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The Cauchy problem for time-fractional linear nonlocal diffusion equations

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Abstract. This manuscript is dedicated to study the Cauchy problem for time-fractional linear nonlocal diffusion problems in the whole $\mathbb{R}^N$, including the existence and uniqueness of solutions, their asymptotic behaviour as t goes to infinity, and the analysis of the corresponding rescaled problems by rescaling the convolution kernel J in some appropriate ways. Two time-fractional models will be considered in our work, one is related to the simplest linear nonlocal diffusion operator of the form $J * u - u$, and the other is proposed as a nonlocal analogy of higher-order evolution equations.

Keywords. Cauchy problem, Existence and uniqueness, Asymptotic behaviour, Rescaled problems, Fractional calculus, Nonlocal diffusion problems.

1. Introduction

As is well known, the fractional calculus has been recognized as a powerful tool for describing the memory effects of power-law kernels [1–3]. Moreover, memory effects can be modelled by using the evolutionary equations of convolution type [4,5]. Therefore, as an indispensable class of evolutionary equations with traditional fractional differential operators (the nonlocal type time derivatives), the time-fractional diffusion equations (with regard to the order α of time-fractional derivative satisfying $0 < \alpha < 1$) and their applications have recently become a hot topic in the fields of mathematics and physics [6–16].

In practical applications, there may be some non-local effects that affect the evolution of a system. In general, for example, we do not have enough information to understand the systems under study and their characteristics at each point. One possible reason for introducing non-local terms into many models or systems is that the actual measurements are not done point by point, but through some local average. Very recently, there are plenty of meaningful works about nonlocal problems and their applications in physics and population dynamics. Among them, we refer to the papers [17–26] and the references therein.

Compared with the classical initial conditions, the nonlocal conditions can describe some physical phenomena better. Although many time-fractional evolutionary equations with various nonlocal conditions have been investigated (see, e.g., [27–30]), most of these existing works only involve the bounded domains in $\mathbb{R}^N$ rather than the whole $\mathbb{R}^N$ and there are no convolution kernels contained in their main equations. In view of these discussions, it is natural and challenging to study the time-fractional nonlocal evolution equations of convolution type in the whole $\mathbb{R}^N$. This can be regarded as the first motivation of our paper.

Besides, the authors Andreu-Vaillo Fuensanta et al presented many theoretical results concerning the nonlocal evolution equations with different boundary conditions in their monograph [31]. They mainly introduced the nonlocal evolution equations of the form

$$\frac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - u(x,t),$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, continuous, and radial (that is, for any $x, y \in \mathbb{R}^N$, it holds $|x| = |y| \Rightarrow J(x) = J(y)$) function satisfying $\int_{\mathbb{R}^N} J(x)dx = 1$. Recently, the mentioned nonlocal diffusion problem of convolution type above and its variants have been widely used to model dispersal processes, here we only refer the readers to the papers [32–35]. As far as we know, however, the study of the nonlocal diffusion equations with convolution terms in the time-fractional case is still limited. This is the second motivation of our paper.

Fourier transform is the basic tool of harmonic analysis, and it plays a very important role in analyzing PDEs. To the best of our knowledge, there are only a few works studying the time-fractional PDEs by utilizing the techniques of Fourier transform or harmonic analysis, we refer to [36–38]. In this paper, for the forthcoming time-fractional nonlocal evolution equations governed by the convolution kernels (see (1.1) and (1.2) below), we attempt to make use of the Fourier transform to investigate the existence and uniqueness of the solutions and their asymptotic behaviour as $t \to +\infty$. In other words, to some extent, the results given in this paper generalize some results of [31] into the time-fractional case. This is the third motivation of the current paper.

Based on the above discussions, in the present paper, we study the Cauchy problem for the following time-fractional linear nonlocal diffusion equation

$$\begin{cases} {}^C\partial_{0,t}^\alpha u(x,t) = J * u(x,t) - u(x,t) \\ \ = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t), \ x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

and the higher-order problem of the form

$$\begin{cases} {}^C\partial_{0,t}^\alpha u(x,t) = (-1)^{n-1}(J * I - 1)^n\big(u(x,t)\big) \\ \ = (-1)^{n-1}\left(\sum_{k=0}^n \binom{n}{k}(-1)^{n-k}(J*)^k(u)\right)(x,t), \ x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$, ${}^C\partial_{0,t}^\alpha u$ denotes the Caputo type time-fractional derivative of the function $u(x,t) = u(x_1, x_2, \cdots, x_N, t)$ of order α (see Definition 1 in Sect. 2), $N \geq 1$, $*$ denotes the convolution, $n \in \mathbb{N}^+$ and $n \geq 2$, I denotes the identity operator, $\binom{n}{k} = C_n^k = \frac{n!}{(n-k)!k!}$, $(J*)^k(u) = J * \underbrace{\cdots}_{k} * J * u$, the convolution kernel $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function and satisfies the following hypothesis, which will be imposed throughout this paper:

(H): $J \in C(\mathbb{R}^N, \mathbb{R}^+)$ is a radial function with $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\mathrm{d}x = 1$.

The remainder of this paper is organized as follows. Section 2 collects some necessary preliminaries. In Sect. 3, we deal with the Cauchy problem for (1.1). The Laplace transform and the Fourier transform help us obtain the explicit expression of the solutions in Fourier variables. After establishing the existence and uniqueness result, we get the asymptotic behaviour of the unique solution and prove that this asymptotic behaviour is same with the one for the unique solution of a space-time fractional diffusion equation given by the same time-fractional derivative and the fractional Laplacian. The rescaled problem then be considered in the end of Sect. 3 by rescaling the convolution kernel J in an appropriate way. Finally, in Sect. 4, the higher-order problem (1.2) will be treated in exactly the same process as Sect. 3.

2. Preliminaries

In this section, we give some necessary preliminaries involving the fractional calculus, the Fourier transform, the Laplace transform, the fractional Laplacian and some properties of the function J under the hypothesis (H).

Definition 1. (See [1–3]) Let $\varphi : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$, $\alpha \in (0, 1)$. Then the fractional integral of the function φ with the lower terminal $t = 0$ of order α is defined by

$$I^{\alpha}_{0,t}\varphi(x,t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(x,s)\mathrm{d}s,\ t > 0,$$

where $\Gamma(\cdot)$ denotes the usual Gamma function.

The Caputo fractional derivative of the function φ with the lower terminal $t = 0$ of order α is defined by

$${}^C\partial^{\alpha}_{0,t}\varphi(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\frac{\partial\varphi(x,s)}{\partial s}\mathrm{d}s,\ t > 0.$$

Lemma 1. *(See [2]) Let $\alpha \in (0, 1)$. If $\varphi \in C\big([0, +\infty); \mathbb{R}\big)$, then*

$$I^{\alpha}_{0,t}\big({}^C\partial^{\alpha}_{0,t}\varphi(t)\big) = \varphi(t) - \varphi(0).$$

Lemma 2. *(See [39,40]) For the Mittag-Leffler function $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{C}$, $\alpha > 0$, if $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, then there exist positive constants m_α, M_α depending only on α_1, α_2 such that*
(i) $E_{\alpha,1}(-z) > 0$, for any $z > 0$; (ii) $\frac{m_\alpha}{1+z} \le E_{\alpha,1}(-z) \le \frac{M_\alpha}{1+z}$, for all $z > 0$.

