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Exponential stability of a geometric nonlinear beam with a nonlinear delay term in boundary feedbacks

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Abstract. This paper is concerned with the stabilization of a geometric nonlinear beam with a nonlinear delay term in boundary control. The well-posedness of the closed-loop system where a nonlinear damping and a nonlinear delay damping are applied at the boundary is examined using the Faedo–Galerkin approximation method. Constructing a novel energy-like function to handle the nonlinear delay, the explicit exponential decay rate of the closed-loop system is established with a generalized Gronwall-type integral inequality and the integral-type multiplier method.

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1. Introduction

Over the last few decades, axially moving systems were investigated because of their wide range of applications in engineering practices, such as power conveyor belts, aerial cable tram ways, belt saws, lift cables, and robotic arms. The vibration of axially moving systems with respect to the flexibility and geometric parameters is generally described by string and beam equations. Suppressing the vibration of the system is a main way in improving the work efficiency, and feedback control at the boundary is one of the most effective methods due to the ease of implementation in practices. There is rich literature on the stabilization of linear beam systems, such as Euler–Bernoulli beams [1,2], viscoelastic Timoshenko beams [3,32], and linear thermoelastic beam [4].

In this article, we consider the stabilization of a geometric nonlinear beam described by the following PDEs (partial differential equations),

$$\begin{aligned}&\rho A(z_{tt}(x,t)+2vz_{xt}(x,t)+v^2z_{xx}(x,t))+EIz_{xxxx}(x,t)\\&\quad=\left[(EA+\frac{P-EA}{\sqrt{1+z_x^2(x,t)}})z_x(x,t)\right]_x\end{aligned}\tag{1.1}$$

where $z(x,t)$ is the transversal deflection at the position x and at time t, $[\cdot]_t$ represents the derivative with respect to t, $[\cdot]_x$ denotes the derivative with respect to x, v is the constant transport velocity, and P, A, ρ, E, I denote the initial axial tension, the cross-sectional area of beam, the mass per unit area, the Young modulus, and the moment of inertia, respectively. The term $T(z_x) := EA + \frac{P-EA}{\sqrt{1+z_x^2}}$ of (1.1) is referred to the nonlinear tension derived by the nonlinear geometric relation [22,23]. For beam systems with limited but small amplitude, the nonlinear tension $T(z_x)$ of (1.1) can be reduced to $P - \frac{P-EA}{2}z_x^2$ since $\frac{1}{\sqrt{1+s^2}} \approx 1 - \frac{s^2}{2}$ as $s^2 \ll 1$. In this case, the approximate system of the nonlinear beam Eq. (1.1),

$$\rho A(z_{tt}(x,t) + 2vz_{xt}(x,t) + v^2 z_{xx}(x,t)) + EIz_{xxxx}(x,t) = \left[(P - \frac{P-EA}{2} z_x^2(x,t))z_x(x,t)\right]_x \tag{1.2}$$

has been analyzed by [5–7], and [8,9] for the string model ($EI = 0$). We are concerned whether the geometric nonlinear beam (1.1) remains exponentially stable when a nonlinear damping and a nonlinear time-delay damping implemented on the free end of the beam are considered.

Time delay is a universal phenomenon in engineering practices, for example, in electromechanical engineering, chemical, physical, etc. In fact, the existence of time delay reduces the productivity, optimization, and stability of the system (see [10,11]). Nevertheless, sometimes it may have a beneficial effect on the system's performance as well (see [12]). It is therefore essential to consider time delay when discussing the control of a system. In [13], Morgül presented a dynamic feedback controller to inhibit small delays in boundary feedback of the wave equation. Liang et al. in [14] proposed the modified Smith predictor to deal with the delayed boundary measurements of the Euler–Bernoulli beam. The exponential stability of wave equations with bounded or internally distributed time delay was derived by Nicaise and Pignotti [15]. The exponential stability of Euler–Bernoulli beams with boundary input delays was investigated by a type of predictor presented in [16]. The time delay was a known term in the above-mentioned works and that the methods of tackling stabilization relied on the parameters determining these time delays. When the time delay in the actual system is an unknown term, how to design the controller to stabilize beam systems is an interesting problem. It should be noted that Li-Xu-Han [17] studied the internal feedback stability of the Euler–Bernoulli beam

$$\begin{cases} z_{tt} + z_{xxxx} + \mathcal{U}(x,t) = 0, & x \in (0,1), \quad t > 0, \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, & t > 0, \\ z_{xxx}(1,t) = \beta z_t(1, t-\tau), & t > 0, \\ z(x,0) = z_0(x), & z_t(x,0) = z_1(x), \ x \in (0,1), \\ z_t(1,\theta) = \eta(\theta), & \theta \in (-\tau, 0), \end{cases}$$

where, here and throughout this paper, $z_x(x,t)$ and $z_t(x,t)$ are replaced by z_x and z_t for notational brevity, $\mathcal{U}(x,t)$ is the control input, and $\beta z_t(1, t-\tau)$ is the boundary time-delay disturbance. In [18], the exponential stability of the following Euler–Bernoulli beam

$$\begin{cases} z_{tt} + z_{xxxx} = 0, \ 0 < x < 1, \ t > 0, \\ z_{xxx}(1,t) = \alpha\mathcal{U}(t) + \beta\mathcal{U}(t-\tau), \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \\ z(x,0) = w_0(x), \ z_t(x,0) = w_1(x), \\ U(\theta) = f(\theta), \ \theta \in (-\tau, 0), \end{cases}$$

where $\alpha\mathcal{U}(t) + \beta\mathcal{U}(t-\tau)$ is the boundary control, was established by applying the dynamic control strategy based on the classical Smith predictor. In fact, similar results have appeared in wave equations, see, e.g., [19,20], where the wave equation is exponentially stable for $\alpha > \beta > 0$, but the wave systems with the same control law are instable if $0 < \alpha < \beta$. However, there are only a few papers where the stability analysis of geometric nonlinear beams with delay in boundary control is considered except for some special cases on Kirchhoff systems [21]. Introduce non-dimensional variables

$$t^* = t\sqrt{\frac{P}{\rho A}}, \ v^* = v\sqrt{\frac{\rho A}{P}}, \ a = \frac{EA}{P}, \zeta = \frac{EI}{P},$$

to rewrite Eq. (1.1), and then, the following non-dimensional forms of the system (1.1) for brevity is provided by

$$z_{tt} + 2vz_{xt} + \zeta z_{xxxx} - \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) z_x\right]_x = 0, \tag{1.3}$$

which is subject to the boundary conditions

$$\begin{cases} \zeta z_{xxx}(1,t) + v z_t(1,t) - \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) z_x(1,t) = \mathcal{U}(t), \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \end{cases} \tag{1.4}$$

and the initial conditions

$$\begin{cases} z(x,0) = h(x),\ z_t(x,0) = g(x), \\ z_t(1,t-\tau) = g_0(1,t-\tau),\ t \in (0,\tau), \end{cases} \tag{1.5}$$

for all $x \in (0,1)$, where $\mathcal{U}(t)$ is control input applied at $x = 1$, $\zeta > 0, \tau > 0$, g_0, h, and g are the time delay, the initial displacement, the initial velocity, and the given value of the system, respectively. From a physical point of view, the velocity of a geometric nonlinear beam does not surpass a critical value and the tensile stiffness is usually much larger than the initial tensile force ($P \leq EA$) for the beam (1.1), and then, it is easy to see that $|v| < 1 \leq a$ in dimensionless form (1.3). When the bending stiffness is not considered, i.e., $\zeta = 0$, a geometric nonlinear string obtained by (1.3) was investigated in [24], where the exponential stability is obtained under the linear feedback control ($\mathcal{U}(t) = kz_t(1,t)$ with $k > 0$).

