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On Liouville-type theorems for the stationary nematic liquid crystal equations

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Abstract. This paper is concerned with 3D steady compressible nematic liquid crystal flows. We establish a Liouville-type theorem when the smooth solution (ρ, u, d) satisfies some suitable conditions.

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1. Introduction

In this paper, we consider the following simplified version of Ericksen–Leslie system modeling the hydrodynamic flow of stationary compressible nematic liquid crystals in $\mathbb{R}^3$([8,12])

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla P(\rho) = -\Delta d \cdot \nabla d, \\ u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \end{cases} \tag{1.1}$$

where ρ, u and d denote the density, the velocity field and the macroscopic average of the nematic liquid crystal orientation field, respectively. The shear viscosity coefficient μ and the bulk viscosity coefficient λ satisfy the physical conditions: $\mu > 0$, $2\mu + 3\lambda > 0$. The $P(\rho)$ is the pressure, which satisfies the so-called γ-law:

$$P(\rho) = a\rho^\gamma \quad with \quad a > 0, \quad \gamma > 1, \tag{1.2}$$

where a is a physical constant and γ is the adiabatic exponent.

When $\nabla d = 0$, the system (1.1) simplifies to the stationary Navier–Stokes equations, and significant progress has been made in studying the Liouville-type problems associated with these equations.

For the stationary incompressible Navier–Stokes equations:

$$\begin{cases} u \cdot \nabla u - \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.3}$$

A well-known result on the Liouville theorem is presented in G.Galdi's book [9]. Galdi established that if $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$, then it holds that $u = 0$. Later, Chae showed an interesting result in [3], stating that the condition $\Delta u \in L^{\frac{6}{5}}(\mathbb{R}^3)$ is sufficient for $u = 0$ in $\mathbb{R}^3$. Furthermore, G.Seregin [18] provided an improved condition $u \in L^6(\mathbb{R}^3) \cap BMO^{-1}$. Recently, Chae-Wolf [4] building upon a refined Caccioppoli-type inequality, achieved a logarithmic enhancement of the Liouville result, under the assumption that:

$$N(u) := \int_{\mathbb{R}^3} |u|^{\frac{9}{2}} \{ln(2 + 1/|u|)\}^{-1} dx < \infty.$$

There exist numerous significant results concerning Liouville-type results in the study of incompressible fluids, we can refer to [5,11,18,21] and the references therein.

For stationary compressible Navier–Stokes equations:

$$\begin{cases} \mathrm{div}(\rho u)=0, \\ \mathrm{div}(\rho u\otimes u)-\mu\Delta u-(\lambda+\mu)\nabla \mathrm{div} u+\nabla P(\rho)=0, \\ P(\rho)=a\rho^\gamma, \quad a>0,\gamma>1. \end{cases} \tag{1.4}$$

Chae [2] was the first to establish Liouville-type results for stationary solutions in the context of the compressible Navier–Stokes equations in $\mathbb{R}^d, d\geq 2$. He demonstrated that if the smooth solution (ρ,u) satisfies:

$$\begin{cases} \|\rho\|_{L^\infty(\mathbb{R}^d)}+\|\nabla u\|_{L^2(\mathbb{R}^d)}+\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}<\infty, \quad when \quad d\leq 6, \\ \|\rho\|_{L^\infty(\mathbb{R}^d)}+\|\nabla u\|_{L^2(\mathbb{R}^d)}+\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}+\|u\|_{L^{\frac{3d}{d-1}}(\mathbb{R}^d)}<\infty, \quad when \quad d\leq 7, \end{cases}$$

then the system (1.4) only have a trivial solution that $u=0$, $\rho=constant$. Later, Li and Yu in their work [14] got the same conclusion under the condition

$$\|\rho\|_{L^\infty(\mathbb{R}^d)}+\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}+\|u\|_{L^{\frac{3d}{d-1}}(\mathbb{R}^d)}<\infty, \quad d\geq 2.$$