Lemma 3. *(See [41]) For the Wright-type function*

$$\mathcal{M}_\alpha(\theta) := \sum_{k=0}^{\infty} \frac{\theta^k}{k!\Gamma(-\alpha k + 1 - \alpha)},\ 0 < \alpha < 1,$$

it holds

$$\mathcal{M}_\alpha(\theta) \ge 0,\ \theta > 0;\ \int_0^\infty \theta^\delta \mathcal{M}_\alpha(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \delta > -1.$$

Moreover, the following relationship holds

$$E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(\theta)e^{-z\theta}\mathrm{d}s,\ z \in \mathbb{C}.$$

Definition 2. (See [31]) Let $f \in L^1(\mathbb{R}^N)$. Then the Fourier transform of f is defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^N} e^{-i(x,\xi)}f(x)\mathrm{d}x,$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^N$. The inverse Fourier transform of f is defined by

$$\mathcal{F}^{-1}[f](x) = \check{f}(x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i(x,\xi)}f(\xi)\mathrm{d}\xi.$$

Moreover, the inverse of the Fourier transform operator $\mathcal{F}$ is given by the inverse formula

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i(x,\xi)}\hat{f}(\xi)\mathrm{d}\xi. \tag{2.1}$$

Lemma 4. *(See [31,42])*

(i) Let $f, \hat{f} \in L^1(\mathbb{R}^N)$. Then (2.1) holds for a.e. $x \in \mathbb{R}^N$.

(ii) *Let* $f, g \in L^1(\mathbb{R}^N)$. *Then* $f * g \in L^1(\mathbb{R}^N)$ *and* $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
(iii) *For a nonzero real number* b, *it holds* $\widehat{f(bx)} = \frac{1}{|b|}\hat{f}(\frac{\xi}{b})$.

Definition 3. (See [43]) Consider the Schwarz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decaying C^∞ functions in $\mathbb{R}^N$. Let $f \in \mathcal{S}(\mathbb{R}^N)$ and $s \in (0,1)$. Then the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s f(x) := C(N,s) P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2s}} \mathrm{d}y,$$

where P.V. is a commonly used abbreviation for "in the principal value sense" and $C(N,s)$ is a dimensional constant that depends on N and s, precisely given by

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|x-y|^{N+2s}} \mathrm{d}\zeta \right)^{-1},$$

where $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N)$.

Lemma 5. (See [31,43]) *Let* $0 < \lambda \leq 2$ *and* $f \in \mathcal{S}(\mathbb{R}^N)$. *Then the Fourier transform of the fractional Laplacian of* f *is given by*

$$\mathcal{F}[(-\Delta)^{\frac{\lambda}{2}} f](\xi) = |\xi|^\lambda \hat{f}(\xi).$$

Remark 1. More generally, let $0 < \lambda \leq 2$ and $f \in \mathcal{S}(\mathbb{R}^N)$, then for fixed positive integer $n \in \mathbb{N}^+$, it holds

$$\begin{aligned} \mathcal{F}[(-\Delta)^{\frac{n\lambda}{2}} f](\xi) &= |\xi|^\lambda \mathcal{F}[(-\Delta)^{\frac{(n-1)\lambda}{2}} f](\xi) \\ &= |\xi|^{2\lambda} \mathcal{F}[(-\Delta)^{\frac{(n-2)\lambda}{2}} f](\xi) \\ &= \cdots = |\xi|^{(n-1)\lambda} \mathcal{F}[(-\Delta)^{\frac{\lambda}{2}} f](\xi) \\ &= |\xi|^{n\lambda} \hat{f}(\xi). \end{aligned} \tag{2.2}$$

Lemma 6. (See [1]) *Let* $\alpha \in (0,1)$. *Assume that* f *is a continuous (or piecewise continuous) function defined on* $[0,\infty)$ *of exponential order* γ, *which means that there exist positive constants* M *and* T *such that* $e^{-\gamma t}|f(t)| \leq M$ *for all* $t > T$. *Then it holds*

$$(i)\ \mathcal{L}\{{}^C\partial_{0,t}^\alpha f(t)\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(0);$$

$$(ii)\ \mathcal{L}\{E_{\alpha,1}(\pm\lambda t^\alpha\}(s) = \frac{s^{\alpha-1}}{s^\alpha \mp \lambda},$$

where $\mathcal{L}\{f(t)\}(s)$ *denotes the Laplace transform of* f.

Lemma 7. (See [31]) *Under the hypothesis* (H) *for the function* J, *it holds*

(i) $\hat{J}(\xi)$ *is real for all* $\xi \in \mathbb{R}^N$ *due to the symmetry of* J.
(ii) $|\hat{J}(\xi)| \leq 1$, $\hat{J}(0) = 1$.
(iii) *Assume that there exist* $A > 0$ *and* $0 < \beta \leq 2$ *such that*

$$\hat{J}(\xi) = 1 - A|\xi|^\beta + o(|\xi|^\beta),\ \xi \to 0, \tag{2.3}$$

then

$$\left|\hat{J}(\xi) - 1 + A|\xi|^\beta\right| \leq |\xi|^\beta h(\xi),$$

where $h \geq 0$ *is bounded and satisfies* $h(\xi) \to 0$ *as* $\xi \to 0$.

Moreover, for a given constant $B > 0$, *there exist* $a > 0$ *and* $0 < \delta < 1$ *such that*

$$\left|\hat{J}(\xi) - 1 + A|\xi|^\beta\right| \leq B|\xi|^\beta,\ |\xi| \leq a,$$

and

$$|\hat{J}(\xi)| \leq 1-\delta, \ \ |\xi| \geq a.$$

Remark 2. In Lemma 7 (iii), the notation $f(\xi) = g(\xi) + o(|\xi|^{\beta})$ as $\xi \to 0$ means that

$$\lim_{\xi \to 0} \frac{f(\xi) - g(\xi)}{|\xi|^{\beta}} = 0.$$

3. The Cauchy problem for (1.1)

There are three tasks in this section, the first one is to discuss the existence and uniqueness of the solution of (1.1) by Fourier variable, the second one is to investigate the asymptotic behaviour of the unique solution as $t \to +\infty$ and the last one is to analyze the rescaled problem.

3.1. Existence and uniqueness

In this subsection, we utilize the techniques of the Laplace transform and the Fourier transform to obtain the existence and uniqueness of the solution of (1.1). Before this, we first present the following definition concerning the concept of the solution of (1.1).

Definition 4. By a mild solution of the time-fractional nonlocal diffusion problem (1.1), we mean that a function $u \in C\big([0,+\infty); L^1(\mathbb{R}^N)\big)$ satisfying (1.1) in the sense of integration, that is, u satisfies

$$u(t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{\mathbb{R}^N} J(x-y)u(y,s)\mathrm{d}y - u(x,s)\mathrm{d}s.$$

Next, we prove a useful lemma that will play an important role in the main result in this subsection.

