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Stability and large time decay for the three-dimensional magneto-micropolar equations with mixed partial viscosity

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Abstract. This paper focuses on the stability problem and decay estimates for two classes of three-dimensional (3D) magneto-micropolar equations with mixed partial viscosity. When $\mu_3 = \nu_3 = \gamma_1 = \gamma_2 = 0$ and $\chi\Delta u$ replaced by χu in (1.2), by fully exploiting the structure of the system, and the method of bootstrapping argument, we prove the global stability of solutions to this system with initial data small in $H^2(\mathbb{R}^3)$. Furthermore, for these global solutions with initial data in $H^s(\mathbb{R}^3)$ $(s \geq 3)$ being large, we obtain their global $H^s(\mathbb{R}^3)$ $(s \geq 3)$ bound which is independent of time. In addition, we obtain the global existence of solutions for small initial data and the decay estimates of these solutions to 3D magneto-micropolar equations with mixed partial viscosity [namely $\mu_3 = \gamma_1 = \gamma_2 = 0$ and $\chi\Delta u$ replaced by χu in (1.2)].

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1. Introduction

The three-dimensional (3D) incompressible magneto-micropolar fluid equations can be written as

$$\begin{cases} \partial_t u - (\mu + \chi)\Delta u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t b - \nu \Delta b + u \cdot \nabla b = b \cdot \nabla u, \\ \partial_t \omega - \gamma \Delta \omega + u \cdot \nabla \omega - \kappa \nabla \nabla \cdot \omega + 4\chi\omega = 2\chi \nabla \times u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ denote the velocity of fluid, $b = (b_1(x,t), b_2(x,t), b_3(x,t))$ the magnetic field, $\omega = (\omega_1(x,t), \omega_2(x,t), \omega_3(x,t))$ the microrotational velocity, $\pi = \pi(x,t)$ the pressure, and μ, χ and $\frac{1}{\nu}$ are, respectively, kinematic viscosity, vortex viscosity and magnetic Reynolds number. γ and κ are angular viscosities. The magneto-micropolar fluid equations usually describe the motion of aggregates of small solid ferromagnetic particles relative to viscous magnetic fluids under the action of magnetic fields, such as saltwater, ester and fluorocarbon [1,2]. It has attracted the attention of many physicists and mathematicians due to its important physical background, rich phenomena, mathematical complexity and challenges.

To the full magneto-micropolar fluid equations (1.1), Galdi and Rionero [3] stated the theorem of existence and uniqueness of strong solutions, but without proof. Ahmadi and Shaninpoor [2] studied the stability of solutions for the system. By using the spectral Galerkin method, Rojas-Medar [4] established the local existence and uniqueness of strong solutions. Ortega-Torres and Rojas-Medar [5] proved the global existence of strong solutions with small initial data. For the weak solution, Rojas-Medar and Boldrini [6] established the local existence in two and three dimensions by using the Galerkin method and also proved the uniqueness in the 2D case. The global existence of smooth solutions and the global regularity of weak solutions are important topics in the study field of magneto-micropolar fluid equations.

The blow-up criteria for smooth solutions and regularity criteria of weak solutions were obtained in different function spaces, such as Morrey–Campanato space, Besov space and homogeneous Besov space, which we may refer to [7–11]. Recently, based on Serrin's type non-blow-up criterion established by [7], Wang and Gu [12] have proved the global existence of a class of smooth solutions, which ensures the L^3 norm is large.

In this paper, we aim at the following three-dimensional (3D) magneto-micropolar fluid equations with mixed partial viscosity

$$\begin{cases} \partial_t u - \mu_1\partial_{11}u - \mu_2\partial_{22}u - \mu_3\partial_{33}u - \chi\Delta u + u\cdot\nabla u + \nabla\pi = b\cdot\nabla b + 2\chi\nabla\times\omega, \\ \partial_t b - \nu_1\partial_{11}b - \nu_2\partial_{22}b - \nu_3\partial_{33}b + u\cdot\nabla b = b\cdot\nabla u, \\ \partial_t\omega - \gamma_1\partial_{11}\omega - \gamma_2\partial_{22}\omega - \gamma_3\partial_{33}\omega + u\cdot\nabla\omega - \kappa\nabla\nabla\cdot\omega + 4\chi\omega = 2\chi\nabla\times u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

If

$$\mu_1 = \mu_2 = \mu_3 = \mu, \quad \nu_1 = \nu_2 = \nu_3 = \nu, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma,$$

then the (1.2) reduces to the standard 3D magneto-micropolar equations (1.1). For notational convenience, we will write ∂_1, ∂_2 and ∂_3 for ∂_{x_1}, ∂_{x_2} and ∂_{x_3}, respectively.

The focus on this paper will be on the global stability problem of 3D magneto-micropolar equations (1.2) with mixed partial viscosity. Firstly, we consider the case: $\mu_1 = \mu_2 = \mu$, $\nu_1 = \nu_2 = \nu$, $\gamma_3 = \gamma$, $\mu_3 = \nu_3 = \gamma_1 = \gamma_2 = 0$, and $\chi\Delta u$ replaced by χu. Therefore, (1.2) reduce to

$$\begin{cases} \partial_t u - \mu\Delta_h u + \chi u + u\cdot\nabla u + \nabla\pi = b\cdot\nabla b + 2\chi\nabla\times\omega, \\ \partial_t b - \nu\Delta_h b + u\cdot\nabla b = b\cdot\nabla u, \\ \partial_t\omega - \gamma\partial_{33}\omega + u\cdot\nabla\omega - \kappa\nabla\nabla\cdot\omega + 4\chi\omega = 2\chi\nabla\times u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x). \end{cases} \tag{1.3}$$

Here $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$, and we use $\nabla_h := (\partial_1, \partial_2)$, $u_h := (u_1, u_2)$ and $b_h := (b_1, b_2)$ for the horizontal gradient, horizontal components of velocity and magnetic field, respectively.

So far, to our best knowledge, the previous well-posedness results on (1.3) mainly focus on the two-dimensional system. Liu [13] proved the global regularity result for the 2D incompressible magneto-micropolar system with the horizontal kinematic dissipation by $\partial_{x_1x_1}$, the horizontal magnetic diffusion by $\partial_{x_1x_1}$ and the spin dissipation by the fractional operator $(-\Delta)^\gamma (\gamma > 1)$. For more well-posedness results to the 2D magneto-micropolar equations with partial dissipation, one refers to [14–18] and the references therein. In addition, when $b = 0$ and $\chi = 0$, then system (1.1) becomes magnetohydrodynamic (MHD) equations. The existence of small data global solutions to the 3D MHD equations with mixed partial dissipation and magnetic diffusion was obtained by Wang and Wang [19]. Wu and Zhu in [20] proved the global stability of perturbations near the steady solution to the 3D MHD equations with only horizontal velocity dissipation and vertical magnetic diffusion. Recently, Wang and Wang [21] have investigated the global existence of smooth solutions to 3D magneto-micropolar fluid equations with mixed partial viscosity (namely $\mu_1, \mu_2, \gamma_2, \gamma_3, \nu_1, \nu_2 > 0$, $\mu_3 = \gamma_1 = \nu_3 = 0$, or $\mu_1, \mu_2, \gamma_2, \gamma_3, \nu_2, \nu_3 > 0$, $\mu_3 = \gamma_1 = \nu_1 = 0$, or $\mu_1, \mu_2, \gamma_1, \gamma_2, \nu_1, \nu_2 > 0$, $\mu_3 = \gamma_3 = \nu_3 = 0$, or $\mu_1, \mu_2, \gamma_1, \gamma_2, \nu_2, \nu_3 > 0$, $\mu_3 = \gamma_3 = \nu_1 = 0$ in (1.2)); Wang and Li [22] proved the global well-posedness of 3D magneto-micropolar equations with mixed partial viscosity near an equilibrium. Shang and Zhai [23] proved the stability problem to the 3D anisotropic magnetohydrodynamic (MHD) equations (namely (1.3) with $\omega = 0$ and $\chi = 0$).

Motivated by the ideas in [23], we focus on the global stability of the system (1.3), and the first result is the global stability of solutions the system (1.3) in $H^2(\mathbb{R}^3)$ and the uniform boundedness for time of these global solutions in $H^s(\mathbb{R}^3)$, as stated in the following theorem.

Theorem 1.1. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume $(u_0, b_0, \omega_0) \in H^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then there exists $\epsilon > 0$ such that, if*

$$\|u_0\|_{H^2(\mathbb{R}^3)} + \|b_0\|_{H^2(\mathbb{R}^3)} + \|\omega_0\|_{H^2(\mathbb{R}^3)} \le \epsilon, \tag{1.4}$$

and $\mu > 4\chi$ and $\gamma > 4\chi$. Then (1.3) has a unique global solution $(u, b, \omega) \in L^\infty(0, \infty; H^2(\mathbb{R}^3))$ satisfying

$$\|u(t)\|_{H^2(\mathbb{R}^3)} + \|b(t)\|_{H^2(\mathbb{R}^3)} + \|\omega(t)\|_{H^2(\mathbb{R}^3)} \le C\epsilon, \tag{1.5}$$

and

$$\begin{aligned}&\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \|\omega(t)\|_{H^2}^2\\&\quad + \int_0^t \left(\|\nabla_h u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^2}^2 + \|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2\right) d\tau\\&\quad \le C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2),\end{aligned} \tag{1.6}$$

for some constant $C > 0$ and for all $t > 0$.