Owing to $L^{\frac{9}{2}}(\mathbb{R}^3)\hookrightarrow L^{\frac{9}{2},\infty}(\mathbb{R}^3)$, Zhong in [22] used the Lorentz space properties to improve the result of Li and Yu [14], here he assumed that

$$\|\rho\|_{L^\infty(\mathbb{R}^3)}+\|\nabla u\|_{L^2(\mathbb{R}^3)}+\|u\|_{L^{\frac{9}{2},\infty}(\mathbb{R}^3)}<\infty.$$

However, to further weaken the condition $u\in L^{\frac{9}{2}}(\mathbb{R}^3)$, Li and Niu [15] proved a Liouville-type theorem by deriving an a priori estimate that

$$\begin{aligned} &\mu\int\limits_{|x|\leq R}|\nabla u|^2dx+\frac{(\lambda+\mu)}{2}\int\limits_{|x|\leq R}|\mathrm{div}u|^2dx \\ &\leq C(R^{\frac{1}{2}-\frac{3}{p}}\|u\|_{L^{p,q}(R\leq|x|\leq 2R)}+R^{1-\frac{6}{p}}\|u\|^2_{L^{p,q}(R\leq|x|\leq 2R)}+R^{2-\frac{9}{p}}\|u\|^3_{L^{p,q}(R\leq|x|\leq 2R)}). \end{aligned}$$

under the hypotheses

$$\|\rho\|_{L^\infty(\mathbb{R}^3)}+\|\nabla u\|_{L^2(\mathbb{R}^3)}+\|u\|_{L^{p,q}(\mathbb{R}^3)}<\infty,$$

where $3<p<\frac{9}{2}$, $3\leq q\leq\infty$ or $p=q=3$. In the case of $p\geq\frac{9}{2}$, additional conditions are required:

$$\liminf_{R\to\infty}R^{\frac{2}{3}-\frac{3}{p}}\|u\|_{L^{p,q}(R\leq|x|\leq 2R)}<\infty,\ \liminf_{R\to\infty}R^{2-\frac{9}{p}}\|u\|^3_{L^{p,q}(R\leq|x|\leq 2R)}<\delta\int\limits_{\mathbb{R}^3}|\nabla u|^2dx,$$

where δ is a sufficiently small positive constant. These conditions are derived from the ideas presented in the works [11] and [19].

Now, we study the Liouville-type theorems for compressible nematic liquid crystal equations in $\mathbb{R}^3$, which have limited existing results and can be regarded as the Navier–Stokes system with the external force $-\Delta d\cdot\nabla d$. When $u=0$, $(1.1)_3$ represents the heat flow of the harmonic map, we present the following results:

Theorem 1.1. *Let (ρ,u,d) be a smooth solution to (1.1)–(1.2) with $\rho\in L^\infty(\mathbb{R}^3)$ and $(u,\nabla d)\in L^{\frac{9}{2},q}(\mathbb{R}^3)$ $(0<q<\infty)$. Then, $u=0$, $\rho=constant$, and d satisfies the harmonic map equation of*

$$\Delta d+|\nabla d|^2d=0. \tag{1.5}$$

Theorem 1.2. *Let $d\colon \mathbb{R}^3 \to \mathbb{S}^2$ be a solution of (1.5), and $\nabla d \in L^p(2 \le p \le 3)$, then d is a constant map.*

Remark 1.3. [11] Based on Lebesgue's dominated convergence theorem, we know when $h \in L^{\frac{9}{2},q}$, then $\lim_{R\to\infty} \|h\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)} = 0 (0 < q < \infty)$, but we do not know whether

$$\lim_{R\to\infty} \|h\|_{L^{\frac{9}{2},\infty}(R\le|x|\le 2R)} = 0.$$

Such as $h \in C^\infty(\mathbb{R}^3)$ and behaves like

$$|h(x)| = O(|x|^{-2/3}),\ as|x| \to \infty.$$

Combining the definition of weak norm, we get that $h(x) \in L^{\frac{9}{2},\infty}$, but

$$\begin{aligned}\|h(x)\|_{L^{\frac{9}{2},\infty}(R\le|x|\le 2R)} &= \sup_{\alpha>0} \alpha|\{|h(x)| > \alpha, x \in [R, 2R]\}|^{2/9}\\ &\ge O(1)|h(2R)| \cdot ((2R)^3)^{2/9}\\ &= O(1).\end{aligned}$$