Lemma 8. *For any fixed $t > 0$, the Mittag-Leffler function $E_{\alpha,1}\big(-(1-\hat{J}(\cdot))t^{\alpha}\big)$ is continuous and bounded on the whole $\mathbb{R}^N$.*

Proof. With the consideration of Lemma 7 (i), the continuity then follows immediately by recalling that $E_{\alpha,1}(z)$ $(z \in \mathbb{C})$ is an entire function and noting that $-(1-\hat{J}(\cdot))t^{\alpha}$ is continuous. As for the boundedness, it is easy to observe from Lemma 2 (ii) and Lemma 7 (ii) that

$$E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^{\alpha}\big) \leq \frac{M_{\alpha}}{1+(1-\hat{J}(\xi))t^{\alpha}} \leq M_{\alpha}$$

holds true for all $\xi \in \mathbb{R}^N$. □

Now, we are ready to give the existence and uniqueness result.

Theorem 1. *Let $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$. Then (1.1) has a unique solution $u \in C\big([0,+\infty); L^1(\mathbb{R}^N)\big)$, which can be given in the form of Fourier variable by*

$$\hat{u}(\xi,t) = E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^{\alpha}\big) \cdot \widehat{u_0}(\xi). \tag{3.1}$$

Proof. Applying the spatial Fourier transform to (1.1) and using Lemma 4 (ii), we have

$$\begin{cases} {}^{C}\partial_{0,t}^{\alpha}\hat{u}(\xi,t) = \hat{J}(\xi)\cdot\hat{u}(\xi,t) - \hat{u}(\xi,t) = (\hat{J}(\xi)-1)\hat{u}(\xi,t), \ t > 0, \\ \hat{u}(\xi,0) = \widehat{u_0}(\xi). \end{cases} \tag{3.2}$$

By taking the temporal Laplace transform to the first equation of (3.2) and using Lemma 6 (i), we get

$$\mathcal{L}\{\hat{u}(\xi,t)\}(s) = \frac{s^{\alpha-1}\widehat{u_0}(\xi)}{s^\alpha + (1-\hat{J}(\xi))}. \tag{3.3}$$

Next, we take the inverse Laplace transform to (3.3) and make use of Lemma 6 (ii), it then yields that

$$\hat{u}(\xi,t) = E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big)\cdot\widehat{u_0}(\xi). \tag{3.4}$$

In view of Lemma 8 and $\widehat{u_0} \in L^1(\mathbb{R}^N)$, the inverse Fourier transform then can be utilized to (3.4) to guarantee the existence. Finally, as for uniqueness, we recall that the Fourier transform operator $\mathcal{F}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^N)$ onto $\mathcal{S}(\mathbb{R}^N)$, which indicates that its inverse exists and is unique. □

3.2. Asymptotic behaviour

We focus our attention in this subsection on the analysis of the asymptotic behaviour of the unique solution of (1.1). We shall prove that the asymptotic behaviour is same as the one that holds for the unique solution of the following space-time fractional diffusion equation

$${}^C\partial^\alpha_{0,t}v(x,t) = -A(-\Delta)^{\frac{\beta}{2}}v(x,t),\ x\in\mathbb{R}^N,\ t>0, \tag{3.5}$$

which is equipped with the same initial datum with (1.1), i.e.,

$$v(x,0) = u_0(x),\ x\in\mathbb{R}^N, \tag{3.6}$$

where the definition of the fractional Laplacian is from Definition 3 and the constants A,β can be found if (2.3) holds.

Remark 3. Obviously, if $\beta = 2$, then (3.5) becomes a standard time-fractional heat equation, which has been considered as a hot topic in mathematical physics researches in recent years. For more related works and further information about the time-fractional heat equations and their applications, one can refer to [44–49] and the references therein.

Firstly, by using the similar ideas as in the proof of Theorem 1, we give the explicit formula of the solution to (3.5)-(3.6) in Fourier variable.

Theorem 2. *Let $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$. Then, in Fourier variable, the unique solution of (3.5)-(3.6) can be expressed by*

$$\hat{v}(\xi,t) = E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\cdot\widehat{u_0}(\xi). \tag{3.7}$$

Proof. By the Fourier transform and Lemma 5, it follows that

$$\begin{cases} {}^C\partial^\alpha_{0,t}\hat{v}(\xi,t) = -A|\xi|^\beta\hat{v}(\xi,t),\ t>0,\\ \hat{v}(\xi,0) = \widehat{u_0}(\xi). \end{cases} \tag{3.8}$$

Acting the Laplace transform on the first equation of (3.8) and using Lemma 6 (i), we then obtain

$$\mathcal{L}\{\hat{v}(\xi,t)\}(s) = \frac{s^{\alpha-1}\widehat{u_0}(\xi)}{s^\alpha + A|\xi|^\beta}. \tag{3.9}$$

Furthermore, taking the inverse Laplace transform to (3.9) and using Lemma 6 (ii), we have

$$\hat{v}(\xi,t) = E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\cdot\widehat{u_0}(\xi).$$

Observe that $E_{\alpha,1}\big(-A|\cdot|^\beta t^\alpha\big)$ is bounded and continuous on the whole $\mathbb{R}^N$ (similar to Lemma 8) and $\widehat{u_0} \in L^1(\mathbb{R}^N)$, the desired result follows automatically by the inverse Fourier transform. □

Before showing our main result in this section, it is essential to emphasize the feasibility of the following basic fact.

Lemma 9. *Let $\alpha \in (0,1), \beta \in (0,2]$. Then there must exist a real-valued function r satisfying*

$$\lim_{t\to+\infty} r(t) = 0, \ \lim_{t\to+\infty} r(t)t^{\frac{\alpha}{\beta}} = +\infty. \tag{3.10}$$

Proof. This result is obvious. In fact, we can find such functions r of power type, like $r(t) = t^{\mu}$ with $-\frac{\alpha}{\beta} < \mu < 0$. □

Now, we give the following asymptotic behaviour result for the particular case $N = 1$.

Theorem 3. *Suppose that there exist $A > 0$ and $1 < \beta \le 2$ such that (2.3) holds. Let $u_0, \widehat{u_0} \in L^1(\mathbb{R})$. Then the asymptotic behaviour of the unique solution of (1.1) is given by*

$$\lim_{t\to+\infty} t^{\frac{\alpha}{\beta}} \max_{x\in\mathbb{R}} |u(x,t) - v(x,t)| = 0, \tag{3.11}$$

where v is the unique solution of (3.5)-(3.6). Moreover, there exists a constant $C_1 > 0$ such that

$$\big\|u(\cdot,t) - v(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \le C_1 t^{-\frac{\alpha}{\beta}}. \tag{3.12}$$

Proof. By (3.1) and (3.7), we have in Fourier variable that

$$\begin{aligned}
&\int_{\mathbb{R}} |\hat{u} - \hat{v}|(\xi,t)d\xi \\
&= \int_{\mathbb{R}} \left| \Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big) - E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\Big)\widehat{u_0}(\xi)\right| d\xi \\
&\le \underbrace{\int_{|\xi|\ge r(t)} \left| \Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big) - E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\Big)\widehat{u_0}(\xi)\right| \mathrm{d}\xi}_{I_1} \\
&\quad + \underbrace{\int_{|\xi|< r(t)} \left| \Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big) - E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\Big)\widehat{u_0}(\xi)\right| \mathrm{d}\xi}_{I_2},
\end{aligned}$$

where the function r can be found by Lemma 9. To deal with I_1, we note that it holds

$$\begin{aligned}
I_1 \le &\underbrace{\int_{|\xi|\ge r(t)} \big|E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\big| \cdot |\widehat{u_0}(\xi)| \mathrm{d}\xi}_{I_{11}} \\
&+ \underbrace{\int_{|\xi|\ge r(t)} \big|E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big)\big| \cdot |\widehat{u_0}(\xi)| \mathrm{d}\xi}_{I_{12}}.
\end{aligned}$$