The main concern of this paper is to establish the well-posedness and exponential stability of solutions for Eqs. (1.3)–(1.5) under the following boundary control

$$\mathcal{U}(t) = U(z_t(1,t)) + D(z_t(1,t-\tau)), \tag{1.6}$$

where the nonlinear function U satisfies the slope-restricted condition stated in Sect. 2, and the nonlinear function D is Lipschitz continuous. It is observed that nonlinear boundary control is actually a practical method, because when dealing with large deformation or saturation and using intelligent materials, the controller needs to use the nonlinear behavior of actuators and sensors. To the best of our knowledge, no relaxed results are available for the stabilization problem of the geometric nonlinear beam (1.3) with a nonlinear time-delay term in boundary control. The novelties and key difficulties of the present article can be summarized as follows:

(i) Due to the nonlinear geometric relation and nonlinear feedback, some commonly used approaches such as frequency domain methods and linear semigroups used in [29] are hardly applicable to establish well-posedness of the geometric nonlinear beam (1.3) which is more accurate than the model considered in [5–7]. Therefore, the Faedo–Galerkin approximation is used to prove the existence and uniqueness of the solution for the closed-loop geometric nonlinear beam system, in which two important estimates are completed by applying the properties of nonlinear functions and the slope-restricted condition. Furthermore, the existence of the solutions is guaranteed to be continuously dependent on the initial value.

(ii) Utilizing a multiplier-based integral inequality instead of the perturbed energy method used in literature [7,28], the exponential stability of the resulting closed-loop system is obtained, in which a novel energy-like function is constructed. One of the main characteristics of this method is that the lower regularity of the integrand function is required.

The content of this paper is arranged as follows. The well-posedness of the resulting closed-loop system is developed in Sect. 2 using the Faedo–Galerkin approximation method. In Sect. 3, the global stability analysis is carried out with the integral-type multiplier method and a generalized Gronwall-type integral inequality. The paper concludes in Sect. 4 with a summary.

2. Well-posedness of the closed-loop system

The current objective in this section is to set up the well-posedness of the resulting closed-loop system

$$\begin{cases} z_{tt} + 2vz_{xt} + \zeta z_{xxxx} - \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) z_x\right]_x = 0, \\ \zeta z_{xxx}(1,t) + vz_t(1,t) - \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) z_x(1,t) \\ = U(z_t(1,t)) + D(z_t(1,t-\tau)), \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \\ z(x,0) = h(x),\ z_t(x,0) = g(x), \\ z_t(1,t-\tau) = g_0(1,t-\tau),\ t \in (0,\tau), \end{cases} \tag{2.1}$$

by substituting (1.6) into (1.4), and noting (1.3) with (1.5) for any $t > 0$ and $x \in (0,1)$. We state the following assumptions on functions U and D that will be needed in our analysis.

($\mathcal{H}_1$) $U : R \to R$ is a continuous function and satisfies the slope-restricted condition

$$U(0) = 0,\ k_1 \le \frac{U(s_1) - U(s_2)}{s_1 - s_2} \le k_2, \quad \forall\ s_1 \ne s_2 \in R, \tag{2.2}$$

for any given constants $k_2 \ge k_1 > 0$;

($\mathcal{H}_2$) $D : R \to R$ is a continuous function satisfying

$$D(0) = 0,\ |D(s_1) - D(s_2)| \le k_3|s_1 - s_2|, \quad \forall\ s_1, s_2 \in R, \tag{2.3}$$

in which $0 < k_3 < \frac{2k_1}{e^{2\tau}+1}$ with constant k_1 given in ($\mathcal{H}_1$).

Remark 2.1. The slope-restricted condition (2.2), which is regarded as a control design criterion in the sense of absolute stability for ODE systems, is present in [25,26], so that a more flexible actuator selection is possible in real dynamic systems. In fact, it is easy to find many nonlinear functions that satisfy these two assumptions. In addition, it is worth emphasizing that we need this assumption that the minimum growth rate of the nonlinear term without delay is greater than the maximum rate of growth of the nonlinear delay, which is consistent with linear beam equations [27] or wave equations [19,20].

We introduce a new variable as in [30]

$$u(\rho,t) = z_t(1, t - \tau\rho),\ \rho \in (0,1),$$

which implies that

$$\tau u_t(\rho,t) + u_\rho(\rho,t) = 0,\ \rho \in (0,1),\ t > 0. \tag{2.4}$$

Hence, the closed-loop system (2.1) is equivalent to

$$\begin{cases} z_{tt} - \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) z_x\right]_x + 2vz_{xt} + \zeta z_{xxxx} = 0, \\ \tau u_t(\rho,t) + u_\rho(\rho,t) = 0,\ \rho \in (0,1), \\ \zeta z_{xxx}(1,t) + vz_t(1,t) - \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) z_x(1,t) \\ = U(u(0,t)) + D(u(1,t)) \\ z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \\ z(x,0) = h(x),\ z_t(x,0) = g(x), \\ u(0,t) = z_t(1,t), \\ u(\rho,0) = g_0(1,-\rho\tau),\ \rho \in (0,1), \end{cases} \tag{2.5}$$

for all $t > 0$ and $x \in (0,1)$. The definition of the energy-like function relevant to system (2.5) is given by

$$\begin{aligned}E(t) &= \frac{1}{2}\int_0^1 z_t^2\mathrm{d}x + \frac{a-v^2}{2}\int_0^1 z_x^2\mathrm{d}x + (1-a)\int_0^1 \sqrt{1+z_x^2}\mathrm{d}x + \frac{\zeta}{2}\int_0^1 z_{xx}^2\mathrm{d}x\\ &\quad+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho\\ &= \frac{1}{2}\int_0^1 z_t^2\mathrm{d}x + \frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x + \frac{\zeta}{2}\int_0^1 z_{xx}^2\mathrm{d}x\\ &\quad+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho, \end{aligned}\tag{2.6}$$

where $e^{2\tau}\tau k_3 < \gamma < \tau(2k_1 - k_3)$ with the constants k_1, k_3 given in assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$.

In what follows, a global existence result (well-posedness) of the system (2.5) is established using the Faedo–Galerkin method. To begin with, we borrow some standard notation from PDEs, e.g., $L^2(0,1), H^1(0,1)$, $H^2(0,1)$ and $H^4(0,1)$. Set

$$\begin{aligned}\Omega_1 &= \{z \in H^2(0,1) : z(0) = z_x(0) = 0\},\\ \Omega_2 &= \{z \in H^4(0,1)\cap\Omega_1 : z_{xx}(1) = 0\},\end{aligned}$$

which are closed subspaces of $H^2(0,1)$ and $H^4(0,1)$, respectively.

Theorem 2.1. *Let $h, g \in \Omega_2$ and $g_0 \in H^1(0,1)$. Suppose $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and the following compatible condition*

$$\zeta h_{xxx}(1) + vg(1) - \left(a - v^2 + \frac{1-a}{\sqrt{1+h_x^2(1)}}\right)h_x(1) = U(g(1)) + D(g_0(1)) \tag{2.7}$$

hold. Then, the system (2.5) admits a unique global solution z in the sense that for any time $T>0$,

$$z \in L^\infty([0,T),\Omega_2),\ z_t \in L^\infty([0,T),\Omega_1),\ z_{tt} \in L^\infty([0,T),L^2(0,1)).$$

Moreover, the existence of the solution is continuously dependent on the initial value condition.

Proof. Multiply the first equation of (2.5) by w and integrate over $x \in (0,1)$ by parts to obtainS

$$\begin{aligned}&\int_0^1 z_{tt}w\mathrm{d}x + \zeta\int_0^1 z_{xx}w_{xx}\mathrm{d}x + \int_0^1\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right)z_xw_x\mathrm{d}x + 2v\int_0^1 z_{xt}w\mathrm{d}x\\ &\quad= [vz_t(1,t) - U(u(0,t)) - D(u(1,t))]w(1),\end{aligned}\tag{2.8}$$

for any $w \in \Omega_1$. Assume that $\{w_i\}_{i=1}^\infty$ is an orthogonal basis on Ω_2. Since $h, g \in \Omega_2$, we may assume without loss of generality that $h, g \in Span\{w_1, w_2\}$. For each $m \in N$ and $m \geq 2$, let $\Xi_m := Span\{w_1, \cdot\cdot\cdot, w_m\}$. We find the Galerkin approximation solution z^m to Eq. (2.5)

$$\begin{cases} z^m(x,t) := \sum_{j=1}^m q_{jm}(t)w_j(x),\\ u^m(\rho,t) = z_t^m(1, t-\tau\rho), \end{cases}\tag{2.9}$$

which satisfies

$$\begin{cases} \int_0^1 z_{tt}^m w \mathrm{d}x + \zeta \int_0^1 z_{xx}^m w_{xx} \mathrm{d}x + 2v \int_0^1 z_{xt}^m w \mathrm{d}x + \int_0^1 \left(a - v^2 + \frac{1-a}{\sqrt{1+[z_x^m]^2}} \right) z_x^m w_x \mathrm{d}x \\ = [v z_t^m(1,t) - U(u^m(0,t)) - D(u^m(1,t))] w(1), \\ z^m(x,t) = \sum_{j=1}^m q_{jm}(t) w_j(x) \in \Xi_m, \\ \tau u_t^m(\rho,t) + u_\rho^m(\rho,t) = 0, \ \rho \in (0,1), \\ z^m(x,0) = h(x), \ z_t^m(x,0) = g(x), \ u^m(\rho,0) = g_0(1,-\rho\tau), \end{cases} \tag{2.10}$$

for all $w \in \Xi_m$. It can be derived directly that there exist local solutions in the interval $[0, t_m)$ for the ODE system driven by $q_{jm}(t)$ in (2.10) because of the Lipschitz continuity of functions U and D, which can be extended to the whole interval $[0, T)$ for any $T > 0$ by two important estimates as follows. □