Furthermore, if $(u_0, b_0, \omega_0) \in H^s(\mathbb{R}^3)$, $s \ge 3$ and (1.4) hold, then the global solution (u, b, ω) obeys for all $t > 0$,

$$\begin{aligned}&\|u(t)\|_{H^s(\mathbb{R}^3)}^2 + \|b(t)\|_{H^s(\mathbb{R}^3)}^2 + \|\omega(t)\|_{H^s(\mathbb{R}^3)}^2\\&\quad + c_1 \int_0^t \left(\|\nabla_h u(\tau)\|_{H^s(\mathbb{R}^3)}^2 + \|\nabla_h b(\tau)\|_{H^s(\mathbb{R}^3)}^2 + \|\partial_3 \omega(\tau)\|_{H^s(\mathbb{R}^3)}^2\right) d\tau\\&\quad \le C_s \left(\|u_0\|_{H^s(\mathbb{R}^3)}^2 + \|b_0\|_{H^s(\mathbb{R}^3)}^2 + \|\omega_0\|_{H^s(\mathbb{R}^3)}^2\right),\end{aligned} \tag{1.7}$$

where $c_1 = \min\{\frac{\mu}{2}, \nu, \gamma - 4\chi, 4\chi - \frac{16\chi^2}{\mu}\}$ and the constant C_s depends on μ, χ, ν, κ, γ and the initial data.

Compared with the magnitude of research conducted on the well-posedness problem for the 3D Magneto-micropolar equations with partial dissipation, the large time behavior for the partial dissipation cases also has attracted considerable attention from the community of mathematical fields (see, e.g., [24–33]). This is an important issue in the fields of partial differential equations. It is well known that the L^2 decay problem of weak solutions to the 3D Navier–Stokes equations, i.e., (1.1) with $\omega = 0$ and $b = 0$, was proposed by the celebrated work of Leray [34]. By introducing the elegant method of Fourier splitting, Schonbek [35,36] successfully established the optimal time decay rate of weak solutions of the Navier–Stokes equations, see also [37,38]. Recently, by the structure to (1.1) and the Fourier splitting method, Li and Shang [39] have established the decay estimates for weak solutions of (1.1) and obtained the same rates as those of the 3D Navier–Stokes equations.

Recently, Shang and Gu [40,41] have proved the 2D magneto-micropolar equations with only micro-rotational dissipation and magneto diffusion. Li [42] proved the L^2-decay estimates for global solution and their derivative for 2D magneto-micropolar equation with partial dissipation (namely $\nu\Delta b$ replace by $\nu\partial_{yy}b_1 + \nu\partial_{xx}b_2$). Niu and Shang [43] proved the magneto-micropolar equations only have velocity dissipation and magnetic diffusion (namely (1.1) with $\gamma = 0$).

Next, we consider the case: $\mu_1 = \mu_2 = \mu$, $\nu_1 = \nu_2 = \nu_3 = \nu$, $\gamma_3 = \gamma$, $\mu_3 = \gamma_1 = \gamma_2 = 0$ and $\chi\Delta u$ replaced by χu. Therefore, (1.2) reduce to

$$\begin{cases}\partial_t u - \mu\Delta_h u + \chi u + u \cdot \nabla u + \nabla \pi = b \cdot \nabla b + 2\chi \nabla \times \omega,\\ \partial_t b - \nu\Delta b + u \cdot \nabla b = b \cdot \nabla u,\\ \partial_t \omega - \gamma\partial_{33}\omega + u \cdot \nabla \omega - \kappa \nabla\nabla \cdot \omega + 4\chi\omega = 2\chi \nabla \times u,\\ \nabla \cdot u = 0,\ \nabla \cdot b = 0,\\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), b(x,0) = b_0(x),\end{cases} \tag{1.8}$$

Before giving the decay results of the solutions for (1.8), we first consider the global stability of solutions to system (1.8), as the following theorem.

Theorem 1.2. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume $(u_0, b_0, \omega_0) \in H^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then there exists $\epsilon > 0$ such that, if*

$$\|u_0\|_{H^2(\mathbb{R}^3)} + \|b_0\|_{H^2(\mathbb{R}^3)} + \|\omega_0\|_{H^2(\mathbb{R}^3)} \le \epsilon, \tag{1.9}$$

and $\mu > 4\chi$ and $\gamma > 4\chi$. Then (1.8) has a unique global solution $(u, b, \omega) \in L^\infty(0, \infty; H^2(\mathbb{R}^3))$ satisfying

$$\|u(t)\|_{H^2(\mathbb{R}^3)} + \|b(t)\|_{H^2(\mathbb{R}^3)} + \|\omega(t)\|_{H^2(\mathbb{R}^3)} \le C\epsilon, \tag{1.10}$$

and

$$\begin{aligned} &\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \|\omega(t)\|_{H^2}^2 \\ &\quad + \int_0^t \left(\|\nabla_h u(\tau)\|_{H^2}^2 + \|u(\tau)\|_{H^2}^2 + \|\nabla b(\tau)\|_{H^2}^2 + \|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2\right) d\tau \\ &\le C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2), \end{aligned} \tag{1.11}$$

for some constant $C > 0$ and for all $t > 0$.

Furthermore, if $(u_0, b_0, \omega_0) \in H^s(\mathbb{R}^3)$, $s \ge 3$ and (1.9) hold, then the global solution (u, b, ω) obeys for all $t > 0$,

$$\begin{aligned} &\|u(t)\|_{H^s(\mathbb{R}^3)}^2 + \|b(t)\|_{H^s(\mathbb{R}^3)}^2 + \|\omega(t)\|_{H^s(\mathbb{R}^3)}^2 \\ &\quad + c_2 \int_0^t \left(\|\nabla_h u(\tau)\|_{H^s(\mathbb{R}^3)}^2 + \|u(\tau)\|_{H^s(\mathbb{R}^3)}^2 + \|\nabla b(\tau)\|_{H^s(\mathbb{R}^3)}^2 + \|\partial_3 \omega(\tau)\|_{H^s(\mathbb{R}^3)}^2\right) d\tau \\ &\le C_s \left(\|u_0\|_{H^s(\mathbb{R}^3)}^2 + \|b_0\|_{H^s(\mathbb{R}^3)}^2 + \|\omega_0\|_{H^s(\mathbb{R}^3)}^2\right), \end{aligned} \tag{1.12}$$

where $c_2 = \min\{\frac{\mu}{2}, \frac{\chi}{2}, \nu, \gamma - 8\chi, 4\chi - \frac{16\chi^2}{\mu}\}$ and the constant C_s depends on μ, χ, ν, κ, γ and the initial data.

Remark 1.3. The proof of Theorem 1.2 is similar to the proof of Theorem 1.1; then, we omit the details.

At last, using the delicate a priori estimates and the properties of the heat operator, we can establish the following decay results for the global solutions established in Theorem 1.2. More precisely, we have the following theorem.

Theorem 1.4. *Let $(u_0, \omega_0) \in H^2(\mathbb{R}^3)$, $b_0 \in L^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, b, ω) be a global solution to system (1.8), and assume that*

$$\mu > 4\chi, \quad \gamma > \frac{16\chi}{3},$$

and

$$\|u_0\|_{H^2(\mathbb{R}^3)} + \|b_0\|_{H^2(\mathbb{R}^3)} + \|\omega_0\|_{H^2(\mathbb{R}^3)} \le \epsilon. \tag{1.13}$$

Then, the following decay properties hold:

$$\|b(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}}, \quad \|\nabla b(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}. \tag{1.14}$$

Moreover, for any $0 < \alpha < \frac{1}{4}$, the following decay properties hold:

$$\|\nabla^2 b(t)\|_{L^2} \le C(1+t)^{-\frac{65}{48} + \frac{17}{12}\alpha}, \tag{1.15}$$

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \le C(1+t)^{-\frac{157}{64} + \frac{17}{16}\alpha}, \tag{1.16}$$

$$\|\nabla u(t)\|_{L^2} + \|\nabla \omega(t)\| \le C(1+t)^{-\frac{471}{512} + \frac{51}{128}\alpha}, \tag{1.17}$$

where the constant C depends on μ, χ, ν, κ, γ and the initial data.

The rest of this paper is divided into two sections. Sect. 2 proves Theorem 1.1 while Sect. 1.4 is devoted to verifying Theorem 1.4. Throughout this manuscript, to simplify the notations, we will write $\int f\mathrm{d}x$ for $\int_{\mathbb{R}^3} f\mathrm{d}x$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^3)}$. We write $\|f\|_{L^p_{x_j}}$ with $j = 1, 2, 3$ for the L^p-norm with respect to x_j on $\mathbb{R}$ and $\|f\|_{L^p_{x_j x_k}}$ with $j, k = 1, 2, 3$ for the L^p-norm with respect to (x_j, x_k) on $\mathbb{R}^2$. We also write $\|f\|_{L^p_h}$ for the $\|f\|_{L^p_{x_1 x_2}}$ to shorten the notation. In addition, the anisotropic norm $\|f\|_{L^p_h L^q_{x_3}} := \|\|f\|_{L^q_{x_3}}\|_{L^p_h}$ is also frequently used.

2. The proof of Theorem 1.1

This section focuses its attention on the proof of Theorem 1.1. As preparations, we state three important lemmas, which are frequently used. The first lemma provides an upper bound for the L^p-norm of a one-dimensional (1D) function, which serves as a basic ingredient for an anisotropic upper bound (see, e.g., [44]).

