, by (m_1) and $(f_{0,2})$ we have

$$I_0(u_k) - \frac{1}{\theta}I_0'(u_k)u_k \geq \left(\frac{1}{2} - \frac{\sigma}{\theta}\right) M(\|u_k\|_{X_0}^2).$$

Now we have to consider two cases. Either $\inf_{k\in\mathbb{N}}\|u_k\|_{X_0} = d > 0$ or $\inf_{k\in\mathbb{N}}\|u_k\|_{X_0} = 0$. If $\inf_{k\in\mathbb{N}}\|u_k\|_{X_0} = d > 0$, we may assume that d does not depend on Palais–Smale sequence considered, from (m_1) and (m_2), with $\tau = d^2$, there exists $\eta_0 > 0$ such that

$$M(\|u_k\|_{X_0}^2) \geq \eta_0\|u_k\|_{X_0}^2 \quad \text{for all} \quad k \in \mathbb{N}.$$

So we have

$$I_0(u_k) - \frac{1}{\theta}I_0'(u_k)u_k \geq \eta_0\left(\frac{1}{2} - \frac{\sigma}{\theta}\right)\|u_k\|_{X_0}^2 \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.7}$$

By (3.6) and (3.7), there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, it holds

$$\left(\frac{1}{2} - \frac{\sigma}{\theta}\right) \eta_0 \|u_k\|_{X_0}^2 \leq C + \|u_k\|_{X_0}.$$

Since $\theta > 2\sigma$, this implies that $\|u_k\|_{X_0} \leq C_1$. If $\inf_{k\in\mathbb{N}} \|u_k\|_{X_0} = 0$, when 0 is an accumulation point for real sequence $(\|u_k\|_{X_0})$, we may conclude that $u_0 = 0$, so $0 = I_0(u_0) = c > 0$, which is impossible. Consequently, 0 is an isolated point of sequence $(\|u_k\|_{X_0})$, then there exists a subsequence, denoted also by $(\|u_k\|_{X_0})$, such that $\inf_{k\in\mathbb{N}} \|u_k\|_{X_0} = d > 0$ and we may proceed as before. Thus, in both cases, we have that this sequence is bounded. □

4. Minimax level for the periodic problem

As already mentioned, we highlight that the main difficulty in our work is the lack of compactness typical for elliptic problems in unbounded domains with nonlinearities with critical growth. To recover this, we will make use of assumptions $(f_{0,4})$ or $(\tilde{f}_{0,4})$ together with (K_1) to control the minimax level in a suitable range where we are able to recover some compactness. For this purpose, in the first case, we need a version of Lions's lemma. In the second case, let us consider the Moser's functions sequence supported in a ball with an appropriated radius, which depends on (K_1). Besides this, we lead with a general Kirchhoff function $M(t)$, which become this difficulty harder. To overcome this obstacle, we need to make some estimates depending on this term.

For this, observe that, by definition, $M(0) = 0$, and since m is a continuous function, it follows that M is continuous. Thus, by theses facts, together with Lemma 3.1, there exists $0 < \varrho' < \min\{1, \rho\}$ such that

$$I_0(t) \leq \frac{\mu}{2} \quad \text{for all} \quad 0 \leq t < \rho'. \tag{4.1}$$

By (m_1),

$$M(t) \leq C_\sigma t^\sigma \quad \text{for all} \quad t \geq \rho', \quad \text{where} \quad C_\sigma = \frac{M(\rho')}{(\rho')^\sigma}. \tag{4.2}$$

Now, observe that as a consequence of Lemmas 3.1 and 3.2, the minimax level

$$c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_0(g(t))$$

is positive.

Moreover, we may compare the minimax levels relying on $(f_{0,4})$ or $(\tilde{f}_{0,4})$ and (K_1) and show that they are real numbers different. So we may obtain distinct solutions for Problem (1.9).

4.1. Minimax estimative of Theorem 1.1

In order to provide an estimate to the minimax level of the functional associated to (1.9), in [14], the authors proved a version of a Lions's result (see Lions [32]) for critical growth in $\mathbb{R}$, more specifically in [14, Lemma 3.1], for a bounded sequence in a suitable space they guarantee that if

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |u_n(x)|^2 \,\mathrm{d}x = 0, \tag{4.3}$$

for some $R > 0$, then $u_n \to 0$ strongly in $L^q(\mathbb{R})$ for $2 < q < \infty$. This result is also available for bounded sequence in X_0.

Now we consider the embedding constant, given by

$$S_p := \inf_{\substack{u\in X_0\\ \|u\|_p=1}} \left(\int_{\mathbb{R}^2} |u(x)-u(y)|^2 K(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V_0(x)u^2(x)\,\mathrm{d}x \right)^{1/2},$$

which is achieved by a nonnegative function u_p in X_0. For more details see [14, Lemma 3.2]. From this, we may estimate the level.