Estimate 1. $\sup_{m \in N} E_m(t) \leq E(0)$ *for almost all* $t \geq 0$*, where*

$$E_m(t) = \frac{1}{2} \int_0^1 [z_t^m]^2 \mathrm{d}x + \frac{1}{2} \int_0^1 \int_0^{[z_x^m]^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}} \right) \mathrm{d}s \mathrm{d}x$$
$$+ \frac{\zeta}{2} \int_0^1 [z_{xx}^m]^2 \mathrm{d}x + \frac{\gamma}{2} \int_0^1 e^{-2\rho\tau} [u^m(\rho,t)]^2 \mathrm{d}\rho. \tag{2.11}$$

From (2.11), we have that

$$\frac{dE_m(t)}{\mathrm{d}t} = \int_0^1 z_{tt}^m z_t^m \mathrm{d}x + \int_0^1 \left(a - v^2 + \frac{1-a}{\sqrt{1+[z_x^m]^2}} \right) z_x^m z_{xt}^m \mathrm{d}x + \zeta \int_0^1 z_{xxt}^m z_{xx}^m \mathrm{d}x$$
$$- \frac{\gamma}{2\tau} \left(e^{-2\tau} [u^m(1,t)]^2 - [u^m(0,t)]^2 \right) - \gamma \int_0^1 e^{-2\rho\tau} [u^m(\rho,t)]^2 \mathrm{d}\rho, \tag{2.12}$$

where we apply by (2.4) the fact that

$$\frac{d}{\mathrm{d}t} \left\{ \frac{\gamma}{2} \int_0^1 e^{-2\rho\tau} [u^m(\rho,t)]^2 \mathrm{d}\rho \right\} = \gamma \int_0^1 e^{-2\rho\tau} u^m(\rho,t) u_t^m(\rho,t) \mathrm{d}\rho$$
$$= -\frac{\gamma}{2\tau} \int_0^1 e^{-2\rho\tau} \frac{\partial}{\partial\rho} [u^m(\rho,t)]^2 \mathrm{d}\rho$$
$$= -\frac{\gamma}{2\tau} (e^{-2\tau} [u^m(1,t)]^2 - [u^m(0,t)]^2)$$
$$- \gamma \int_0^1 e^{-2\rho\tau} [u^m(\rho,t)]^2 \mathrm{d}\rho. \tag{2.13}$$

Taking $w = z_t^m$ *in (2.10) and applying (2.12), we obtain*

$$\frac{dE_m(t)}{\mathrm{d}t} = [-U(u^m(0,t)) - D(u^m(1,t))] z_t^m(1,t) - \frac{\gamma}{2\tau} \left(e^{-2\tau} [u^m(1,t)]^2 - [u^m(0,t)]^2 \right)$$
$$- \gamma \int_0^1 e^{-2\rho\tau} [u^m(\rho,t)]^2 \mathrm{d}\rho - 2v \int_0^1 z_{xt}^m z_t^m \mathrm{d}x + v [z_t^m(1,t)]^2. \tag{2.14}$$

Based on the slope-restricted condition ($\mathcal{H}_1$) and ($\mathcal{H}_2$), we can deduce that

$$\begin{aligned} &U(u^m(0,t))u^m(0,t) \geq k_1[u^m(0,t)]^2, \\ &|D(u^m(1,t))| \leq k_3|u^m(1,t)|. \end{aligned} \tag{2.15}$$

Using the Young's inequality, (2.15) and $z_t^m(1,t) = u^m(0,t)$ and we have

$$\begin{aligned} &[-U(u^m(0,t)) - D(u^m(1,t))]u^m(0,t) \\ &\leq -(k_1 - \frac{k_3}{2})[u^m(0,t)]^2 + \frac{k_3}{2}[u^m(1,t)]^2. \end{aligned} \tag{2.16}$$

Since $z^m(0,t) = 0$, then $z_t^m(0,t) = 0$, which gives

$$2v \int_0^1 z_{xt}^m z_t^m \mathrm{d}x = v[z_t^m(1,t)]^2. \tag{2.17}$$

Substitute (2.16) and (2.17) into (2.14) to obtain

$$\begin{aligned} \frac{dE_m(t)}{\mathrm{d}t} &\leq -(k_1 - \frac{k_3}{2})[u^m(0,t)]^2 - \frac{\gamma}{2\tau}(e^{-2\tau}[u^m(1,t)]^2 - [u^m(0,t)]^2) \\ &\quad + \frac{k_3}{2}[u^m(1,t)]^2 - \gamma \int_0^1 e^{-2\rho\tau}[u^m(\rho,t)]^2 \mathrm{d}\rho \\ &\leq -K_1[u^m(0,t)]^2 - K_2[u^m(1,t)]^2 - \gamma \int_0^1 e^{-2\rho\tau}[u^m(\rho,t)]^2 \mathrm{d}\rho, \end{aligned} \tag{2.18}$$

where $K_1 = k_1 - \frac{k_3}{2} - \frac{\gamma}{2\tau} > 0$ and $K_2 = \frac{\gamma}{2\tau}e^{-2\tau} - \frac{k_3}{2} > 0$ owing to $e^{2\tau}\tau k_3 < \gamma < \tau(2k_1 - k_3)$ and $0 < k_3 < \frac{2k_1}{e^{2\tau}+1}$. As a result, we have

$$\begin{aligned} E_m(t) &\leq E_m(0) \\ &= \frac{1}{2}\int_0^1 g^2 \mathrm{d}x + \int_0^1 \int_0^{h_x^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) \mathrm{d}s\mathrm{d}x \\ &\quad + \frac{\zeta}{2}\int_0^1 h_{xx}^2 \mathrm{d}x + \frac{\gamma}{2}\int_0^1 e^{-2\rho\tau} g_0^2 \mathrm{d}\rho, \end{aligned} \tag{2.19}$$

so estimate 1 follows.

Estimate 2. *For any $T > 0$, there exists a constant $\mathcal{C}_T$ such that*

$$\sup_{m\in N}\{\|z_{tt}^m\|^2 + \zeta\|z_{xxt}^m\|^2\} \leq \mathcal{C}_T, \tag{2.20}$$

for $t > 0$ a.e.. First of all, let us estimate $\|z_{tt}^m(\cdot,0)\|^2 < \infty$. Considering the variational structure of (2.10) and the compatibility condition, by setting $t = 0$ in (2.10), it follows that

$$\begin{aligned} &\int_0^1 z_{tt}^m(x,0) w \mathrm{d}x - \int_0^1 \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+[z_x^m(x,0)]^2}}\right) z_x^m(x,0)\right]_x w\mathrm{d}x \\ &\quad + \zeta \int_0^1 w z_{xxxx}^m(x,0)\mathrm{d}x + 2v\int_0^1 w z_{xt}^m(x,0)\mathrm{d}x = 0, \end{aligned} \tag{2.21}$$

for any $w \in \Omega_1$. *Taking* $w = z_{tt}^m(0)$ *in (2.21) and using the initial value condition, one gets*

$$\|z_{tt}^m(\cdot,0)\|^2 = \int_0^1 \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+h_x^2(x)}}\right) h_x(x)\right]_x z_{tt}^m(x,0)\mathrm{d}x$$
$$-\zeta \int_0^1 z_{tt}^m(x,0)h_{xxxx}(x)\mathrm{d}x - 2v\int_0^1 z_{tt}^m(x,0)g_x\mathrm{d}x. \tag{2.22}$$

