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Sign-changing solutions for Kirchhoff-type equations with indefinite nonlinearities

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Abstract. We are interested in the existence of sign-changing solutions for the following Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=\left(h^+(x)+\lambda h^-(x)\right)|u|^{p-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $a,b>0$, $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, the potential $h:\overline{\Omega}\to\mathbb{R}$ is a sign-changing continuous function, and $\lambda>0$ is a parameter. If $p\in(4,6)$, we prove the existence of least energy sign-changing solution $u_{b,\lambda}$, the asymptotic behavior of $u_{b,\lambda}$ as $b\to0^+$ or $\lambda\to+\infty$ are also analyzed. Moreover, if the set $\{x\in\Omega:\ h(x)>0\}$ possesses several disjoint components, we also prove the existence of multi-bump sign-changing solutions.

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1. Introduction

In the past decades, the following Kirchhoff-type equations

$$-\left(a+b\int\limits_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^3, \tag{1.1}$$

has been investigated by many authors, where $V:\mathbb{R}^3\to\mathbb{R}$, $f\in\mathcal{C}(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and $a,b>0$ are constants. If $V(x)\equiv0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega\subset\mathbb{R}^3$ in (1.1), we then obtain the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a+b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases} \tag{1.2}$$

Equation (1.2) is related to the stationary analogue of the following equation

$$\rho\frac{\partial^2u}{\partial^2t}-\left(\frac{P_0}{h}+\frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2u}{\partial^2x}=0,$$

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which is proposed by Kirchhoff in [19] as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. After the pioneer work of Lions [20], where a functional analysis approach was proposed to the equation

$$\begin{cases} u_{tt} - \left(a + b \int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x\right) \Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

equation (1.3) began to call attention of several researchers, see [2,5,8] and the references therein.

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.2), u denotes the displacement, $f(x,u)$ is the external force, b is the initial tension and a is related to the intrinsic properties of the string. We point out that such nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself, for example, population density. For more mathematical and physical background of (1.2), we refer the reader to the papers [1,2,5,15,16,19,21] and the references therein.

Mathematically, Eq. (1.1) is a nonlocal problem as the appearance of the nonlocal term $\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \Delta u$, which implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.1) particularly interesting. A lot of interesting results on the existence of positive solutions, multiple solutions, semiclassical state solutions and sign-changing solutions for (1.1) are obtained in last decade, see for examples, [6,9,11,12,15–18,21,22,24,26–28] and the references therein.

In particular, Chen, Kuo and Wu [6] studied the following nonlinear Kirchhoff-type equation with indefinite nonlinearity

$$\begin{cases} -\left(a + b \int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x\right) \Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, \ x \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a, b > 0$, Ω is a smooth bounded domain in $\mathbb{R}^N$ with $1 < q < 2 < p < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = +\infty$ if $N = 1, 2$), $\lambda > 0$ is a parameter, the weight functions $f, g \in \mathcal{C}(\Omega)$ satisfy $f^+(x) := \max\{f(x), 0\} \not\equiv 0$ and $g^+(x) := \max\{g(x), 0\} \not\equiv 0$. By using Nehari manifold and fibering map, the authors proved the existence of multiple positive solutions for Eq. (1.4). We point out that Kirchoff-type equations with potential well and indefinite nonlinearities were also investigated in [26,30].

Recently, Figueiredo et al [13] investigated ground states of elliptic problems over cones. As an application, the authors [13] proved the following Kirchhoff-type equation

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x\right) \Delta u = b(x)|u|^{r-2}u, \ x \in \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.5}$$

has a positive ground state solution provided $b^+(x) := \max\{b(x), 0\} \not\equiv 0$ and $r \in (4, 6)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $M : [0, +\infty) \to [0, +\infty)$ is a monotone increasing $\mathcal{C}^1$ function such that $M(0) := m_0 > 0$ and $t \mapsto \frac{M(t)}{t}$ is increasing on $(0, +\infty)$.

Based on the above results, a natural question is whether Eq. (1.5) has sign-changing solutions with $b(x)$ is a sign-changing function. The present paper is devoted to this aspect and partially answers this question. More precisely, we devoted to study the existence of sign-changing solutions for the following Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u=(h^+(x)+\lambda h^-(x))\,|u|^{p-2}u, & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.6}$$

where $a,\ b>0$, $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, the potential $h:\overline{\Omega}\to\mathbb{R}$ is a sign-changing continuous function, $\lambda>0$ is a parameter, and

$$h^+(x)=\max\{h(x),0\},\quad h^-(x)=\min\{h(x),0\}.$$

Throughout this paper, we denote $H_0^1(\Omega)$ the usual Sobolev space equipped with the inner product and norm

$$(u,v)=\int\limits_{\Omega}\nabla u\nabla v\mathrm{d}x,\quad \|u\|=(u,u)^{1/2}.$$

Define the energy functional $I_{b,\lambda}:H_0^1(\Omega)\to\mathbb{R}$ by

$$I_{b,\lambda}(u):=\frac{a}{2}\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x+\frac{b}{4}\left(\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)^2-\frac{1}{p}\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^p\mathrm{d}x. \tag{1.7}$$

Obviously, the functional $I_{b,\lambda}$ is well-defined and belongs to $\mathcal{C}^1(H_0^1(\Omega),\mathbb{R})$. Moreover, for any $u,\varphi\in H_0^1(\Omega)$, we have

$$\langle I'_\lambda(u),\varphi\rangle=a\int\limits_{\Omega}\nabla u\nabla\varphi\mathrm{d}x+b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\int\limits_{\Omega}\nabla u\nabla\varphi\mathrm{d}x-\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^{p-2}u\varphi\mathrm{d}x. \tag{1.8}$$

In the case $h(x)\equiv 1$, by constrained minimization method, Figueiredo and Nascimento [12] and Shuai [25] proved the existence of least energy sign-changing solution for Eq. (1.6). The authors first proved the following set

$$\mathcal{M}_{b,\lambda}=\left\{u\in H_0^1(\Omega),\ u^\pm\neq 0\ \text{ and }\ \langle I'_{b,\lambda}(u),u^+\rangle=\langle I'_{b,\lambda}(u),u^-\rangle=0\right\} \tag{1.9}$$

is nonempty, which is a crucial step. Then, the authors sought a minimizer of the energy functional $I_{b,\lambda}$ restricted on $\mathcal{M}_{b,\lambda}$ and proved the minimizer is a sign-changing solution of (1.6) by quantitative deformation lemma. In the first step, the authors proved that, *for each* $u\in H_0^1(\Omega)$ with $u^\pm\neq 0$, *there exists a unique pair* $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$ *such that* $su^++tu^-\in\mathcal{M}_{b,\lambda}$, see Lemma 2.3 in [12] and Lemma 2.1 in [25]. However, if $h(x)$ is a sign-changing continuous function, this fact does not hold for all $u\in H_0^1(\Omega)$ with $u^\pm\neq 0$, but rather in some part of it. A direct observation is that, a necessary condition for $u\in\mathcal{M}_{b,\lambda}$ is $u^+,u^-\in\mathcal{A}$, where

$$\mathcal{A}:=\left\{u\in H_0^1(\Omega)\setminus\{0\}:\ \int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^p\mathrm{d}x>0\right\}. \tag{1.10}$$

Thus, the method that used in [12,25] cannot be applied to Eq. (1.6), we need some crucial modifications.

Our first main result can be stated as follows.

Theorem 1.1. *Assume* $h:\overline{\Omega}\to\mathbb{R}$ *is a sign-changing continuous function,* $p\in(4,6)$ *and* $\lambda>0$*, then Eq.* (1.6) *possesses one least energy sign-changing solution* $u_{b,\lambda}$*, which has precisely two nodal domains. Moreover,* $I_{b,\lambda}(u_{b,\lambda})>2c_{b,\lambda}$*, where*

$$c_{b,\lambda}:=\inf_{u\in\mathcal{N}_{b,\lambda}}I_{b,\lambda}(u) \tag{1.11}$$

and

$$\mathcal{N}_{b,\lambda}:=\left\{u\in H_0^1(\Omega)\setminus\{0\}:\langle I'_{b,\lambda}(u),u\rangle=0\right\}. \tag{1.12}$$

Theorem 1.1 implies that, the energy of any sign-changing solution of Eq. (1.6) is larger than two times the least energy, this property is called energy doubling by Weth in [29]. It is obvious that the least energy of the sign-changing solution $u_{b,\lambda}$ obtained in Theorem 1.1 depends on b and λ. We next focus on the convergence property of $u_{b,\lambda}$ as $b \to 0^+$ or $\lambda \to +\infty$. Our main results in this direction can be stated as follows.

Theorem 1.2. *If the assumptions of Theorem 1.1 hold, for any sequence $\{b_n\}$ with $b_n \to 0^+$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that $u_{b_n,\lambda} \to u_{0,\lambda}$ strongly in $H_0^1(\Omega)$ as $n \to \infty$, where $u_{b_n,\lambda}$ denote the least energy sign-changing solution of Eq. (1.6) with $b = b_n$ obtained by Theorem 1.1, and $u_{0,\lambda}$ is a least energy sign-changing solution of the following equation*

$$\begin{cases} -a\Delta u = (h^+(x) + \lambda h^-(x))\,|u|^{p-2}u, \; x \in \Omega, \\ u = 0, \qquad \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

which changes sign only once.

The proof of Theorem 1.2 includes three steps, we first prove $\{u_{b_n,\lambda}\}$ is bounded in $H_0^1(\Omega)$, then we prove $u_{b_n,\lambda} \to u_{0,\lambda}$ strongly in $H_0^1(\Omega)$, and we finally prove that $u_{0,\lambda}$ is just a least energy sign-changing solution of (1.13).

Theorem 1.3. *If the assumptions of Theorem 1.1 hold, for any sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $u_{b,\lambda_n} \to \bar{u}$ strongly in $H_0^1(\Omega)$ as $n \to \infty$, where u_{b,λ_n} denote the least energy sign-changing solution of Eq. (1.6) with $\lambda = \lambda_n$ obtained by Theorem 1.1, and $\bar{u}$ is a least energy sign-changing solution of following equation*

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = h^+(x)|u|^{p-2}u, \; x \in \Omega \setminus \Omega^-, \\ u = 0, \qquad x \in \Omega^-, \\ u = 0, \qquad x \in \partial\Omega. \end{cases} \tag{1.14}$$

which changes sign only once, here $\Omega^- := \{x \in \Omega \mid h(x) < 0\}$.

Next, we study the existence of multi-bump sign-changing solutions for Eq. (1.6). We now assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function satisfying

(h_1) $\Omega^+ := \{x \in \Omega \mid h(x) > 0\} = \Omega \backslash \overline{\Omega^-}$;

(h_2) the set Ω^+ is the union of k ($k \geq 2$) open connected and disjoint Lipschitz components, that is

$$\Omega^+ = \cup_{i=1}^k \Omega_i \quad \text{and} \quad dist(\Omega_i, \Omega_j) > 0 \;\; \text{for} \;\; i \neq j; \;\; i, j = 1, 2, \ldots, k. \tag{1.15}$$

Theorem 1.4. *Assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and (h_1)–(h_2) hold. If $p \in (4, 6)$, then, for any non-empty subset $\Gamma \subset \{1, 2, \ldots, k\}$ with*

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \quad \text{and} \quad \Gamma_i \cap \Gamma_j = \varnothing \quad \text{for} \quad i \neq j, \; i, j = 1, 2, 3, \tag{1.16}$$

there exists a constant $\Lambda_\Gamma > 0$ such that for $\lambda \geq \Lambda_\Gamma$, Eq. (1.6) has a sign-changing multi-bump solution $u_{b,\lambda}$, which possesses the following property: for any sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $u_{b,\lambda_n} \to u$ strongly in $H_0^1(\Omega)$ as $n \to \infty$, where u solves the following equation

$$\begin{cases} -\left(a + b\displaystyle\int_{\Omega_\Gamma} |\nabla u|^2 \mathrm{d}x\right)\Delta u = h^+(x)|u|^{p-2}u, \; x \in \Omega_\Gamma = \cup_{i\in\Gamma}\Omega_i, \\ u = 0, \qquad x \in \Omega \setminus \Omega_\Gamma, \\ u = 0, \qquad x \in \partial\Omega. \end{cases} \tag{1.17}$$

Moreover, $u|_{\Omega_i}$ is positive for $i \in \Gamma_1$, $u|_{\Omega_i}$ is negative for $i \in \Gamma_2$, and $u|_{\Omega_i}$ changes sign exactly once for $i \in \Gamma_3$.