Lemma 2.1. *Let $2 \le p \le \infty$ and $\Lambda = (-\Delta)^{-\frac{1}{2}}$ be the Zygmund operator. Then there exists a positive constant C such that, for any 1D functions $f \in H^s(\mathbb{R})$ with $s > \frac{1}{2} - \frac{1}{p}$,*

$$\|f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}(\frac{1}{2}-\frac{1}{p})}\|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}(\frac{1}{2}-\frac{1}{p})}.$$

In particular, if $p = \infty$ and $s = 1$, then any $f = f(x_3) \in H^1(\mathbb{R})$ satisfies

$$\|f\|_{L^\infty(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|\partial_{x_3} f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

The second lemma provides an anisotropic upper bound for the integral of a triple product (see, e.g., [20]). It is a very powerful tool in dealing with anisotropic equations.

Lemma 2.2. *Assume that f, $\partial_1 f$, $\partial_2 f$, $\partial_1\partial_2 f$, g, $\partial_2 g$, h, $\partial_3 h \in L^2(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} |fgh|\mathrm{d}x \le C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 h\|_{L^2}^{\frac{1}{2}}.$$

$$\int_{\mathbb{R}^3} |fgh|\mathrm{d}x \le C\|f\|_{L^2}^{\frac{1}{4}}\|\partial_1 f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_3 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}.$$

The third lemma states Minkowski's inequality. It is an elementary tool that allows us to estimate the Lebesgue norm with a large index first followed by the Lebesgue norm with a smaller index (see, e.g., [45,46]).

Lemma 2.3. *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces. Let f be a nonnegative measurable function over $X_1 \times X_2$. For all $1 \le p \le q \le \infty$, we have*

$$\left\|\|f(\cdot, x_2)\|_{L^p(X_1,\mu_1)}\right\|_{L^q(X_2,\mu_2)} \le \left\|\|f(x_1, \cdot)\|_{L^q(X_2,\mu_2)}\right\|_{L^p(X_1,\mu_1)}.$$

In particular, for a nonnegative measurable function f over $\mathbb{R}^m \times \mathbb{R}^n$ and for $1 \le p \le q \le \infty$,

$$\left\|\|f\|_{L^p(\mathbb{R}^m)}\right\|_{L^q(\mathbb{R}^n)} \le \left\|\|f\|_{L^q(\mathbb{R}^n)}\right\|_{L^p(\mathbb{R}^m)}.$$

Next, we are ready to prove Theorem 1.1. The general approach to establish the global existence and regularity results consists of two main steps. The first step assesses the local (in time) well-posedness while the second extends the local solution into a global one by obtaining global (in time) a priori bounds. For the system (1.3) concerned here, the local well-posedness follows from a standard approach such as the contraction mapping principle or successive approximations (see, e.g., [47]). Therefore, the effort of proving the Theorem 1.1 is mainly devoted to the global a priori bounds for $\|(u,b,\omega)\|_{H^3}$ and $\|(u,b,\omega)\|_{H^s}$ with $s \geq 3$.

Proof of Theorem 1.1. For the sake of clarity, we divide the proof into two parts.

(1) This part proves (1.5) of Theorem 1.1. Firstly, we examine the global *a priori* L^2- bound as follows.

Proposition 2.4. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume $(u_0, b_0, \omega_0) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. If $\mu > 4\chi$ and $\gamma > 4\chi$, then the corresponding solution (u, b, ω) of (1.3) obeys the following uniform bounds, for any $t > 0$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + \mu\|\nabla_h u\|_{L^2}^2 + 2\nu\|\nabla_h b\|_{L^2}^2 + 2(\gamma - 4\chi)\|\partial_3\omega\|_{L^2}^2 + 2(4\chi - \frac{16\chi^2}{\mu})\|\omega\|_{L^2}^2 \leq 0, \tag{2.1}$$

or

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + 2c_1\int_0^t \left(\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2 + \|\partial_3\omega(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2\right) d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2, \tag{2.2}$$

where $c_1 = \min\{\frac{\mu}{2}, \nu, \gamma - 4\chi, 4\chi - \frac{16\chi^2}{\mu}\}$.

Proof of Proposition 2.4. Taking the L^2-inner product to (1.3) with (u, b, ω), by integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + \mu\|\nabla_h u\|_{L^2}^2 + \chi\|u\|_{L^2}^2 + \nu\|\nabla_h b\|_{L^2}^2 \\
&\quad + \gamma\|\partial_3\omega\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\cdot\omega\|_{L^2}^2 \\
&= 4\chi\int \nabla\times u\cdot\omega \mathrm{d}x \\
&= 4\chi\int (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)\cdot\omega \mathrm{d}x \\
&= 4\chi\int \left((\partial_2 u_3 - \partial_3 u_2)\omega_1 + (\partial_3 u_1 - \partial_1 u_3)\omega_2 + (\partial_1 u_2 - \partial_2 u_1)\omega_3\right) \mathrm{d}x \\
&= 4\chi\int (\partial_2 u_3\omega_1 - \partial_1 u_3\omega_2 + \partial_1 u_2\omega_3 - \partial_2 u_1\omega_3)\mathrm{d}x - 4\chi\int (u_1\partial_3\omega_2 - u_2\partial_3\omega_1)\mathrm{d}x \\
&\leq \frac{\mu}{2}(\|\partial_2 u_3\|_{L^2}^2 + \|\partial_1 u_3\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2) + \frac{8\chi^2}{\mu}(\|\omega_1\|_{L^2}^2 + \|\omega_2\|_{L^2}^2 + 2\|\omega_3\|_{L^2}^2) \\
&\quad + \chi(\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2) + 4\chi(\|\partial_3\omega_2\|_{L^2}^2 + \|\partial_3\omega_1\|_{L^2}^2) \\
&\leq \frac{\mu}{2}\|\nabla_h u\|_{L^2}^2 + \frac{16\chi^2}{\mu}\|\omega\|_{L^2}^2 + \chi\|u\|_{L^2}^2 + 4\chi\|\partial_3\omega\|_{L^2}^2,
\end{aligned}$$

where we have used the facts that

$$\int \nabla\times u\cdot\omega \mathrm{d}x = \int \nabla\times\omega\cdot u \mathrm{d}x,$$

$$\int b\cdot\nabla b\cdot u \mathrm{d}x + \int b\cdot\nabla u\cdot b \mathrm{d}x = 0.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right)+\mu\|\nabla_h u\|_{L^2}^2+2\nu\|\nabla_h b\|_{L^2}^2+2(\gamma-4\chi)\|\partial_3\omega\|_{L^2}^2$$
$$+2(4\chi-\frac{16\chi^2}{\mu})\|\omega\|_{L^2}^2\le 0,$$

which immediately yields (2.1). Integrating the above inequality in [0, t], we can get (2.2) immediately. □

Applying ∂_k^2 with $k=1,2,3$ to $(1.3)_1$, $(1.3)_2$ and $(1.3)_3$, dotting the results by $\partial_k^2 u$, $\partial_k^2 b$ and $\partial_k^2\omega$, respectively, integrating in space domain and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{3}\left(\|\partial_k^2 u(t)\|_{L^2}^2+\|\partial_k^2 b(t)\|_{L^2}^2+\|\partial_k^2\omega(t)\|_{L^2}^2\right)+\mu\sum_{k=1}^{3}\|\nabla_h\partial_k^2 u\|_{L^2}^2+\chi\sum_{k=1}^{3}\|\partial_k^2 u\|_{L^2}^2\\
&\quad+\nu\sum_{k=1}^{3}\|\nabla_h\partial_k^2 b\|_{L^2}^2+\gamma\sum_{k=1}^{3}\|\partial_3\partial_k^2\omega\|_{L^2}^2+4\chi\sum_{k=1}^{3}\|\partial_k^2\omega\|_{L^2}^2+\kappa\sum_{k=1}^{3}\|\partial_k^2\nabla\cdot\omega\|_{L^2}^2\\
&=-\sum_{k=1}^{3}\int\partial_k^2(u\cdot\nabla u)\cdot\partial_k^2 u\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k^2(b\cdot\nabla b)\cdot\partial_k^2 u\mathrm{d}x+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k^2\omega\cdot\partial_k^2 u\mathrm{d}x\\
&\quad-\sum_{k=1}^{3}\int\partial_k^2(u\cdot\nabla b)\cdot\partial_k^2 b\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k^2(b\cdot\nabla u)\cdot\partial_k^2 b\mathrm{d}x-\sum_{k=1}^{3}\int\partial_k^2(u\cdot\nabla\omega)\cdot\partial_k^2\omega\mathrm{d}x\\
&\quad+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k^2 u\cdot\partial_k^2\omega\mathrm{d}x\\
&:=M_1+M_2+M_3+M_4+M_5+M_6+M_7.
\end{aligned}\tag{2.3}$$

Due to the divergence free condition $\nabla\cdot u=0$, we have

$$\sum_{k=1}^{3}\int u\cdot\nabla\partial_k^2 u\cdot\partial_k^2 u\mathrm{d}x=0.$$

Thus, we can rewrite M_1 as

$$\begin{aligned}
M_1&=-\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int\partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}u\cdot\partial_k^2 u\mathrm{d}x\\
&=-\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int\partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}u\cdot\partial_k^2 u\mathrm{d}x-\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int\partial_3^\alpha u_j\partial_j\partial_3^{2-\alpha}u\cdot\partial_3^2 u\mathrm{d}x\\
&\quad-\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int\partial_3^\alpha u_3\partial_3\partial_3^{2-\alpha}u\cdot\partial_3^2 u\mathrm{d}x\\
&:=M_{11}+M_{12}+M_{13},
\end{aligned}$$

where $\binom{2}{\alpha}=\frac{2!}{\alpha!(2-\alpha)!}$ is the binomial coefficient.