Application of the Cauchy–Schwarz inequality on the first term of (2.22) yields

$$\int_0^1 \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+h_x^2(x)}}\right) h_x(x)\right]_x z_{tt}^m(x,0)\mathrm{d}x$$
$$= \int_0^1 \left[a - v^2 - \frac{a-1}{\sqrt{1+h_x^2(x)}} + \frac{(a-1)h_x^2(x)}{(\sqrt{1+h_x^2(x)})^3}\right] h_{xx}(x) z_{tt}^m(x,0)\mathrm{d}x$$
$$\le \int_0^1 (2a - v^2 - 1)|h_{xx}(x)z_{tt}^m(x,0)|\mathrm{d}x$$
$$\le (2a - v^2 - 1)\|h_{xx}\|\|z_{tt}^m(\cdot,0)\|.$$

Likewise, one sees immediately that

$$\zeta\int_0^1 z_{tt}^m(x,0)h_{xxxx}(x)\mathrm{d}x \le \zeta\|z_{tt}^m(\cdot,0)\|\|h_{xxxx}\|,$$
$$\int_0^1 z_{tt}^m(x,0)g_x\mathrm{d}x \le \|z_{tt}^m(\cdot,0)\|\|g_x\|.$$

Substitute the above estimates into (2.22) to get

$$\|z_{tt}^m(\cdot,0)\| \le \zeta\|h_{xxxx}\| + (2a - v^2 - 1)\|h_{xx}\| + 2v\|g_x\|. \tag{2.23}$$

Now, the variational structure (2.10) shows

$$\int_0^1 z_{tt}^m w\mathrm{d}x + \int_0^1 \left(a - v^2 + \frac{1-a}{\sqrt{1+[z_x^m]^2}}\right) z_x^m w_x\mathrm{d}x$$
$$+\zeta\int_0^1 z_{xx}^m w_{xx}\mathrm{d}x + [U(u^m(0,t) + D(u^m(1,t))]w(1)$$
$$+2v\int_0^1 z_{xt}^m w\mathrm{d}x - vz_t(1,t)w(1) = 0, \tag{2.24}$$

for any $w \in \Omega_1$. *Fix* $t, \delta > 0$ *such that* $\delta < T - t$. *Replacing* t *by* $t + \delta$ *and subtracting (2.24), one obtains*

$$\int_0^1 (z_{tt}^m(x,t+\delta) - z_{tt}^m(x,t))w(x)\mathrm{d}x + \zeta\int_0^1 (z_{xx}^m(x,t+\delta) - z_{xx}^m(x,t))w_{xx}(x)\mathrm{d}x$$
$$+[U(u^m(0,t+\delta)) + D(u^m(1,t+\delta))]w(1) - [U(u^m(0,t)) + D(u^m(1,t)]w(1)$$

$$+2v\int_0^1 (z_{xt}^m(x,t+\delta)-z_{xt}^m(x,t))w(x)\mathrm{d}x - v[z_t^m(1,t+\delta)-z_t^m(1,t)]w(1)$$
$$+\int_0^1 \left(a-v^2+\frac{1-a}{\sqrt{1+[z_x^m(x,t+\delta)]^2}}\right) z_x^m(x,t+\delta)w_x(x)\mathrm{d}x$$
$$-\int_0^1 \left(a-v^2+\frac{1-a}{\sqrt{1+[z_x^m(x,t)]^2}}\right) z_x^m(x,t)w_x(x)\mathrm{d}x = 0. \tag{2.25}$$

Taking $w = z_t^m(x,t+\delta) - z_t^m(x,t)$ in (2.25), one has

$$\frac{1}{2}\frac{d\Phi(\delta,t)}{\mathrm{d}t} + P_1 + P_2 + P_3 = 0, \tag{2.26}$$

where

$$\Phi(\delta,t) := \zeta\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(\cdot,t)\|^2 + \|z_t^m(\cdot,t+\delta)-z_t^m(\cdot,t)\|^2,$$
$$P_1 = \int_0^1 \left(a-v^2+\frac{1-a}{\sqrt{1+[z_x^m(x,t+\delta)]^2}}\right) z_x^m(x,t+\delta)[z_{xt}^m(x,t+\delta)-z_{xt}^m(x,t)]\mathrm{d}x$$
$$-\int_0^1 \left(a-v^2+\frac{1-a}{\sqrt{1+[z_x^m(x,t)]^2}}\right) z_x^m(x,t)[z_{xt}^m(x,t+\delta)-z_{xt}^m(x,t)]\mathrm{d}x$$
$$P_2 = [U(u^m(0,t+\delta)) + D(u^m(1,t+\delta))][u^m(0,t+\delta)-u^m(0,t)]$$
$$-[U(u^m(0,t)) + D(u^m(1,t)][u^m(0,t+\delta)-u^m(0,t)]$$
$$P_3 = 2v\int_0^1 (z_{xt}^m(x,t+\delta)-z_{xt}^m(x,t))(z_t^m(x,t+\delta)-z_t^m(x,t))\mathrm{d}x$$
$$-v[z_t^m(1,t+\delta)-z_t^m(1,t)]^2. \tag{2.27}$$

For simplicity, a continuous differentiable function $\varphi : R_+ \to R_+$ is defined by

$$\phi(s) := s[a-v^2-\frac{a-1}{\sqrt{1+s^2}}], \quad \forall s\in R, \tag{2.28}$$

where $a \geq 1 > |v| > 0$. By taking the derivative of $\phi(s)$, it is easy to see that

$$\frac{d\phi}{ds} = a-v^2-\frac{a-1}{\sqrt{1+s^2}} + \frac{(a-1)s^2}{(\sqrt{1+s^2})^3}, \tag{2.29}$$

which gives

$$1-v^2 < \frac{d\phi}{ds} < 2a-v^2-1, \quad \forall s\in R. \tag{2.30}$$

Now, let us estimate P_1. Integration by parts reveals

$$P_1 = [\phi(z_x^m(1,t+\delta)) - \phi(z_x^m(1,t))][u^m(0,t+\delta)-u^m(0,t)]$$
$$-\int_0^1 \frac{\partial[\phi(z_x^m(x,t+\delta))-\phi(z_x^m(x,t))]}{\partial x}[z_t^m(x,t+\delta)-z_t^m(x,t)]\mathrm{d}x$$
$$= [\phi(z_x^m(1,t+\delta)) - \phi(z_x^m(1,t))][u^m(0,t+\delta)-u^m(0,t)]$$

$$
\begin{aligned}
&-\int_0^1 \frac{d\phi}{ds}|_{s=z_x^m(x,t+\delta)} z_{xx}^m(x,t+\delta)[z_t^m(x,t+\delta)-z_t^m(x,t)]\mathrm{d}x\\
&+\int_0^1 \frac{d\phi}{ds}|_{s=z_x^m(x,t)} z_{xx}^m(x,t)[z_t^m(x,t+\delta)-z_t^m(x,t)]\mathrm{d}x\\
&= Q_1 - Q_2 - Q_3, \qquad (2.31)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 &= [\phi(z_x^m(1,t+\delta)) - \phi(z_x^m(1,t))][u^m(0,t+\delta)-u^m(0,t)],\\
Q_2 &= \int_0^1 \frac{d\phi}{ds}|_{s=z_x^m(x,t+\delta)}[z_{xx}^m(x,t+\delta)-z_{xx}^m(x,t)][z_t^m(x,t+\delta)-z_t^m(x,t)]\mathrm{d}x,\\
Q_3 &= \int_0^1 \left[\frac{d\phi}{ds}|_{s=z_x^m(x,t+\delta)} - \frac{d\phi}{ds}|_{s=z_x^m(x,t)}\right] z_{xx}^m(x,t)[z_t^m(x,t+\delta)-z_t^m(x,t)]\mathrm{d}x.
\end{aligned}
$$

Next, using the mean value theorem, Young's inequality, (2.30) and $|z_x(1,t)|^2 \le \|z_{xx}(\cdot,t)\|^2$ *on* Q_1*, it follows that*

$$
\begin{aligned}
Q_1 &\le (2a-v^2-1)|z_x^m(1,t+\delta)-z_x^m(1,t)||u^m(0,t+\delta)-u^m(0,t)|\\
&\le \frac{2a-v^2-1}{4\eta}|z_x^m(1,t+\delta)-z_x^m(1,t)|^2\\
&\quad +(2a-v^2-1)\eta|u^m(0,t+\delta)-u^m(0,t)|^2\\
&\le \frac{2a-v^2-1}{4\eta}\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(\cdot,t)\|^2\\
&\quad +(2a-v^2-1)\eta|u^m(0,t+\delta)-u^m(0,t)|^2. \qquad (2.32)
\end{aligned}
$$