Proposition 4.1. *Suppose that* $(f_{0,4})$ *holds. Then*

$$\mu \le c_0 < \frac{\eta_0(\theta-2\sigma)w}{2\theta\pi}.$$

Proof. Let $u_p \in X_0$ such that $\|u_p\|^2_{X_0} = S_p$ and $\|u_p\|_p = 1$. Then

$$c_0 \le \max_{t\ge 0} I_0(tu_p) = \max_{t\ge 0}\left\{ \frac{1}{2}M(t^2\|u_p\|^2_{X_0}) - \int_{\mathbb{R}} F_0(x,tu_p)\mathrm{d}x \right\}.$$

When

$$t < \frac{\sqrt{\rho'}}{S_p} = \rho_p, \tag{4.4}$$

we have $\|tu_p\|^2_{X_0} < \rho'$. So, using the estimate (4.1), we get

$$I_0(tu_p) \le \frac{\mu}{2}.$$

By (4.2) and $(f_{0,4})$, we obtain

$$\begin{aligned} c_0 &\le \max_{t\ge\rho_p}\left\{ \frac{1}{2}M(t^2\|u_p\|^2_{X_0}) - \int_{\mathbb{R}} F_0(x,tu_p)\mathrm{d}x \right\} \\ &\le \max_{t\ge\rho_p}\left\{ \frac{1}{2}C_\sigma t^{2\sigma} S_p^{2\sigma} - \frac{C_p}{p}t^p \right\}. \end{aligned}$$

Observe that $t_0 = \left(\frac{\sigma C_\sigma S_p^{2\sigma}}{C_p}\right)^{\frac{1}{p-2\sigma}}$, where the maximum is achieved, satisfies the estimate (4.4) if $C_p < \dfrac{\sigma C_\sigma S_p^p}{(\rho')^{\frac{p-2\sigma}{2}}}$, what occurs as $\rho' < C_\sigma^{\frac{1}{\sigma}}\left(\dfrac{pw\eta_0(\theta-2\sigma)}{(p-2\sigma)\theta\pi}\right)^{\frac{1}{\sigma}}$. Consequently,

$$\begin{aligned} c_0 &\le \frac{1}{2}C_\sigma S_p^{2\sigma}\left[\frac{\sigma C_\sigma S_p^{2\sigma}}{C_p}\right]^{\frac{2\sigma}{p-2\sigma}} - \frac{C_p}{p}\left[\frac{\sigma C_\sigma S_p^{2\sigma}}{C_p}\right]^{\frac{p}{p-2\sigma}} \\ &= \frac{C_\sigma^{\frac{p}{p-2\sigma}} S_p^{\frac{2p\sigma}{p-2\sigma}} \sigma^{\frac{2\sigma}{p-2\sigma}}(p-2\sigma)}{2pC_p^{\frac{2\sigma}{p-2\sigma}}}. \end{aligned}$$

Then, taking

$$C_p \ge C_\sigma^{\frac{p}{2\sigma}} S_p^p \sigma \left[\frac{(p-2\sigma)\theta}{p\eta_0(\theta-2\sigma)w}\right]^{\frac{p-2\sigma}{2\sigma}},$$

we have

$$c_0 < \frac{\eta_0(\theta-2\sigma)w}{2\theta\pi}.$$

□

4.2. Minimax estimative of Theorem 1.2

In this section, in order to estimate the minimax level replacing $(f_{0,4})$ by $(\tilde{f}_{0,4})$, we need the additional hypothesis (K_1) under the kernel.

In order to control the minimax level when we assume a hypothesis kind of $(\tilde{f}_{0,4})$, it is usual to consider the following sequence of nonnegative functions given by

$$v_n(y) = \begin{cases} (\ln n)^{1/2}, & 0 \le |y| < \frac{1}{n}, \\ \frac{\ln \frac{1}{|y|}}{(\ln n)^{1/2}}, & \frac{1}{n} \le |y| \le 1, \\ 0, & |y| \ge 1, \end{cases}$$

well-known as Moser's sequence.

By changing of variable, we obtain the following sequence of functions supported in $(x_0 - r_0, x_0 + r_0)$ given by

$$u_n(x) = \begin{cases} (\ln n)^{1/2}, & 0 \le |x - x_0| < \frac{r_0}{n}, \\ \frac{\ln \frac{r_0}{|x-x_0|}}{(\ln n)^{1/2}}, & \frac{r_0}{n} \le |x - x_0| \le r_0, \\ 0, & |x - x_0| \ge r_0, \end{cases}$$

with $r_0 = \min\left\{ r, \dfrac{\lambda_0}{2\pi}, \dfrac{\lambda_0 \eta_0 (\theta - 2\sigma)\omega}{2\theta C_\sigma \pi^2} \right\}$. Notice that the restriction of u_n to $(x_0 - r_0, x_0 + r_0)$ belongs to $H^{1/2}((x_0 - r_0, x_0 + r_0))$ (see [33]). The following lemma deals with the asymptotic estimate on Moser's sequence.