If $b=0$, Eq. (1.6) does not depend on the nonlocal term $\int_\Omega |\nabla u|^2 \mathrm{d}x \Delta u$ any more. In this case, Eq. (1.6) becomes to the following semilinear elliptic equation

$$\begin{cases} -a\Delta u = (h^+(x) + \lambda h^-(x))\,|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.18}$$

Under the conditions (h_1)–(h_2), separate the components of Ω^+ arbitrarily into three families, i.e.,

$$\Omega^+ = \left(\cup_{i=1}^{I} \widetilde{\omega}_i\right) \cup \left(\cup_{j=1}^{J} \widehat{\omega}_j\right) \cup \left(\cup_{i=k}^{K} \overline{\omega}_k\right),$$

by using constrained minimization method, Girão and Gomes [14] proved the existence of multi-bump nodal solution u_λ for Eq. (1.18) if $\lambda > 0$ large enough. Moreover, for any sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $u_{\lambda_n} \to u$ strongly in $H_0^1(\Omega)$ as $n \to \infty$, where u solves the following equation

$$\begin{cases} -a\Delta u = h^+(x)|u|^{p-2}u, & x \in \Omega^+, \\ u = 0, & x \in \Omega \setminus \Omega^+, \end{cases} \tag{1.19}$$

here $u|_{\widetilde{\omega}_i}$ changes sign exactly once for $i = 1, 2, \ldots, I$, $u|_{\widehat{\omega}_j}$ is positive for $j = 1, 2, \ldots, J$, $u|_{\overline{\omega}_k} \equiv 0$ for $k = 1, 2, \ldots, K$. We refer the reader to [4] for multiple positive solutions for Eq. (1.18).

However, we cannot apply the same method that used in [14] to Eq. (1.6), because Kirchhoff-type equation depends on the global information of its solution. Different from the method used in [14], we first construct a special minimax value of the energy functional; Then, by careful analysis of the deformation flow to the energy functional, we prove the existence of multi-bump sign-changing solutions for Eq. (1.6); Finally, we show that the multi-bump sign-changing solutions are localized near the components of Ω^+ and converge to the solutions (1.17) with prescribed sign properties. We remark that our method also can be used to study the existence of multi-bump sign-changing solutions for Eq. (1.18).

The paper is organized as follows. In Sect. 2, we give some primarily results. In Sect. 3, we prove Theorems 1.1–1.3. In Sect. 4 and 5, we devote to proving Theorem 1.4.

2. Some preliminary results

In this section, we give some preliminary results.

Lemma 2.1. *Assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and $p \in (4, 6)$. If $u \in \mathcal{A}$, then there exists a unique $t > 0$ such that $tu \in \mathcal{N}_{b,\lambda}$, where $\mathcal{A}$ is defined by (1.10), $\mathcal{N}_{b,\lambda}$ is defined by (1.12).*

Proof. For $u \in \mathcal{A}$, we define

$$\begin{aligned} V_u(t) = \langle I'_{b,\lambda}(tu), tu\rangle = at^2 \int_\Omega |\nabla u|^2 \mathrm{d}x + bt^4 \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^2 \\ - t^p \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u|^p \mathrm{d}x. \end{aligned}$$

Since $u \in \mathcal{A}$, then

$$\int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u|^p \mathrm{d}x > 0.$$

Therefore, $V_u(t) > 0$ for $t > 0$ small enough and $V_u(t) < 0$ for $t < 0$ large enough, since $p \in (4, 6)$. Thus, there exists $t_0 > 0$ such that $I'_{b,\lambda}(t_0 u) = 0$, that is $t_0 u \in \mathcal{N}_{b,\lambda}$.

Assume $t_1,\, t_2>0$ such that $t_1 u,\ t_2 u\in\mathcal{N}_{b,\lambda}$, that is

$$at_1^2\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x+bt_1^4\left(\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)^2-t_1^p\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^p\mathrm{d}x=0 \tag{2.1}$$

and

$$at_2^2\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x+bt_2^4\left(\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)^2-t_2^p\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^p\mathrm{d}x=0. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$a\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x=(t_1^{p-4}-t_2^{p-4})\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u|^p\mathrm{d}x,$$

which implies $t_1=t_2$. □

Lemma 2.2. *Assume $h:\overline{\Omega}\to\mathbb{R}$ is a sign-changing continuous function and $p\in(4,6)$, if $u\in H_0^1(\Omega)$ with $u^\pm\in\mathcal{A}$, then there is a unique pair (s_u,t_u) of positive numbers such that $s_u u^+ + t_u u^-\in\mathcal{M}_{b,\lambda}$.*

Proof. We prove the lemma by two steps.

Step 1: Define $\overrightarrow{F}(s,t):=(f_1(s,t),f_2(s,t))$, where

$$\begin{cases} f_1(s,t)=as^2\|u^+\|^2+bs^4\|u^+\|^4+bs^2t^2\|u^+\|^2\|u^-\|^2-s^p\displaystyle\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^+|^p\mathrm{d}x,\\ f_2(s,t)=at^2\|u^-\|^2+bt^4\|u^-\|^4+bs^2t^2\|u^+\|^2\|u^-\|^2-t^p\displaystyle\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^-|^p\mathrm{d}x. \end{cases} \tag{2.3}$$

Since $u^\pm\in\mathcal{A}$, then

$$\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^+|^p\mathrm{d}x>0\ \text{ and }\ \int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^-|^p\mathrm{d}x>0.$$

We deduce that there exist $0<r<R$ such that

$$\begin{cases} f_1(r,t)>0\ \text{ and }\ f_1(R,t)<0\ \text{ for all }\ t\in[r,R],\\ f_2(s,r)>0\ \text{ and }\ f_2(s,R)<0\ \text{ for all }\ s\in[r,R], \end{cases} \tag{2.4}$$

since $p\in(4,6)$. Then, by using Miranda lemma [23], we conclude that there exists $(s_u,t_u)\in\mathbb{R}_+\times\mathbb{R}_+$ such that

$$f_1(s_u,t_u)=0\ \text{ and }\ f_2(s_u,t_u)=0,$$

which implies that $s_u u^+ + t_u u^-\in\mathcal{M}_{b,\lambda}$.

Step 2: We prove (s_u,t_u) is unique.

Case 1: $u\in\mathcal{M}_{b,\lambda}$. Suppose $(\bar{s},\bar{t})\neq(1,1)$ be another pair of positive numbers such that $\bar{s}u^+ + \bar{t}u^-\in\mathcal{M}_{b,\lambda}$, then

$$\begin{cases} a\bar{s}^2\|u^+\|^2+b\bar{s}^4\|u^+\|^4+b\bar{s}^2\bar{t}^2\|u^+\|^2\|u^-\|^2=\bar{s}^p\displaystyle\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^+|^p\mathrm{d}x,\\ a\bar{t}^2\|u^-\|^2+b\bar{t}^4\|u^-\|^4+b\bar{s}^2\bar{t}^2\|u^+\|^2\|u^-\|^2=\bar{t}^p\displaystyle\int\limits_{\Omega}\left(h^+(x)+\lambda h^-(x)\right)|u^-|^p\mathrm{d}x. \end{cases}$$

Without loss of generality, we assume $\bar{s} \geq \bar{t} > 0$, then

$$\begin{cases} a\frac{1}{\bar{s}^2}\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \geq \bar{s}^{p-4} \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^+|^p \mathrm{d}x, \\ a\frac{1}{\bar{t}^2}\|u^-\|^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \bar{t}^{p-4} \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^-|^p \mathrm{d}x. \end{cases}$$

Since $u \in \mathcal{M}_{b,\lambda}$, we have

$$\begin{cases} a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 = \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^+|^p \mathrm{d}x, \\ a\|u^-\|^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 = \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^-|^p \mathrm{d}x. \end{cases}$$

Thus, we conclude that

$$\begin{cases} a\left(\frac{1}{\bar{s}^2} - 1\right)\|u^+\|^2 \geq \left(\bar{s}^{p-4} - 1\right) \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^+|^p \mathrm{d}x, \\ a\left(\frac{1}{\bar{t}^2} - 1\right)\|u^-\|^2 \leq \left(\bar{t}^{p-4} - 1\right) \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^-|^p \mathrm{d}x, \end{cases}$$

which implies $1 \geq \bar{s} \geq \bar{t} \geq 1$. Thus, $(\bar{s}, \bar{t}) = (1, 1)$.

Case 2: $u \notin \mathcal{M}_{b,\lambda}$ but $u^{\pm} \in \mathcal{A}$, then by Step 1, we know that there exists $(s_u, t_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_{b,\lambda}$. Assume that $(s'_u, t'_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ also satisfying $s'_u u^+ + t'_u u^- \in \mathcal{M}_{b,\lambda}$. Hence we have

$$\frac{s'_u}{s_u} s_u u^+ + \frac{t'_u}{t_u} t_u u^- \in \mathcal{M}_{b,\lambda}. \tag{2.5}$$

Since $s_u u^+ + t_u u^- \in \mathcal{M}_{b,\lambda}$, by the arguments of case 1, we deduce that

$$\frac{s'_u}{s_u} = \frac{t'_u}{t_u} = 1.$$

Thus, $s'_u = s_u$ and $t'_u = t_u$. □

Lemma 2.3. *Assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and $p \in (4, 6)$, suppose that $u^{\pm} \in \mathcal{A}$ such that*

$$\begin{cases} a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^+|^p \mathrm{d}x, \\ a\|u^-\|^2 + b\|u^-\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \int\limits_\Omega \left(h^+(x) + \lambda h^-(x)\right)|u^-|^p \mathrm{d}x. \end{cases}$$

Then the unique pair (s_u, t_u) of positive numbers obtained in Lemma 2.2 satisfies $0 < s_u, t_u \leq 1$.

Proof. Suppose that $s_u \geq t_u > 0$, since $s_u u^+ + t_u u^- \in \mathcal{M}_b$, then we have

$$\begin{aligned} &a s_u^2\|u^+\|^2 + b s_u^4\left(\int\limits_\Omega |\nabla u^+|^2 \mathrm{d}x\right)^2 + b s_u^4 \int\limits_\Omega |\nabla u^+|^2 \mathrm{d}x \int\limits_\Omega |\nabla u^-|^2 \mathrm{d}x \\ &\geq a s_u^2\|u^+\|^2 + b s_u^4\left(\int\limits_\Omega |\nabla u^+|^2 \mathrm{d}x\right)^2 + b s_u^2 t_u^2 \int\limits_\Omega |\nabla u^+|^2 \mathrm{d}x \int\limits_\Omega |\nabla u^-|^2 \mathrm{d}x \end{aligned}$$

$$= s_u^p \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u^+|^p \mathrm{d}x. \tag{2.6}$$

On the other hand,

$$a\|u^+\|^2 + b\left(\int_\Omega |\nabla u^+|^2 \mathrm{d}x\right)^2 + b\int_\Omega |\nabla u^+|^2 \mathrm{d}x \int_\Omega |\nabla u^-|^2 \mathrm{d}x \le \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u^+|^p \mathrm{d}x. \tag{2.7}$$

Combine (2.6) and (2.7), we then get

$$\left(\frac{1}{s_u^2} - 1\right) a\|u^+\|^2 \ge (s_u^{p-4} - 1) \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u^+|^p \mathrm{d}x.$$

Therefore, we must have $s_u \le 1$. Then the proof is completed. □

Lemma 2.4. *Assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and $p \in (4, 6)$. If $u^\pm \in \mathcal{A}$, then the vector (s_u, t_u) which obtained in Lemma 2.2 is the unique maximum point of the function $\phi : (\mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ defined by $\phi(s,t) := I_{b,\lambda}(su^+ + tu^-)$.*

Proof. From the proof of Lemma 2.2, (s_u, t_u) is the unique critical point of ϕ in $\mathbb{R}_+ \times \mathbb{R}_+$. Since $p \in (4, 6)$, we deduce that $\phi(s,t) \to -\infty$ uniformly as $|(s,t)| \to +\infty$, so it is sufficient to check that the maximum point is not achieved on the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$.