Furthermore, in the light of (2.30), and applying Young's inequality on Q_2*, we can find*

$$
\begin{aligned}
Q_2 &\le (2a-v^2-1)\int_0^1 |z_{xx}^m(x,t+\delta)-z_{xx}^m(x,t)||z_t^m(x,t+\delta)-z_t^m(x,t)|\mathrm{d}x\\
&\le \frac{2a-v^2-1}{2}\{\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(\cdot,t)\|^2 + \|z_t^m(\cdot,t+\delta)-z_t^m(\cdot,t)\|^2\}. \qquad (2.33)
\end{aligned}
$$

According to (2.29), it is easy to get that

$$
\frac{d^2\phi}{ds^2} = 3(a-1)s(1+s^2)^{-3/2}\left[1-\frac{s^2}{1+s^2}\right], \qquad (2.34)
$$

which implies

$$
|\frac{d^2\phi}{ds^2}| \le 3(a-1), \quad \forall s \in R. \qquad (2.35)
$$

From (2.35), the Sobolev inequality $\sup_{x\in[0,1]}|z_x| \le \|z_{xx}\|$ *and the estimate 1 (*$\zeta\|z_{xx}^m\|^2 \le 2E(0)$*), we have that*

$$
\begin{aligned}
Q_3 &\le 3(a-1)\int_0^1 |z_{xx}^m||z_x^m(x,t+\delta)-z_x^m(x,t)||z_t^m(x,t+\delta)-z_t^m(x,t)|\mathrm{d}x \\
&\le \frac{3(a-1)}{2}\int_0^1 |z_{xx}^m|^2|z_x^m(x,t+\delta)-z_x^m(x,t)|^2\mathrm{d}x \\
&\quad+\frac{3(a-1)}{2}\int_0^1 |z_t^m(x,t+\delta)-z_t^m(x,t)|^2\mathrm{d}x \\
&\le \frac{3(a-1)}{2}\sup_{x\in[0,1]}|z_x^m(x,t+\delta)-z_x^m(x,t)|^2\int_0^1 |z_{xx}^m|^2\mathrm{d}x \\
&\quad+\frac{3(a-1)}{2}\int_0^1 |z_t^m(x,t+\delta)-z_t^m(x,t)|^2\mathrm{d}x \\
&\le \frac{3(a-1)E(0)}{\zeta}\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(\cdot,t)\|^2 \\
&\quad+\frac{3(a-1)}{2}\|z_t^m(\cdot,t+\delta)-z_t^m(\cdot,t)\|^2. \qquad (2.36)
\end{aligned}
$$

Then, substitute (2.32), (2.33) and (2.36) into (2.31) and we have

$$
\begin{aligned}
|P_1| \le\ & C_1\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(x,t)\|^2+C_2\|z_t^m(\cdot,t+\delta)-z_t^m(x,t)\|^2 \\
&+(2a-v^2-1)\eta|u^m(0,t+\delta)-u^m(0,t)|^2, \qquad (2.37)
\end{aligned}
$$

where $C_1, C_2>0$ are two positive constants. The slope-restricted condition ($\mathcal{H}_1$) leads to

$$
\begin{aligned}
&[U(u^m(0,t+\delta))-U(u^m(0,t))][u^m(0,t+\delta)-u^m(0,t)] \\
&\quad\ge k_1|u^m(0,t+\delta)-u^m(0,t)|^2 \qquad (2.38)
\end{aligned}
$$

for $t\in[0,T]$ a.e.. Hence, for P_2 in (2.27), it follows from (2.38) that

$$
\begin{aligned}
P_2 \ge\ & k_1|u^m(0,t+\delta)-u^m(0,t)|^2+[D(u^m(1,t+\delta))-D(u^m(1,t))] \\
&\times[u^m(0,t+\delta)-u^m(0,t)], \qquad (2.39)
\end{aligned}
$$

which with the Young's inequality together gives

$$
\begin{aligned}
-P_2 \le\ & -k_1|u^m(0,t+\delta)-u^m(0,t)|^2+\frac{k_3}{2}[u^m(1,t+\delta)-u^m(1,t)]^2 \\
&+\frac{k_3}{2}[u^m(0,t+\delta)-u^m(0,t)]^2 \qquad (2.40)
\end{aligned}
$$

for $t\in[0,T]$ a.e.. Similar to (2.17), **it is easy** *to show $P_3=0$. Putting (2.37), (2.40) and $P_3=0$ into (2.26) implies*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\left\{\Phi(\delta,t)+\xi\int_0^1 e^{-2\rho\tau}[u^m(\rho,t+\delta)-u^m(\rho,t)]^2\mathrm{d}\rho\right\} \\
&\le C_1\|z_{xx}^m(\cdot,t+\delta)-z_{xx}^m(\cdot,t)\|^2+C_2\|z_t^m(\cdot,t+\delta)-z_t^m(\cdot,t)\|^2 \\
&\quad+(2b-v^2-1)\eta|u^m(0,t+\delta)-u^m(0,t)|^2-k_1|u^m(0,t+\delta)-u^m(0,t)|^2 \\
&\quad+\frac{k_3}{2}[u^m(1,t+\delta)-u^m(1,t)]^2+\frac{k_3}{2}[u^m(0,t+\delta)-u^m(0,t)]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma}{2\tau}[u^m(0,t+\delta)-u^m(0,t)]^2-\frac{\gamma}{2\tau}e^{-2\tau}[u^m(1,t+\delta)-u^m(1,t)]^2\\
&-\gamma\int_0^1 e^{-2\rho\tau}[u^m(\rho,t+\delta)-u^m(\rho,t)]^2\mathrm{d}\rho\\
\le{}& C_1\|z^m_{xx}(\cdot,t+\delta)-z^m_{xx}(\cdot,t)\|^2+C_2\|z^m_t(\cdot,t+\delta)-z^m_t(\cdot,t)\|^2\\
&+(2b-v^2-1)\eta|u^m(0,t+\delta)-u^m(0,t)|^2-K_1|u^m(0,t+\delta)-u^m(0,t)|^2\\
&-K_2|u^m(1,t+\delta)-u^m(1,t)|^2,
\end{aligned}\tag{2.41}
$$

where $K_1=k_1-\frac{k_3}{2}-\frac{\gamma}{2\tau}>0$ *and* $K_2=\frac{\gamma}{2\tau}e^{-2\tau}-\frac{k_3}{2}>0$ *by (2.18). Since the Young's parameter* $\eta>0$ *is arbitrary, take* $\eta=\frac{K_1}{2b-v^2-1}>0$*; therefore, it follows for (2.41) that*

$$
\frac{d\Lambda(\delta,t)}{\mathrm{d}t}\le 2C_3\Lambda(\delta,t),\tag{2.42}
$$

where

$$
\begin{aligned}
\Lambda(\delta,t):={}&\zeta\|z^m_{xx}(\cdot,t+\delta)-z^m_{xx}(\cdot,t)\|^2+\|z^m_t(\cdot,t+\delta)-z^m_t(\cdot,t)\|^2\\
&+\gamma\int_0^1 e^{-2\rho\tau}[u^m(\rho,t+\delta)-u^m(\rho,t)]^2\mathrm{d}\rho,
\end{aligned}
$$

$C_3>0$ *is a constant.* **This allows** *us to get*

$$
\Phi(\delta,t)\le\Lambda(\delta,t)\le\Lambda(\delta,0)e^{2C_3T},\tag{2.43}
$$

where the function Φ *is defined in (2.27). Divide the above inequality by* δ^2 *and pass to the limit as* $\delta\to0$ *to obtain*

$$
\|z^m_{tt}(t)\|^2+\zeta\|z^m_{xxt}\|^2\le[\|z^m_{tt}(0)\|^2+\zeta\|g_{xx}\|^2+\frac{\gamma}{\tau^2}\|g_{0\rho}\|^2]e^{2C_3T},\tag{2.44}
$$

which with (2.23) together gives estimate 2.