Remark 1.4. When studying the harmonic mapping from an open subset of $\mathbb{R}^3$ into a sphere $\mathbb{S}^2$, various results have been obtained under certain geometric conditions. For instance, Choi [6] demonstrated that if the image of the harmonic map d lies in a hemisphere, then d is a constant map. Additionally, Jin [17] proved that when ∇d decays rapidly with respect to spatial variations, the Liouville theorem for d can be derived. Taking inspiration from their work, we speculate that if d decays rapidly at infinity, a Liouville theorem for d may also be established. Therefore, by combining the techniques proposed in [7,16,20] for minimizing the energy functional of harmonic mappings, we can obtain the Liouville theorem for d under the condition $\nabla d \in L^p(2 \le p \le 3)$.

Remark 1.5. When d is a constant vector, Theorem 1.1 can be viewed as an extension of the works presented in [15] and [22].

The rest of this paper is organized as follows: In Sect. 2, we prepare some elementary facts, which are important for the proof. Finally, we will give the proofs of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

We will recall some definitions and lemmas that will be used later.

Definition 2.1. Given $1 \le p < \infty$, $1 \le q < \infty$, the Lorentz space $L^{p,q}(\mathbb{R}^3)$ consists of all measurable functions f for which the quantity

$$\|f\|_{L^{p,q}(\mathbb{R}^3)} := \begin{cases} (\int_0^\infty t^{q-1}|\{x \in \mathbb{R}^3 : |f(x)| > t\}|^{\frac{q}{p}} dt)^{\frac{1}{q}}, & q < +\infty,\\ sup_{t>0} t|\{x \in \mathbb{R}^3 : |f(x)| > t\}|^{\frac{1}{p}}, & q = +\infty,\end{cases}$$

is finite.

Lemma 2.1. [10,13] *Let $1 < p < \infty$ and $1 \le s \le \infty$ with $\frac{1}{p'} + \frac{1}{p} = 1$ and $\frac{1}{s'} + \frac{1}{s} = 1$. Then, pointwise multiplication is a bounded bilinear operator:*

(i) from $L^{p,s}(\mathbb{R}^3) \times L^\infty(\mathbb{R}^3)$ to $L^{p,s}(\mathbb{R}^3)$;
(ii) from $L^{p,s}(\mathbb{R}^3) \times L^{p',s'}(\mathbb{R}^3)$ to $L^1(\mathbb{R}^3)$;
(iii) from $L^{p_1,s_1}(\mathbb{R}^3) \times L^{p_2,s_2}(\mathbb{R}^3)$ to $L^{p,s}(\mathbb{R}^3)$ for $1 < p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 \le s_1, s_2 \le \infty$ with $s = min\{s_1, s_2\}$.

Lemma 2.2. [11] *Let $1 \leq r < \infty$, $1 \leq s \leq \infty$ and $f \in L^{r,s}(\mathbb{R}^3)$. For a parameter $R > 0$, we put $f_R(x) = f(\frac{x}{R})$. Then, it holds that $\|f_R\|_{L^{r,s}} = R^{\frac{3}{r}}\|f\|_{L^{r,s}}$.*

Lemma 2.3. [14] *Let $P \in L^\infty(\mathbb{R}^3)$, $p_1 \in L^{r_1}(\mathbb{R}^3)$, $p_2 \in L^{r_2}(\mathbb{R}^3)$ with $1 \leq r_1, r_2 < \infty$. Assuming that $P - p_1 - p_2$ is weakly harmonic, that is $\Delta(P - p_1 - p_2) = 0$ in the sense of distribution, there is a constant c such that $P - p_1 - p_2 = c$, for a.e. $x \in \mathbb{R}^3$.*

If more $P(x) \geq 0$ a.e., then we get $c \geq 0$ too.

Lemma 2.4. [1] *Let Ω be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, $1 < q \leq \infty$ and $f \in L^{p,q}(\Omega)$. Then,*

$$\|\nabla^2(-\Delta)^{-1}f\|_{L^{p,q}(\Omega)} \leq C\|f\|_{L^{p,q}(\Omega)},$$

where the constant $C > 0$ is independent of Ω.