Fix $\bar{t} > 0$, since

$$\begin{aligned}
\phi(s,\bar{t}) &= I_{b,\lambda}(su^+ + \bar{t}u^-) \\
&= \frac{as^2}{2}\int_\Omega |\nabla u^+|^2 \mathrm{d}x + \frac{bs^4}{4}\left(\int_\Omega |\nabla u^+|^2 \mathrm{d}x\right)^2 - s^p \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u^+|^p \mathrm{d}x \\
&\quad + \frac{bs^2\bar{t}^2}{2}\int_\Omega |\nabla u^+|^2 \mathrm{d}x \int_\Omega |\nabla u^-|^2 \mathrm{d}x \\
&\quad + \frac{a\bar{t}^2}{2}\int_\Omega |\nabla u^-|^2 \mathrm{d}x + \frac{b\bar{t}^4}{4}\left(\int_\Omega |\nabla u^+|^2 \mathrm{d}x\right)^2 - \bar{t}^p \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u^-|^p \mathrm{d}x
\end{aligned}$$

is an increasing function with respect to s if $s > 0$ small enough, therefore the pair $(0, \bar{t})$ is not a maximum point of ϕ in $\mathbb{R}_+ \times \mathbb{R}_+$. □

By Lemma 2.2, we now define

$$m_{b,\lambda} := \inf\Big\{I_{b,\lambda}(u) : u \in \mathcal{M}_{b,\lambda}\Big\}. \tag{2.8}$$

Lemma 2.5. *Assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and $p \in (4, 6)$, then $m_{b,\lambda} > 0$ is achieved.*

Proof. For every $u \in \mathcal{M}_{b,\lambda}$, we have $\langle I'_{b,\lambda}(u), u\rangle = 0$. Then, by using Sobolev embedding theorem, one gets

$$\begin{aligned}
a\|u\|^2 &\le a\int_\Omega |\nabla u|^2 \mathrm{d}x + b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 = \int_\Omega \big(h^+(x) + \lambda h^-(x)\big) |u|^p \mathrm{d}x \\
&\le \int_\Omega h^+(x)|u|^p \mathrm{d}x \le \|h^+(x)\|_{L^\infty(\Omega)} \int_\Omega |u|^p \mathrm{d}x
\end{aligned}$$

$$\leq C\|u\|^p. \tag{2.9}$$

Thus, there exists a constant $\alpha > 0$ such that $\|u\|^2 \geq \alpha$. Therefore

$$I_{b,\lambda}(u) = I_{b,\lambda}(u) - \frac{1}{p}\langle I'_{b,\lambda}(u), u\rangle \geq \left(\frac{1}{2} - \frac{1}{p}\right) a\|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) a\alpha, \quad \text{for each} \quad u \in \mathcal{M}_{b,\lambda},$$

which implies $m_{b,\lambda} \geq \left(\frac{1}{2} - \frac{1}{p}\right) a\alpha > 0$.

Let $\{u_n\} \subset \mathcal{M}_{b,\lambda}$ be a sequence such that $I_{b,\lambda}(u_n) \to m_{b,\lambda}$. Then $\{u_n\}$ is bounded in $H_0^1(\Omega)$, up to a subsequence, still denote by $\{u_n\}$, such that $u_n^\pm \rightharpoonup u_{b,\lambda}^\pm$ weakly in $H_0^1(\Omega)$. Since $u_n \in \mathcal{M}_{b,\lambda}$, we have $\langle I'_{b,\lambda}(u_n), u_n^\pm\rangle = 0$, that is

$$a\int_\Omega |\nabla u_n^\pm|^2 \mathrm{d}x + b\int_\Omega |\nabla u_n|^2 \mathrm{d}x \int_\Omega |\nabla u_n^\pm|^2 \mathrm{d}x = \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_n^\pm|^p \mathrm{d}x. \tag{2.10}$$

Similar as (2.9) there exist a constant $\mu > 0$ such that $\|u_n^\pm\|^2 \geq \mu$ for all $n \in \mathbb{N}$. Since $u_n \in \mathcal{M}_{b,\lambda}$, thus

$$\mu \leq \|u_n^\pm\|^2 < \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_n^\pm|^p \mathrm{d}x \leq \int_\Omega h^+(x)|u_n^\pm|^p \mathrm{d}x.$$

By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q < 6$, we get

$$\int_\Omega h^+(x)|u_{b,\lambda}^\pm|^p \mathrm{d}x \geq \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_{b,\lambda}^\pm|^p \mathrm{d}x \geq \mu. \tag{2.11}$$

Hence, $u_{b,\lambda}^\pm \in \mathcal{A}$. By the weak semicontinuity of norm, we have

$$a\|u_{b,\lambda}^\pm\|^2 + b\int_\Omega |\nabla u_{b,\lambda}|^2 \mathrm{d}x \int_\Omega |\nabla u_{b,\lambda}^\pm|^2 \mathrm{d}x \leq \liminf_{n\to\infty} \left\{a\|u_n^\pm\|^2 + b\int_\Omega |\nabla u_n|^2 \mathrm{d}x \int_\Omega |\nabla u_n^\pm|^2 \mathrm{d}x\right\}. \tag{2.12}$$

It follows from (2.10) that

$$a\|u_{b,\lambda}^\pm\|^2 + b\int_\Omega |\nabla u_{b,\lambda}|^2 \mathrm{d}x \int_\Omega |\nabla u_{b,\lambda}^\pm|^2 \mathrm{d}x \leq \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_{b,\lambda}^\pm|^p \mathrm{d}x. \tag{2.13}$$

From (2.13) and Lemma 2.3, there exists $(\bar{s}, \bar{t}) \in (0,1] \times (0,1]$ such that

$$\overline{u}_{b,\lambda} := \bar{s}u_{b,\lambda}^+ + \bar{t}u_{b,\lambda}^- \in \mathcal{M}_{b,\lambda}.$$

Hence

$$\begin{aligned}
m_{b,\lambda} &\leq I_{b,\lambda}(\overline{u}_{b,\lambda}) = I_{b,\lambda}(\overline{u}_{b,\lambda}) - \frac{1}{p}\langle I'_{b,\lambda}(\overline{u}_{b,\lambda}), \overline{u}_{b,\lambda}\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) a\int_\Omega |\overline{u}_{b,\lambda}|^2 \mathrm{d}x + \left(\frac{1}{4} - \frac{1}{p}\right) b\left(\int_\Omega |\overline{u}_{b,\lambda}|^2 \mathrm{d}x\right)^2 \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) a\left[\|\bar{s}u_{b,\lambda}^+\|^2 + \|\bar{t}u_{b,\lambda}^-\|^2\right] + \left(\frac{1}{4} - \frac{1}{p}\right) b\left[\|\bar{s}u_{b,\lambda}^+\|^2 + \|\bar{t}u_{b,\lambda}^-\|^2\right]^2 \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right) a\left[\|u_{b,\lambda}^+\|^2 + \|u_{b,\lambda}^-\|^2\right] + \left(\frac{1}{4} - \frac{1}{p}\right) b\left[\|u_{b,\lambda}^+\|^2 + \|u_{b,\lambda}^-\|^2\right]^2 \\
&\leq \liminf_{n\to\infty} \left[I_{b,\lambda}(u_n) - \frac{1}{p}\langle I'_{b,\lambda}(u_n), u_n\rangle\right] = m_{b,\lambda},
\end{aligned} \tag{2.14}$$

which implies that $\bar{s} = \bar{t} = 1$. Thus, $\overline{u}_{b,\lambda} = u_{b,\lambda}$ and $I_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda}$. □

3. Proof of Theorems 1.1–1.3.

The main aim of this section is to prove Theorems 1.1–1.3. We first prove that the minimizer $u_{b,\lambda}$ to the minimization problem (2.8) is indeed a sign-changing solution of Eq. (1.6), that is, $I'_{b,\lambda}(u_{b,\lambda}) = 0$.

Proof of Theorem 1.1. Using the quantitative deformation lemma, we prove that $I'_{b,\lambda}(u_{b,\lambda}) = 0$.

It is clear that $\langle I'_{b,\lambda}(u_{b,\lambda}), u^+_{b,\lambda}\rangle = 0 = \langle I'_{b,\lambda}(u_{b,\lambda}), u^-_{b,\lambda}\rangle$. If $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $(s,t) \neq (1,1)$, it follows from Lemma 2.4 that

$$I_{b,\lambda}(su^+_{b,\lambda} + tu^-_{b,\lambda}) < I_{b,\lambda}(u^+_{b,\lambda} + u^-_{b,\lambda}) = m_{b,\lambda}. \tag{3.1}$$

If $I'_{b,\lambda}(u_{b,\lambda}) \neq 0$, then there exist $\delta > 0$ and $\rho > 0$ such that

$$\|I'_{b,\lambda}(v)\| \geq \rho, \quad \text{for all} \quad \|v - u_{b,\lambda}\| \leq 3\delta.$$

Let $D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and $g(s,t) := su^+_{b,\lambda} + tu^-_{b,\lambda}$. It follows from Lemma 2.4 again that

$$\bar{m}_{b,\lambda} := \max_{\partial D} I_{b,\lambda} \circ g < m_{b,\lambda} \tag{3.2}$$

For $\varepsilon := \min\{(m_{b,\lambda} - \bar{m}_{b,\lambda})/2, \rho\delta/8\}$ and $S := B(u_{b,\lambda}, \delta)$, [see [31], Lemma 2.3] yields a deformation η such that

(a) $\eta(1,u) = u$ if $u \notin I^{-1}_{b,\lambda}([m_{b,\lambda} - 2\varepsilon, m_{b,\lambda} + 2\varepsilon]) \cap S_{2\delta}$;

(b) $\eta(1, I^{m_{b,\lambda}+\varepsilon}_{b,\lambda} \cap S) \subset I^{m_{b,\lambda}-\varepsilon}_{b,\lambda}$;

(c) $I_{b,\lambda}(\eta(1,u)) \leq I_{b,\lambda}(u)$ for all $u \in H^1_0(\Omega)$.

It is clear that

$$\max_{(s,t)\in\bar{D}} I_{b,\lambda}(\eta(1, g(s,t))) < m_{b,\lambda}. \tag{3.3}$$

We now prove that $\eta(1, g(D)) \cap \mathcal{M}_{b,\lambda} \neq \varnothing$, contradicting to the definition of $m_{b,\lambda}$. Let us define $h(s,t) := \eta(1, g(s,t))$ and

$$\Psi_0(s,t) := \left(I'_{b,\lambda}(su^+_{b,\lambda} + tu^-_{b,\lambda})u^+_{b,\lambda}, I'_{b,\lambda}(su^+_{b,\lambda} + tu^-_{b,\lambda})u^-_{b,\lambda}\right),$$
$$\Psi_1(s,t) := \left(\frac{1}{s} I'_{b,\lambda}(h(s,t))\, h^+(s,t), \frac{1}{t} I'_{b,\lambda}(h(s,t))\, h^-(s,t)\right).$$

Lemma 2.2 and the the degree theory now yields $\deg(\Psi_0, D, 0) = 1$. It follows from (3.2) that $g = h$ on ∂D. Consequently, we obtain $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Therefore, $\Psi_1(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, so that $\eta(1, g(s_0, t_0)) = h(s_0, t_0) \in \mathcal{M}_{b,\lambda}$, which is a contradiction. From this, $u_{b,\lambda}$ is a critical point of $I_{b,\lambda}$, and so, a sign-changing solution for equation (1.6).

Now, we show that $u_{b,\lambda}$ has exactly two nodal domains. The proof on the number of nodal domains follows the arguments in Bartsch [3] and Castro et al. [7]. To this end, we assume by contradiction that

$$u_{b,\lambda} = u_1 + u_2 + u_3$$

with

$$u_i \neq 0, \ u_1 \geq 0, \ u_2 \leq 0 \ \text{ and } \ \mathrm{suppt}(u_i) \cap \mathrm{suppt}(u_j) = \varnothing, \quad \text{for } i \neq j, \quad i,j = 1,2,3$$

and

$$\langle I'_{b,\lambda}(u_{b,\lambda}), u_i\rangle = 0, \quad \text{for } i = 1,2,3. \tag{3.4}$$

Setting $v := u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e. $v^\pm \neq 0$. Then, we can conclude $v^\pm \in \mathcal{A}$. By Lemma 2.2, there exists a unique pair (s_v, t_v) of positive numbers such that

$$s_v v^+ + t_v v^- \in \mathcal{M}_{b,\lambda},$$

or equivalently,

$$s_v u_1 + t_v u_2 \in \mathcal{M}_{b,\lambda}.$$

And so,

$$I_{b,\lambda}(s_v u_1 + t_v u_2) \geq m_{b,\lambda}. \tag{3.5}$$

Moreover, using the fact that $\langle I'_{b,\lambda}(u_{b,\lambda}), u_i\rangle = 0$ for $i = 1, 2, 3$, it follows that

$$\langle I'_{b,\lambda}(v), v^{\pm}\rangle < 0.$$

From Lemma 2.3, we have that

$$(s_v, t_v) \in (0, 1] \times (0, 1].$$

On the other hand,

$$\begin{aligned}
0 = \frac{1}{4}\langle I'_{b,\lambda}(u_{b,\lambda}), u_3\rangle &= \frac{a}{4}\int_\Omega |\nabla u_3|^2 \mathrm{d}x + \frac{b}{4}\left(\int_\Omega |\nabla u_3|^2 \mathrm{d}x\right)^2 + \frac{b}{4}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x \\
&\quad + \frac{b}{4}\int_\Omega |\nabla u_2|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x - \frac{1}{4}\int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_3|^p \mathrm{d}x \\
&< I_{b,\lambda}(u_3) + \frac{b}{4}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x + \frac{b}{4}\int_\Omega |\nabla u_2|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x.
\end{aligned} \tag{3.6}$$

Then, by using (3.4), we can calculate that

$$\begin{aligned}
I_{b,\lambda}(s_v u_1 + t_v u_2) &= \frac{a s_v^2}{4}\|u_1\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) s_v^p \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_1|^p \mathrm{d}x + \frac{a t_v^2}{4}\|u_2\|^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{p}\right) t_v^p \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_2|^p \mathrm{d}x \\
&\leq \frac{a}{4}\|u_1\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_1|^p \mathrm{d}x + \frac{a}{4}\|u_2\|^2 \\
&\quad + \left(\frac{1}{4} - \frac{1}{p}\right) \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |u_2|^p \mathrm{d}x \\
&= I_{b,\lambda}(u_1) + I_{b,\lambda}(u_2) + \frac{b}{2}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_2|^2 \mathrm{d}x \\
&\quad + \frac{b}{4}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x + \frac{b}{4}\int_\Omega |\nabla u_2|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x.
\end{aligned} \tag{3.7}$$

Then, from (3.5), (3.6) and (3.7), we have

$$\begin{aligned}
m_{b,\lambda} \leq I_{b,\lambda}(s_v u_1 + t_v u_2) &< I_{b,\lambda}(u_1) + I_{b,\lambda}(u_2) + I_{b,\lambda}(u_3) + \frac{b}{2}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_2|^2 \mathrm{d}x \\
&\quad + \frac{b}{2}\int_\Omega |\nabla u_1|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x + \frac{b}{2}\int_\Omega |\nabla u_2|^2 \mathrm{d}x \int_\Omega |\nabla u_3|^2 \mathrm{d}x \\
&= I_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda},
\end{aligned}$$

which is a contradiction. This way, $u_3 = 0$, and $u_{b,\lambda}$ has exactly two nodal domains.