The estimates 1,2 guarantee

$$
\begin{aligned}
&\{z^m\}_{m\ge1}\text{ is bounded in }L^\infty([0,T);\Omega_1),\\
&\{z^m_t\}_{m\ge1}\text{ is bounded in }L^\infty([0,T);\Omega_1),\\
&\{z^m_{tt}\}_{m\ge1}\text{ is bounded in }L^\infty([0,T);L^2(0,1)).
\end{aligned}
$$

In light of the Lions lemma, we conclude a subsequence from $\{z^m\}_{m\ge1}\in L^\infty([0,T);\Omega_1)$*, still denoted by* $\{z^m\}_{m\ge1}$*, and* $z\in L^\infty([0,T);\Omega_1)$ *satisfying*

$$
\begin{aligned}
&z^m\to z\text{ in }L^\infty([0,T);\Omega_1)\text{ weak}^*,\\
&z^m_t\to z_t\text{ in }L^\infty([0,T);\Omega_1)\text{ weak}^*,\\
&z^m_{tt}\to z_{tt}\text{ in }L^\infty([0,T);L^2(0,1))\text{ weak}^*.
\end{aligned}
$$

From estimate 1, we have that $\{z^m\}_{m=1}^\infty$ *is bounded in* $H^2(0,1)$*, which together with the compact embedding* $(H^2(0,1)\subset H^1(0,1))$ *implies that* $\{z^m_x\}_{m=1}^\infty$ *is compact in* $L^2(0,1)$*. Thus, we can find a subsequence of* $\{z^m_x\}_{m=1}^\infty$ *(denoted by itself) such that* $z^m_x\to z_x$*. Due to (2.28), one has* $\phi(z^m_x)\to\phi(z_x)$ *in* $L^2(0,1)$ *for a.e.* $t\in[0,T)$*. An application of estimate 2 means that* $\{z^m_{xxt}(\cdot,t)\}_{m=1}^\infty$ *is bounded in* $L^2(0,1)$*, which yields that* $\{z^m_{xt}(\cdot,t)\}_{m=1}^\infty$ *is compact in* $L^2(0,1)$*. This, together with* $z^m_t(1,t)=\int_0^1 z^m_{xt}\mathrm{d}x$*, implies that there is a subsequence of* $z^m_t(1,t)$ *(denoted by itself), such that* $z^m_t(1,t)\to z_t(1,t)$ *for a.e.* $t\in[0,T)$*. Applying the*

Lebesgue dominated convergence theorem with the continuity of U and D gives $U(z_t^m(1,t)) \to U(z_t(1,t))$ and $D(z_t^m(1,t-\tau)) \to D(z_t(1,t-\tau))$. By the Aubin–Lions theorem, we obtain that

$$\int_0^1 z_{tt} w \mathrm{d}x + \zeta \int_0^1 z_{xx} w_{xx} \mathrm{d}x + 2v \int_0^1 z_{xt} w \mathrm{d}x + \int_0^1 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}} \right) z_x w_x \mathrm{d}x$$
$$= [v z_t(1,t) - U(u(0,t)) - D(u(1,t))] w(1), \ \forall w \in \Omega_1, \text{ and } t \in [0,T) \text{ a.e.} \tag{2.45}$$

by passing to the limit as $m \to \infty$ to (2.10). Set $w \in \mathcal{W}_0 := \{w \in \Omega_1 : w(1) = 0\}$. Based on (2.45), we arrive at

$$\int_0^1 z_{tt} w \mathrm{d}x + \zeta \int_0^1 z_{xx} w_{xx} \mathrm{d}x + 2v \int_0^1 z_{xt} w \mathrm{d}x = - \int_0^1 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}} \right) z_x w_x \mathrm{d}x \tag{2.46}$$

for almost every $t \in [0,T)$. This implies that the existence of generalized derivatives z_{xxxx} is obtained, namely $z \in \Omega_2$, and

$$z_{tt} + \zeta z_{xxxx} + 2v z_{xt} = \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}} \right) z_x \right]_x \in L^2(0,1). \tag{2.47}$$

Integrating (2.45) by parts leads to

$$\int_0^1 z_{tt} w \mathrm{d}x + \zeta \int_0^1 z_{xxxx} w \mathrm{d}x + 2v \int_0^1 z_{xt} w \mathrm{d}x - \int_0^1 \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}} \right) z_x \right]_x w \mathrm{d}x$$
$$+ \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}} \right) z_x(1,t) w(1) - \zeta z_{xxx}(1,t) w(1)$$
$$= [v z_t(1,t) - U(u(0,t)) - D(u(1,t))] w(1). \tag{2.48}$$

Invoking (2.47) then shows

$$\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}} \right) z_x(1,t) - \zeta z_{xxx}(1,t) = v z_t(1,t) - U(u(0,t)) - D(u(1,t)). \tag{2.49}$$

Thus, the existence of the global solution for the closed-loop system (2.5) follows in $[0,T)$, for all $T > 0$.

In what follows, we demonstrate the uniqueness of the solution. Let $z, \tilde{z}$ be two solutions of the closed-loop system (2.5). Substitute $\tilde{z}$ for z in (2.8) and subtract (2.8) to obtain

$$\int_0^1 (\tilde{z}_{tt}(x,t) - z_{tt}(x,t)) w(x) \mathrm{d}x + \zeta \int_0^1 (\tilde{z}_{xx}(x,t) - z_{xx}(x,t)) w_{xx}(x) \mathrm{d}x$$
$$+ \int_0^1 \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+\tilde{z}_x^2}} \right) \tilde{z}_x - \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}} \right) z_x \right] w_x \mathrm{d}x$$
$$+ [U(\tilde{u}(0,t)) + D(\tilde{u}(1,t))] w(1) - [U(u(0,t)) + D(u(1,t)] w(1)$$
$$+ 2v \int_0^1 (\tilde{z}_{xt}(x,t) - z_{xt}(x,t)) w(x) \mathrm{d}x - v[\tilde{z}_t(1,t) - z_t(1,t)] w(1) = 0. \tag{2.50}$$

Taking $w = \tilde{z} - z$ and arguing as in (2.26), we can obtain that

$$\frac{d\Gamma(t)}{\mathrm{d}t} \le 2C_3 \Gamma(t), \tag{2.51}$$

where C_3 *is given in (2.42) and*

$$\Gamma(t)=\zeta\|\tilde{z}_{xx}(\cdot,t)-z_{xx}(\cdot,t)\|^2+\|\tilde{z}_t(\cdot,t)-z_t(\cdot,t)\|^2+\gamma\int_0^1 e^{-2\rho\tau}[\tilde{u}(\rho,t)-u(\rho,t)]^2\mathrm{d}\rho,$$

which with $\Gamma(0)=0$ *gives the uniqueness of the solution.*

Lastly, let us show the continuous dependence of the solution on the initial functions. Let z^n *be the solution of the closed-loop system (2.5) with initial value* $(h^n,g^n,g_0^n)\in\Omega_2\times\Omega_2\times H^1(0,1)$ *satisfying* $h^n\to h$, $g^n\to g$ *in* Ω_2 *and* $g_0^n\to g_0$ *in* $H^1(0,1)$. *Then in a similar fashion, it follows from (2.51) that*

$$\chi(t)\le\chi(0)e^{2C_3t}, \tag{2.52}$$

where

$$\chi(t)=\zeta\|z^n_{xx}(\cdot,t)-z_{xx}(\cdot,t)\|^2+\|z^n_t(\cdot,t)-z_t(\cdot,t)\|^2+\gamma\int_0^1 e^{-2\rho\tau}[u^n(\rho,t)-u(\rho,t)]^2\mathrm{d}\rho,$$

and

$$\chi(0)=\zeta\|h^n_{xx}-h_{xx}\|^2+\|g^n-g\|^2+\gamma\int_0^1 e^{-2\rho\tau}[g_0^n-g_0]^2\mathrm{d}\rho,$$

which means that $z^n\to z$ *in* $H^2(0,1)$ *for any* $t>0$ *as* $n\to\infty$. *The proof of Theorem 2.1 is completed.*

3. Stability analysis of the closed-loop system

In this section, we complete the stability analysis of the displacement response and energy-like function of the nonlinear beam system (2.5). For this, first a lemma which indicates that the energy-like function $E(t)$ is non-increasing, follows easily from estimate 1 in Theorem 2.1.