Lemma 4.2. *Suppose that (K_1) holds, then there exist convergente sequences $\tilde{C}_n$ and δ_n such that*

$$\|u_n\|^2_{X_0} \le 2\pi \tilde{C}_n + \delta_n.$$

Proof. Note that $\|(-\Delta)^{1/4} u_n\|_2^2 = r_0 \|(-\Delta)^{1/4} v_n\|_2^2$, so by Takahashi [18], we have

$$\|(-\Delta)^{1/4} u_n\|_2^2 \le \pi r_0 \left(1 + \frac{1}{C \ln(n)}\right).$$

From (K_1),

$$\frac{1}{2\pi}[u_n]_{1/2,K} \le \frac{1}{\lambda_0}\|(-\Delta)^{1/4} u_n\|_2^2 \le \frac{\pi r_0}{\lambda_0}\left(1 + \frac{1}{C \ln(n)}\right) := \widetilde{C}_n. \tag{4.5}$$

Thus, for n large enough, we have

$$\|u_n\|^2_{X_0} \le 2\pi \widetilde{C}_n + 2V r_0 \left[\int_{-\frac{1}{n}}^{\frac{1}{n}} \ln(n)\, \mathrm{d}x + \frac{1}{\ln(n)} \int_{-1}^{-\frac{1}{n}} (\ln|x|)^2\, \mathrm{d}x \right] + 2V r_0 \left[\frac{1}{\ln(n)} \int_{\frac{1}{n}}^{1} (\ln|x|)^2\, \mathrm{d}x \right],$$

where $V := \max_{x \in \mathbb{R}} V_0(x)$, which implies that $\|u_n\|^2_{X_0} \le 2\pi \widetilde{C}_n + \delta_n$, with

$$\delta_n := 4V r_0 \left(\frac{n - 1 - \ln(n)}{n \ln(n)} \right).$$

Notice that

$$\delta_n \to 0 \quad \text{and} \quad \tilde{C}_n \to \frac{\pi r_0}{\lambda_0} \quad \text{as} \;\; n \to +\infty. \tag{4.6}$$

□

Now we considered $\omega_n = \frac{u_n}{\sqrt{2\pi \tilde{C}_n + \delta_n}}$, so by Lemma 4.2 $\|\omega_n\|_{X_0} \leq 1$, which will help us to estimate the minimax level.

Proposition 4.3. *Suppose that (K_1) and $(\tilde{f}_{0,4})$ are satisfied, then*

$$\mu \leq c_0 < \frac{\pi r_0}{\lambda_0} C_\sigma,$$

where C_σ is given in (4.2).

Proof. By applying Lemma 3.1, we have that $c_0 \geq \mu$. In order to get an upper estimate, it is enough to prove that there exists a function $\omega \in X_0$, $\|\omega\|_{X_0} \leq 1$, such that

$$\max_{t \in [0,1]} I_0(t\omega) < \frac{\pi r_0}{\lambda_0} C_\sigma.$$

Let us argue by contradiction and suppose that for all $n \in \mathbb{N}$ there exists $t_n > 0$ such that

$$I_0(t_n \omega_n) = \max_{t \in [0,+\infty)} I_0(t\omega_n) \geq \frac{\pi r_0}{\lambda_0} C_\sigma. \tag{4.7}$$

Note still, that $t_n \geq \varrho'$ for all $n \in \mathbb{N}$, with ρ' is given in (4.1), because, otherwise, there exists $n_0 \in \mathbb{N}$ such that $t_{n_0} < \varrho' \leq 1$, then

$$\|t_{n_0} \omega_{n_0}\|_{X_0}^2 = t_{n_0}^2 \|\omega_{n_0}\|_{X_0}^2 \leq t_{n_0}^2 < t_{n_0} < \varrho'.$$

Consequently, by (4.1), $I_0(t_{n_0}\omega_{n_0}) < \dfrac{\mu}{2}$. On another hand, as $c_0 \geq \mu$, we have $I_0(t_{n_0}\omega_{n_0}) \geq \mu$, which is a contradiction. Since $t_n \geq \varrho'$ for all $n \in \mathbb{N}$, from (4.7), we have

$$\max_{t \in [0,+\infty)} I_0(t\omega_n) = \max_{t \in [\varrho',+\infty)} I_0(t\omega_n).$$

So, by (4.2), for n sufficiently large we obtain

$$\frac{C_\sigma}{2} \|t_n \omega_n\|_{X_0}^{2\sigma} - \int_{\mathbb{R}} F_0(x, t_n \omega_n)\, \mathrm{d}x \geq \frac{\pi r_0}{\lambda_0} C_\sigma.$$

As $r_0 \leq \dfrac{\lambda_0}{2\pi}$, we obtain

$$t_n^2 \geq \left(\frac{2\pi r_0}{\lambda_0} \right)^{1/\sigma} \geq \frac{2\pi r_0}{\lambda_0}. \tag{4.8}$$