On the one hand, by changing the variable $\eta = \xi t^{\frac{\alpha}{\beta}}$ and using Lemma 2, we get

$$\begin{aligned}
I_{11} &\le t^{-\frac{\alpha}{\beta}} \int_{|\eta|\ge r(t)t^{\frac{\alpha}{\beta}}} E_{\alpha,1}\big(-A|\eta|^\beta\big)|\widehat{u_0}(\eta t^{-\frac{\alpha}{\beta}})| d\eta \\
&\le t^{-\frac{\alpha}{\beta}} M_\alpha \int_{|\eta|\ge r(t)t^{\frac{\alpha}{\beta}}} \frac{1}{1 + A|\eta|^\beta} |\widehat{u_0}(\eta t^{-\frac{\alpha}{\beta}})| \mathrm{d}\eta,
\end{aligned}$$

which implies that

$$t^{\frac{\alpha}{\beta}} I_{11} \leq M_\alpha \|\widehat{u_0}\|_{L^\infty(\mathbb{R})} \int\limits_{|\eta|\geq r(t)t^{\frac{\alpha}{\beta}}} \frac{1}{1+A|\eta|^\beta} \mathrm{d}\eta \to 0,\ t\to +\infty. \tag{3.13}$$

On the other hand, let a, δ be as in Lemma 7 (iii) for $B = \frac{A}{2}$, then we can decompose I_{12} into two parts as follows:

$$I_{12} \leq \underbrace{\int\limits_{a\geq|\xi|\geq r(t)} E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big)|\widehat{u_0}(\xi)|d\xi}_{I_{12}^1} + \underbrace{\int\limits_{|\xi|\geq a} E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big)|\widehat{u_0}(\xi)|\mathrm{d}\xi}_{I_{12}^2}.$$

For I_{12}^1, it follows from Lemma 7 (iii), the inequality $ye^{-y} \leq 1$ for $y > 0$, Lemma 3 and Lemma 2 (ii) that

$$\begin{aligned}
I_{12}^1 &= \int\limits_{a\geq|\xi|\geq r(t)} \Bigg(\int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-(1-\hat{J}(\xi))t^\alpha\theta}\mathrm{d}\theta\Bigg)|\widehat{u_0}(\xi)|d\xi \\
&= \int\limits_{a\geq|\xi|\geq r(t)} \Bigg(\int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-A|\xi|^\beta t^\alpha\theta + [\hat{J}(\xi)-1+A|\xi|^\beta]t^\alpha\theta}\mathrm{d}\theta\Bigg)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&\leq \int\limits_{a\geq|\xi|\geq r(t)} \Bigg(\int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-A|\xi|^\beta t^\alpha\theta} e^{\left|\hat{J}(\xi)-1+A|\xi|^\beta\right|t^\alpha\theta} d\theta\Bigg)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&\leq \int\limits_{a\geq|\xi|\geq r(t)} \Bigg(\int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-\frac{A}{2}|\xi|^\beta t^\alpha\theta} d\theta\Bigg)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&= \int\limits_{a\geq|\xi|\geq r(t)} E_{\alpha,1}\Big(-\frac{A}{2}|\xi|^\beta t^\alpha\Big)|\widehat{u_0}(\xi)|d\xi \\
&\leq M_\alpha \int\limits_{a\geq|\xi|\geq r(t)} \frac{1}{1+\frac{A}{2}|\xi|^\beta t^\alpha}|\widehat{u_0}(\xi)|\mathrm{d}\xi.
\end{aligned}$$

Setting $\eta = \xi t^{\frac{\alpha}{\beta}}$ again, we then obtain

$$\begin{aligned}
t^{\frac{\alpha}{\beta}} I_{12}^1 &\leq M_\alpha \int\limits_{at^{\frac{\alpha}{\beta}}\geq|\eta|\geq r(t)t^{\frac{\alpha}{\beta}}} \frac{1}{1+\frac{A}{2}|\eta|^\beta}|\widehat{u_0}(\eta t^{-\frac{\alpha}{\beta}})|d\eta \\
&\leq M_\alpha \|\widehat{u_0}\|_{L^\infty(\mathbb{R})} \int\limits_{|\eta|\geq r(t)t^{\frac{\alpha}{\beta}}} \frac{1}{1+\frac{A}{2}|\eta|^\beta} d\eta \to 0,\ t\to +\infty.
\end{aligned} \tag{3.14}$$

Next, by Lemma 7 (iii) and Lemma 2 (ii) again, we estimate I_{12}^2 as follows:

$$I_{12}^2 = \int\limits_{|\xi|\geq a} E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big)|\widehat{u_0}(\xi)|d\xi$$

$$
\begin{aligned}
&= \int\limits_{|\xi|\geq a} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{(\hat{J}(\xi)-1)t^\alpha\theta} \mathrm{d}\theta\Big) |\widehat{u_0}(\xi)| \mathrm{d}\xi \\
&\leq \int\limits_{|\xi|\geq a} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-t^\alpha\theta} e^{|\hat{J}(\xi)|t^\alpha\theta} \mathrm{d}\theta\Big) |\widehat{u_0}(\xi)| d\xi \\
&\leq \int\limits_{|\xi|\geq a} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-\delta t^\alpha\theta} \mathrm{d}\theta\Big) |\widehat{u_0}(\xi)| d\xi \\
&= \int\limits_{|\xi|\geq a} E_{\alpha,1}\big(-\delta t^\alpha\big) |\widehat{u_0}(\xi)| \mathrm{d}\xi \leq M_\alpha \int\limits_{|\xi|\geq a} \frac{1}{\delta t^\alpha} |\widehat{u_0}(\xi)| \mathrm{d}\xi,
\end{aligned}
$$

which means that

$$
\begin{aligned}
t^{\frac{\alpha}{\beta}} I_{12}^2 &\leq \frac{M_\alpha}{\delta} \int\limits_{|\xi|\geq a} \frac{1}{t^{\alpha(1-\frac{1}{\beta})}} |\widehat{u_0}(\xi)| \mathrm{d}\xi \\
&\leq \frac{M_\alpha}{\delta t^{\alpha(1-\frac{1}{\beta})}} \|\widehat{u_0}\|_{L^1(\mathbb{R})} \to 0,\ t \to +\infty. \qquad (3.15)
\end{aligned}
$$