Lemma 3.1. *Let* z *be the solution provided by Eq. (2.5). Then, the energy-like function* $E(t)$ *defined in (2.6) satisfies*

$$\begin{aligned}\frac{dE(t)}{\mathrm{d}t}&\le -K_1u^2(0,t)-K_2u^2(1,t)-\gamma\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho\\&\le-\min\{K_1,K_2,1\}\left[u^2(0,t)+u^2(1,t)+\gamma\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho\right],\end{aligned} \tag{3.1}$$

where $K_1=k_1-\frac{k_3}{2}-\frac{\gamma}{2\tau}>0$ *and* $K_2=\frac{\gamma}{2\tau}e^{-2\tau}-\frac{k_3}{2}>0$ *given by* (2.18).

A generalized Gronwall-type integral inequality [31, p. 103] shown as Lemma 3.2 is needed to build up the exponential stability of the closed-loop system (2.5).

Lemma 3.2. *Suppose* $\mathcal{N}:[0,+\infty)\to[0,+\infty)$ *is a non-increasing real-valued function with a constant* $\alpha>0$ *such that*

$$\int_T^\infty \mathcal{N}(s)ds\le\frac{1}{\alpha}\mathcal{N}(T)\ \textit{for all}\ T\ge0. \tag{3.2}$$

Then, the following estimate is valid

$$\mathcal{N}(t) \le \mathcal{N}(0)e^{1-\alpha t}. \tag{3.3}$$

Now, let us state the absolute stability of the closed-loop system (2.5).

Theorem 3.1. *Under the assumptions of Theorem 2.1, the energy-like function $E(t)$ defined by (2.6) decays uniformly exponentially, i.e.,*

$$E(t) \le e^{1-\alpha t}E(0) \tag{3.4}$$

for all $t > 0$, where

$$\alpha^{-1} = 2\hat{C} + \frac{C_*}{\min\{K_1, K_2, 1\}} \tag{3.5}$$

with $\hat{C} = \max\left\{1, \frac{1+2v}{1-v^2}\right\}$ and $C_* = \max\left\{\frac{2(v+k_2)^2+(1-v^2)}{2(1-v^2)}, \frac{k_3^2}{1-v^2}\right\}$.

Proof. Take the inner product with xz_x on both sides of the first equation in the closed-loop system (2.5) to yield

$$H_1 + H_2 + H_3 = H_4 \tag{3.6}$$

where

$$H_1 = \langle xz_x, z_{tt}\rangle,\ H_2 = \langle xz_x, 2vz_{xt}\rangle,\ H_3 = \zeta\langle xz_x, z_{xxxx}\rangle,$$
$$H_4 = \left\langle xz_x, \left[\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right)z_x\right]_x\right\rangle.$$

By the law of derivation and the boundary value condition $z_t(0,t) = 0$, for H_1 we have

$$\begin{aligned} H_1 &= \int_0^1 [xz_xz_t]_t \mathrm{d}x - \int_0^1 xz_{xt}z_t \mathrm{d}x \\ &= \int_0^1 [xz_xz_t]_t \mathrm{d}x - \frac{1}{2}\int_0^1 [xz_t^2]_x \mathrm{d}x + \frac{1}{2}\int_0^1 z_t^2 \mathrm{d}x \\ &= \int_0^1 [xz_xz_t]_t \mathrm{d}x + \frac{1}{2}\int_0^1 z_t^2 \mathrm{d}x - \frac{1}{2}z_t^2(1,t). \end{aligned} \tag{3.7}$$

Likewise, one gets

$$H_2 = 2v\int_0^1 xz_xz_{xt}\mathrm{d}x = \int_0^1 [vxz_x^2]_t \mathrm{d}x. \tag{3.8}$$

After integration by parts, we have that

$$\begin{aligned} H_3 &= \zeta z_x(1,t)z_{xxx}(1,t) - \zeta\int_0^1 z_{xxx}(z_x + xz_{xx})\mathrm{d}x \\ &= \zeta z_x(1,t)z_{xxx}(1,t) + \zeta\int_0^1 z_{xx}(2z_{xx} + xz_{xxx})\mathrm{d}x \end{aligned}$$

$$= \zeta z_x(1,t) z_{xxx}(1,t) + 2\zeta \int_0^1 z_{xx}^2 \mathrm{d}x + \frac{\zeta}{2} \int_0^1 \left(x z_{xx}^2\right)_x \mathrm{d}x - \frac{\zeta}{2} \int_0^1 z_{xx}^2 \mathrm{d}x$$

$$= \zeta z_x(1,t) z_{xxx}(1,t) + \frac{3\zeta}{2} \int_0^1 z_{xx}^2 \mathrm{d}x, \tag{3.9}$$

and

$$H_4 = z_x^2(1,t)\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) - \int_0^1 z_x^2 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) \mathrm{d}x$$

$$- \int_0^1 x z_x z_{xx} \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) \mathrm{d}x$$

$$= z_x^2(1,t)\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) - \int_0^1 z_x^2 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) \mathrm{d}x$$

$$- \frac{1}{2} \int_0^1 \left[x \int_0^{z_x^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) ds \right]_x \mathrm{d}x$$

$$+ \frac{1}{2} \int_0^1 \int_0^{z_x^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) \mathrm{d}s \mathrm{d}x$$

$$= z_x^2(1,t)\left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right) - \int_0^1 z_x^2 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) \mathrm{d}x$$

$$- \frac{1}{2} \int_0^{z_x^2(1,t)} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) ds$$

$$+ \frac{1}{2} \int_0^1 \int_0^{z_x^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) \mathrm{d}s \mathrm{d}x. \tag{3.10}$$

Owing to the monotonicity of the function $a - v^2 + \frac{1-a}{\sqrt{1+s}}$ for any $s \geq 0$, then

$$z_x^2 \left(a - v^2 + \frac{1-a}{\sqrt{1+z_x^2}}\right) \geq \int_0^{z_x^2} \left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right) ds.$$

This together with (3.10) gives

$$H_4 \leq z_x^2(1,t)\left(a-v^2+\frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right)-\frac{1}{2}\int_0^{z_x^2(1,t)}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)ds$$

$$-\frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x. \tag{3.11}$$

Inserting (3.7)–(3.9) and (3.11) into (3.6) easily leads to

$$\frac{1}{2}\int_0^1 z_t^2\mathrm{d}x+\frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x+\frac{3\zeta}{2}\int_0^1 z_{xx}^2\mathrm{d}x$$

$$\leq -\int_0^1[xz_xz_t+vxz_x^2]_t\mathrm{d}x+\frac{1}{2}z_t^2(1,t)-\zeta z_x(1,t)z_{xxx}(1,t)$$

$$+z_x^2(1,t)\left(a-v^2+\frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right)$$

$$-\frac{1}{2}\int_0^{z_x^2(1,t)}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)ds. \tag{3.12}$$

Following the boundary value condition (2.5), the slope-restricted condition $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and Young's inequality, we immediately obtain that

$$\left|-\zeta z_x(1,t)z_{xxx}(1,t)+z_x^2(1,t)\left(a-v^2+\frac{1-a}{\sqrt{1+z_x^2(1,t)}}\right)\right|$$

$$\leq |z_x(1,t)[vz_t(1,t)-U(u(0,t))-D(u(1,t))]|$$

$$\leq |z_x(1,t)|[v|u(0,t)|+k_2|u(0,t)|+k_3|u(1,t)|]$$

$$\leq (v+k_2)|z_x(1,t)u(0,t)|+k_3|z_x(1,t)u(1,t)|$$

$$\leq (\eta_1+\eta_2)z_x^2(1,t)+\frac{(v+k_2)^2}{4\eta_1}u^2(0,t)+\frac{k_3^2}{4\eta_2}u^2(1,t), \tag{3.13}$$

where $\eta_1,\eta_2>0$ are Young's coefficients. According to the definition of the energy function $E(t)$ defined by (2.6), it can be deduced from (3.12) and (3.13) that

$$E(t)=\frac{1}{2}\int_0^1 z_t^2\mathrm{d}x+\frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x+\frac{\zeta}{2}\int_0^1 z_{xx}^2\mathrm{d}x$$

$$+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho$$

$$\leq -\int_0^1[xz_xz_t+vxz_x^2]_t\mathrm{d}x+(\eta_1+\eta_2)z_x^2(1,t)+\frac{(v+k_2)^2+2\eta_1}{4\eta_1}u^2(0,t)$$

$$+\frac{k_3^2}{4\eta_2}u^2(1,t)-\frac{1}{2}\int_0^{z_x^2(1,t)}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)ds$$

$$+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho. \tag{3.14}$$

Since $\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)ds \geq (1-v^2)z_x^2$ due to $a \geq 1 > |v| > 0$, inserting this into (3.14) produces

$$\begin{aligned}E(t) \leq & -\int_0^1 [xz_xz_t+vxz_x^2]_t\mathrm{d}x+(\eta_1+\eta_2)z_x^2(1,t)+\frac{(v+k_2)^2+2\eta_1}{4\eta_1}u^2(0,t)\\ &+\frac{k_3^2}{4\eta_2}u^2(1,t)-\frac{1-v^2}{2}z_x^2(1,t)+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho. \end{aligned}\tag{3.15}$$