Since t_n satisfies

$$\frac{d}{dt} I_0(t\omega_n) \bigg|_{t=t_n} = 0,$$

by using (m_1), it follows that

$$\sigma C_\sigma t_n^{2\sigma} \geq \int_{\mathbb{R}} t_n \omega_n f_0(x, t_n \omega_n). \tag{4.9}$$

For $n \in \mathbb{N}$ large enough, we can use (4.9) in order to obtain

$$
\begin{aligned}
t_n^{2\sigma} &\geq \frac{1}{\sigma C_\sigma} \int_{x_0-\frac{r_0}{n}}^{x_0+\frac{r_0}{n}} \frac{t_n\sqrt{\ln n}}{\sqrt{2\pi\widetilde{C}_n+\delta_n}} f_0\left(x, \frac{t_n\sqrt{\ln n}}{\sqrt{2\pi\widetilde{C}_n+\delta_n}}\right) \mathrm{d}x \\
&\geq \frac{2r_0}{\sigma C_\sigma} \exp\left[\left(\frac{\pi t_n^2}{2\pi\widetilde{C}_n+\delta_n}-1\right)\ln(n)\right] \frac{f_0\left(x_1, \frac{t_n\sqrt{\ln n}}{\sqrt{2\pi\widetilde{C}_n+\delta_n}}\right)}{\exp\left(\frac{\pi t_n^2 \ln n}{2\pi\widetilde{C}_n+\delta_n}\right)},
\end{aligned}
\tag{4.10}
$$

where x_1 is a minimum point of f_0 in $[x_0-\frac{r_0}{n}, x_0+\frac{r_0}{n}]$. This implies that t_n^2 is bounded. Moreover, from (4.6) and (4.8), we get

$$
t_n^2 \to \frac{2\pi r_0}{\lambda_0}. \tag{4.11}
$$

Thus, (4.10) together with assumption $(\tilde{f}_{0,4})$ contradict Eq. (4.8). □

Remark 4.4. It is important to point out that as $r_0 \leq \dfrac{\lambda_0\eta_0(\theta-2\sigma)\omega}{2\theta\pi^2 C_\sigma}$, we have

$$
\frac{\pi r_0}{\lambda_0} C_\sigma \leq \frac{\eta_0(\theta-2\sigma)w}{2\theta\pi}.
$$

So in both cases, in Propositions 4.1 and 4.3, we have

$$
c_0 < \frac{\eta_0(\theta-2\sigma)w}{2\theta\pi}. \tag{4.12}
$$

5. Existence of a solution for the periodic problem

In the Sect. 3.2, we guarantee that a Palais Smale sequence is bounded in X_0. Since X_0 is a Hilbert space, up to a subsequence, we can assume that there exists $u_0 \in X_0$ such that

$$
\begin{cases}
u_k \rightharpoonup u_0 \ \text{ weakly in } X_0, \\
u_k \to u_0 \ \text{ in } L^q_{loc}(\mathbb{R}) \ \text{ for all } q \geq 1, \\
u_k(x) \to u_0(x) \ \text{ almost everywhere in } \ \mathbb{R}.
\end{cases}
$$

In order to ensure that the weak limit of the Palais Smale sequence is a solution of (1.9), we need the following auxiliary results, which hold under all hypotheses already cited.

Lemma 5.1.

$$
\int_{\mathbb{R}} f_0(x,u_k)v\,\mathrm{d}x \to \int_{\mathbb{R}} f_0(x,u_0)v\,\mathrm{d}x, \quad \textit{for all} \quad v \in C_0^\infty(\mathbb{R}). \tag{5.1}
$$

Proof. Note that combining (3.6) and (3.7), we reach

$$
c_0 \geq \frac{(\theta-2\sigma)\eta_0}{2\theta} \limsup \|u_k\|^2_{X_0}.
$$

Thus, by Proposition 4.1 (or Proposition 4.3) and (4.12) we obtain

$$
\limsup \|u_k\|^2_{X_0} < \frac{\omega}{\pi}.
$$

This implies $\pi\|u_k\|_{X_0}^2 < \omega$ for k enough large. Hence, we can choose $p > 1$ sufficiently close to 1 and $\delta > 0$ small enough such that $p(\pi + \delta)\|u_k\|_{X_0}^2 < \omega$ for k sufficiently large. Consequently, by (2.3) there exists $C > 0$ such that

$$\int_{\mathbb{R}} (e^{p(\alpha_0+\delta)\|u_k\|_{X_0}^2\left(\frac{u_k}{\|u_k\|_{X_0}}\right)^2} - 1)\,\mathrm{d}x \le C. \tag{5.2}$$