Combining (3.13),(3.14) with (3.15), we deduce that $t^{\frac{\alpha}{\beta}} I_1 \to 0$ as $t \to +\infty$, and our next task is to check that $t^{\frac{\alpha}{\beta}} I_2 \to 0$ ($t \to +\infty$). In fact, from Lemma 3, Lemma 7 (iii), the elementary inequalities $ye^{-y} \leq 1$ for $y > 0$ and $|e^y - 1| \leq C|y|$ for $|y|$ bounded and some $C > 0$, we derive that

$$
\begin{aligned}
I_2 &= \int\limits_{|\xi|<r(t)} \Big|\Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))t^\alpha\big) - E_{\alpha,1}\big(-A|\xi|^\beta t^\alpha\big)\Big)\widehat{u_0}(\xi)\Big| d\xi \\
&= \int\limits_{|\xi|<r(t)} \Big|\int_0^\infty \mathcal{M}_\alpha(\theta)\Big(e^{-(1-\hat{J}(\xi))t^\alpha\theta} - e^{-A|\xi|^\beta t^\alpha\theta}\Big) \mathrm{d}\theta \cdot \widehat{u_0}(\xi)\Big| \mathrm{d}\xi \\
&= \int\limits_{|\xi|<r(t)} \Big|\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-A|\xi|^\beta t^\alpha\theta}\Big(e^{(\hat{J}(\xi)-1+A|\xi|^\beta)t^\alpha\theta} - 1\Big) d\theta \cdot \widehat{u_0}(\xi)\Big| \mathrm{d}\xi \\
&\leq C \int\limits_{|\xi|<r(t)} \Big|\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-A|\xi|^\beta t^\alpha\theta} h(\xi)|\xi|^\beta t^\alpha \theta \mathrm{d}\theta \cdot \widehat{u_0}(\xi)\Big| \mathrm{d}\xi \\
&\leq C \int\limits_{|\xi|<r(t)} \frac{|\xi|^\beta t^\alpha h(\xi)}{A|\xi|^\beta t^\alpha} \big(\int_0^\infty \mathcal{M}_\alpha(\theta) d\theta\big) |\widehat{u_0}(\xi)| \mathrm{d}\xi \\
&= \frac{C}{A} \int\limits_{|\xi|<r(t)} h(\xi)|\widehat{u_0}(\xi)| \mathrm{d}\xi \leq \frac{C\|\widehat{u_0}\|_{L^\infty(\mathbb{R})}}{A} \int\limits_{|\xi|<r(t)} h(\xi) \mathrm{d}\xi \to 0,\ t \to +\infty.
\end{aligned}
$$

Consequently, the above arguments show that

$$
\lim_{t\to+\infty} t^{\frac{\alpha}{\beta}} \big\|\hat{u}(\cdot,t) - \hat{v}(\cdot,t)\big\|_{L^1(\mathbb{R})} = \lim_{t\to+\infty} t^{\frac{\alpha}{\beta}} \int\limits_{\mathbb{R}} |\hat{u} - \hat{v}|(\xi,t) \mathrm{d}\xi = 0,
$$

which further implies that

$$t^{\frac{\alpha}{\beta}}|u(x,t)-v(x,t)| = t^{\frac{\alpha}{\beta}}\left|\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x,\xi)}\big(\hat{u}(\xi,t)-\hat{v}(\xi,t)\big)\mathrm{d}\xi\right|$$
$$\le t^{\frac{\alpha}{\beta}}\frac{1}{2\pi}\int_{\mathbb{R}}\big|\hat{u}(\xi,t)-\hat{v}(\xi,t)\big|\mathrm{d}\xi \to 0,\ t\to+\infty.$$

This means that the asymptotic behaviour (3.11) is satisfied.

Moreover, it holds

$$\begin{aligned} t^{\frac{\alpha}{\beta}}\big\|u-v\big\|_{L^\infty(\mathbb{R})} &= t^{\frac{\alpha}{\beta}}\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \\ &= t^{\frac{\alpha}{\beta}}\, ess\sup_{x\in\mathbb{R}}\big|u(x,t)-v(x,t)\big| \\ &\le t^{\frac{\alpha}{\beta}}\max_{x\in\mathbb{R}}\big|u(x,t)-v(x,t)\big| \to 0,\ t\to+\infty, \end{aligned}$$

which tells us that there exist $C_1>0$ and $T>0$ such that

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \le C_1 t^{-\frac{\alpha}{\beta}},\ \forall t>T.$$

This ensures the convergence of $u-v$ in $L^\infty(\mathbb{R})$ and the proof is completed. □

Remark 4. Clearly, Theorem 3 indicates that the decay rate is $t^{-\frac{\alpha}{\beta}}$, which depends on not only α (the order of time-fractional derivative) but also β (the order of spatial fractional Laplacian).

3.3. Analysis of the rescaled problem

In this subsection, by rescaling the kernel J in (1.1) via the way $J_\varepsilon(x)=\frac{1}{\varepsilon}J(\frac{x}{\varepsilon})$, we show that the time-fractional heat problem ${}^C\partial^\alpha_{0,t}v(x,t)=\Delta v(x,t)$ can be approximated by a rescaled problem

$$\begin{cases} {}^C\partial^\alpha_{0,t}u_\varepsilon(x,t)=\frac{1}{\varepsilon^2}\big(J_\varepsilon * u_\varepsilon(x,t)-u_\varepsilon(x,t)\big),\ x\in\mathbb{R},\ t>0, \\ u_\varepsilon(x,0)=u_0(x),\ x\in\mathbb{R}. \end{cases} \tag{3.16}$$

Theorem 4. *Assume that (2.3) holds for $A=1,\beta=2$. Let u_ε be the unique solution of (3.16). Then, for every $T>0$, it holds*

$$\lim_{\varepsilon\to0}\|u_\varepsilon-v\|_{L^\infty(\mathbb{R}\times(0,T))}=0, \tag{3.17}$$

where v is the unique solution of the following time-fractional heat equation

$${}^C\partial^\alpha_{0,t}v(x,t)=\Delta v(x,t),\ x\in\mathbb{R},\ t>0, \tag{3.18}$$

which has the same initial condition with (3.16), that is,

$$v(x,0)=u_0(x),\ x\in\mathbb{R}. \tag{3.19}$$

Proof. Using the same methods as in Theorem 1 and Theorem 2, noting that $\int_{\mathbb{R}^N}J_\varepsilon(x)\mathrm{d}x=1$ and $\widehat{J_\varepsilon}(\xi)=\hat{J}(\varepsilon\xi)$ are satisfied by the condition (H) and Lemma 4 (iii) respectively, we can obtain the explicit formulas for the solutions of (3.16) and (3.18)-(3.19) in Fourier variables as follows:

$$\widehat{u_\varepsilon}(\xi,t)=E_{\alpha,1}\big(-\frac{1}{\varepsilon^2}(1-\hat{J}(\varepsilon\xi))t^\alpha\big)\cdot\widehat{u_0}(\xi),$$
$$\hat{v}(\xi,t)=E_{\alpha,1}\big(-|\xi|^2t^\alpha\big)\cdot\widehat{u_0}(\xi).$$