Recall that $c_{b,\lambda}$ and $\mathcal{N}_{b,\lambda}$ are defined by (1.11) and (1.12), respectively. Then, similar as the proof of Lemma 2.5, for each $b > 0$, we can deduce that there exists $v_{b,\lambda} \in \mathcal{N}_{b,\lambda}$ such that $I_{b,\lambda}(v_{b,\lambda}) = c_{b,\lambda} > 0$. By

Corollary 2.9 in [15], the critical points of the functional $I_{b,\lambda}$ on $\mathcal{N}_{b,\lambda}$ are critical points of $I_{b,\lambda}$ in $H_0^1(\Omega)$, we conclude that $I'_{b,\lambda}(v_{b,\lambda})=0$. Thus, $v_{b,\lambda}$ is a ground state solution of (1.6).

On the other hand, suppose that $u_{b,\lambda}=u_{b,\lambda}^{+}+u_{b,\lambda}^{-}$ is a least energy sign-changing solution for Eq. (1.6). By Lemma 2.1, there is unique $\bar{s}>0$, $\bar{t}>0$ such that

$$\bar{s}u_{b,\lambda}^{+}\in\mathcal{N}_{b,\lambda}\ \text{ and }\ \bar{t}u_{b,\lambda}^{+}\in\mathcal{N}_{b,\lambda}.$$

Then, by Lemma 2.4, we get

$$2c_{b,\lambda}\leq I_{b,\lambda}(\bar{s}u_{b,\lambda}^{+})+I_{b,\lambda}(\bar{t}u_{b,\lambda}^{-})<I_{b,\lambda}(\bar{s}u_{b,\lambda}^{+}+\bar{t}u_{b,\lambda}^{-})\leq I_{b,\lambda}(u_{b,\lambda}^{+}+u_{b,\lambda}^{-})=m_{b,\lambda},$$

that is $m_{b,\lambda}>2c_{b,\lambda}$. This completes the proof. □

Now, we are in a situation to prove Theorem 1.2. In the following, we regard $b>0$ as a parameter in equation (1.6). We shall analyze the convergence property of $u_{b,\lambda}$ as $b\to 0^{+}$.

Proof of Theorem 1.2. For any $b>0$ and $\lambda>0$, denote $u_{b,\lambda}\in H_0^1(\Omega)$ the least energy sign-changing solution of (1.6) obtained in Theorem 1.1, which changes sign only once.

Step 1. We claim that, for any sequence $\{b_n\}$ with $b_n\to 0^{+}$ as $n\to\infty$, $\{u_{b_n,\lambda}\}$ is bounded in $H_0^1(\Omega)$.

Choose a nonzero function $\varphi\in\mathcal{C}_0^{\infty}(\Omega)$ with $\varphi^{\pm}\in\mathcal{A}$. Since $p\in(4,6)$, then, for any $b\in[0,1]$, there exists a pair (τ_1,τ_2) of positive numbers, which does not depend on b, such that

$$\begin{cases} a\tau_1^2\|\varphi^{+}\|^2+b\tau_1^4\left(\int\limits_{\Omega}|\nabla\varphi^{+}|^2\mathrm{d}x\right)^2+bB_{\varphi}\tau_1^2\tau_2^2-\tau_1^p\int\limits_{\Omega}\left(h^{+}(x)+\lambda h^{-}(x)\right)|\varphi^{+}|^p\mathrm{d}x<0,\\ a\tau_2^2\|\varphi^{-}\|^2+b\tau_2^4\left(\int\limits_{\Omega}|\nabla\varphi^{-}|^2\mathrm{d}x\right)^2+bB_{\varphi}\tau_1^2\tau_2^2-\tau_2^p\int\limits_{\Omega}\left(h^{+}(x)+\lambda h^{-}(x)\right)|\varphi^{-}|^p\mathrm{d}x<0,\end{cases}$$

where $B_{\varphi}=\int\limits_{\Omega}|\nabla\varphi^{+}|^2\mathrm{d}x\int\limits_{\Omega}|\nabla\varphi^{-}|^2\mathrm{d}x$. In view of Lemma 2.2 and Lemma 2.3, for any $b\in[0,1]$, there exists a unique pair $(s_{\varphi}(b),t_{\varphi}(b))\in(0,1]\times(0,1]$ such that

$$\bar{\varphi}:=s_{\varphi}(b)\tau_1\varphi^{+}+t_{\varphi}(b)\tau_2\varphi^{-}\in\mathcal{M}_{b,\lambda}. \tag{3.8}$$

Thus, for any $b\in[0,1]$, we have

$$\begin{aligned} I_{b,\lambda}(u_{b,\lambda})&\leq I_{b,\lambda}(\bar{\varphi})=I_{b,\lambda}(\bar{\varphi})-\frac{1}{4}\langle I'_{b,\lambda}(\bar{\varphi}),\bar{\varphi}\rangle\\ &=\frac{a}{4}\|\bar{\varphi}\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\int\limits_{\Omega}\left(h^{+}(x)+\lambda h^{-}(x)\right)|\bar{\varphi}|^p\mathrm{d}x\\ &\leq\frac{a}{4}\|\bar{\varphi}\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\int\limits_{\Omega}h^{+}(x)|\bar{\varphi}|^p\mathrm{d}x\\ &\leq\frac{a}{4}\|\tau_1\varphi^{+}\|^2+\frac{a}{4}\|\tau_2\varphi^{-}\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\int\limits_{\Omega}h^{+}(x)\left(\tau_1^p|\varphi^{+}|^p+\tau_2^p|\varphi^{-}|^p\right)\mathrm{d}x\\ &:=C_0, \end{aligned}\tag{3.9}$$

where C_0 does not depend on b. For n large enough, it follows that

$$C_0+1\geq I_{b_n,\lambda}(u_{b_n,\lambda})=I_{b_n,\lambda}(u_{b_n,\lambda})-\frac{1}{4}\langle I'_{b_n,\lambda}(u_{b_n,\lambda}),u_{b_n,\lambda}\rangle\geq\frac{a}{4}\|u_{b_n,\lambda}\|^2, \tag{3.10}$$

which implies $\{u_{b_n,\lambda}\}$ is bounded in $H_0^1(\Omega)$.

Step 2. There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that

$$u_{b_n,\lambda}\rightharpoonup u_{0,\lambda}\ \text{weakly in } H_0^1(\Omega).$$

Then, $u_{0,\lambda}$ is a weak solution of (1.13). Since $u_{b_n,\lambda}$ is the least energy sign-changing solution of (1.6) with $b = b_n$, then by the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q < 6$, we deduce that $u_{b_n,\lambda} \to u_{0,\lambda}$ strongly in $H_0^1(\Omega)$ as $n \to \infty$. In fact,

$$\|u_{b_n,\lambda} - u_{0,\lambda}\|^2 = \langle I'_{b_n,\lambda}(u_{b_n,\lambda}) - I'_{0,\lambda}(u_{0,\lambda}), u_{b_n,\lambda} - u_{0,\lambda}\rangle - b_n \int_\Omega |\nabla u_{b_n,\lambda}|^2 \mathrm{d}x \int_\Omega \nabla u_{b_n,\lambda} (\nabla u_{b_n,\lambda} - \nabla u_{0,\lambda})\, \mathrm{d}x$$
$$+ \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) \left[|u_{b_n,\lambda}|^{p-2} u_{b_n,\lambda} - |u_{0,\lambda}|^{p-2} u_{0,\lambda}\right] (u_{b_n,\lambda} - u_{0,\lambda})\, \mathrm{d}x,$$

and the right hand of last equality tend to zero as $n \to \infty$. Then, by the same arguments as (2.11), we conclude $u_{0,\lambda}^{\pm} \neq 0$, hence $u_{0,\lambda}$ is sign-changing solution of equation (1.13).

Step 3. Suppose that v_0 is a least energy sign-changing solution of (1.13), the existence of v_0 was proved by Vladimir in [32]. By Lemma 2.2, for each $b_n > 0$, there is a unique pair (s_{b_n}, t_{b_n}) of positive numbers such that

$$s_{b_n} v_0^+ + t_{b_n} v_0^- \in \mathcal{M}_{b_n,\lambda}.$$

Then, we have

$$a(s_{b_n})^2 \|v_0^+\|^2 + b_n (s_{b_n})^4 \left(\int_\Omega |\nabla v_0^+|^2 \mathrm{d}x\right)^2 + b_n (s_{b_n} t_{b_n})^2 \int_\Omega |\nabla v_0^+|^2 \mathrm{d}x \int_\Omega |\nabla v_0^-|^2 \mathrm{d}x$$
$$= (s_{b_n})^p \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |v_0^+|^p \mathrm{d}x \tag{3.11}$$

and

$$a(t_{b_n})^2 \|v_0^-\|^2 + b_n (t_{b_n})^4 \left(\int_\Omega |\nabla v_0^-|^2 \mathrm{d}x\right)^2 + b_n (s_{b_n} t_{b_n})^2 \int_\Omega |\nabla v_0^+|^2 \mathrm{d}x \int_\Omega |\nabla v_0^-|^2 \mathrm{d}x$$
$$= (t_{b_n})^p \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |v_0^-|^p \mathrm{d}x. \tag{3.12}$$

Recall that $v_0^{\pm}$ satisfies

$$a\|v_0^+\|^2 = \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |v_0^+|^p \mathrm{d}x$$

and

$$a\|v_0^-\|^2 = \int_\Omega \left(h^+(x) + \lambda h^-(x)\right) |v_0^-|^p \mathrm{d}x.$$

Up to a subsequence, one can easily deduce that

$$(s_{b_n}, t_{b_n}) \to (1, 1), \quad \text{as } n \to \infty. \tag{3.13}$$

It follows from (3.13) and Lemma 2.4 that

$$I_{0,\lambda}(v_0) \leq I_{0,\lambda}(u_{0,\lambda}) = \lim_{n\to\infty} I_{b_n,\lambda}(u_{b_n,\lambda}) = m_{b_n,\lambda}$$
$$\leq \lim_{n\to\infty} I_{b_n,\lambda}\left(s_{b_n} v_0^+ + t_{b_n} v_0^-\right) = I_{0,\lambda}(v_0^+ + v_0^-) = I_{0,\lambda}(v_0), \tag{3.14}$$

which implies $u_{0,\lambda}$ is a least energy sign-changing solution of Eq. (1.13). This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. For arbitrary $b>0$, let $u_{b,\lambda_n}\in H_0^1(\Omega)$ is a least energy sign-changing solution for Eq. (1.6) with $\lambda=\lambda_n$, which is obtained by Theorem 1.1. Obviously,

$$m_{b,0}\geq m_{b,\lambda},\quad \text{for each}\quad \lambda>0. \tag{3.15}$$

Therefore

$$\begin{aligned} m_{b,0}\geq m_{b,\lambda_n} &= I_{b,\lambda_n}(u_{b,\lambda_n})\\ &= I_{b,\lambda_n}(u_{b,\lambda_n})-\frac{1}{p}\langle I'_{b,\lambda_n}(u_{b,\lambda_n}),u_{b,\lambda_n}\rangle\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)a\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{p}\right)b\left(\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2\mathrm{d}x\right)^2, \end{aligned}$$

which implies that $\{u_{b,\lambda_n}\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, we may suppose there exists $u_{b,0}\in H_0^1(\Omega)$ such that $u_{b,\lambda_n}\rightharpoonup u_{b,0}$ weakly in $H_0^1(\Omega)$.