Combining (3.2) and Hölder's inequality for $p' = p/(p-1) > 2$, we get

$$\begin{aligned}\int_{\mathbb{R}} f_0(x,u_k)u_k\,\mathrm{d}x &\le (M(1)-\varepsilon)\int_{\mathbb{R}} u_k^{2\sigma}\,\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} (e^{(\alpha_0+\delta)u_k^2} - 1)|u_k|^q\,\mathrm{d}x \\ &\le (M(1)-\varepsilon)C \\ &\quad + C_\varepsilon \|u_k\|_{p'q}^q \left(\int_{\mathbb{R}} (e^{p(\alpha_0+\delta)\|u_k\|_{X_0}^2\left(\frac{u_k}{\|u_k\|_{X_0}}\right)^2} - 1)\,\mathrm{d}x\right)^{1/p}.\end{aligned} \tag{5.3}$$

Hence, by (5.2), we have

$$\int_{\mathbb{R}} f_0(x,u_k)u_k\,\mathrm{d}x \le C.$$

Consequently, thanks to Lemma 2.1 in [24], we reach

$$f_0(x,u_k) \to f_0(x,u_0) \ \text{ in } \ L^1_{loc}(\mathbb{R}),$$

which implies (5.1). □

Lemma 5.2. *Assume* $(f_{0,3})$, *then for all* $x \in \mathbb{R}$

$$f_0(x,t)t - 2\sigma F_0(t,x) \quad \textit{is increasing for} \quad t > 0 \quad \textit{and decreasing for} \quad t < 0.$$

Proof. Let $0 < t_1 < t_2$ be fixed. By $(f_{0,3})$, it follows

$$f_0(x,t_1)t_1 - 2\sigma F_0(x,t_1) < \frac{f_0(x,t_2)}{t_2^{2\sigma-1}} t_1^{2\sigma} - 2\sigma F_0(x,t_2) + 2\sigma \int_{t_1}^{t_2} f_0(x,\kappa)\mathrm{d}\kappa. \tag{5.4}$$

Again from $(f_{0,3})$, we obtain

$$2\sigma \int_{t_1}^{t_2} f_0(x,\kappa)\mathrm{d}\kappa < 2\sigma \frac{f_0(x,t_2)}{t_2^{2\sigma-1}} \int_{t_1}^{t_2} \kappa^{2\sigma-1}\mathrm{d}\kappa = \frac{f_0(x,t_2)}{t_2^{2\sigma-1}}(t_2^{2\sigma} - t_1^{2\sigma}). \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$f_0(x,t_1)t_1 - 2\sigma F_0(x,t_1) < f_0(x,t_2)t_2 - 2\sigma F_0(x,t_2).$$

Analogously, we obtain the result for $t < 0$. □

Proposition 5.3. *Let* (u_k) *Palais Smale sequence for* I_0. *Then there exists* $u_0 \in X_0$ *such that* $m(\|u_k\|^2) \to m(\|u_0\|^2)$. *In particular,* $I_0'(u_0) = 0$, *this is,* u_0 *is a weak solution of* (P_0).

Proof. By Lemma 3.3, we have that (u_k) is a bounded sequence in X_0. Since X_0 is a reflexive Hilbert space, up to a subsequence, we can assume that there exists $u_0 \in X_0$ such that

$$\begin{cases} u_k \rightharpoonup u_0 \ \text{ weakly in } X_0, \\ u_k \to u_0 \ \text{ in } L^q_{loc}(\mathbb{R}) \ \text{ for all } q \ge 1, \\ u_k(x) \to u_0(x) \ \text{ almost everywhere in } \ \mathbb{R}. \end{cases}$$

On other hand, $(\|u_k\|_{X_0})_k$ is a bounded sequence in $\mathbb{R}$. Thus, up a subsequence, we have that there exists $L \in \mathbb{R}$ such that $\|u_k\|_{X_0} \to L$. Moreover, as the norm is weakly lower semicontinuous, we have

$$\|u_0\|^2_{X_0} \leq \liminf \|u_k\|^2_{X_0}.$$

Since m is nondecreasing, we get

$$m(\|u_0\|^2_{X_0}) \leq m(\liminf \|u_k\|^2_{X_0}),$$

consequently,

$$m(\|u_0\|^2_{X_0}) \leq m(L^2).$$

Since $I_0'(u_k) \to 0$ we obtain that

$$m(\|u_k\|^2_{X_0})\langle u_k, v\rangle - \int_{\mathbb{R}} f_0(x, u_k) v \mathrm{d}x \to 0 \quad \forall v \in C_0^\infty(\mathbb{R}).$$

Consequently, by Lemma 5.1,

$$m(L^2)\langle u_0, v\rangle - \int_{\mathbb{R}} f_0(x, u_0) v = 0 \quad \forall v \in C_0^\infty(\mathbb{R}).$$