Then, we have in Fourier variable that

$$
\begin{aligned}
&\int_{\mathbb{R}} |\widehat{u_\varepsilon} - \hat{v}|(\xi,t)\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \Big|\Big(E_{\alpha,1}\big(-\frac{1}{\varepsilon^2}(1-\hat{J}(\varepsilon\xi))t^\alpha\big) - E_{\alpha,1}\big(-|\xi|^2 t^\alpha\big)\Big)\widehat{u_0}(\xi)\Big|\mathrm{d}\xi \\
&\le \int_{|\xi|\ge r(\varepsilon)} \Big|\Big(E_{\alpha,1}\big(-\frac{1}{\varepsilon^2}(1-\hat{J}(\varepsilon\xi))t^\alpha\big) - E_{\alpha,1}\big(-|\xi|^2 t^\alpha\big)\Big)\widehat{u_0}(\xi)\Big|\mathrm{d}\xi \\
&\quad + \int_{|\xi|< r(\varepsilon)} \Big|\Big(E_{\alpha,1}\big(-\frac{1}{\varepsilon^2}(1-\hat{J}(\varepsilon\xi))t^\alpha\big) - E_{\alpha,1}\big(-|\xi|^2 t^\alpha\big)\Big)\widehat{u_0}(\xi)\Big|\mathrm{d}\xi,
\end{aligned}
$$

where the function r satisfies $r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Obviously, for $t \in [0, T]$, we can proceed as in the proof of Theorem 3 to derive that

$$
\begin{aligned}
|u_\varepsilon(x,t) - v(x,t)| &= \Big|\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x,\xi)}\big(\widehat{u_\varepsilon}(\xi,t) - \hat{v}(\xi,t)\big)\mathrm{d}\xi\Big| \\
&\le \frac{1}{2\pi}\int_{\mathbb{R}} \big|\widehat{u_\varepsilon}(\xi,t) - \hat{v}(\xi,t)\big|\mathrm{d}\xi \to 0,\ \varepsilon \to 0,
\end{aligned}
$$

which verifies that

$$
\max_{x\in\mathbb{R}, t\in[0,T]} \big|u_\varepsilon(x,t) - v(x,t)\big| \to 0,\ \varepsilon \to 0.
$$

Hence, it must hold

$$
\begin{aligned}
\|u_\varepsilon - v\|_{L^\infty(\mathbb{R}\times(0,T))} &= ess\sup_{(x,t)\in\mathbb{R}\times(0,T)} \big|u_\varepsilon(x,t) - v(x,t)\big| \\
&\le \max_{x\in\mathbb{R}, t\in[0,T]} \big|u_\varepsilon(x,t) - v(x,t)\big| \to 0,\ \varepsilon \to 0.
\end{aligned}
$$

This completes the proof. □

4. Higher-order problem (1.2)

In this section, we consider (1.2), a time-fractional nonlocal diffusion problem with a nonlocal diffusion operator of higher order.

4.1. Existence and uniqueness

Firstly, as it was done for (1.1) in Subsect. 3.1, we will investigate the existence and uniqueness of the solution of (1.2), which should be also understood in the sense of integration.

Theorem 5. *Let $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$. Then (1.2) possesses a unique solution $u \in C\big([0, +\infty); L^1(\mathbb{R}^N)\big)$, which is given in Fourier variable by the explicit formula*

$$
\hat{u}(\xi,t) = E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big)\cdot\widehat{u_0}(\xi). \tag{4.1}
$$

Proof. Applying the Fourier transform to (1.2) and using Lemma 4 (ii), we have

$$\begin{cases} {}^C\partial_{0,t}^{\alpha}\hat{u}(\xi,t) = (-1)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\big(J(\xi)\big)^k\hat{u}(\xi,t) \\ = (-1)^{n-1}\big(J(\xi)-1\big)^n\hat{u}(\xi,t),\ t>0, \\ \hat{u}(\xi,0) = \widehat{u_0}(\xi). \end{cases} \tag{4.2}$$

Taking the Laplace transform to the first equation of (4.2) and using Lemma 6 (i), it yields that

$$\mathcal{L}\{\hat{u}(\xi,t)\}(s) = \frac{s^{\alpha-1}\widehat{u_0}(\xi)}{s^{\alpha}-(-1)^{n-1}\big(\hat{J}(\xi)-1\big)^n}. \tag{4.3}$$

The inverse Laplace transform then can be used to derive

$$\hat{u}(\xi,t) = E_{\alpha,1}\big((-1)^{n-1}\big(\hat{J}(\xi)-1\big)^n t^{\alpha}\big)\cdot\widehat{u_0}(\xi) = E_{\alpha,1}\big(-\big(1-\hat{J}(\xi)\big)^n t^{\alpha}\big)\cdot\widehat{u_0}(\xi).$$

On account that $\hat{u}_0 \in L^1(\mathbb{R}^N)$ and the Mittag-Leffler function $E_{\alpha,1}\big(-\big(1-\hat{J}(\xi)\big)^n t^{\alpha}\big)$ is bounded and continuous, we deduce that $\hat{u}(\cdot,t) \in L^1(\mathbb{R}^N)$ and hence the result follows by conducting the inverse Fourier transform. □

4.2. Asymptotic behaviour

This subsection deals with the asymptotic behaviour of the unique solution of (1.2) as $t \to +\infty$. Before obtaining the main result, we give a result concerning the explicit expression of the solution to the following time-fractional diffusion equation with a high order fractional Laplacian of the form

$${}^C\partial_{0,t}^{\alpha}v(x,t) = -A^n(-\Delta)^{\frac{n\beta}{2}}v(x,t),\ x\in\mathbb{R}^N,\ t>0, \tag{4.4}$$

subjected to the same initial condition with (1.2), i.e.,

$$v(x,0) = u_0(x),\ x\in\mathbb{R}^N. \tag{4.5}$$

Similarly, the constants $A>0$ and $\beta\in(0,2]$ here are assumed to fulfill (2.3).

Theorem 6. *Let $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$. Then, in Fourier variable, the unique solution of (4.4)-(4.5) can be expressed by*

$$\hat{v}(\xi,t) = E_{\alpha,1}\big(-A^n|\xi|^{n\beta}t^{\alpha}\big)\cdot\widehat{u_0}(\xi). \tag{4.6}$$

Proof. The proof can be easily done by using (2.2) and is quite similar to Theorem 2, so we omit it here. □

Theorem 7. *Suppose that there exist $A>0$ and $0<\beta\le 2$ such that (2.3) holds. Let $u_0, \widehat{u_0} \in L^1(\mathbb{R}^N)$ and $N<n\beta$. Then the asymptotic behaviour of the unique solution of (1.2) is given by*

$$\lim_{t\to+\infty} t^{\frac{N\alpha}{n\beta}}\max_{x\in\mathbb{R}^N}|u(x,t)-v(x,t)| = 0,$$

where v is the unique solution of (4.4)-(4.5). Moreover, there exists a constant $C_2>0$ such that

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^{\infty}(\mathbb{R}^N)} \le C_2 t^{-\frac{N\alpha}{n\beta}}.$$

Proof. By combining (4.1) with (4.6), we have in Fourier variable that

$$\begin{aligned} &\int_{\mathbb{R}^N}|\hat{u}-\hat{v}|(\xi,t)\mathrm{d}\xi \\ &= \int_{\mathbb{R}^N}\Big|\Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^{\alpha}\big) - E_{\alpha,1}\big(-A^n|\xi|^{n\beta}t^{\alpha}\big)\Big)\widehat{u_0}(\xi)\Big|\mathrm{d}\xi \end{aligned}$$

$$\le \underbrace{\int\limits_{|\xi|\ge r(t)} \left|\Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big) - E_{\alpha,1}\big(-A^n|\xi|^{n\beta}t^\alpha\big)\Big)\widehat{u_0}(\xi)\right|\mathrm{d}\xi}_{[I_1]}$$

$$+ \underbrace{\int\limits_{|\xi|< r(t)} \left|\Big(E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big) - E_{\alpha,1}\big(-A^n|\xi|^{n\beta}t^\alpha\big)\Big)\widehat{u_0}(\xi)\right|\mathrm{d}\xi}_{[I_2]},$$

where the function r satisfies $r(t)\to 0$ as $t\to+\infty$ and will be determined later.