Since $\{u_{b,\lambda_n}\}$ is bounded in $H_0^1(\Omega)$, it follows from (3.15) that

$$\begin{aligned} -\frac{\lambda_n}{p}\int\limits_\Omega h^-(x)|u_{b,\lambda_n}|^p dx &= I_{b,\lambda_n}(u_{b,\lambda_n})-\frac{a}{2}\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2dx-\frac{b}{4}\left(\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2dx\right)^2\\ &\quad+\frac{1}{p}\int\limits_\Omega h^+(x)\,|u_{b,\lambda_n}|^p\,dx\\ &\leq C. \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{p}\int\limits_\Omega h^-(x)|u_{b,0}|^p dx &= \liminf_{n\to\infty}\left[-\frac{1}{p}\int\limits_\Omega h^-(x)|u_{b,\lambda_n}|^p dx\right]\\ &= \liminf_{n\to\infty}\left[\frac{1}{\lambda_n}\left(-\frac{\lambda_n}{p}\int\limits_\Omega h^-(x)|u_{b,\lambda_n}|^p dx\right)\right]=0, \end{aligned}$$

which implies $u_{b,0}=0$ on Ω^-.

On the other hand, since $\langle I'_{b,\lambda_n}(u_{b,\lambda n})-I'_{b,0}(u_{b,0}),u_{b,\lambda_n}-u_{b,0}\rangle=0$, then

$$\begin{aligned} & a\int\limits_\Omega|\nabla u_{b,\lambda_n}-\nabla u_{b,0}|^2dx+b\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2dx\int\limits_\Omega|\nabla u_{b,\lambda_n}-\nabla u_{b,0}|^2dx\\ &= b\left(\int\limits_\Omega|\nabla u_{b,0}|^2dx-\int\limits_\Omega|\nabla u_{b,\lambda_n}|^2dx\right)\int\limits_\Omega\nabla u_{b,0}\left(\nabla u_{b,\lambda_n}-\nabla u_{b,0}\right)dx\\ &\quad+\int\limits_\Omega\left(h^+(x)+\lambda h^-(x)\right)\left(|u_{b,\lambda_n}|^{p-2}u_{b,\lambda_n}-|u_{b,0}|^{p-2}u_{b,0}\right)(u_{b,\lambda_n}-u_{b,0})\,dx, \end{aligned} \tag{3.16}$$

the right hand of (3.16) tend to zero as $n\to\infty$ since $u_{b,\lambda_n}\rightharpoonup u_{b,0}$ weakly in $H_0^1(\Omega)$, which implies $u_{b,n}\to u_{b,0}$ strongly in $H_0^1(\Omega)$. Therefore

$$\langle I'_{b,0}(u_{b,0}),\varphi\rangle=\liminf_{n\to\infty}\langle I'_{b,\lambda_n}(u_{b,\lambda n}),\varphi\rangle=0,\quad\text{for each}\quad\varphi\in H_0^1(\Omega),$$

which implies $u_{b,0}$ is a solution of Eq. (1.14). By a similar method that used in [25], one can prove the existence of least energy sign-changing solution for equation (1.14). Suppose $v_{b,0}$ is a least energy sign-changing solution for Eq. (1.14), by Lemma 2.2, for each $\lambda_n > 0$, there exist a unique pair of positive numbers $(s_{\lambda_n}, t_{\lambda_n})$ such that

$$s_{\lambda_n} v_{b,0}^+ + t_{\lambda_n} v_{b,0}^- \in \mathcal{M}_{b,\lambda_n}.$$

That is

$$\begin{aligned} a(s_{\lambda_n})^2\|v_{b,0}^+\|^2 + b\,(s_{\lambda_n})^4\left(\int_\Omega |\nabla v_{b,0}^+|^2 \mathrm{d}x\right)^2 + b\,(s_{\lambda_n} t_{\lambda_n})^2 \int_\Omega \left|\nabla v_{b,0}^+\right|^2 dx \int_\Omega \left|\nabla v_{b,0}^-\right|^2 dx \\ = s_{\lambda_n}^p \int_\Omega \left(h^+(x) + \lambda_n h^-(x)\right) |v_{b,0}^+|^p dx, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} a(t_{\lambda_n})^2\|v_{b,0}^-\|^2 + b\,(t_{\lambda_n})^4\left(\int_\Omega |\nabla v_{b,0}^-|^2 \mathrm{d}x\right)^2 + b\,(s_{\lambda_n} t_{\lambda_n})^2 \int_\Omega \left|\nabla v_{b,0}^+\right|^2 dx \int_\Omega \left|\nabla v_{b,0}^-\right|^2 dx \\ = t_{\lambda_n}^p \int_\Omega \left(h^+(x) + \lambda_n h^-(x)\right) |v_{b,0}^-|^p dx, \end{aligned} \tag{3.18}$$

Recall that $v_{b,0}^\pm$ satisfying

$$a\|v_{0,\lambda}^+\|^2 + b\|v_{0,\lambda}^+\|^4 = \int_\Omega h^+(x)|v_{b,0}^+|^p dx \quad \text{and} \quad a\|v_{0,\lambda}^-\|^2 + b\|v_{0,\lambda}^-\|^4 = \int_\Omega h^+(x)|v_{b,0}^-|^p dx. \tag{3.19}$$

It follows from (3.17)–(3.19) that

$$(s_{\lambda_n}, t_{\lambda_n}) \to (1,1), \quad \text{as } n \to \infty. \tag{3.20}$$

Therefore, by (3.20) and Lemma 2.4, we can deduce that

$$\begin{aligned} I_{b,0}(v_{b,0}) &\le I_{b,0}(u_{b,0}) = \lim_{n\to\infty} I_{b,\lambda_n}(u_{b,\lambda_n}) \\ &\le \lim_{n\to\infty} I_{b,\lambda_n}\left(s_{\lambda_n} v_{b,0}^+ + t_{\lambda_n} v_{b,0}^-\right) = I_{b,0}\left(v_{b,0}^+ + v_{b,0}^-\right) = I_{b,0}(v_{b,0}). \end{aligned} \tag{3.21}$$

Therefore, we conclude that $u_{b,0}$ is a least energy sign-changing solution for Eq. (1.14), which changes sign once. The proof is completed. □

4. A special minimax value for the energy functional

In this section, we assume $h : \overline{\Omega} \to \mathbb{R}$ is a sign-changing continuous function and (h_1)–(h_2) hold.

We first state a result on the existence of solutions for Eq. (1.17).

Theorem 4.1. *(Theorem1.2, [10]) Suppose that $4 < p < 6$ and (h_1)–(h_2) hold. Then, for any non-empty subset $\Gamma \subset \{1, 2, \ldots, k\}$ satisfies (1.16), Eq. (1.17) has a nontrivial solution $u \in H_0^1(\Omega)$ with $u|_{\Omega_i}$ is positive for $i \in \Gamma_1$, $u|_{\Omega_i}$ is negative for $i \in \Gamma_2$, $u|_{\Omega_i}$ changes sign exactly once for $i \in \Gamma_3$, and $u \equiv 0$ on $\Omega \setminus \Omega_\Gamma$. Furthermore, u is the least energy solution among all solutions with these sign properties, that is, u achieves the following extremum*

$$m_\Gamma := \inf\left\{ I_\Gamma(u) \left| \begin{array}{l} u \text{ is a solution of (1.17) with } u|_{\Omega_i} \text{ is positive for } i \in \Gamma_1, u|_{\Omega_i} \text{is} \\ \text{negative for } i \in \Gamma_2 \text{ and } u|_{\Omega_i} \text{ changes sign exactly once for } i \in \Gamma_3. \end{array} \right.\right\} \tag{4.1}$$

The functional $I_\Gamma : H_0^1(\Omega_\Gamma) \to \mathbb{R}$ *is defined by*

$$I_\Gamma(u) := \frac{1}{2}\int\limits_{\Omega_\Gamma} a|\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int\limits_{\Omega_\Gamma} |\nabla u|^2 \mathrm{d}x\right)^2 - \int\limits_{\Omega_\Gamma} h^+(x)|u|^p \mathrm{d}x. \tag{4.2}$$

Without loss of generality, we next only consider the case $\Gamma_1 = \{1\}, \Gamma_2 = \{2\}, \Gamma_3 = \{3\}$ for simplicity. In this case, $\Gamma = \cup_{i=1}^3 \Gamma_i = \{1, 2, 3\}$ and

$$\Omega_\Gamma = \cup_{i=1}^3 \Omega_i \ \text{ with } \ dist(\Omega_i, \Omega_j) > 0 \ \text{ for } \ i \neq j, \quad i, j = 1, 2, 3.$$

We can choose open sets $\Omega_i^\rho := \{x \in \Omega \ \ dist(x, \Omega_i) < \rho\}$ for $i = 1, 2, 3$ with smooth boundary such that

$$\Omega_i \subset\subset \Omega_i^\rho \ \text{ and } \ dist(\Omega_i^\rho, \Omega_j^\rho) > 0 \ \text{ for } \ i \neq j, \quad i, j = 1, 2, 3.$$

We denote $\Omega^\rho := \cup_{i=1}^3 \Omega_i^\rho$ and define

$$\widehat{I}_{b,\lambda}(u) := \frac{a}{2}\int\limits_{\Omega^\rho} |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int\limits_{\Omega^\rho} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{p}\int\limits_{\Omega^\rho} \left(h^+(x) + \lambda h^-(x)\right)|u|^p \mathrm{d}x, \quad u \in H_0^1(\Omega^\rho). \tag{4.3}$$

Now, we consider the following constraint minimization problem

$$\widehat{m}_\lambda := \inf_{u \in \widehat{\mathcal{M}}_{b,\lambda}} \widehat{I}_{b,\lambda}(u),$$

where

$$\widehat{\mathcal{M}}_{b,\lambda} := \Big\{u \in H_0^1(\Omega^\rho) \mid \langle \widehat{I}'_{b,\lambda}(u), u_i\rangle = 0 \text{ for } i = 1, 2, \ u_1^+ \neq 0, u_2^- \neq 0 \\ \text{and } \langle \widehat{I}'_{b,\lambda}(u), u_3^\pm\rangle = 0, u_3^\pm \neq 0\Big\}.$$

Combining the approach applied in Sect. 2 in [10] and that used in the proof of Theorem 1.1, we deduce that there exists $v_\lambda \in H_0^1(\Omega^\rho)$ such that

$$\widehat{I}_{b,\lambda}(v_\lambda) = \widehat{m}_\lambda \ \text{ and } \ \widehat{I}'_{b,\lambda}(v_\lambda) = 0.$$

Proposition 4.2. *Suppose* $\lambda_n \to +\infty$ *as* $n \to \infty$ *and* $\{v_{\lambda_n}\} \subset H_0^1(\Omega^\rho)$ *satisfying*

$$\widehat{I}_{b,\lambda_n}(v_{\lambda_n}) = \widehat{m}_{\lambda_n} \ \text{ and } \ \widehat{I}'_{b,\lambda_n}(v_{\lambda_n}) = 0.$$

then, up to a subsequence, there exists $v \in H_0^1(\Omega^\rho)$ *such that*

(*i*) $v_n \to v$ *strongly in* $H_0^1(\Omega^\rho)$, *where we write* v_{λ_n} *as* v_n *for simplicity;*

(*ii*) $v = 0$ *in* $\Omega^\rho \backslash \Omega_\Gamma$ *and* v *is a solution to Eq.* (1.17);

(*iii*) $\widehat{I}_{b,\lambda_n}(v_n) \to \widehat{I}_{b,0}(v) = \frac{a}{2}\int\limits_{\Omega_\Gamma} |\nabla v|^2 \mathrm{d}x + \frac{b}{4}\left(\int\limits_{\Omega_\Gamma} |\nabla v|^2 \mathrm{d}x\right)^2 - \int\limits_{\Omega_\Gamma} h^+(x)|v|^p \mathrm{d}x.$

Proof. It is easy to prove that $\{v_n\}$ is bounded in $H_0^1(\Omega^\rho)$, since $\widehat{m}_{\lambda_n} \leq m_\Gamma$. Then, up to a subsequence, there exists $v \in H_0^1(\Omega^\rho)$ such that

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } \ H_0^1(\Omega^\rho), \\ v_n \to v \text{ strongly in } \ L^q(\Omega^\rho) \ \text{ for } \ 2 \leq q < 6, \\ v_n \to v \text{ for a.e. } \ x \in \Omega^\rho. \end{cases} \tag{4.4}$$

We first prove $v = 0$ in $\Omega^\rho \setminus \Omega_\Gamma$. Set $\Omega_-^\rho = \{x \in \Omega^\rho : h(x) < 0\}$, since $\{v_{\lambda_n}\}$ is bounded in $H_0^1(\Omega^\rho)$, then

$$-\frac{1}{p}\int\limits_{\Omega_-^\rho}\lambda_n h^-(x)|v_n|^p\mathrm{d}x = \widehat{I}_{b,\lambda_n}(v_n) - \frac{a}{2}\int\limits_{\Omega^\rho}|\nabla v_n|^2\mathrm{d}x - \frac{b}{4}\left(\int\limits_{\Omega^\rho}|\nabla v_n|^2\mathrm{d}x\right)^2 + \frac{1}{p}\int\limits_{\Omega^\rho}h^+(x)|v_n|^p\mathrm{d}x \le C. \tag{4.5}$$

Therefore

$$-\int\limits_{\Omega_-^\rho}h^-(x)|v|^p\mathrm{d}x = \liminf_{n\to\infty}\left[-\frac{1}{\lambda_n}\int\limits_{\Omega_-^\rho}\lambda_n h^-(x)|v_n|^p\mathrm{d}x\right] = 0,$$

which indicates that $v=0$ on Ω_-^ρ. Thus, we conclude $v=0$ in $\Omega^\rho\setminus\Omega_\Gamma$.