3. The proof of Theorem 1.1

Let $\phi \in C_c^\infty(\mathbb{R}^3)$ be a radial cutoff function such that

$$\phi(|x|)\begin{cases} = 1, \ |x| < 1, \\ = 0, \ |x| \geq 2. \end{cases}$$

For each $R > 0$, let

$$\phi_R(|x|) \triangleq \phi(\frac{|x|}{R}). \tag{3.1}$$

Moreover, there exists a constant C independent on R and $k = 0, 1, 2, 3$ on $\mathbb{R}^3$ such that

$$|\nabla^k\phi_R| \leq \frac{C}{R^k}. \tag{3.2}$$

Firstly, taking the inner product of $(1.1)_2$ with $u\phi_R^2$ and integrating by parts over $\mathbb{R}^3$, we get

$$\begin{aligned} &\mu\int_{\mathbb{R}^3}|\nabla u|^2\phi_R^2dx + (\lambda+\mu)\int_{\mathbb{R}^3}|\mathrm{div}u|^2\phi_R^2dx \\ =&\mu\int_{\mathbb{R}^3}|u|^2(\phi_R\Delta\phi_R + |\nabla\phi_R^2|)dx - 2(\lambda+\mu)\int_{\mathbb{R}^3}\mathrm{div}u(u\cdot\nabla\phi_R\cdot\phi_R)dx \\ &-\int_{\mathbb{R}^3}\nabla P\cdot u\phi_R^2dx - \int_{\mathbb{R}^3}\rho(u\cdot\nabla u)\phi_R^2dx - \int_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot u\phi_R^2dx. \end{aligned} \tag{3.3}$$

Similarly, taking the inner product of $(1.1)_3$ with $(\Delta d + |\nabla d|^2 d)\phi_R^2$, we have

$$\int_{\mathbb{R}^3}|\Delta d + |\nabla d|^2 d|^2\phi_R^2dx = \int_{\mathbb{R}^3}(u\cdot\nabla d)(\Delta d + |\nabla d|^2 d)\phi_R^2dx. \tag{3.4}$$

Combining (3.3) and (3.4) yields

$$\begin{aligned}
&\mu\int\limits_{\mathbb{R}^3}|\nabla u|^2\phi_R^2dx+(\lambda+\mu)\int\limits_{\mathbb{R}^3}|\mathrm{div}u|^2\phi_R^2dx+\int\limits_{\mathbb{R}^3}|\Delta d+|\nabla d|^2d|^2\phi_R^2dx\\
&=\mu\int\limits_{\mathbb{R}^3}|u|^2(\phi_R\Delta\phi_R+|\nabla\phi_R^2|)dx-2(\lambda+\mu)\int\limits_{\mathbb{R}^3}\mathrm{div}u(u\cdot\nabla\phi_R\cdot\phi_R)dx\\
&\quad-\int\limits_{\mathbb{R}^3}\nabla P\cdot u\phi_R^2dx-\int\limits_{\mathbb{R}^3}\rho(u\cdot\nabla u)u\phi_R^2dx-\int\limits_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot u\phi_R^2dx\\
&\quad+\int\limits_{\mathbb{R}^3}(u\cdot\nabla d)(\Delta d+|\nabla d|^2d)\phi_R^2dx\\
&\triangleq\sum_{i=1}^{6}J_i(t).
\end{aligned}\tag{3.5}$$

Secondly, we will estimate J_i one by one. It follows from Lemma 2.1, Lemma 2.2 and (3.2) that

$$\begin{aligned}
J_1&\le C(\mu)\int\limits_{R\le|x|\le 2R}|u|^2(|\phi_R\Delta\phi_R|+|\nabla\phi_R^2|)dx\\
&\le CR^{-2}\||u|^2\|_{L^{\frac{9}{4},\frac{q}{2}}(R\le|x|\le 2R)}\|1\|_{L^{\frac{9}{5},\frac{q}{q-2}}(R\le|x|\le 2R)}\\
&\le CR^{-2+\frac{5}{3}}\|u\|^2_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\\
&\le CR^{-\frac{1}{3}}\|u\|^2_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\to 0\quad as\quad R\to\infty.
\end{aligned}\tag{3.6}$$