We first bound M_{11}. To do this, applying Lemma 2.2, we drive that

$$\begin{aligned} M_{11} &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}u\cdot\partial_k^2 u\mathrm{d}x \\ &\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\|\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_3\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_k^{2-\alpha}u\|_{L^2}^{\frac12}\|\partial_2\partial_j\partial_k^{2-\alpha}u\|_{L^2}^{\frac12}\|\partial_k^2 u\|_{L^2}^{\frac12}\|\partial_1\partial_k^2 u\|_{L^2}^{\frac12} \\ &\le C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2. \end{aligned}$$

Again applying Lemma 2.2, yields

$$\begin{aligned} M_{12} &= -\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{2-\alpha}u\cdot\partial_3^2 u\mathrm{d}x \\ &\le C\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\|\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_3^{2-\alpha}u\|_{L^2}^{\frac12}\|\partial_3\partial_j\partial_3^{2-\alpha}u\|_{L^2}^{\frac12}\|\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 u\|_{L^2}^{\frac12} \\ &\le C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2. \end{aligned}$$

We cannot estimate the term M_{13} directly. To bound it, the strategy is to use the divergence free condition $\nabla\cdot u = 0$ to convert $\partial_3 u_3$ to $-\nabla_h\cdot u_h$ and using Lemma 2.2. Then it can be bounded by

$$\begin{aligned} M_{13} &= -\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_3\partial_3\partial_3^{2-\alpha}u\cdot\partial_3^2 u\mathrm{d}x \\ &= -\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^{\alpha-1}\nabla_h\cdot u_h\partial_3\partial_3^{2-\alpha}u\cdot\partial_3^2 u\mathrm{d}x \\ &\le C\sum_{\alpha=1}^{2}\|\partial_3^{\alpha-1}\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3^\alpha\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3^{3-\alpha}u\|_{L^2}^{\frac12}\|\partial_1\partial_3^{3-\alpha}u\|_{L^2}^{\frac12}\|\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 u\|_{L^2}^{\frac12} \\ &\le C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2. \end{aligned}$$

Therefore, we obtain

$$M_1 \le C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.$$

By integrating by parts, Hölder's inequality and the Young inequality, we have

$$\begin{aligned} M_3 + M_7 &= 2\chi\sum_{k=1}^{3}\int \nabla\times\partial_k^2\omega\cdot\partial_k^2 u\mathrm{d}x + 2\chi\sum_{k=1}^{3}\int \nabla\times\partial_k^2 u\cdot\partial_k^2\omega\mathrm{d}x \\ &= 4\chi\sum_{k=1}^{3}\int \nabla\times\partial_k^2 u\cdot\partial_k^2\omega\mathrm{d}x \\ &\le \frac{\mu}{2}\sum_{k=1}^{3}\|\nabla_h\partial_k^2 u\|_{L^2}^2 + \frac{16\chi^2}{\mu}\sum_{k=1}^{3}\|\partial_k^2\omega\|_{L^2}^2 + \chi\sum_{k=1}^{3}\|\partial_k^2 u\|_{L^2}^2 + 4\chi\sum_{k=1}^{3}\|\partial_3\partial_k^2\omega\|_{L^2}^2. \end{aligned}$$

By using the same method as M_1, we have

$$\begin{aligned}
M_4 =& -\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}b\cdot\partial_k^2 b\mathrm{d}x\\
=& -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}b\cdot\partial_k^2 b\mathrm{d}x - \sum_{j=1}^{2}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{2-\alpha}b\cdot\partial_3^2 b\mathrm{d}x\\
&-\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_3\partial_3\partial_3^{2-\alpha}b\cdot\partial_3^2 b\mathrm{d}x\\
\le& C\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\|\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_3\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_k^{2-\alpha}b\|_{L^2}^{\frac12}\|\partial_2\partial_j\partial_k^{2-\alpha}b\|_{L^2}^{\frac12}\|\partial_k^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_k^2 b\|_{L^2}^{\frac12}\\
&+C\sum_{j=1}^{2}\sum_{\alpha=1}^{2}\|\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_3^{2-\alpha}b\|_{L^2}^{\frac12}\|\partial_3\partial_j\partial_3^{2-\alpha}b\|_{L^2}^{\frac12}\|\partial_3^2 b\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 b\|_{L^2}^{\frac12}\\
&+C\sum_{\alpha=1}^{2}\|\partial_3^{\alpha-1}\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3^{\alpha}\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3^{3-\alpha}b\|_{L^2}^{\frac12}\|\partial_1\partial_3^{3-\alpha}b\|_{L^2}^{\frac12}\|\partial_3^2 b\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 b\|_{L^2}^{\frac12}\\
\le& C\|b\|_{H^2}\|\nabla_h u\|_{H^2}\|\nabla_h b\|_{H^2} + C\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12}\|\nabla_h u\|_{H^2}^{\frac12}\|\nabla_h b\|_{H^2}^{\frac32}\\
\le& C(\|u\|_{H^2}+\|b\|_{H^2})(\|\nabla_h u\|_{H^2}^2+\|\nabla_h b\|_{H^2}^2).
\end{aligned}$$

Note that

$$\sum_{k=1}^{3}\int b\cdot\nabla\partial_k^2 b\cdot\partial_k^2 u\mathrm{d}x+\sum_{k=1}^{3}\int b\cdot\nabla\partial_k^2 u\cdot\partial_k^2 b\mathrm{d}x=0,$$

then the estimates similar as M_1, we have

$$\begin{aligned}
M_2+M_5 =& \sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\left(\int \partial_k^\alpha b_j\partial_j\partial_k^{2-\alpha}b\cdot\partial_k^2 u\mathrm{d}x+\int \partial_k^\alpha b_j\partial_j\partial_k^{2-\alpha}u\cdot\partial_k^2 b\mathrm{d}x\right)\\
\le& C(\|u\|_{H^2}+\|b\|_{H^2})(\|\nabla_h u\|_{H^2}^2+\|\nabla_h b\|_{H^2}^2).
\end{aligned}$$

Finally, we consider M_6,

$$\begin{aligned}
M_6 =& -\sum_{k=1}^{3}\int \partial_k^2(u\cdot\nabla\omega)\cdot\partial_k^2\omega\mathrm{d}x\\
=& -\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}\omega\cdot\partial_k^2\omega\mathrm{d}x\\
=& -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}\omega\cdot\partial_k^2\omega\mathrm{d}x-\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{2-\alpha}\omega\cdot\partial_3^2\omega\mathrm{d}x\\
:=& M_{61}+M_{62}.
\end{aligned}$$

Applying the Young inequality, Lemma 2.2 and the fact that $\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{H^2(\mathbb{R}^3)}$, we obtain

$$\begin{aligned}
M_{61} &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{2-\alpha}\omega\cdot\partial_k^2\omega \mathrm{d}x \\
&= -2\sum_{k=1}^{2}\sum_{j=1}^{3}\int \partial_k u_j\partial_j\partial_k\omega\cdot\partial_k^2\omega\mathrm{d}x - \sum_{k=1}^{2}\sum_{j=1}^{3}\int \partial_k^2 u_j\partial_j\omega\cdot\partial_k^2\omega\mathrm{d}x \\
&\leq C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^\infty}\|\partial_j\partial_k\omega\|_{L^2}\|\partial_k^2\omega\|_{L^2} \\
&\quad + C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k^2 u_j\|_{L^2}^{\frac{1}{2}}\|\partial_k^2\partial_1 u_j\|_{L^2}^{\frac{1}{2}}\|\partial_j\omega\|_{L^2}^{\frac{1}{2}}\|\partial_j\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_k^2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_k^2\partial_3\omega\|_{L^2}^{\frac{1}{2}} \\
&\leq C\|\nabla_h u\|_{H^2}\|\omega\|_{H^2}^2 + C\|\omega\|_{H^2}(\|\nabla_h u\|_{H^2}\|\omega\|_{H^2}^{\frac{1}{2}}\|\partial_3\omega\|_{H^2}^{\frac{1}{2}}) \\
&\leq C\|\omega\|_{H^2}(\|\nabla_h u\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\partial_3\omega\|_{H^2}^2).
\end{aligned}$$

Similarly as M_{61}, by Lemma 2.2, the Young inequality and Hölder's inequality yield

$$\begin{aligned}
M_{62} &= -\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\binom{2}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{2-\alpha}\omega\cdot\partial_3^2\omega\mathrm{d}x \\
&= -2\int \partial_3 u\cdot\nabla\partial_3\omega\cdot\partial_3^2\omega\mathrm{d}x - \int \partial_3^2 u\cdot\nabla\omega\cdot\partial_3^2\omega\mathrm{d}x \\
&\leq C\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\omega\|^{\frac{1}{2}}\|\partial_3^3\omega\|^{\frac{1}{2}} \\
&\quad + C\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3\omega\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\omega\|^{\frac{1}{2}}\|\partial_3^2\partial_2\omega\|^{\frac{1}{2}} \\
&\leq C\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\omega\|_{H^2}^{\frac{1}{2}}\|\partial_3\omega\|_{H^2}\|\omega\|_{H^2}^{\frac{1}{2}} + C\|u\|_{H^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^2}^{\frac{1}{2}}\|\omega\|_{H^2}^{\frac{1}{2}}\|\partial_3\omega\|_{H^2}^{\frac{1}{2}}\|\omega\|_{H^2} \\
&\leq C(\|u\|_{H^2} + \|\omega\|_{H^2})(\|\partial_3\omega\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\nabla_h u\|_{H^2}^2).
\end{aligned}$$

Then we have

$$M_6 \leq C(\|u\|_{H^2} + \|\omega\|_{H^2})(\|\partial_3\omega\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\nabla_h u\|_{H^2}^2).$$