By using the fact $\langle \widehat{I}'_{b,\lambda_n}(v_n) - \widehat{I}'_{b,0}(v), v_n - v\rangle = 0$ that

$$\begin{aligned} &a\int\limits_{\Omega^\rho}|\nabla v_n-\nabla v|^2\mathrm{d}x + b\int\limits_{\Omega^\rho}|\nabla v_n|^2\mathrm{d}x\int\limits_{\Omega^\rho}|\nabla v_n-\nabla v|^2\mathrm{d}x \\ &= b\left(\int\limits_{\Omega^\rho}|\nabla v|^2\mathrm{d}x - \int\limits_{\Omega^\rho}|\nabla v_n|^2\mathrm{d}x\right)\int\limits_{\Omega^\rho}\nabla v(\nabla v_n - \nabla v)\mathrm{d}x \\ &\quad + \int\limits_{\Omega^\rho}h^+(x)\left(|v_n|^{p-2}v_n - |v|^{p-2}v\right)(v_n-v)\mathrm{d}x + \int\limits_{\Omega^\rho}\lambda_n h^-(x)|v_n|^{p-2}v_n(v_n-v)\mathrm{d}x. \end{aligned}$$

Obviously, the right hand of the last equality tend to zero as $n\to\infty$, since $\{v_n\}$ is bounded in $H_0^1(\Omega^\rho)$ and $v=0$ in $\Omega^\rho\backslash\Omega_\Gamma$. Thus, $v_n\to v$ strongly in $H_0^1(\Omega^\rho)$, and hence v is a solution of (1.17).

Finally, it is easy to conclude that (iii) from (i)–(ii). □

Moreover, we have the following asymptotic behavior for $\widehat{m}_\lambda$ as $\lambda\to+\infty$.

Lemma 4.3. *There holds that*

(i) $0<\widehat{m}_\lambda\le m_\Gamma$, *for all* $\lambda\ge 0$;

(ii) $\widehat{m}_\lambda\to m_\Gamma$, *as* $\lambda\to+\infty$.

Proof. The proof of point (i) is trivial, so we omit the detail.

Now, we are going to prove point (ii). Let $\{\lambda_n\}$ be a sequence with $\lambda_n\to+\infty$ as $n\to+\infty$. For each λ_n, there exists $v_{\lambda_n}\in H_0^1(\Omega^\rho)$ with

$$\widehat{I}_{b,\lambda_n}(v_{\lambda_n}) = \widehat{m}_{b,\lambda_n} \quad\text{and}\quad \widehat{I}'_{b,\lambda_n}(v_{\lambda_n}) = 0. \tag{4.6}$$

We suppose, up to a subsequence, $\{\widehat{I}_{b,\lambda_n}(v_{\lambda_n})\}$ converges, since $\widehat{m}_{b,\lambda_n}\le m_\Gamma$. By using similar arguments as in Proposition 4.2, we know that there exists $v\in H_0^1(\Omega^\rho)$ such that

$$v_{\lambda_n}\to v \quad\text{strongly in}\quad H_0^1(\Omega^\rho) \quad\text{as}\quad n\to+\infty,$$

and $(v|_{\Omega_1})^+$, $(v|_{\Omega_2})^-$, $(v|_{\Omega_3})^\pm\ne 0$. Moreover,

$$\widehat{m}_{b,\lambda_n} = \widehat{I}_{b,\lambda_n}(v_{\lambda_n}) \to \widehat{I}_{b,0}(v), \tag{4.7}$$

and

$$0 = \widehat{I}'_{b,\lambda_n}(v_{\lambda_n}) \to \widehat{I}'_{b,0}(v). \tag{4.8}$$

By the definition of m_Γ, we have that

$$\lim_{n_i\to+\infty}\widehat{m}_{b,\lambda_n} = \widehat{I}_{b,0}(v) \ge m_\Gamma. \tag{4.9}$$

By conclusion (i) of this Lemma, we know that $\widehat{m}_{b,\lambda_n}\to m_\Gamma$ as $n\to\infty$. □

Next, we denote the solution of (1.17) given in Theorem 4.1 by $v \in H_0^1(\Omega)$, that is

$$v \in H_0^1(\Omega_\Gamma),\ I_\Gamma(v) = m_\Gamma,\ I'_\Gamma(v) = 0, \tag{4.10}$$

and $v_1 = v|_{\Omega_1}$ is positive, $v_2 = v|_{\Omega_2}$ is negative, $v_3 = v|_{\Omega_3}$ changes sign exactly once. Obviously, there exist constants $\tau_2 > \tau_1 > 0$ such that

$$\tau_1 \le \|v_1\|,\ \ \|v_2\|,\ \ \|v_3^+\|,\ \ \|v_4^-\| \le \tau_2. \tag{4.11}$$

We now define $\gamma_0 : [\frac{1}{2}, \frac{3}{2}]^4 \to H_0^1(\Omega)$ by

$$\gamma_0(t_1, t_2, t_3, t_4) := t_1 v_1 + t_2 v_2 + t_3 v_3^+ + t_4 v_3^- \tag{4.12}$$

and

$$m_\lambda := \inf_{\gamma \in \Sigma_\lambda} \max_{\mathbf{t} \in [\frac{1}{2}, \frac{3}{2}]^4} I_{b,\lambda}(\gamma(\mathbf{t})), \tag{4.13}$$

where

$$\begin{aligned}\Sigma_\lambda : = \Bigg\{\gamma \in \mathcal{C}\left(\left[\frac{1}{2}, \frac{3}{2}\right]^4, H_0^1(\Omega)\right) : \|\gamma(\mathbf{t})\| \le 6\tau_2 + \tau_1,\ (\gamma|_{\Omega_1^\rho})^+,\ (\gamma|_{\Omega_2^\rho})^-,\ (\gamma|_{\Omega_3^\rho})^\pm \neq 0 \\ \text{and}\ \ \gamma = \gamma_0\ \ \text{on}\ \ \partial\left[\frac{1}{2}, \frac{3}{2}\right]^4\Bigg\}.\end{aligned} \tag{4.14}$$

Obviously, $\gamma_0 \in \Sigma_\lambda$, so $\Sigma_\lambda \neq \varnothing$. Thus m_λ is well-defined.

Lemma 4.4. *For any $\gamma \in \Sigma_\lambda$, there exists an 4-tuple $\mathbf{t}^* = (t_1^*, t_2^*, t_3^*, t_4^*) \in D = (\frac{1}{2}, \frac{3}{2})^4$ such that*

$$\langle \widehat{I}'_{b,\lambda}(\gamma(\mathbf{t}^*)|_{\Omega^\rho}), \gamma_1^+(\mathbf{t}^*)\rangle = \langle \widehat{I}'_{b,\lambda}(\gamma(\mathbf{t}^*)|_{\Omega^\rho}), \gamma_2^-(\mathbf{t}^*)\rangle = 0 \ \ \textit{and} \ \ \langle \widehat{I}'_{b,\lambda}(\gamma(\mathbf{t}^*)|_{\Omega^\rho}), \gamma_3^\pm(\mathbf{t}^*)\rangle = 0,$$

where $\gamma_i(\mathbf{t}) = \gamma(\mathbf{t})|_{\Omega_i^\rho}$ for $i = 1, 2, 3$.

Proof. For each $\gamma \in \Sigma_\lambda$, let us define $\Psi : [\frac{1}{2}, \frac{3}{2}]^4 \to \mathbb{R}^4$ given by

$$\Psi(\mathbf{t}) = \left(\widehat{I}'_{b,\lambda}\left(\gamma(\mathbf{t})|_{\Omega^\rho}\right)\gamma_1^+(\mathbf{t}),\ \widehat{I}'_{b,\lambda}\left(\gamma(\mathbf{t})|_{\Omega^\rho}\right)\gamma_2^-(\mathbf{t}),\ \widehat{I}'_{b,\lambda}\left(\gamma(\mathbf{t})|_{\Omega^\rho}\right)\gamma_3^+(\mathbf{t}),\ \widehat{I}'_{b,\lambda}\left(\gamma(\mathbf{t})|_{\Omega^\rho}\right)\gamma_3^-(\mathbf{t})\right).$$

Denote

$$\Psi_0(\mathbf{t}) = \left(\widehat{I}'_{b,\lambda}\left(\gamma_0(\mathbf{t})\right)t_1 v_1,\ \widehat{I}'_{b,\lambda}\left(\gamma_0(\mathbf{t})\right)t_2 v_2,\ \widehat{I}'_{b,\lambda}\left(\gamma_0(\mathbf{t})\right)t_3 v_3^+,\ \widehat{I}'_{b,\lambda}\left(\gamma_0(\mathbf{t})\right)t_4 v_3^-\right).$$

Obviously,

$$\Psi(\mathbf{t}) = \Psi_0(\mathbf{t}) \neq 0, \ \ \text{for each} \ \ \mathbf{t} \in \partial\left(\frac{1}{2}, \frac{3}{2}\right)^4.$$

Therefore, we can verify that

$$deg(\Psi, D, 0) = deg(\Psi_0, D, 0) = 1.$$

This implies that there exists $\mathbf{t}^* \in (\frac{1}{2}, \frac{3}{2})^4$ such that $\Psi(\mathbf{t}^*) = 0$. □

Lemma 4.5. *There holds that*

(i) $\widehat{m}_\lambda \le m_\lambda \le m_\Gamma$ *for all* $\lambda \ge 1$*;*

(ii) $m_\lambda \to m_\Gamma$ *as* $\lambda \to +\infty$*;*

(iii) There exists $\varepsilon_0 > 0$ *such that* $I_{b,\lambda}(\gamma(\mathbf{t})) < m_\Gamma - \varepsilon_0$ *for all* $\lambda > 0$, $\gamma \in \Sigma_\lambda$ *and* $\mathbf{t} = (t_1, t_2, t_3, t_4) \in \partial[\frac{1}{2}, \frac{3}{2}]^4$.

Proof. (i) Since $\gamma_0 \in \Sigma_\lambda$, we have

$$m_\lambda \le \max_{\mathbf{t}\in[\frac12,\frac32]^4} I_{b,\lambda}(\gamma_0(\mathbf{t})) = I_{b,\lambda}(\gamma_0(1,1,1,1)) = m_\Gamma,$$

where we have used Lemma 2.2 in [10]. Recall that

$$\widehat{m}_\lambda := \inf_{u\in\widehat{\mathcal{M}}_{b,\lambda}} \widehat{I}_{b,\lambda}(u).$$

For each $\gamma \in \Sigma_\lambda$, fix $\mathbf{t}^* \in (\frac12,\frac32)^4$ given by Lemma 4.4, then

$$\widehat{m}_\lambda \le \widehat{I}_{b,\lambda}(\gamma(\mathbf{t}^*)|_{\Omega^\rho}).$$

Therefore,

$$\max_{\mathbf{t}\in[\frac12,\frac32]^4} I_{b,\lambda}(\gamma(\mathbf{t})) \ge \widehat{I}_{b,\lambda}(\gamma(\mathbf{t}^*)|_{\Omega^\rho}) \ge \widehat{m}_\lambda, \quad \text{for each} \quad \gamma \in \Sigma_\lambda.$$

Thus,

$$m_\lambda \ge \widehat{m}_\lambda.$$

(ii) Since $\widehat{m}_\lambda \to m_\Gamma$ by Lemma 4.3 (ii), we have

$$m_\lambda \to m_\Gamma \quad \text{as} \quad \lambda \to +\infty.$$

(iii) For $\mathbf{t} = (t_1,t_2,t_3,t_4) \in \partial[\frac12,\frac32]^4$, it holds $\gamma(\mathbf{t}) = \gamma_0(\mathbf{t})$ and hence

$$I_{b,\lambda}(\gamma(\mathbf{t})) = I_{b,\lambda}(\gamma_0(\mathbf{t})) \quad \text{for} \quad \mathbf{t} = (t_1,t_2,t_3,t_4) \in \partial\left[\frac12,\frac32\right]^4.$$

By Lemma 2.2 in [10], we know that $(1,1,1,1)$ is the unique maximum point of $\varphi(\mathbf{t}) = I_{b,0}(\gamma_0(\mathbf{t}))$, which gives that

$$I_{b,\lambda}(\gamma(\mathbf{t})) < m - \varepsilon_0 \quad \text{for} \quad \mathbf{t} = (t_1,t_2,t_3,t_4) \in \partial\left[\frac12,\frac32\right]^4.$$

where $\varepsilon_0 > 0$ is a small constant. □

5. Proof of Theorem 1.4.

In this section, we prove Theorem 1.4. More precisely, we show that the existence of sign-changing multi-bump solutions to Eq. (1.6) for large λ, which converges to solutions of (1.17) with prescribed sign properties as $\lambda \to +\infty$.