The estimate of J_2 is similar to that of J_1 and we get

$$\begin{aligned}
J_2=&-2(\lambda+\mu)\int\limits_{\mathbb{R}^3}\mathrm{div}u(u\cdot\nabla\phi_R\cdot\phi_R)dx\\
&\le C(\lambda+\mu)R^{-1}\int\limits_{R\le|x|\le 2R}|\phi_R\mathrm{div}u||u|dx\\
&\le C(\lambda+\mu)R^{-1}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|\phi_R\mathrm{div}u\|_{L^{2,q}(R\le|x|\le 2R)}\|1\|_{L^{\frac{18}{5},\frac{q}{q-2}}(R\le|x|\le 2R)}\\
&\le\frac{(\lambda+\mu)}{4}\|\phi_R\mathrm{div}u\|^2_{L^{2,q}(R\le|x|\le 2R)}+CR^{-\frac{1}{3}}\|u\|^2_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\\
&\to\frac{(\lambda+\mu)}{4}\|\phi_R\mathrm{div}u\|^2_{L^{2,q}(R\le|x|\le 2R)}\quad as\quad R\to\infty.
\end{aligned}\tag{3.7}$$

As for J_3, we incorporate some ideas from [14]. Taking divergence on both sides of $(1.1)_2$, we get

$$\Delta(P-p_1-p_2)=0,$$

where

$$p_1\triangleq(-\Delta)^{-1}\partial_i\partial_j(\rho u_iu_j),\quad p_2\triangleq(\lambda+2\mu)\mathrm{div}u-\partial_id\cdot\partial_jd.\tag{3.8}$$

According to Lemma 2.3, there exists a nonnegative constant c such that

$$a\rho^\gamma=P=c+p_1+p_2,\quad a.e\quad x\in\mathbb{R}^3.$$

Setting

$$P_1 \triangleq \rho^{\gamma-1} - (\frac{c}{a})^{\frac{\gamma-1}{\gamma}} = (\frac{c+p_1+p_2}{a})^{\frac{\gamma-1}{\gamma}} - (\frac{c}{a})^{\frac{\gamma-1}{\gamma}}. \tag{3.9}$$

By combining (1.2) and (3.9), we can deduce that

$$\nabla P = \nabla(a\rho^\gamma) = \frac{a\gamma}{\gamma-1}\rho\nabla(\rho^{\gamma-1}) = \frac{a\gamma}{\gamma-1}\rho\nabla P_1,$$

and

$$|P_1\rho| \le C(a, \|\rho\|_{L^\infty})(|p_1|+|p_2|). \tag{3.10}$$

Therefore, by applying integration by parts and combining $(1.1)_1$ and (3.10), we can conclude that

$$\begin{aligned} J_3 =& -\int_{\mathbb{R}^3} \nabla P\cdot u\phi_R^2 dx \\ =& -\int_{\mathbb{R}^3} \frac{a\gamma}{\gamma-1}\rho\nabla P_1\cdot u\phi_R^2 dx \\ =& \ 2\frac{a\gamma}{\gamma-1}\int_{\mathbb{R}^3} P_1\rho u\cdot\phi_R\nabla\phi_R dx \\ \le& \ C(a,\gamma)R^{-1}\int_{\mathbb{R}^3} |P_1\rho||u||\phi_R| dx \\ \le& \ C(a,\gamma,\|\rho\|_{L^\infty})R^{-1}\int_{\mathbb{R}^3} (|p_1||u|+|p_2||u|)|\phi_R| dx \\ \triangleq& I_1+I_2. \end{aligned} \tag{3.11}$$