Due to the arbitrariness of parameters, letting $\eta_1=\eta_2=\frac{1-v^2}{4}>0$, with (3.15) and Lemma 3.1 implies that

$$\begin{aligned}E(t) &\leq -\int_0^1 [xz_xz_t+vxz_x^2]_t\mathrm{d}x+\frac{2(v+k_2)^2+(1-v^2)}{2(1-v^2)}u^2(0,t)+\frac{k_3^2}{1-v^2}u^2(1,t)\\ &\quad+\frac{\gamma}{2}\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho\\ &\leq C_*\left[u^2(0,t)+u^2(1,t)+\gamma\int_0^1 e^{-2\rho\tau}u^2(\rho,t)\mathrm{d}\rho\right]-\int_0^1[xz_xz_t+vxz_x^2]_t\mathrm{d}x\\ &\leq -\int_0^1[xz_xz_t+vxz_x^2]_t\mathrm{d}x-\frac{C_*\dot{E}(t)}{\min\{K_1,K_2,1\}}, \end{aligned}\tag{3.16}$$

where $C_*=\max\left\{\frac{2(v+k_2)^2+(1-v^2)}{2(1-v^2)},\frac{k_3^2}{1-v^2}\right\}$, $K_1=k_1-\frac{k_3}{2}-\frac{\gamma}{2\tau}>0$, and $K_2=\frac{\gamma}{2\tau}e^{-2\tau}-\frac{k_3}{2}>0$. On the other hand, we can find from the definition of $E(t)$ given in (2.6) that

$$\begin{aligned}\int_0^1 xz_tz_x\mathrm{d}x+v\int_0^1 xz_x^2\mathrm{d}x &\leq \frac{1}{2}\int_0^1 z_t^2\mathrm{d}x+\frac{1}{2}\int_0^1 z_x^2\mathrm{d}x+v\int_0^1 z_x^2\mathrm{d}x\\ &\leq \frac{1}{2}\int_0^1 z_t^2\mathrm{d}x+\frac{1+2v}{2(1-v^2)}\int_0^1(1-v^2)z_x^2\mathrm{d}x\\ &\leq \hat{C}\left(\frac{1}{2}\int_0^1 z_t^2\mathrm{d}x+\frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a-v^2+\frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x\right)\\ &\leq \hat{C}E(t), \end{aligned}\tag{3.17}$$

with $\hat{C}=\max\left\{1,\frac{1+2v}{1-v^2}\right\}$, for all $t \geq 0$. This together with Lemma 3.1 shows that

$$\begin{aligned}\left|\int_T^S\int_0^1[xz_xz_t+vxz_x^2]_t\mathrm{d}x dt\right| &\leq \left|\int_0^1[xz_t(x,S)z_x(x,S)+vxz_x^2(x,S)]\mathrm{d}x\right|\\ &\quad+\left|\int_0^1[xz_t(x,T)z_x(x,T)+vxz_x^2(x,T)]\mathrm{d}x\right| \end{aligned}$$

$$\leq \hat{C}(E(T)+E(S))$$
$$\leq 2\hat{C}E(T). \tag{3.18}$$

Integrating (3.16) from T to S ($S \geq T$) and substituting (3.18) into it yield

$$\int_T^S E(t)dt \leq 2\hat{C}E(T) + \frac{C_* E(T)}{\min\{K_1, K_2, 1\}} \leq \frac{1}{\alpha}E(T), \tag{3.19}$$

where $\hat{C}$, C_* and α are given in (3.5). Passing to the limit as $S \to +\infty$ gives

$$\int_T^{+\infty} E(t)dt \leq \frac{1}{\alpha}E(T). \tag{3.20}$$

Then, the inequality (3.4) follows by invoking Lemma 3.2, which completes the proof of Theorem 3.1. □

Finally, we further analyze that the displacement response of the beam system is also exponentially stable and the control input U, D for the closed-loop system (2.5) belongs to $L^2(0,\infty)$.

Corollary 3.1. *If the assumptions of Lemma 3.1 are satisfied, the displacement response z of the closed-loop system (2.5) decays exponentially and*

$$\int_0^{+\infty} U^2(u(0,t))dt + \int_0^{+\infty} D^2(u(1,t))dt \leq \frac{k_2^2}{\min\{K_1, K_2, 1\}}E(0), \tag{3.21}$$

where k_2 are given by (2.2) and $K_1, K_2 > 0$ are given by (2.18).

Proof. Thanks to $z(0,t) = 0$, for all $t \geq 0$, it follows that

$$|z(x,t)| = \left|\int_0^x z_x(s,t)ds\right| \leq \int_0^1 |z_x(x,t)|\mathrm{d}x \leq \|z_x(\cdot,t)\| \leq \sqrt{\frac{2}{1-v^2}E(t)}, \tag{3.22}$$

for all $t \geq 0$ and $x \in [0,1]$, where the following estimate

$$\frac{1}{2}\int_0^1\int_0^{z_x^2}\left(a - v^2 + \frac{1-a}{\sqrt{1+s}}\right)\mathrm{d}s\mathrm{d}x \geq \frac{1-v^2}{2}\|z_x\|^2$$

is applied. Inserting the conclusion of Theorem 3.1 to (3.22) concludes our desired result.

For any $p > 0$, integrating over $(0,p)$ on (3.1) of Lemma 3.1 gives

$$\min\{K_1, K_2, 1\}\int_0^p [u^2(0,t) + u^2(1,t)]ds \leq E(0) - E(p) \leq E(0),$$

which with the slope-restricted condition $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ implies that

$$\int_0^{+\infty} U^2(u(0,t))dt + \int_0^{+\infty} D^2(u(1,t))dt \leq \frac{k_2^2}{\min\{K_1, K_2, 1\}}E(0).$$

□

Remark 3.1. When the linear feedback control, i.e., $U(y_t(1,t)) = k_1 y_t(1,t),\ D(y_t(1,t-\tau)) = k_3 y_t(1,t-\tau)$ for $0 < k_3 < \frac{2k_1}{e^{2\tau}+1}$, is implemented on the free end of the beam, the exponential stability of the closed-loop system (2.5) is easily derived by the same method adopted in this paper. From the conclusion of Theorem 3.1 and Corollary 3.1, the explicit exponential decay rate of the closed-loop system α is related to the upper bounds k_2, k_3 of growth coefficients in assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and the moving speed v. Consequently, distinct from the Lyapunov direct method ([33]), the explicit exponential decay rate not relevant to initial energy can be guaranteed by the generalized Gronwall-type integral inequality (Lemma 3.2).

4. Conclusion

This paper investigates the stability of a geometric nonlinear beams when a nonlinear damping and a nonlinear delay damping are applied at the free end of boundary. The emergence of time-delay term brings some complexity to the analysis of the system. In order to deal with the time-delay term, an innovative energy-like function is constructed to complete two important estimates. Then, the well-posedness of the closed-loop system is completed by invoking the Faedo–Galerkin approximation approach, where the existence of the solution is continuously dependent on the initial value. The uniform exponential stability of the closed-loop system is demonstrated, for which the integral-type multiplier method and a generalized Gronwall-type integral inequality are used, instead of the direct Lyapunov method applied in the literature [7,28], to handle the nonlinearities derived by the nonlinear geometric relation and the nonlinear feedbacks. If only the time-delay controller is implemented at the boundary, whether the system can continue to maintain stability is an interesting and open problem.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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