Inserting these above estimates in (2.3), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{3}\left(\|\partial_k^2 u(t)\|_{L^2}^2 + \|\partial_k^2 b(t)\|_{L^2}^2 + \|\partial_k^2\omega(t)\|_{L^2}^2\right) \\
&\qquad + c_1\sum_{k=1}^{3}\left(\|\nabla_h\partial_k^2 u\|_{L^2}^2 + \|\nabla_h\partial_k^2 b\|_{L^2}^2 + \|\partial_3\partial_k^2\omega\|_{L^2}^2 + \|\partial_k^2\omega\|_{L^2}^2\right) \\
&\leq C(\|u\|_{H^2} + \|b\|_{H^2} + \|\omega\|_{H^2})(\|\partial_3\omega\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\nabla_h u\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2),
\end{aligned} \tag{2.4}$$

where $c_1 = \min\{\frac{\mu}{2}, \nu, \gamma - 4\chi, 4\chi - \frac{16\chi^2}{\mu}\}$.
Adding (2.1) and (2.4) up, and integrating the result in $[0, t]$, we find

$$\begin{aligned}
&\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \|\omega(t)\|_{H^2}^2 \\
&\quad + 2c_1 \int_0^t \left(\|\nabla_h u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^2}^2 + \|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2\right) d\tau \\
&\leq \|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2 + C \int_0^t (\|u(\tau)\|_{H^2} + \|b(\tau)\|_{H^2} + \|\omega(\tau)\|_{H^2}) \\
&\quad \times \left(\|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2 + \|\nabla_h u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^2}^2\right) d\tau. \qquad (2.5)
\end{aligned}$$

Let

$$\begin{aligned}
E(t) = &\sup_{0 \leq \tau \leq t} \{\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2\} \\
&+ 2c_1 \int_0^t \left(\|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2 + \|\nabla_h u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^2}^2\right) d\tau. \qquad (2.6)
\end{aligned}$$

Then (2.5) implies

$$E(t) \leq E(0) + C_0 E^{\frac{3}{2}}(t). \qquad (2.7)$$

Now we can start to show (1.5), by using the bootstrapping argument. By choosing $\epsilon \leq \frac{1}{4\sqrt{2}C_0}$ in (1.4), we have

$$E(0) \leq \frac{1}{32 C_0^2}. \qquad (2.8)$$

To initial the bootstrapping argument, we assume that

$$E(t) \leq \frac{1}{4 C_0^2}. \qquad (2.9)$$

Then (2.7)–(2.9) imply

$$E(t) \leq \frac{5}{32 C_0^2}. \qquad (2.10)$$

This completes the proof (1.5). Substituting (1.5) with $\epsilon < \frac{2c_1}{C}$ into (2.5), we have

$$\begin{aligned}
&\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \|\omega(t)\|_{H^2}^2 \\
&\quad + \int_0^t \left(\|\nabla_h u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^2}^2 + \|\partial_3 \omega(\tau)\|_{H^2}^2 + \|\omega(\tau)\|_{H^2}^2\right) d\tau \\
&\leq C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2). \qquad (2.11)
\end{aligned}$$

(2) Next we start to show (1.7). At the beginning of the proof, we show the uniform bound for these global solutions in $H^3(\mathbb{R}^3)$ as follows.

Proposition 2.5. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume $(u_0, b_0, \omega_0) \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$ and (1.4) hold. If $\mu > 4\chi$ and $\gamma > 4\chi$, then the global solution (u, b, ω) of (1.3) obeys for all*

$t > 0$,

$$
\begin{aligned}
&\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \|\omega(t)\|_{H^3}^2 \\
&\qquad + \int_0^t \left(\|\nabla_h u(\tau)\|_{H^3}^2 + \|\nabla_h b(\tau)\|_{H^3}^2 + \|\partial_3 \omega(\tau)\|_{H^3}^2 + \|\omega(\tau)\|_{H^3}^2\right) d\tau \\
&\quad \le (\|u_0\|_{H^3}^2 + \|b_0\|_{H^3}^2 + \|\omega_0\|_{H^3}^2) e^{C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2)} \\
&\quad \le C_3(\|u_0\|_{H^3}^2 + \|b_0\|_{H^3}^2 + \|\omega_0\|_{H^3}^2)
\end{aligned}
\tag{2.12}
$$

proof of Proposition 2.5. Applying ∂_k^3 with $k = 1, 2, 3$ to $(1.3)_1$, $(1.3)_2$ and $(1.3)_3$, taking the L^2-inner product with $\partial_k^3 u$, $\partial_k^3 b$ and $\partial_k^3 \omega$, respectively, and adding the results up, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{k=1}^3 \left(\|\partial_k^3 u(t)\|_{L^2}^2 + \|\partial_k^3 b(t)\|_{L^2}^2 + \|\partial_k^3 \omega(t)\|_{L^2}^2\right) + \mu \sum_{k=1}^3 \|\nabla_h \partial_k^3 u\|_{L^2}^2 + \chi \sum_{k=1}^3 \|\partial_k^3 u\|_{L^2}^2 \\
&\quad + \nu \sum_{k=1}^3 \|\nabla_h \partial_k^3 b\|_{L^2}^2 + \gamma \sum_{k=1}^3 \|\partial_3 \partial_k^3 \omega\|_{L^2}^2 + 4\chi \sum_{k=1}^3 \|\partial_k^3 \omega\|_{L^2}^2 + \kappa \sum_{k=1}^3 \|\partial_k^3 \nabla \cdot \omega\|_{L^2}^2 \\
&= -\sum_{k=1}^3 \int \partial_k^3 (u \cdot \nabla u) \cdot \partial_k^3 u \mathrm{d}x + \sum_{k=1}^3 \int \partial_k^3 (b \cdot \nabla b) \cdot \partial_k^3 u \mathrm{d}x + 2\chi \sum_{k=1}^3 \int \nabla \times \partial_k^3 \omega \cdot \partial_k^3 u \mathrm{d}x \\
&\quad - \sum_{k=1}^3 \int \partial_k^3 (u \cdot \nabla b) \cdot \partial_k^3 b \mathrm{d}x + \sum_{k=1}^3 \int \partial_k^3 (b \cdot \nabla u) \cdot \partial_k^3 b \mathrm{d}x - \sum_{k=1}^3 \int \partial_k^3 (u \cdot \nabla \omega) \cdot \partial_k^3 \omega \mathrm{d}x \\
&\quad + 2\chi \sum_{k=1}^3 \int \nabla \times \partial_k^3 u \cdot \partial_k^3 \omega \mathrm{d}x \\
&:= N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7.
\end{aligned}
\tag{2.13}
$$

Thus, we can rewrite N_1 as

$$
\begin{aligned}
N_1 &= -\sum_{k=1}^3 \int \partial_k^3 (u \cdot \nabla u) \cdot \partial_k^3 u \mathrm{d}x \\
&= -\sum_{k=1}^2 \sum_{j=1}^3 \sum_{\alpha=1}^3 \binom{3}{\alpha} \int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} u \cdot \partial_k^3 u \mathrm{d}x - \sum_{j=1}^2 \sum_{\alpha=1}^3 \binom{3}{\alpha} \int \partial_3^\alpha u_j \partial_j \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&\quad - \sum_{\alpha=1}^3 \binom{3}{\alpha} \int \partial_3^\alpha u_3 \partial_3 \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&:= N_{11} + N_{12} + N_{13}.
\end{aligned}
$$

Applying the Young inequality, Lemma 2.2 and the fact that $\|f\|_{L^\infty(\mathbb{R}^3)} \le C\|f\|_{H^2(\mathbb{R}^3)}$, we obtain

$$
\begin{aligned}
N_{11} &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{3}\binom{3}{\alpha}\int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} u \cdot \partial_k^3 u \mathrm{d}x \\
&= -\sum_{k=1}^{2}\sum_{j=1}^{3}\binom{3}{1}\int \partial_k u_j \partial_j \partial_k^2 u \cdot \partial_k^3 u \mathrm{d}x - \sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} u \cdot \partial_k^3 u \mathrm{d}x \\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^\infty}\|\partial_j\partial_k^2 u\|_{L^2}\|\partial_k^3 u\|_{L^2} \\
&\quad + C\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=2}^{3}\|\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_3\partial_k^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_k^{3-\alpha}u\|_{L^2}^{\frac12}\|\partial_2\partial_j\partial_k^{3-\alpha}u\|_{L^2}^{\frac12}\|\partial_k^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_k^3 u\|_{L^2}^{\frac12} \\
&\le C\|u\|_{H^3}\|\nabla_h u\|_{H^2}\|\nabla_h u\|_{H^3} \\
&\le \frac{c_1}{128}\|\nabla_h u\|_{H^3}^2 + C\|\nabla_h u\|_{H^2}^2\|u\|_{H^3}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
N_{12} &= \sum_{j=1}^{2}\sum_{\alpha=1}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_j \partial_j \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&= -3\sum_{j=1}^{2}\int \partial_3 u_j \partial_j \partial_3^2 u \cdot \partial_3^3 u \mathrm{d}x - \sum_{j=1}^{2}\sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_j \partial_j \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&\le C\sum_{j=1}^{2}\|\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_j\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_j\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12} \\
&\quad + C\sum_{j=1}^{2}\sum_{\alpha=2}^{3}\|\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_j\partial_3^{3-\alpha}u\|_{L^2}^{\frac12}\|\partial_j\partial_3^{4-\alpha}u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12} \\
&\le \frac{c_1}{128}\|\nabla_h u\|_{H^3}^2 + C\|\nabla_h u\|_{H^2}^2\|u\|_{H^3}^2.
\end{aligned}
$$