By $\rho\in L^\infty$, Lemma 2.1, Lemma 2.2 and Lemma 2.4, we get that

$$\begin{aligned} I_1 \le& C(a,\gamma,\|\rho\|_{L^\infty})R^{-1}\|p_1\|_{L^{\frac{9}{4},\frac{q}{2}}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|1\|_{L^{3,\frac{q}{q-3}}(R\le|x|\le 2R)} \\ \le& C\|(-\Delta)^{-1}\partial_i\partial_j\rho u_i u_j\|_{L^{\frac{9}{4},\frac{q}{2}}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)} \\ \le& C\|u\|^3_{L^{\frac{9}{2},q}(R\le|x|\le 2R)} \to 0 \quad as \quad R\to\infty. \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_2 \le& C(a,\gamma,\|\rho\|_{L^\infty})R^{-1}\int_{\mathbb{R}^3}(\lambda+2\mu)|\mathrm{div}u||u||\phi_R| + |\partial_i d\cdot\partial_j d||u||\phi_R| dx \\ \le& C(a,\gamma,\mu,\|\rho\|_{L^\infty})R^{-1}\int_{(R\le|x|\le 2R)}(|\phi_R\mathrm{div}u||u|+|\nabla d|^2|u|)dx \\ \le& CR^{-1}\|\phi_R\mathrm{div}u\|_{L^{2,q}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|1\|_{L^{\frac{18}{5},\frac{q}{q-2}}(R\le|x|\le 2R)} \\ &+ CR^{-1}\|\nabla d\|^2_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|1\|_{L^{3,\frac{q}{q-3}}(R\le|x|\le 2R)} \\ \le& CR^{-\frac{1}{6}}\|\phi_R\mathrm{div}u\|_{L^{2,q}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)} \\ &+ C\|\nabla d\|^2_{L^{\frac{9}{2},q}(R\le|x|\le 2R)}\|u\|_{L^{\frac{9}{2},q}(R\le|x|\le 2R)} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\lambda+\mu)}{4}\|\phi_R \mathrm{div} u\|^2_{L^{2,q}(R\leq|x|\leq 2R)} + CR^{-\frac{1}{3}}\|u\|^2_{L^{\frac{9}{2},q}(R\leq|x|\leq 2R)} \\
&\quad + C\|\nabla d\|^2_{L^{\frac{9}{2},q}(R\leq|x|\leq 2R)}\|u\|_{L^{\frac{9}{2},q}(R\leq|x|\leq 2R)} \\
&\to \frac{(\lambda+\mu)}{4}\|\phi_R \mathrm{div} u\|^2_{L^{2,q}(R\leq|x|\leq 2R)} \quad as \quad R\to\infty.
\end{aligned}
\tag{3.13}
$$

As for J_4, integrating by parts and utilizing $(1.1)_1$, Lemma 2.1 and Lemma 2.3, we get the following results that

$$
\begin{aligned}
J_4 &= -\int_{\mathbb{R}^3} \rho u\cdot\nabla u\cdot u\phi_R^2 dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^3} \rho u\cdot\nabla |u|^2\cdot\phi_R^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \mathrm{div}(\rho u)|u|^2\cdot\phi_R^2 + \rho|u|^3\cdot\nabla(\phi_R^2) dx \\
&= \int_{\mathbb{R}^3} \rho|u|^3\cdot\phi_R\nabla\phi_R dx \\
&\leq CR^{-1}\||u|^3\|_{L^{\frac{3}{2},\frac{q}{3}}(R\leq|x|\leq 2R)}\|1\|_{L^{3,\frac{q}{q-3}}(R\leq|x|\leq 2R)} \\
&\leq C\|u\|^3_{L^{\frac{9}{2},q}(R\leq|x|\leq 2R)} \to 0 \quad as \quad R\to\infty.
\end{aligned}
\tag{3.14}
$$

For the estimation of J_5+J_6, direct computation yields

$$
\begin{aligned}
J_5+J_6 &= -\int_{\mathbb{R}^3}\Delta d\cdot\nabla d\cdot u\phi_R^2 dx + \int_{\mathbb{R}^3}(u\cdot\nabla d)(\Delta d+|\nabla d|^2 d)\phi_R^2 dx \\
&= \int_{\mathbb{R}^3} u\cdot\nabla d\cdot d|\nabla d|^2 d\phi_R^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} u\cdot\nabla|d|^2|\nabla d|^2 d\phi_R^2 dx \\
&= 0,
\end{aligned}
\tag{3.15}
$$

where we have used $|d|=1$. Inserting (3.6), (3.7), (3.11), (3.12), (3.13), (3.14) and (3.15) into (3.5), we deduce that

$$
\mu\int_{\mathbb{R}^3}|\nabla u|^2 dx + (\lambda+\mu)\int_{\mathbb{R}^3}|\mathrm{div} u|^2 dx + \int_{\mathbb{R}^3}|\Delta d+|\nabla d|^2 d|^2 dx = 0.
$$