By density arguments, in particular, for $v = u_0$ we have

$$m(L^2)\|u_0\|^2_{X_0} - \int_{\mathbb{R}} f_0(x, u_0) u_0 = 0.$$

Note that

$$I_0'(u_0)u_0 = (m(\|u_0\|^2_{X_0}) - m(L^2))\|u_0\|^2_{X_0}. \tag{5.6}$$

We claim that $m(\|u_0\|^2_{X_0}) = m(L^2)$. In order to show this, we suppose, by contradiction, that $m(\|u_0\|^2_{X_0}) < m(L^2)$. Thus, it follows from (5.6) that $I_0'(u_0)u_0 < 0$. However, by using $(f_{0,1})$ and critical exponential growth, proceeding as in the demonstration of the Lemma 3.1, we have that

$$I_0'(t_0u_0)t_0u_0 > 0$$

for t sufficiently small. Therefore, $I_0'(u_0)u_0 < 0$ e $I_0'(tu_0)tu_0 > 0$. Then there exists $t_0 \in (0, 1)$ such that

$$I_0'(t_0u_0)t_0u_0 = 0 \quad \text{and} \quad \max_{t\in[0,1]} I_0(tu_0) = I_0(t_0u_0).$$

By Lemma 5.2, (m_3) and Fatou's lemma, we conclude

$$\begin{aligned}
c_0 &\leq I_0(t_0u_0) - \frac{1}{2\sigma} I_0'(t_0u_0)t_0u_0 \\
&= \frac{1}{2} M(t_0^2\|u_0\|^2_{X_0}) - \int_{\mathbb{R}} F(x, t_0u_0) \mathrm{d}x \\
&\quad - \frac{1}{2\sigma}\left[m(t_0^2\|u_0\|^2_{X_0})t_0^2\|u_0\|^2_{X_0} + \int_{\mathbb{R}} f_0(x, t_0u_0)t_0u_0 \mathrm{d}x \right] \\
&= \frac{1}{2\sigma}\left[\sigma M(t_0^2\|u_0\|^2_{X_0}) - m(t_0^2\|u_0\|^2_{X_0})t_0^2\|u_0\|^2_{X_0}\right] \\
&\quad + \frac{1}{2\sigma}\int_{\mathbb{R}} \left[f(x, t_0u_0)t_0u_0 - 2\sigma F(x, t_0u_0)\right] \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{2\sigma}\left[\sigma M(\|u_0\|^2_{X_0}) - m(\|u_0\|^2_{X_0})\|u_0\|^2_{X_0}\right] + \frac{1}{2\sigma}\int\limits_{\mathbb{R}} [f(x,u_0)u_0 - 2\sigma F(x,u_0)]\,\mathrm{d}x \\
&\le \frac{1}{2\sigma}\left[\sigma M(\liminf \|u_k\|^2_{X_0}) - m(\liminf \|u_k\|^2_{X_0})\liminf \|u_0\|^2_{X_0}\right] \\
&\quad + \frac{1}{2\sigma}\int\limits_{\mathbb{R}} [f(x,u_0)u_0 - 2\sigma F(x,u_0)]\,\mathrm{d}x \\
&= \liminf\left[I_0(u_k) - \frac{1}{2\sigma}I_0'(u_k)u_0\right] = c_0.
\end{aligned}
$$

But this is a contradiction. Therefore $m(\|u_0\|^2_{X_0}) = m(L^2)$. So we conclude that u_0 is a weak solution of (1.9).

□

Remark 5.4. By (3.5), we can take $0 < \tilde{\varrho} < \rho'$ such that $I_0(u) \ge \tilde{\mu}$ when $\|u\|_{X_0} = \tilde{\rho}$ for some $\tilde{\mu} < \min\left\{\frac{\pi r_0}{\lambda_0}\tilde{C}_\sigma, \mu\right\}$, where ρ' and μ are given in (4.1). Moreover, as $M(0) = 0$ and I_0 is continuous, we have

$$
I_0(t) \le \frac{\tilde{\mu}}{2} \quad \text{for all} \quad 0 \le t < \tilde{\rho}.
$$

By (m_1),

$$
M(t) \le \tilde{C}_\sigma t^\sigma \quad \text{for all} \quad t \ge \tilde{\rho}, \quad \text{where} \quad \tilde{C}_\sigma = \frac{M(\tilde{\rho})}{(\tilde{\rho})^\sigma}.
$$

So we may define

$$
\tilde{c}_0 = \inf_{g\in\Gamma}\max_{t\in[0,1]} I_0(g(t)),
$$

where $\Gamma = \{g \in C([0,1], X_0) : g(0) = 0 \text{ and } g(1) = e\}$ and we conclude, as in Proposition 4.3, that

$$
\tilde{\mu} \le \tilde{c}_0 < \frac{\pi r_0}{\lambda_0}\tilde{C}_\sigma,
$$

Since here $\tilde{\mu} < \mu \le c_0$, we can take $\frac{\tilde{\mu}\lambda_0}{\pi\tilde{C}_\sigma} < r_0 < \frac{c_0\lambda_0}{\pi\tilde{C}_\sigma}$, and, consequently, we get

$$
\tilde{c}_0 < c_0.
$$

Thus, we observe that solutions which will obtained in Theorems 1.1 and 1.2 can be different.