Using the divergence free condition $\nabla \cdot u = 0$, together with Lemma 2.2 and the Young inequality, we have

$$
\begin{aligned}
N_{13} &= -\sum_{\alpha=1}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_3 \partial_3 \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&= 3\int \partial_h u_h \partial_3^3 u \cdot \partial_3^3 u \mathrm{d}x - \sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_3 \partial_3 \partial_3^{3-\alpha} u \cdot \partial_3^3 u \mathrm{d}x \\
&\le C\|\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12} \\
&\quad + C\sum_{\alpha=2}^{3}\|\partial_3^{\alpha-1}\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3\partial_3^{\alpha-1}\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3^{4-\alpha}u\|_{L^2}^{\frac12}\|\partial_2\partial_3^{4-\alpha}u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12} \\
&\le \frac{c_1}{128}\|\nabla_h u\|_{H^3}^2 + C\|\nabla_h u\|_{H^2}^2\|u\|_{H^3}^2.
\end{aligned}
$$

Then we have

$$
N_1 \le \frac{c_1}{32}\|\nabla_h u\|_{H^3}^2 + C\|\nabla_h u\|_{H^2}^2\|u\|_{H^3}^2.
$$

Due to the divergence free condition $\nabla \cdot b = 0$, we have

$$\sum_{k=1}^{3} \int b \cdot \nabla \partial_k^3 b \cdot \partial_k^3 u \mathrm{d}x + \sum_{k=1}^{3} \int b \cdot \nabla \partial_k^3 u \cdot \partial_k^3 b \mathrm{d}x = 0.$$

Then applying the same strategy as above, we get

$$\begin{aligned} N_2 + N_5 &= \sum_{k=1}^{3} \int \partial_k^3 (b \cdot \nabla b) \cdot \partial_k^3 u \mathrm{d}x + \sum_{k=1}^{3} \int \partial_k^3 (b \cdot \nabla u) \cdot \partial_k^3 b \mathrm{d}x \\ &= \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{\alpha=1}^{3} \binom{3}{\alpha} \left(\int \partial_k^\alpha b_j \partial_j \partial_k^{3-\alpha} b \cdot \partial_k^3 u \mathrm{d}x + \int \partial_k^\alpha b_j \partial_j \partial_k^{3-\alpha} u \cdot \partial_k^3 b \mathrm{d}x \right) \\ &\le \frac{c_1}{16} (\|\nabla_h u\|_{H^3}^2 + \|\nabla_h b\|_{H^3}^2) + C(\|\nabla_h u\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2)(\|u\|_{H^3}^2 + \|b\|_{H^3}^2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} N_4 &= -\sum_{k=1}^{3} \int \partial_k^3 (u \cdot \nabla b) \cdot \partial_k^3 b \mathrm{d}x \\ &= -\sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{\alpha=1}^{3} \binom{3}{\alpha} \int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} b \cdot \partial_k^3 b \mathrm{d}x \\ &\le \frac{c_1}{16} (\|\nabla_h u\|_{H^3}^2 + \|\nabla_h b\|_{H^3}^2) + C(\|\nabla_h u\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2)(\|u\|_{H^3}^2 + \|b\|_{H^3}^2). \end{aligned}$$

By the Young inequality and Hölder's inequality, we get

$$\begin{aligned} N_3 + N_7 &= 2\chi \sum_{k=1}^{3} \int \nabla \times \partial_k^3 \omega \cdot \partial_k^3 u \mathrm{d}x + 2\chi \sum_{k=1}^{3} \int \nabla \times \partial_k^3 u \cdot \partial_k^3 \omega \mathrm{d}x \\ &= 4\chi \sum_{k=1}^{3} \int \nabla \times \partial_k^3 u \cdot \partial_k^3 \omega \mathrm{d}x \\ &\le \frac{\mu}{2} \sum_{k=1}^{3} \|\nabla_h \partial_k^3 u\|_{L^2}^2 + \frac{16\chi^2}{\mu} \sum_{k=1}^{3} \|\partial_k^3 \omega\|_{L^2}^2 + \chi \sum_{k=1}^{3} \|\partial_k^3 u\|_{L^2}^2 + 4\chi \sum_{k=1}^{3} \|\partial_3 \partial_k^3 \omega\|_{L^2}^2. \end{aligned}$$

Next, we consider N_6,

$$\begin{aligned} N_6 &= -\sum_{k=1}^{3} \int \partial_k^3 (u \cdot \nabla \omega) \cdot \partial_k^3 \omega \mathrm{d}x \\ &= -\sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{\alpha=1}^{3} \binom{3}{\alpha} \int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} \omega \cdot \partial_k^3 \omega \mathrm{d}x \\ &= -\sum_{k=1}^{2} \sum_{j=1}^{3} \sum_{\alpha=1}^{3} \binom{3}{\alpha} \int \partial_k^\alpha u_j \partial_j \partial_k^{3-\alpha} \omega \cdot \partial_k^3 \omega \mathrm{d}x - \sum_{j=1}^{3} \sum_{\alpha=1}^{3} \binom{3}{\alpha} \int \partial_3^\alpha u_j \partial_j \partial_3^{3-\alpha} \omega \cdot \partial_3^3 \omega \mathrm{d}x \\ &:= N_{61} + N_{62}. \end{aligned}$$

We can rewrite N_{61} as:

$$\begin{aligned} N_{61} &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=1}^{3}\binom{3}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{3-\alpha}\omega\cdot\partial_k^3\omega \mathrm{d}x \\ &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\binom{3}{1}\int \partial_k u_j\partial_j\partial_k^{2}\omega\cdot\partial_k^3\omega \mathrm{d}x - \sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{3-\alpha}\omega\cdot\partial_k^3\omega \mathrm{d}x \\ &= N_{611} + N_{612}. \end{aligned}$$

For N_{611}, by the Young inequality and Hölder inequality, we have

$$\begin{aligned} N_{611} &= -3\sum_{k=1}^{2}\sum_{j=1}^{3}\int \partial_k u_j\partial_j\partial_k^{2}\omega\cdot\partial_k^3\omega \mathrm{d}x \\ &\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^\infty}\|\partial_j\partial_k^2\omega\|_{L^2}\|\partial_k^3\omega\|_{L^2} \\ &\le C\|\nabla_h u\|_{H^2}\|\omega\|_{H^3}^2 \\ &\le \frac{c_1}{64}\|\omega\|_{H^3}^2 + C(\|\nabla_h u\|_{H^2}^2\|\omega\|_{H^3}^2), \end{aligned}$$

where we also used the fact that $\|f\|_{L^\infty(\mathbb{R}^3)} \le C\|f\|_{H^2(\mathbb{R}^3)}$. By Lemma 2.2 and the Young inequality, we have

$$\begin{aligned} N_{612} &= -\sum_{k=1}^{2}\sum_{j=1}^{3}\sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_k^\alpha u_j\partial_j\partial_k^{3-\alpha}\omega\cdot\partial_k^3\omega \mathrm{d}x \\ &\le C\sum_{k=1}^{2}\sum_{\alpha=2}^{3}\|\partial_k^\alpha u\|_{L^2}^{\frac12}\|\partial_1\partial_k^\alpha u\|_{L^2}^{\frac12}\|\nabla\partial_k^{3-\alpha}\omega\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_k^{3-\alpha}\omega\|_{L^2}^{\frac12}\|\partial_k^3\omega\|_{L^2}^{\frac12}\|\partial_3\partial_k^3\omega\|_{L^2}^{\frac12} \\ &\le C\|\nabla_h u\|_{H^2}^{\frac12}\|\nabla_h u\|_{H^3}^{\frac12}\|\omega\|_{H^2}^{\frac12}\|\omega\|_{H^3}^{\frac12}\|\omega\|_{H^3}^{\frac12}\|\partial_3\omega\|_{H^3}^{\frac12} \\ &\le C(\|\nabla_h u\|_{H^2}^2 + \|\omega\|_{H^2}^2)\|\omega\|_{H^3}^2 + \frac{c_1}{64}(\|\partial_3\omega\|_{H^3}^2 + \|\nabla_h u\|_{H^3}^2). \end{aligned}$$

Then, we have

$$N_{61} \le C(\|\nabla_h u\|_{H^2}^2 + \|\omega\|_{H^2}^2)\|\omega\|_{H^3}^2 + \frac{c_1}{32}(\|\omega\|_{H^3}^2 + \|\partial_3\omega\|_{H^3}^2 + \|\nabla_h u\|_{H^3}^2).$$

Similarly,

$$\begin{aligned} N_{62} &= -\sum_{j=1}^{3}\sum_{\alpha=1}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{3-\alpha}\omega\cdot\partial_3^3\omega \mathrm{d}x \\ &= -\sum_{j=1}^{3}\sum_{\alpha=2}^{3}\binom{3}{\alpha}\int \partial_3^\alpha u_j\partial_j\partial_3^{3-\alpha}\omega\cdot\partial_3^3\omega \mathrm{d}x - 3\sum_{j=1}^{3}\int \partial_3 u_j\partial_j\partial_3^{2}\omega\cdot\partial_3^3\omega \mathrm{d}x \\ &\le C\sum_{\alpha=2}^{3}\|\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3^\alpha u_j\|_{L^2}^{\frac12}\|\nabla\partial_3^{3-\alpha}\omega\|_{L^2}^{\frac12}\|\partial_3\nabla\partial_3^{3-\alpha}\omega\|_{L^2}^{\frac12}\|\partial_3^3\omega\|_{L^2}^{\frac12}\|\partial_2\partial_3^3\omega\|_{L^2}^{\frac12} \\ &\quad + C\|u\|_{H^3}\|\partial_3\omega\|_{H^2}\|\partial_3\omega\|_{H^3} \\ &\le C(\|u\|_{H^3}^2 + \|\omega\|_{H^3}^2)(\|\omega\|_{H^2}^2 + \|\partial_3\omega\|_{H^2}^2) + \frac{c_1}{32}(\|\partial_3\omega\|_{H^3}^2 + \|\nabla_h u\|_{H^3}^2). \end{aligned}$$