For $[I_1]$, it holds

$$[I_1] \le \underbrace{\int\limits_{|\xi|\ge r(t)} \left|E_{\alpha,1}\big(-A^n|\xi|^{n\beta}t^\alpha\big)\right|\cdot|\widehat{u_0}(\xi)|\mathrm{d}\xi}_{[I_{11}]}$$

$$+ \underbrace{\int\limits_{|\xi|\ge r(t)} \left|E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big)\right|\cdot|\widehat{u_0}(\xi)|\mathrm{d}\xi}_{[I_{12}]}.$$

Firstly, let us consider $[I_{11}]$. Setting $\eta=\xi t^{\frac{\alpha}{n\beta}}$, then it follows from Lemma 2 that

$$[I_{11}] \le t^{-\frac{N\alpha}{n\beta}} \int\limits_{|\eta|\ge r(t)t^{\frac{\alpha}{n\beta}}} E_{\alpha,1}\big(-A^n|\eta|^{n\beta}\big)|\widehat{u_0}(\eta t^{-\frac{\alpha}{n\beta}})|\mathrm{d}\eta$$

$$\le t^{-\frac{N\alpha}{n\beta}}\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} M_\alpha \int\limits_{|\eta|\ge r(t)t^{\frac{\alpha}{n\beta}}} \frac{1}{1+A^n|\eta|^{n\beta}}\mathrm{d}\eta,$$

which implies that

$$t^{\frac{N\alpha}{n\beta}}[I_{11}] \to 0 \ (t\to+\infty) \tag{4.7}$$

holds true if we assume that

$$r(t)t^{\frac{\alpha}{n\beta}} \to +\infty \ (t\to+\infty). \tag{4.8}$$

Next, we check that

$$t^{\frac{N\alpha}{n\beta}}[I_{12}] \to 0 \ (t\to+\infty). \tag{4.9}$$

On the one hand, by decomposing $[I_{12}]$ into two parts, we obtain

$$[I_{12}] \le \underbrace{\int\limits_{a_0\ge|\xi|\ge r(t)} E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big)|\widehat{u_0}(\xi)|\mathrm{d}\xi}_{[I_{12}^1]}$$

$$+ \underbrace{\int\limits_{|\xi|\ge a_0} E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big)|\widehat{u_0}(\xi)|d\xi}_{[I_{12}^2]},$$

where a_0 is an arbitrary positive constant. Let a and δ be as in Lemma 7 (iii) for $B = \frac{1}{2n}A$ and choose $a_0 = a$. Then it follows from Lemma 2 (ii) and Lemma 3 that

$$
\begin{aligned}
[I_{12}^1] &= \int\limits_{a\ge|\xi|\ge r(t)} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-(1-\hat{J}(\xi))^n t^\alpha \theta} \mathrm{d}\theta\Big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&\le \int\limits_{a\ge|\xi|\ge r(t)} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-(1-\frac{1}{2n})^n A^n |\xi|^{n\beta} t^\alpha \theta} \mathrm{d}\theta\Big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&= \int\limits_{a\ge|\xi|\ge r(t)} E_{\alpha,1}\big(-(1-\frac{1}{2n})^n A^n |\xi|^{n\beta} t^\alpha\big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&\le M_\alpha \int\limits_{a\ge|\xi|\ge r(t)} \frac{1}{1+(1-\frac{1}{2n})^n A^n |\xi|^{n\beta} t^\alpha}|\widehat{u_0}(\xi)|\mathrm{d}\xi.
\end{aligned} \tag{4.10}
$$

Then, by changing the variable in (4.10) as before, i.e., $\eta = \xi t^{\frac{\alpha}{n\beta}}$, we have

$$
\begin{aligned}
t^{\frac{N\alpha}{n\beta}}[I_{12}^1] &\le M_\alpha \int\limits_{at^{\frac{\alpha}{n\beta}}\ge|\eta|\ge r(t)t^{\frac{\alpha}{n\beta}}} \frac{1}{1+(1-\frac{1}{2n})^n A^n |\eta|^{n\beta}}|\widehat{u_0}(\eta t^{-\frac{\alpha}{n\beta}})|d\eta \\
&\le \frac{M_\alpha}{(1-\frac{1}{2n})^n A^n}\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int\limits_{|\eta|\ge r(t)t^{\frac{\alpha}{n\beta}}} \frac{1}{|\eta|^{n\beta}}\mathrm{d}\eta \to 0,
\end{aligned}
$$

as $t\to+\infty$, provided that (4.8) holds. On the other hand, we estimate $[I_{12}^2]$ as follows:

$$
\begin{aligned}
[I_{12}^2] &= \int\limits_{|\xi|\ge a} E_{\alpha,1}\big(-(1-\hat{J}(\xi))^n t^\alpha\big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&= \int\limits_{|\xi|\ge a} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-(1-\hat{J}(\xi))^n t^\alpha \theta} d\theta\Big)|\widehat{u_0}(\xi)|d\xi \\
&\le \int\limits_{|\xi|\ge a} \Big(\int_0^\infty \mathcal{M}_\alpha(\theta) e^{-\delta^n t^\alpha \theta} d\theta\Big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \\
&= \int\limits_{|\xi|\ge a} E_{\alpha,1}\big(-\delta^n t^\alpha\big)|\widehat{u_0}(\xi)|\mathrm{d}\xi \le M_\alpha \int\limits_{|\xi|\ge a} \frac{1}{\delta^n t^\alpha}|\widehat{u_0}(\xi)|\mathrm{d}\xi,
\end{aligned}
$$

which shows that

$$
t^{\frac{N\alpha}{n\beta}}[I_{12}^2] \le \frac{M_\alpha \|\widehat{u_0}\|_{L^1(\mathbb{R}^N)}}{\delta^n t^\alpha (1-\frac{N}{n\beta})} \to 0,\ t\to+\infty. \tag{4.11}
$$

Therefore, (4.9) holds if we impose the condition (4.8).

Now, it remains for us to deal with $[I_2]$. By the relationship $E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(\theta)e^{-z\theta}\mathrm{d}s,\ z\in\mathbb{C}$ again, we can rewrite $[I_2]$ as

$$
[I_2] = \int\limits_{|\xi|<r(t)} \Big|\int_0^\infty \mathcal{M}_\alpha(\theta)\Big(e^{-(1-\hat{J}(\xi))^n t^\alpha \theta} - e^{-A^n|\xi|^{n\beta} t^\alpha \theta}\Big)d\theta \cdot \widehat{u_0}(\xi)\Big|\mathrm{d}\xi
$$

$$= \int\limits_{|\xi|<r(t)} \Big| \int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-A^n|\xi|^{n\beta}t^\alpha\theta} \Big(e^{\left(-(1-\hat{J}(\xi))^n + A^n|\xi|^{n\beta}\right)t^\alpha\theta} - 1 \Big) \mathrm{d}\theta \cdot \widehat{u_0}(\xi) \Big| \mathrm{d}\xi.$$