5.1. Proof of Theorems 1.1 and 1.2

Using Proposition 5.3 we have that u_0 is a weak solution of (1.9), thus if u_0 is nontrivial the theorem is proved. If $u_0 = 0$, we have the following claim: There exist $(y_k) \subset \mathbb{R}$ and $R, a > 0$ such that

$$
\liminf \sup_{y_k\in\mathbb{R}} \int\limits_{B_R(y_k)} |u_k|^2\,\mathrm{d}x > a. \tag{5.7}
$$

Indeed, assume that (5.7) does not hold, then for all sequence $(y_k) \subset \mathbb{R}$ and $R > 0$, we have

$$
\liminf \sup_{y_k\in\mathbb{R}} \int\limits_{B_R(y_k)} |u_k|^2\,\mathrm{d}x = 0. \tag{5.8}
$$

Using (5.8) and the version of Lions' lemma for X_0, we obtain that $u_k \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$. Thus, by applying (5.2) and (5.3) we reach

$$\int_{\mathbb{R}} f_0(x, u_k)u_k \,\mathrm{d}x \to 0.$$

Combining this estimate together with (3.6), we get that $\|u_k\|_{X_0} \to 0$. Furthermore, in view of assumption $(f_{0,2})$ we conclude that

$$\int_{\mathbb{R}} F_0(x, u_k) \,\mathrm{d}x \to 0. \tag{5.9}$$

Combining the convergence $\|u_k\|_{X_0} \to 0$, (5.9) and (3.6), we get that $c_0 = 0$, but this is impossible. Thus, (5.7) holds.

We may assume, without loss of generality, that $(y_k) \subset \mathbb{Z}$. Letting $w_k(x) = u_k(x - y_k)$, since $V_0(\cdot)$, $f_0(\cdot, s)$ and $F_0(\cdot, s)$ are 1-periodic functions, by a careful calculation we obtain

$$\|u_k\|_{X_0} = \|w_k\|_{X_0},\ m(\|u_k\|_{X_0}) = m(\|w_k\|_{X_0}),$$

$$I_0(u_k) = I_0(w_k) \to c_0 \quad \text{and} \quad I_0'(w_k) \to 0.$$

Consequently, by similar arguments made in the previous sections, we obtain that (w_k) is bounded in X_0 and there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ weakly in X_0 and w_0 is a weak solution of the problem (1.9). Moreover, by (5.7), taking a subsequence and R sufficiently large, we get

$$a^{1/2} \le \|w_k\|_{L^2(B_R(0))} \le \|w_k - w_0\|_{L^2(B_R(0))} + \|w_0\|_{L^2(B_R(0))}. \tag{5.10}$$

Thus, by using Corollary 2.2 we conclude that w_0 is nontrivial.

To finalize, notice that if u is a weak solution of (1.9), since $f_0(x, s) = 0$ for all $s \le 0$ and $I_0'(u)v = 0$ for all $v \in X_0$, choosing the test function $v = -u^-$, by using the following inequality $|u^-(x) - u^-(y)|^2 \le (u(x) - u(y))(u^-(y) - u^-(x))$ and the fact that m is nondecreasing we get that $\|u^-\|_{X_0} \le 0$. Thus, u is a nonnegative function. This completes the proof of Theorems 1.1 and 1.2.

6. Existence of a solution for the nonperiodic problem

In the section, we are concerned to find a nonnegative and nontrivial solution for (1.12). For this, we consider the functional $I : X_1 \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}M(\|u\|_{X_1}^2) - \int_{\mathbb{R}} F(x, u) \,\mathrm{d}x.$$

From (f_1) - (f_2), Lemmas 2.3 and 2.4, similarly to Sect. 3, we can see that I is well defined and by using standard arguments $I \in C^1(X_1, \mathbb{R})$ with

$$I'(u)v = m(\|u\|_{X_1}^2)\langle u, v\rangle_1 - \int_{\mathbb{R}} f(x, u)v \,\mathrm{d}x.$$

for all $v \in X_1$. Thus, a critical point of I is a weak solution of (1.12) and reciprocally. Moreover, I has the geometry of the mountain-pass theorem, by analogous steps to Propositions 3.1 and 3.2, we obtain

Proposition 6.1. *If (f_2)-(f_3) and (v_1) hold, then*

(i) *there exist σ_1, $\rho_1 > 0$ such that $I(u) \ge \sigma_1$ if $\|u\|_{X_1} = \rho_1$;*
(ii) *there exists $e_1 \in E$, with $\|e_1\|_{X_1} > \rho_1$, such that $I(e_1) < 0$.*

As a consequence of Proposition 6.1, the minimax level

$$c_1 := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$$

is positive, where $\Gamma = \{\gamma \in C([0,1], X_1) : \gamma(0) = 0 \text{ and } \gamma(1) = e_1\}$.