Combining the estimates N_{61} and N_{62} yields

$$\begin{aligned}N_6 \leq &\frac{c_1}{16}(\|\omega\|_{H^3}^2 + \|\partial_3\omega\|_{H^3}^2 + \|\nabla_h u\|_{H^3}^2)\\&+ C(\|\omega\|_{H^3}^2 + \|u\|_{H^3}^2)(\|\nabla_h u\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\partial_3\omega\|_{H^2}^2).\end{aligned}$$

Inserting the above bounds into (2.13), and combining with (2.1), one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \|\omega(t)\|_{H^3}^2)\\&\quad + c_1\left(\|\nabla_h u\|_{H^3}^2 + \|\nabla_h b\|_{H^3}^2 + \|\partial_3\omega\|_{H^3}^2 + \|\omega\|_{H^3}^2\right)\\&\leq C\left(\|u\|_{H^3}^2 + \|b\|_{H^3}^2 + \|\omega\|_{H^3}^2\right)\left(\|\partial_3\omega\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\nabla_h u\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2\right).\end{aligned} \tag{2.14}$$

This together with Gronwall's inequality and (1.6) implies

$$\begin{aligned}&\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \|\omega(t)\|_{H^3}^2\\&\quad + c_1\int_0^t\left(\|\nabla_h u(\tau)\|_{H^3}^2 + \|\nabla_h b(\tau)\|_{H^3}^2 + \|\partial_3\omega(\tau)\|_{H^3}^2 + \|\omega(\tau)\|_{H^3}^2\right)d\tau\\&\leq (\|u_0\|_{H^3}^2 + \|b_0\|_{H^3}^2 + \|\omega_0\|_{H^3}^2)e^{C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2)}\\&\leq C(\|u_0\|_{H^3}^2 + \|b_0\|_{H^3}^2 + \|\omega_0\|_{H^3}^2).\end{aligned} \tag{2.15}$$

We set $C_3 = C$, which is (2.12).

□

To complete the proof of (1.7), we use the induction for s. The case s = 3 has been proved in Proposition 2.5. Assume that for $s \geq 3$,

$$\begin{aligned}&\|u(t)\|_{H^{s-1}}^2 + \|b(t)\|_{H^{s-1}}^2 + \|\omega(t)\|_{H^{s-1}}^2\\&\quad + c_1\int_0^t\left(\|\nabla_h u(\tau)\|_{H^{s-1}}^2 + \|\nabla_h b(\tau)\|_{H^{s-1}}^2 + \|\partial_3\omega(\tau)\|_{H^{s-1}}^2 + \|\omega(\tau)\|_{H^{s-1}}^2\right)d\tau\\&\leq C_{s-1}(\|u_0\|_{H^{s-1}}^2 + \|b_0\|_{H^{s-1}}^2 + \|\omega_0\|_{H^s-1}^2).\end{aligned} \tag{2.16}$$

If we have

$$\begin{aligned}&\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \|\omega(t)\|_{H^s}^2\\&\quad + c_1\int_0^t\left(\|\nabla_h u(\tau)\|_{H^s}^2 + \|\nabla_h b(\tau)\|_{H^s}^2 + \|\partial_3\omega(\tau)\|_{H^s}^2 + \|\omega(\tau)\|_{H^s}^2\right)d\tau\\&\leq C_s(\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 + \|\omega_0\|_{H^s}^2),\end{aligned} \tag{2.17}$$

then we yield the desired estimate (1.7). Next, we need to verify that (2.17) is correct. Applying ∂_k^s with $k = 1, 2, 3$ to $(1.3)_1$, $(1.3)_2$ and $(1.3)_3$, taking the L^2-inner product with $\partial_k^s u$, $\partial_k^s b$ and $\partial_k^s\omega$, respectively, and adding the results up, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{3}\left(\|\partial_k^s u(t)\|_{L^2}^2+\|\partial_k^s b(t)\|_{L^2}^2+\|\partial_k^s \omega(t)\|_{L^2}^2\right)+\mu\sum_{k=1}^{3}\|\nabla_h\partial_k^s u\|_{L^2}^2+\chi\sum_{k=1}^{3}\|\partial_k^s u\|_{L^2}^2\\
&\quad+\nu\sum_{k=1}^{3}\|\nabla_h\partial_k^s b\|_{L^2}^2+\gamma\sum_{k=1}^{3}\|\partial_3\partial_k^s \omega\|_{L^2}^2+4\chi\sum_{k=1}^{3}\|\partial_k^s \omega\|_{L^2}^2+\kappa\sum_{k=1}^{3}\|\partial_k^s\nabla\cdot\omega\|_{L^2}^2\\
&=-\sum_{k=1}^{3}\int\partial_k^s(u\cdot\nabla u)\cdot\partial_k^s u\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k^s(b\cdot\nabla b)\cdot\partial_k^s u\mathrm{d}x+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k^s\omega\cdot\partial_k^s u\mathrm{d}x\\
&\quad-\sum_{k=1}^{3}\int\partial_k^s(u\cdot\nabla b)\cdot\partial_k^s b\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k^s(b\cdot\nabla u)\cdot\partial_k^s b\mathrm{d}x-\sum_{k=1}^{3}\int\partial_k^s(u\cdot\nabla\omega)\cdot\partial_k^s \omega\mathrm{d}x\\
&\quad+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k^s u\cdot\partial_k^s\omega\mathrm{d}x\\
&:=J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned}\tag{2.18}
$$

Since the estimate method is very similar to N_1 - N_7 in the proof of Proposition 2.5, we omit the details for simplicity and have

$$
\begin{aligned}
&J_1+J_4\le\frac{c_1}{16}(\|\nabla_h u\|_{H^s}^2+\|\nabla_h b\|_{H^s}^2)+C(\|\nabla_h u\|_{H^{s-1}}^2+\|\nabla_h b\|_{H^{s-1}}^2)(\|u\|_{H^s}^2+\|b\|_{H^s}^2),\\
&J_2+J_5\le\frac{c_1}{16}(\|\nabla_h u\|_{H^s}^2+\|\nabla_h b\|_{H^s}^2)+C(\|\nabla_h u\|_{H^{s-1}}^2+\|\nabla_h b\|_{H^{s-1}}^2)(\|u\|_{H^s}^2+\|b\|_{H^s}^2),\\
&J_3+J_7\le\frac{\mu}{2}\sum_{k=1}^{3}\|\nabla_h\partial_k^s u\|_{L^2}^2+\frac{16\chi^2}{\mu}\sum_{k=1}^{3}\|\partial_k^s\omega\|_{L^2}^2+\chi\sum_{k=1}^{3}\|\partial_k^s u\|_{L^2}^2+4\chi\sum_{k=1}^{3}\|\partial_3\partial_k^s\omega\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_6\le&\frac{c_1}{16}(\|\nabla_h u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|\partial_3\omega\|_{H^s}^2)\\
&+C(\|\nabla_h u\|_{H^{s-1}}^2+\|\partial_3\omega\|_{H^{s-1}}^2+\|\omega\|_{H^{s-1}}^2)(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2).
\end{aligned}
$$

Inserting the above bounds into (2.18), and combining with (2.1), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2+\|\omega(t)\|_{H^s}^2)\\
&\quad+c_1\left(\|\nabla_h u\|_{H^s}^2+\|\nabla_h b\|_{H^s}^2+\|\partial_3\omega\|_{H^s}^2+\|\omega\|_{H^s}^2\right)\\
&\le C\left(\|u\|_{H^s}^2+\|b\|_{H^s}^2+\|\omega\|_{H^s}^2\right)\left(\|\partial_3\omega\|_{H^{s-1}}^2+\|\omega\|_{H^{s-1}}^2+\|\nabla_h u\|_{H^{s-1}}^2+\|\nabla_h b\|_{H^{s-1}}^2\right).
\end{aligned}\tag{2.19}
$$

This together with Gronwall's inequality and (2.16) implies

$$
\begin{aligned}
&\|u(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2+\|\omega(t)\|_{H^s}^2\\
&\qquad+c_1\int_0^t\left(\|\nabla_h u(\tau)\|_{H^s}^2+\|\nabla_h b(\tau)\|_{H^s}^2+\|\partial_3\omega(\tau)\|_{H^s}^2+\|\omega(\tau)\|_{H^s}^2\right)d\tau\\
&\quad\le(\|u_0\|_{H^s}^2+\|b_0\|_{H^s}^2+\|\omega_0\|_{H^s}^2)\exp\{CC_{s-1}(\|u_0\|_{H^{s-1}}^2+\|b_0\|_{H^{s-1}}^2+\|\omega_0\|_{H^{s-1}}^2)\}\\
&\quad\le C_s(\|u_0\|_{H^s}^2+\|b_0\|_{H^s}^2+\|\omega_0\|_{H^s}^2),
\end{aligned}
$$

where C_s is dependent on μ, χ, ν, κ and the initial data, which is (2.17).