Define

$$\mathcal{S} := \Big\{ u \in \mathcal{M}_\Gamma \;\Big|\; I_\Gamma(u) = m_\Gamma \Big\},$$

where

$$\mathcal{M}_\Gamma = \Big\{ u \in H_0^1(\Omega_\Gamma) \mid \langle I'_\Gamma(u), u|_{\Omega_i}\rangle = 0, i=1,2, \ (u|_{\Omega_1})^+ \ne 0, (u|_{\Omega_2})^- \ne 0, \\ \text{and } \langle I'_\Gamma(u), (u|_{\Omega_3})^\pm\rangle = 0, (u|_{\Omega_3})^\pm \ne 0 \Big\}.$$

Obviously, $\mathcal{S}$ contains all least energy solutions of (1.17) with $u|_{\Omega_1}$ is positive, $u|_{\Omega_2}$ is negative, $u|_{\Omega_3}$ changes sign exactly once. Moreover, we have the following Lemma.

Lemma 5.1. *$\mathcal{S}$ is compact in $H_0^1(\Omega_\Gamma)$.*

Proof. Let $\{u_n\} \subset \mathcal{S}$, then $\{u_n\}$ is a bounded $(PS)_{m_\Gamma}$ sequence of I_Γ. Since I_Γ satisfies (PS)-condition, up to a subsequence, we may suppose $u_n \to u_\infty$ strongly in $H_0^1(\Omega_\Gamma)$. It follows that $u_\infty \in \mathcal{M}_\Gamma$ and $I_\Gamma(u_\infty) = \lim_{n\to\infty} I_\Gamma(u_n) = m_\Gamma$. Therefore, $u_\infty \in \mathcal{S}$. □

Lemma 5.2. *Let $d > 0$ be a fixed number and let $\{u_n\} \subset \mathcal{S}^d$ be a sequence. Then, up to a subsequence, $u_n \rightharpoonup u_0 \in \mathcal{S}^{2d}$ weakly in $H_0^1(\Omega)$ as $n \to \infty$, where*

$$\mathcal{S}^d := \left\{u \in H_0^1(\Omega) : dist_\lambda(u, \mathcal{S}) \le d\right\}$$

and dist denotes the distance in $H_0^1(\Omega)$.

Proof. Since $\mathcal{S}$ is compact in $H_0^1(\Omega)$, then there exists a sequence $\{\bar{u}_n\} \subset \mathcal{S}$ such that

$$dist\,(u_n, \mathcal{S}) = dist\,(u_n, \bar{u}_n) \le d.$$

By Lemma 5.1, there exists $\bar{u} \in \mathcal{S}$ such that, up to a subsequence, $\bar{u}_n \to \bar{u}$ strongly in $H_0^1(\Omega)$. Hence, $dist\,(\bar{u}_n, \bar{u}) \le d$ for n large enough. Thus, $\{u_n\}$ is bounded and, up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Since $B_{2d}(\bar{u})$ is weakly closed in $H_0^1(\Omega)$, therefore, $u_0 \in B_{2d}(\bar{u}) \subset \mathcal{S}^{2d}$. □

Lemma 5.3. *Let $d \in (0, \tau_1)$, where τ_1 is given by (4.11). Suppose that there exist a sequence $\lambda_n > 0$ with $\lambda_n \to +\infty$, and $\{u_n\} \subset \mathcal{S}^d$ satisfying*

$$\lim_{n\to\infty} I_{b,\lambda_n}(u_n) \le m_\Gamma, \quad \lim_{n\to\infty} I'_{b,\lambda_n}(u_n) = 0.$$

Then, up to a subsequence, $\{u_n\}$ converges strongly in $H_0^1(\Omega)$ to an element $u \in \mathcal{S}$.

Proof. Since $\lim_{n\to\infty} I_{b,\lambda_n}(u_n) \le m_\Gamma$ and $\lim_{n\to\infty} I'_{b,\lambda_n}(u_n) = 0$, we deduce that $\{\|u_n\|\}$ and $\{I_{\lambda_n}(u_n)\}$ are bounded. Up to a subsequence, we may assume that

$$I_{b,\lambda_n}(u_n) \to c \in (-\infty, m_\Gamma].$$

By using Proposition 4.2, there exists $u \in H_0^1(\Omega)$ such that

$$u_n \to u \text{ strongly in } H_0^1(\Omega), \quad u = 0 \text{ in } \Omega \setminus \Omega_\Gamma \text{ and } I_{b,\lambda_n}(u_n) \to I_\Gamma(u). \tag{5.1}$$

Moreover, u is a solution to the following equation

$$\begin{cases} -\left(a + b\displaystyle\int_{\Omega_\Gamma} |\nabla u|^2 \mathrm{d}x\right)\Delta u = h^+(x)|u|^{p-2}u, \, x \in \Omega_\Gamma, \\ u = 0, & x \in \Omega \setminus \Omega_\Gamma, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5.2}$$

Since $\{u_n\} \subset \mathcal{S}^d$ and $d \in (0, \tau_1)$, we deduce that $(u|_{\Omega_1})^+ \ne 0$, $(u|_{\Omega_2})^- \ne 0$ and $(u|_{\Omega_3})^\pm \ne 0$. Consequently, $I_\Gamma(u) \ge m$. The conclusion $I_\Gamma(u) = m$ follows from the fact that $I_{b,\lambda_n}(u_n) \to I_\Gamma(u) \le m_\Gamma$, Thus, $u \in \mathcal{S}$ is proved. □

Lemma 5.4. *Let $\tau_1 > 0$ be as in Lemma 5.3. Then, for $\delta \in (0, d)$, there exist constants $0 < \sigma < 1$ and $\Lambda_1 > 0$ such that $\|I'_{b,\lambda}(u)\|_{H^{-1}} \ge \sigma$ for any $u \in I_{b,\lambda}^{m_\lambda} \cap (\mathcal{S}^\delta \backslash \mathcal{S}^{\frac{\delta}{2}})$ and $\lambda \ge \Lambda_1$.*

Proof. We argue by contradiction. Suppose that there exist a number $\delta_0 \in (0, d)$, a positive sequence $\{\lambda_j\}$ with $\lambda_j \to 0$, and a sequence of function $\{u_j\} \subset I_{b,\lambda_j}^{m_{\lambda_j}} \cap (\mathcal{S}^{\delta_0} \backslash \mathcal{S}^{\frac{\delta_0}{2}})$ such that

$$\lim_{j\to+\infty} I'_{b,\lambda_j}(u_j) = 0.$$

Up to a subsequence, we obtain

$$\{u_j\} \subset \mathcal{S}^{\delta_0}, \quad \lim_{j\to\infty} I_{b,\lambda_j}(u_j) \le m_\Gamma.$$

Hence, we can apply Lemma 5.3 and assert that there exists $u \in \mathcal{S}$ such that $u_j \to u$ strongly in $H_0^1(\Omega)$. As a consequence, $dist\,(u_j, \mathcal{S}) \to 0$ as $j \to +\infty$. This contradict the fact that $u_j \notin \mathcal{S}^{\frac{\delta_0}{2}}$. □

From now on, we fix a small constant $\delta \in (0, d)$ and corresponding constants $0 < \sigma < 1$ and $\Lambda_1 > 0$ such that our Lemma 5.4 hold. For convenient, we next denote $Q := [\frac{1}{2}, \frac{3}{2}]^4$.

Lemma 5.5. *There exist $\Lambda_2 \geq \Lambda_1$ and $\alpha > 0$ such that for any $\lambda \geq \Lambda_2$,*

$$I_{b,\lambda}(\gamma_0(t_1, t_2, t_3, t_4)) \geq m_\lambda - \alpha \text{ implies that } \gamma_0(t_1, t_2, t_3, t_4) \in \mathcal{S}^{\frac{\delta}{2}}. \tag{5.3}$$

Proof. Assume by contradiction that there exist $\lambda_n \to \infty$, $\alpha_n \to 0$ and $(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \in Q$ such that

$$I_{b,\lambda}(\gamma_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)})) \geq m_{\lambda_n} - \alpha_n \text{ and } \gamma_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \notin \mathcal{S}^{\frac{\delta}{2}}. \tag{5.4}$$

Passing to a subsequence, we may assume that $(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) \to (\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) \in Q$. Then, Lemma 4.5 implies that

$$I_\Gamma(\gamma_0(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4)) \geq \lim_{n\to\infty} (m_{\lambda_n} - \alpha_n) = m_\Gamma.$$

From Lemma 2.2 in [10], we can deduce that $(\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4) = (1, 1, 1, 1)$ and hence

$$\lim_{n\to\infty} \|\gamma_0(t_1^{(n)}, t_2^{(n)}, t_3^{(n)}, t_4^{(n)}) - \gamma_0(1, 1, 1, 1)\| = 0.$$

However, $\gamma_0(1, 1, 1, 1) = v \in \mathcal{S}$, which contradicts to (5.4). □

Next, we set

$$\alpha_0 := \min\left\{\frac{\alpha}{2}, \frac{\varepsilon_0}{2}, \frac{1}{8}\delta\sigma^2\right\}, \tag{5.5}$$

where δ, σ are given in Lemma 5.4, α is from Lemma 5.5, ε_0 is from Lemma 4.5 (iii). By Lemma 4.4, there exists $\Lambda_3 \geq \Lambda_2$ such that

$$|m_\lambda - m_\Gamma| < \alpha_0 \text{ for all } \lambda \geq \Lambda_3. \tag{5.6}$$

Proposition 5.6. *For each $\lambda \geq \Lambda_3$, there exists a critical point u_λ of $I_{b,\lambda}$ with $u_\lambda \in \mathcal{S}^\delta \cap I_{b,\lambda}^{m_\Gamma}$.*

Proof. Fix $\lambda \geq \Lambda_3$. Assume by contradiction that there exists $0 < \rho_\lambda < 1$ such that $\|I'_{b,\lambda}(u)\| \geq \rho_\lambda$ on $\mathcal{S}^\delta \cap I_\lambda^{m_\Gamma}$. Then there exists a pseudo-gradient vector field T_λ in $H_0^1(\Omega)$ which is defined on a neighborhood Z_λ of $\mathcal{S}^\delta \cap I_{b,\lambda}^{m_\Gamma}$ such that for any $u \in Z_\lambda$ there holds

$$\|T_\lambda(u)\| \leq 2\min\{1, \|I'_{b,\lambda}(u)\|\},$$
$$\langle I'_{b,\lambda}(u), T_\lambda(u)\rangle \geq \min\{1, \|I'_{b,\lambda}(u)\|\}\|I'_{b,\lambda}(u)\|.$$

Let ψ_λ be a Lipschitz continuous function on $H_0^1(\Omega)$ such that $0 \leq \psi_\lambda \leq 1$, $\psi_\lambda \equiv 1$ on $S^\delta \cap I_{b,\lambda}^{m_\Gamma}$ and $\psi_\lambda \equiv 0$ on $H_0^1(\Omega)\backslash Z_\lambda$. Let ξ_λ be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \leq \xi_\lambda \leq 1$, $\xi_\lambda(t) \equiv 1$ if $|t - m_\Gamma| \leq \frac{\alpha}{2}$ and $\xi_\lambda(t) \equiv 0$ if $|t - m_\Gamma| \geq \alpha$. Define

$$e_\lambda(u) := \begin{cases} -\psi_\lambda(u)\xi_\lambda(I_{b,\lambda}(u))T_\lambda(u), & \text{if } u \in Z_\lambda, \\ 0, & \text{if } u \in H_0^1(\Omega) \setminus Z_\lambda. \end{cases} \tag{5.7}$$

Then there exists a global solution $\eta_\lambda : H_0^1(\Omega) \times [0, +\infty) \to H_0^1(\Omega)$ for the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}\eta_\lambda(u, \theta) = e_\lambda(\eta_\lambda(u, \theta)), \\ \eta_\lambda(u, 0) = u. \end{cases} \tag{5.8}$$

It is easy to see that η_λ has the following properties:

(1) $\eta_\lambda(u, \theta) = u$ if $\theta = 0$ or $u \in H_0^1(\Omega)\backslash Z_\lambda$ or $|I_{b,\lambda}(u) - m_\Gamma| \geq \alpha$.
(2) $\|\frac{\mathrm{d}}{\mathrm{d}\theta}\eta_\lambda(u, \theta)\| \leq 2$.
(3) $\frac{\mathrm{d}}{\mathrm{d}\theta} I_{b,\lambda}(\eta_\lambda(u, \theta)) = \langle I'_{b,\lambda}(\eta_\lambda(u, \theta)), e_\lambda(\eta_\lambda(u, \theta))\rangle \leq 0$. □