Moreover, by applying the mountain-pass theorem without the (PS) condition (see [31]), there exists a sequence $(v_k) \subset X_1$ such that

$$I(v_k) \to c_1 \quad \text{and} \quad I'(v_k) \to 0.$$

Thus, similarly to Lemma 3.3, we obtain that (v_k) is a bounded sequence in X_1. Moreover, by using the arguments made in Proposition 5.3, we get the following result:

Proposition 6.2. *If $(f_{0,4})$ (or $(\tilde{f}_{0,4})$ and (K_1)), (f_1)-(f_3) and (v_1) hold, then $v_k \rightharpoonup v_0$ weakly in X_1 and v_0 is a critical point of functional I.*

Now, in order to prove that there exists a nontrivial critical point of I we need some auxiliary results, among then a lemma of convergence. More specifically, assuming for the sake of contradiction that v_0 is trivial, following the same steps of Lemma 4.3 in [13], we obtain the following result.

Lemma 6.3. *If (v_1), -$(f_{0,2})$ and (f_1)-(f_3) hold, then*

(i) $\int_{\mathbb{R}} [f_0(x, v_k) - f(x, v_k)]v_k \,\mathrm{d}x \to 0$;

(ii) $\int_{\mathbb{R}} [F_0(x, v_k) - F(x, v_k)] \,dx \to 0$

(iii) $\int_{\mathbb{R}} [V_0(x) - V(x)]v_k^2 \,dx \to 0$.

6.1. Proof of Theorem 1.3

Assuming for the sake of contradiction that v_0 is trivial, since m is a continuous function, as a consequence of Lemma 6.3, it follows that

$$|I_0(v_k) - I(v_k)| \to 0 \quad \text{and} \quad \|I_0'(v_k) - I'(v_k)\|_* \to 0. \tag{6.1}$$

Hence,

$$I_0(v_k) \to c_1 \quad \text{and} \quad I_0'(v_k) \to 0.$$

In addition, we obtained a version of Lions's result for a sequence in X_1 as in (4.3). From this, we conclude that there exist $(y_k) \subset \mathbb{Z}$ and $R, a > 0$ such that

$$\liminf \sup_{y_k\in\mathbb{R}} \int_{B_R(y_k)} |v_k|^2 \,\mathrm{d}x > a.$$

Now consider $w_k(x) = v(x - y_k)$, since $V_0(x)$, $f_0(x, s)$ and $F_0(x, s)$ are 1−periodic functions in x, we get

$$\|v_k\|_{X_0} = \|w_k\|_{X_0},\ m(\|v_k\|_{X_0}) = m(\|w_k\|_{X_0}),$$

$$I_0(v_k) = I_0(w_k) \to c_1 \quad \text{and} \quad I_0'(w_k) \to 0.$$

Then, there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ weakly in X_0 and $I_0'(w_0) = 0$. Moreover, using (6.1) and Fatou's lemma, we have

$$I_0(w_0) = I_0(w_0) - \frac{1}{2}I_0'(w_0)w_0$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}} [f_0(x,w_0)w_0 - 2F_0(x,w_0)]\,\mathrm{d}x \\
&\le \liminf \frac{1}{2}\int_{\mathbb{R}} [f_0(x,w_k)w_k - 2F_0(x,w_k)]\,\mathrm{d}x \\
&= \lim[I_0(w_k) - \frac{1}{2}I_0'(w_k)w_k] = c_1.
\end{aligned}
$$

Arguing as in (5.10) we conclude that w_0 is nontrivial. Now, by $(f_{0,3})$, we have that $\max\{I_0(tw_0) : t \ge 0\}$ is unique and then

$$
c_0 \le \max_{t\ge 0} I_0(tw_0) = I_0(w_0) \le c_1. \tag{6.2}
$$

On the other hand, considering u_0 the solution obtained in Theorem 1.1 (or Theorem1.2), as m is nondecreasing, so from (v_1), (f_1), (f_5), (f_4) and $(f_{0,3})$, we have

$$
c_1 \le \max_{t\ge 0} I(tu_0) = I(t_1u_0) < I_0(t_1u_0) \le \max_{t\ge 0} I_0(tu_0) = I_0(u_0) = c_0,
$$

that is, $c_1 < c_0$, which is a contradiction with (6.2). Therefore, v_0 is nontrivial.
To finalize, notice that similarly to proof of Theorems 1.1 and 1.2 if we have a weak solution of (1.12), then it is a nonnegative function. This completes the proof of Theorem 1.3.

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Eudes Barboza, Yane Araújo and Gilson de Carvalho
Departamento de Matemática
Universidade Federal Rural de Pernambuco
Rua Dom Manuel de Medeiros
Recife Pernambuco
Brazil
e-mail: eudes.barboza@ufrpe.br

Yane Araújo
e-mail: yane.araujo@ufrpe.br

Gilson de Carvalho
e-mail: gilson.carvalho@ufrpe.br

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