Since $\mu>0$, $2\mu+3\lambda>0$ and taking $\|u\|_{L^6}\leq\|\nabla u\|_{L^2}$ into consideration, yields

$$
u=0, \quad \Delta d+|\nabla d|^2 d = 0.
$$

Now, based on $(1.1)_2$ and $|d|=1$, the pressure P satisfies

$$
\begin{aligned}
\nabla P &= -\Delta d\nabla d = |\nabla d|^2 d\nabla d \\
&= \frac{1}{2}|\nabla d|^2\nabla|d|^2 \\
&= 0.
\end{aligned}
$$

Thus, we have showed that $\rho = constant$ and finished the proof of Theorem 1.1.

4. The proof of Theorem 1.2

By [7], we know that a function $d \in H^1(\mathbb{R}^3; \mathbb{S}^2)$ is a weakly harmonic mapping of $\mathbb{R}^3$ into $\mathbb{S}^2$ provided:

$$-\Delta d = |\nabla d|^2 d \quad in \quad \mathbb{R}^3, \tag{4.1}$$

now the solution d of this Euler–Lagrange equation satisfies the minima of the energy function

$$E(d) = \int |\nabla d|^2 dy. \tag{4.2}$$

This system (4.1) is to hold in the weak sense, that is,

$$\int_{\mathbb{R}^3} \nabla d : \nabla w dx = \int_{\mathbb{R}^3} |\nabla d|^2 d \cdot w dx, \tag{4.3}$$

for each test function $w \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ having compact support, which shows that d is stationary with respect to variations of the target $\mathbb{S}^2$. So the stationary mapping d satisfies the monotonicity inequalities

$$\frac{1}{r} \int_{B(x,r)} |\nabla d|^2 dy \leq \frac{1}{R} \int_{B(x,R)} |\nabla d|^2 dy, \tag{4.4}$$

for all concentric balls $B(x,r) \subset B(x,R) \subset \mathbb{R}^3$, which was apparently first proved by Price [16]. By (4.4), thus we have

$$(\frac{1}{r} - \frac{1}{R}) \int_{B(x,r)} |\nabla d|^2 dy \leq \frac{1}{R} \int_{B_R \setminus B_r} |\nabla d|^2 dy. \tag{4.5}$$

Now we observe that by $\nabla d \in L^p$ and $H\ddot{o}lder$ inequality

$$\frac{1}{R} \int_{B_R \setminus B_r} |\nabla d|^2 dy \leq \frac{1}{R} \left((\int_{B_R \setminus B_r} |\nabla d|^p dy)^{\frac{2}{p}} \cdot (R^3 - r^3)^{\frac{2}{q}} \right) \leq C R^{\frac{6}{q} - 1} (\int_{B_R \setminus B_r} |\nabla d|^p dy)^{\frac{2}{p}}, \tag{4.6}$$

where p, q are positive values satisfying $\frac{2}{p} + \frac{2}{q} = 1$ and $\frac{6}{q} - 1 \leq 0$. Now, we can establish the following result:

$$6 \leq q, \ 2 \leq p \leq 3.$$

When $R \to \infty$, together with (4.5) and (4.6), which leads to

$$\frac{1}{r} \int_{B(x,r)} |\nabla d|^2 dy \leq C (\int_{\mathbb{R}^3 \setminus B_r} |\nabla d|^p dy)^{\frac{2}{p}}.$$

Because of the monotonicity of u, implies that

$$\frac{1}{r} \int_{B(x,r)} |\nabla d|^2 dy \leq \lim_{r \to \infty} \frac{1}{r} \int_{B(x,r)} |\nabla d|^2 dy \leq \lim_{r \to \infty} C (\int_{\mathbb{R}^3 \setminus B_r} |\nabla d|^p dy)^{\frac{2}{p}} = 0.$$

So $\nabla d \equiv 0$ and d is a constant map. This completes the proof of Theorem 1.2.

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Declarations

Conflict of interest This work does not have any conflicts of interest.

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