Claim 1. *For any $(t_1,t_2,t_3,t_4)\in Q$, there exists $\overline{\theta}=\theta(t_1,t_2,t_3,t_4)\in[0,+\infty)$ such that $\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\overline{\theta})\in I_{b,\lambda}^{m_\Gamma-\alpha_0}$, where α_0 is given by (5.5).*

Assume by contradiction that there exists $(t_1,t_2,t_3,t_4)\in Q$ such that

$$I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))>m_\Gamma-\alpha_0$$

for any $\theta\geq 0$. Note that $\alpha_0<\alpha$, we see, from Lemma 5.5, that $\gamma_0(t_1,t_2,t_3,t_4)\in\mathcal{S}^{\frac{\delta}{2}}$. Moreover, since $I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4))\leq m_\Gamma$, we have, from the property (3) of η_λ, that

$$m_\Gamma-\alpha_0<I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\leq I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4))\leq m_\Gamma$$

for $\theta\geq 0$. This implies that $\xi_\lambda(I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta)))\equiv 1$. If $\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta)\in\mathcal{S}^\delta$ for all $\theta\geq 0$, we can deduce that

$$\psi_\lambda\left(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta)\right)\equiv 1 \quad \text{and} \quad \|I'_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\|\geq\rho_\lambda$$

for all $\theta>0$. It follows that

$$I_{b,\lambda}\big(\eta_\lambda\big(\gamma_0(t_1,t_2,t_3,t_4),\frac{\alpha}{\rho_\lambda^2}\big)\big)\leq m_\Gamma-\int_0^{\frac{\alpha}{\rho_\lambda^2}}\rho_\lambda^2dt\leq m_\Gamma-\alpha,$$

which is a contradiction. Thus, there exists $\theta_3>0$ such that $\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_3)\notin\mathcal{S}^\delta$. Note that $\gamma_0(t_1,t_2,t_3,t_4)\in\mathcal{S}^{\frac{\delta}{2}}$, there exist $0<\theta_1<\theta_2\leq\theta_3$ such that

$$\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_1)\in\partial\mathcal{S}^{\frac{\delta}{2}},\quad \eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_2)\in\partial\mathcal{S}^\delta$$

and

$$\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta)\in\mathcal{S}^\delta\setminus\mathcal{S}^{\frac{\delta}{2}} \quad \text{for all} \quad \theta\in(\theta_1,\theta_2).$$

By Lemma 5.4, we have that

$$\|I'_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\|\geq\sigma \quad \text{for all} \quad \theta\in(\theta_1,\theta_2).$$

By using property (2) of η_λ we have

$$\frac{\delta}{2}\leq\|\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_2)-\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_1)\|\leq 2|\theta_2-\theta_1|.$$

This implies that

$$\begin{aligned}
&I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_2))\\
&\quad\leq I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),0))+\int_0^{\theta_2}\frac{\mathrm{d}}{\mathrm{d}\theta}I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\mathrm{d}\theta\\
&\quad< I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4))+\int_{\theta_1}^{\theta_2}\frac{\mathrm{d}}{\mathrm{d}\theta}I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\mathrm{d}\theta\\
&\quad\leq m_\Gamma-\sigma^2(\theta_2-\theta_1)\leq m_\Gamma-\frac{1}{4}\delta\sigma^2\\
&\quad< m_\Gamma-\alpha_0,
\end{aligned} \tag{5.9}$$

which is a contradiction. Thus, we finish the proof of Claim 1.

Now, we can define

$$T(t_1,t_2,t_3,t_4):=\inf\Big\{\theta\geq 0: I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))\leq m_\Gamma-\alpha_0\Big\}$$

and

$$\widetilde{\gamma}(t_1,t_2,t_3,t_4) := \eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4)).$$

Then $\Phi_\lambda(\widetilde{\gamma}(t_1,t_2,t_3,t_4)) \le m_\Gamma - \alpha_0$ *for all* $(t_1,t_2,t_3,t_4) \in Q$.

Claim 2. $\widetilde{\gamma}(t_1,t_2,t_3,t_4) = \eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4)) \in \Sigma_\lambda$.

For any $(t_1,t_2,t_3,t_4) \in \partial Q$, *by* (5.5)–(5.6), *we have*

$$I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4)) \le I_\Gamma(\gamma_0(t_1,t_2,t_3,t_4)) < m_\Gamma - \varepsilon_0 \le m_\Gamma - \alpha_0,$$

which implies that $T(t_1,t_2,t_3,t_4) = 0$ *and thus* $\widetilde{\gamma}(t_1,t_2,t_3,t_4) = \gamma_0(t_1,t_2,t_3,t_4)$ *for* $(t_1,t_2,t_3,t_4) \in \partial Q$.

By the definition of Σ_λ *in* (4.14), *it suffices to prove that* $\|\widetilde{\gamma}(t_1,t_2,t_3,t_4)\| \le 6\tau_2+\tau_1$ *for all* $(t_1,t_2,t_3,t_4) \in Q$ *and* $T(t_1,t_2,t_3,t_4)$ *is continuous with respect to* (t_1,t_2,t_3,t_4).

For any $(t_1,t_2,t_3,t_4) \in Q$, *we have* $T(t_1,t_2,t_3,t_4) = 0$ *if* $I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4)) \le m_\Gamma - \alpha_0$, *and hence* $\widetilde{\gamma}(t_1,t_2,t_3,t_4) = \gamma_0(t_1,t_2,t_3,t_4)$. *By* (4.11), *we deduce that* $\|\widetilde{\gamma}(t_1,t_2,t_3,t_4)\| \le 6\tau_2 < 6\tau_2 + \tau_1$.

On the other hand, if $I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4)) > m_\Gamma - \alpha_0$, *we can deduce that*

$$I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4)) > m_\lambda - \alpha,$$

thus $\gamma_0(t_1,t_2,t_3,t_4) \in \mathcal{S}^{\frac{\delta}{2}}$ *and*

$$m_\Gamma - \alpha_0 < I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta)) < m_\Gamma + \alpha_0, \ \ \textit{for all} \ \ \theta \in [0, T(t_1,t_2,t_3,t_4)).$$

This implies that

$$\xi_\lambda(I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta))) \equiv 1 \ \ \textit{for all} \ \ \theta \in [0, T(t_1,t_2,t_3,t_4)).$$

Now, we are going to prove that $\widetilde{\gamma}(t_1,t_2,t_3,t_4) \in \mathcal{S}^\delta$. *Otherwise, if* $\widetilde{\gamma}(t_1,t_2,t_3,t_4) \notin \mathcal{S}^\delta$, *similar to the proof of* **Claim 1**, *we can find two constants* $0 < \theta_1 < \theta_2 < T(t_1,t_2,t_3,t_4)$ *such that* (5.9) *hold. This implies that* $I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4),\theta_2)) < m_\Gamma - \alpha_0$ *which contradicts to the definition of* $T(t_1,t_2,t_3,t_4)$. *Therefore,*

$$\widetilde{\gamma}(t_1,t_2,t_3,t_4) = \eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4)) \in \mathcal{S}^\delta.$$

Thus there exists $u \in \mathcal{S}$ *such that* $\|\widetilde{\gamma}(t_1,t_2,t_3,t_4) - u\| \le \delta \le \tau_1$. *It follows from* (4.11) *that*

$$\|\widetilde{\gamma}(t_1,t_2,t_3,t_4)\| \le \|u\| + \tau_1 \le 6\tau_2 + \tau_1.$$

To prove the continuity of $T(t_1,t_2,t_3,t_4)$, *we fix arbitrarily* $(t_1,t_2,t_3,t_4) \in Q$. *First, we assume that* $I_{b,\lambda}(\widetilde{\gamma}(t_1,t_2,t_3,t_4)) < m_\Gamma - \alpha_0$. *In this case, we deduce directly that* $T(t_1,t_2,t_3,t_4) = 0$ *by the definition of* $T(t_1,t_2,t_3,t_4)$, *which gives that*

$$I_{b,\lambda}(\gamma_0(t_1,t_2,t_3,t_4)) < m - \alpha_0.$$

By the continuity of γ_0, *there exists* $r > 0$ *such that for any* $(s_1,s_2,s_3,s_4) \in B_r(t_1,t_2,t_3,t_4) \cap Q$, *we have* $I_{b,\lambda}(\gamma_0(s_1,s_2,s_3,s_4)) < m_\Gamma - \alpha_0$. *Thus,* $T(s_1,s_2,s_3,s_4) = 0$, *and hence* T *is continuous at* (t_1,t_2,t_3,t_4).

Now, we assume that $I_{b,\lambda}(\widetilde{\gamma}(t_1,t_2,t_3,t_4)) = m_\Gamma - \alpha_0$. *From the previous proof we see that* $\widetilde{\gamma}(t_1,t_2,t_3,t_4) = \eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4)) \in \mathcal{S}^\delta$, *and so*

$$\|I'_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4)))\| \ge \rho_\lambda > 0.$$

Thus for any $\omega > 0$, *we have*

$$I_{b,\lambda}(\eta_\lambda(\gamma_0(t_1,t_2,t_3,t_4), T(t_1,t_2,t_3,t_4) + \omega)) < m_\Gamma - \alpha_0.$$

By the continuity of η_λ, *there exists* $r > 0$ *such that*

$$I_{b,\lambda}(\eta_\lambda(\gamma_0(s_1,s_2,s_3,s_4), T(t_1,t_2,t_3,t_4) + \omega))) < m_\Gamma - \alpha_0,$$

for any $(s_1,s_2,s_3,s_4) \in B_r(t_1,t_2,t_3,t_4) \cap Q$. *Thus,* $T(s_1,s_2,s_3,s_4) \le T(t_1,t_2,t_3,t_4) + \omega$. *It follows that*

$$0 \le \limsup_{(s_1,s_2,s_3,s_4)\to(t_1,t_2,t_3,t_4)} T(s_1,s_2,s_3,s_4) \le T(t_1,t_2,t_3,t_4). \tag{5.10}$$

If $T(t_1,t_2,t_3,t_4)=0$, *we immediately implies that*

$$\lim_{(s_1,s_2,s_3,s_4)\to(t_1,t_2,t_3,t_4)} T(s_1,s_2,s_3,s_4)=T(t_1,t_2,t_3,t_4).$$

If $T(t_1,t_2,t_3,t_4)>0$, *we can similarly deduce that*

$$I_{b,\lambda}(\eta_\lambda(\gamma_0(s_1,s_2,s_3,s_4),T(t_1,t_2,t_3,t_4)-\omega))>m_\Gamma-\alpha_0.$$

for any $0<\omega<T(t_1,t_2,t_3,t_4)$.

By the continuity of η_λ *again, we see that*

$$\liminf_{(s_1,s_2,s_3,s_4)\to(t_1,t_2,t_3,t_4)} T(s_1,s_2,s_3,s_4)\geq T(t_1,t_2,t_3,t_4). \tag{5.11}$$

It follows from (5.10)–(5.11) *that* T *is continuous at* (t_1,t_2,t_3,t_4). *This completes the proof of Claim 2.*

Thus, we have proved that $\widetilde{\gamma}(t_1,t_2,t_3,t_4)\in\Sigma_\lambda$ *and*

$$\max_{(t_1,t_2,t_3,t_4)\in Q} I_\lambda(\widetilde{\gamma}(t_1,t_2,t_3,t_4))\leq m_\Gamma-\alpha_0,$$

which contradicts the definition of m_Γ. *This completes the proof.* □

Proof of Theorem 1.4. We still prove Theorem 1.4 with $\Gamma_1=\{1\}$, $\Gamma_2=\{2\}$ and $\Gamma_3=\{3\}$. For the general Γ verifying (1.16), the proof is very similar and just needs a slight modification.

By Proposition 5.6, there exists a solution u_λ for Eq. (1.6) with $u_\lambda\in\mathcal{S}^\delta\cap I_{b,\lambda}^{m_\Gamma}$ for all $\lambda\geq\Lambda_3$. Therefore, for any sequence $\{\lambda_n\}$ with $\lambda_n\to+\infty$ as $n\to\infty$, there exists a sequence $\{u_n\}\subset H_0^1(\Omega)$ such that

$$I_{b,\lambda_n}(u_n)\leq m_\Gamma,\quad I'_{b,\lambda_n}(u_n)=0.$$

By using Lemma 5.3, we can deduce that $u_{\lambda_n}\to u\in\mathcal{S}$ strongly in $H_0^1(\Omega)$. Thus, we complete the proof of Theorem 1.4. □

Author contributions Zhiying Cui and Wei Shuai wrote the main manuscript text together. All authors reviewed the manuscript

Declarations

Conflict of interest The authors declare no conflict of interest.

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