By the binomial formula, it holds

$$\begin{aligned}
& -(1-\hat{J}(\xi))^n + A^n|\xi|^{n\beta} \\
&= (-1)^{n-1}(\hat{J}(\xi)-1)^n + A^n|\xi|^{n\beta} \\
&= (-1)^{n-1}\left[\left(\hat{J}(\xi)-1+A|\xi|^\beta\right) - A|\xi|^\beta\right]^n + A^n|\xi|^{n\beta} \\
&= (-1)^{n-1}\left(\sum_{k=0}^n \binom{n}{k} \left(\hat{J}(\xi)-1+A|\xi|^\beta\right)^k (-A|\xi|^\beta)^{n-k}\right) + A^n|\xi|^{n\beta} \\
&= (-1)^{n-1}\left(\sum_{k=1}^n \binom{n}{k} \left(\hat{J}(\xi)-1+A|\xi|^\beta\right)^k (-A|\xi|^\beta)^{n-k}\right) \\
&\quad + (-1)^{n-1}(-A|\xi|^\beta)^n + A^n|\xi|^{n\beta} \\
&= (-1)^{n-1}\left(\sum_{k=1}^n \binom{n}{k} \left(\hat{J}(\xi)-1+A|\xi|^\beta\right)^k (-A|\xi|^\beta)^{n-k}\right).
\end{aligned}$$

Next, by Lemma 7 (iii), the elementary inequalities $ye^{-y} \le 1$ for $y > 0$ and $|e^y - 1| \le C|y|$ for $|y|$ bounded and some $C > 0$, we have

$$\begin{aligned}
[I_2] &\le C \int\limits_{|\xi|<r(t)} \int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-A^n|\xi|^{n\beta}t^\alpha\theta} t^\alpha\theta \Big(\sum_{k=1}^n \binom{n}{k} \big|\hat{J}(\xi) - 1 + A|\xi|^\beta\big|^k \\
&\quad \times \big| -A|\xi|^\beta\big|^{n-k} \Big) d\theta \cdot \widehat{u_0}(\xi) \mathrm{d}\xi \\
&\le C \int\limits_{|\xi|<r(t)} \int\limits_0^\infty \mathcal{M}_\alpha(\theta) e^{-A^n|\xi|^{n\beta}t^\alpha\theta} t^\alpha\theta \Big(\sum_{k=1}^n \binom{n}{k} A^{n-k} \big(h(\xi)\big)^k \Big) \\
&\quad \times |\xi|^{n\beta} d\theta \cdot \widehat{u_0}(\xi) \mathrm{d}\xi \\
&\le CC_{A,h} \int\limits_{|\xi|<r(t)} \frac{|\xi|^{n\beta}t^\alpha h(\xi)}{A^n|\xi|^{n\beta}t^\alpha} \Big(\int\limits_0^\infty \mathcal{M}_\alpha(\theta) d\theta \Big) |\widehat{u_0}(\xi)| \mathrm{d}\xi \\
&= \frac{CC_{A,h}}{A^n} \int\limits_{|\xi|<r(t)} h(\xi) |\widehat{u_0}(\xi)| d\xi \\
&\le \frac{CC_{A,h}\|\widehat{u_0}\|_{L^\infty(\mathbb{R}^N)}}{A^n} \int\limits_{|\xi|<r(t)} h(\xi)\mathrm{d}\xi \to 0, \ t \to +\infty,
\end{aligned} \tag{4.12}$$

where the constant $C_{A,h} > 0$ satisfies the inequality

$$\Big(\sum_{k=1}^n \binom{n}{k} A^{n-k} \big(h(\xi)\big)^k \Big) \le C_{A,h} h(\xi).$$

Hence, we deduce from (4.7), (4.9) and (4.12) that

$$t^{\frac{N\alpha}{n\beta}} \int\limits_{\mathbb{R}^N} |\hat{u}-\hat{v}|(\xi,t)\mathrm{d}\xi \le t^{\frac{N\alpha}{n\beta}}([I_1]+[I_2]) \to 0,\ t\to+\infty \tag{4.13}$$

holds true if there exists a function r with $r(t)\to 0$ as $t\to+\infty$ such that the condition (4.8) is satisfied. In fact, this can be done by Lemma 9 changing there β by $n\beta$.

By (4.13), we know

$$\lim_{t\to+\infty} t^{\frac{N\alpha}{n\beta}} \big\|\hat{u}(\cdot,t)-\hat{v}(\cdot,t)\big\|_{L^1(\mathbb{R}^N)} = \lim_{t\to+\infty} t^{\frac{N\alpha}{n\beta}} \int\limits_{\mathbb{R}^N} |\hat{u}-\hat{v}|(\xi,t)\mathrm{d}\xi = 0,$$

which shows that

$$\begin{aligned} t^{\frac{N\alpha}{n\beta}} |u(x,t)-v(x,t)| &= t^{\frac{N\alpha}{n\beta}} \left| \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} e^{i(x,\xi)}\big(\hat{u}(\xi,t)-\hat{v}(\xi,t)\big)\mathrm{d}\xi \right| \\ &\le t^{\frac{N\alpha}{n\beta}} (2\pi)^{-N} \int\limits_{\mathbb{R}^N} \big|\hat{u}(\xi,t)-\hat{v}(\xi,t)\big|\mathrm{d}\xi \to 0,\ t\to+\infty. \end{aligned}$$

Therefore, we obtain

$$t^{\frac{N\alpha}{n\beta}} \max_{x\in\mathbb{R}^N} |u(x,t)-v(x,t)| \to 0,\ t\to+\infty.$$

Furthermore, it holds

$$\begin{aligned} t^{\frac{N\alpha}{n\beta}} \big\|u-v\big\|_{L^\infty(\mathbb{R}^N)} &= t^{\frac{N\alpha}{n\beta}} \big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \\ &= t^{\frac{N\alpha}{n\beta}} ess\sup_{x\in\mathbb{R}^N} \big|u(x,t)-v(x,t)\big| \\ &\le t^{\frac{N\alpha}{n\beta}} \max_{x\in\mathbb{R}^N} |u(x,t)-v(x,t)| \to 0,\ t\to+\infty, \end{aligned}$$

which states that there exist $C_2>0$ and $T>0$ such that

$$\big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^\infty(\mathbb{R}^N)} \le C_2 t^{-\frac{N\alpha}{n\beta}},\ \forall t>T.$$

This guarantees the convergence of $u-v$ in $L^\infty(\mathbb{R}^N)$. □

4.3. Rescaling the kernel in (1.2)

With regard to the corresponding rescaled problem for (1.2), we conclude this section with a direct generalization of Subsect. 3.3 by using the same procedure with the proof of Theorem 7.

Corollary 1. *Suppose that all the conditions in Theorem 7 are satisfied. Rescaling the kernel J by*

$$J_\varepsilon(x) = \varepsilon^{-N} J(\frac{x}{\varepsilon}).$$

Let u_ε be the unique solution of the following rescaled higher-order nonlocal problem

$$\begin{cases} {}^C\partial^\alpha_{0,t} u_\varepsilon(x,t) = \frac{(-1)^{n-1}}{\varepsilon^{n\beta}}\big(J_\varepsilon * I - 1\big)^n\big(u_\varepsilon(x,t)\big),\ x\in\mathbb{R}^N,\ t>0, \\ u_\varepsilon(x,0) = u_0(x),\ x\in\mathbb{R}^N. \end{cases} \tag{4.14}$$

Then, for every $T>0$, it holds

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - v\|_{L^\infty(\mathbb{R}^N\times(0,T))} = 0, \tag{4.15}$$

where v is the unique solution of (4.4)-(4.5).

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Declarations

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