We now briefly explain the uniqueness at the H^2-level, which can be quickly established. Let $(u^{(1)}, b^{(1)}, \omega^{(1)})$ and $(u^{(2)}, b^{(2)}, \omega^{(2)})$ be two solutions of (1.3) in the regularity class, for $T > 0$,

$$(u^{(1)}, b^{(1)}, \omega^{(1)}), \quad (u^{(2)}, b^{(2)}, \omega^{(2)}) \in L^\infty(0, T; H^2(\mathbb{R}^3)). \tag{2.20}$$

Their difference $(\tilde{u}, \tilde{b}, \tilde{\omega})$ with

$$\tilde{u} = u^{(1)} - u^{(2)}, \quad \tilde{b} = b^{(1)} - b^{(2)}, \quad \tilde{\omega} = \omega^{(1)} - \omega^{(2)}$$

satisfies

$$\begin{cases} \partial_t \tilde{u} + u^{(1)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(2)} + \chi \tilde{u} = -\nabla \tilde{\pi} + \mu \Delta_h \tilde{u} + b^{(1)} \cdot \nabla \tilde{b} + \tilde{b} \cdot \nabla b^{(2)} + 2\chi \nabla \times \tilde{\omega}, \\ \partial_t \tilde{b} + u^{(1)} \cdot \nabla \tilde{b} + \tilde{u} \cdot \nabla b^{(2)} = \nu \Delta_h \tilde{b} + b^{(1)} \cdot \nabla \tilde{u} + \tilde{b} \cdot \nabla u^{(2)}, \\ \partial_t \tilde{\omega} + u^{(1)} \cdot \nabla \tilde{\omega} + \tilde{u} \cdot \nabla \omega^{(2)} + 4\chi \tilde{\omega} = \gamma \partial_{33} \tilde{\omega} + \kappa \nabla \nabla \cdot \tilde{\omega} + 2\chi \nabla \times \tilde{u}, \\ \nabla \cdot \tilde{u} = 0, \ \nabla \cdot \tilde{b} = 0, \\ \tilde{u}(x, 0) = 0, \tilde{\omega}(x, 0) = 0, \tilde{b}(x, 0) = 0, \end{cases}$$

where $\tilde{\pi} = \pi^{(1)} - \pi^{(2)}$ with $\pi^{(1)}$ and $\pi^{(2)}$ being the pressure corresponding to $u^{(1)}$ and $u^{(2)}$, respectively. Taking the L^2-inner product to the above system with $\tilde{u}$, $\tilde{b}$ and $\tilde{\omega}$, respectively, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\|\tilde{u}(t)\|_{L^2}^2 + \|\tilde{b}(t)\|_{L^2}^2 + \|\tilde{\omega}(t)\|_{L^2}^2) + \mu \|\nabla_h \tilde{u}\|_{L^2}^2 + \nu \|\nabla_h \tilde{b}\|_{L^2}^2 + \gamma \|\partial_3 \tilde{\omega}\|_{L^2}^2 \\ &\qquad + \chi \|\tilde{u}\|_{L^2}^2 + \kappa \|\nabla \cdot \tilde{\omega}\|_{L^2}^2 + 4\chi \|\tilde{\omega}\|_{L^2}^2 \\ &\quad = \int \tilde{b} \cdot \nabla b^{(2)} \cdot \tilde{u} \mathrm{d}x - \int \tilde{u} \cdot \nabla u^{(2)} \cdot \tilde{u} \mathrm{d}x - \int \tilde{u} \cdot \nabla b^{(2)} \cdot \tilde{b} \mathrm{d}x + \int \tilde{b} \cdot \nabla u^{(2)} \cdot \tilde{b} \mathrm{d}x \\ &\qquad - \int \tilde{u} \cdot \nabla \omega^{(2)} \cdot \tilde{\omega} \mathrm{d}x + 2\chi \int \nabla \times \tilde{\omega} \cdot \tilde{u} \mathrm{d}x + 2\chi \int \nabla \times \tilde{u} \cdot \tilde{\omega} \mathrm{d}x \\ &\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{2.21}$$

Applying Hölder's inequality, Lemma 2.1, Lemma 2.2, Gagliardo–Nirenberg inequality and the Young inequality yields

$$\begin{aligned} I_1 &= \int \tilde{b} \cdot \nabla b^{(2)} \cdot \tilde{u} \mathrm{d}x \\ &\le \|\tilde{b}\|_{L_h^4 L_{x_3}^2} \|\nabla b^{(2)}\|_{L_h^2 L_{x_3}^\infty} \|\tilde{u}\|_{L_h^4 L_{x_3}^2} \\ &\le \Big\| \|\tilde{b}\|_{L_h^4} \Big\|_{L_{x_3}^2} \Big\| \|\nabla b^{(2)}\|_{L_{x_3}^\infty} \Big\|_{L_h^2} \Big\| \|\tilde{u}\|_{L_h^4} \Big\|_{L_{x_3}^2} \\ &\le \Big\| \|\tilde{b}\|_{L_h^2}^{\frac{1}{2}} \|\nabla_h \tilde{b}\|_{L_h^2}^{\frac{1}{2}} \Big\|_{L_{x_3}^2} \Big\| \|\nabla b^{(2)}\|_{L_{x_3}^2}^{\frac{1}{2}} \|\partial_3 \nabla b^{(2)}\|_{L_{x_3}^2}^{\frac{1}{2}} \Big\|_{L_h^2} \Big\| \|\tilde{u}\|_{L_h^2}^{\frac{1}{2}} \|\nabla_h \tilde{u}\|_{L_h^2}^{\frac{1}{2}} \Big\|_{L_{x_3}^2} \\ &\le C \|\tilde{b}\|_{L^2}^{\frac{1}{2}} \|\nabla_h \tilde{b}\|_{L^2}^{\frac{1}{2}} \|b^{(2)}\|_{H^2} \|\tilde{u}\|_{L^2}^{\frac{1}{2}} \|\nabla_h \tilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{c_1}{16} (\|\nabla_h \tilde{u}\|_{L^2}^2 + \|\nabla_h \tilde{b}\|_{L^2}^2) + C \|b^{(2)}\|_{H^2}^2 (\|\tilde{b}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= -\int \tilde{u} \cdot \nabla u^{(2)} \cdot \tilde{u} \mathrm{d}x \\ &\le \|\tilde{u}\|_{L_h^4 L_{x_3}^2}^2 \|\nabla u^{(2)}\|_{L_h^2 L_{x_3}^\infty} \\ &\le C \|\tilde{u}\|_{L^2} \|\nabla_h \tilde{u}\|_{L^2} \|\nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u^{(2)}\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{c_1}{16} \|\nabla_h \tilde{u}\|_{L^2}^2 + C \|u^{(2)}\|_{H^2}^2 \|\tilde{u}\|_{L^2}^2, \end{aligned}$$

$$
\begin{aligned}
I_3 &= -\int \tilde{u}\cdot\nabla b^{(2)}\cdot\tilde{b}\mathrm{d}x\\
&\le \|\tilde{u}\|_{L_h^4L_{x_3}^2}\|\nabla b^{(2)}\|_{L_h^2L_{x_3}^\infty}\|\tilde{b}\|_{L_h^4L_{x_3}^2}\\
&\le C\|\tilde{u}\|_{L^2}^{\frac12}\|\nabla_h\tilde{u}\|_{L^2}^{\frac12}\|b^{(2)}\|_{H^2}\|\tilde{b}\|_{L^2}^{\frac12}\|\nabla_h\tilde{b}\|_{L^2}^{\frac12}\\
&\le \frac{c_1}{16}(\|\nabla_h\tilde{u}\|_{L^2}^2+\|\nabla_h\tilde{b}\|_{L^2}^2)+C\|b^{(2)}\|_{H^2}^2(\|\tilde{b}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2),\\
I_4 &= \int \tilde{b}\cdot\nabla u^{(2)}\cdot\tilde{b}\mathrm{d}x\\
&\le \|\tilde{b}\|_{L_h^4L_{x_3}^2}\|\nabla u^{(2)}\|_{L_h^2L_{x_3}^\infty}\|\tilde{b}\|_{L_h^4L_{x_3}^2}\\
&\le C\|\tilde{b}\|_{L^2}^{\frac12}\|\nabla_h\tilde{b}\|_{L^2}^{\frac12}\|u^{(2)}\|_{H^2}\|\tilde{b}\|_{L^2}^{\frac12}\|\nabla_h\tilde{b}\|_{L^2}^{\frac12}\\
&\le \frac{c_1}{16}\|\nabla_h\tilde{b}\|_{L^2}^2+C\|u^{(2)}\|_{H^2}^2\|\tilde{b}\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_5 &= -\int \tilde{u}\cdot\nabla\omega^{(2)}\cdot\tilde{\omega}\mathrm{d}x\\
&\le \|\tilde{u}\|_{L_h^4L_{x_3}^2}\|\nabla\omega^{(2)}\|_{L_h^4L_{x_3}^2}\|\tilde{\omega}\|_{L_h^2L_{x_3}^\infty}\\
&\le C\|\tilde{u}\|_{L^2}^{\frac12}\|\nabla_h\tilde{u}\|_{L^2}^{\frac12}\|\nabla\omega^{(2)}\|_{L^2}^{\frac12}\|\nabla\nabla_h\omega^{(2)}\|_{L^2}^{\frac12}\|\tilde{\omega}\|_{L^2}^{\frac12}\|\partial_3\tilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\|\tilde{u}\|_{L^2}^{\frac12}\|\omega^{(2)}\|_{H^2}(\|\nabla_h\tilde{u}\|_{L^2}^{\frac32}+\|\tilde{\omega}\|_{L^2}^{\frac32}+\|\partial_3\tilde{\omega}\|_{L^2}^{\frac32})\\
&\le \frac{c_1}{16}(\|\nabla_h\tilde{u}\|_{L^2}^2+\|\partial_3\tilde{\omega}\|_{L^2}^2+\|\tilde{\omega}\|_{L^2}^2)+C\|\tilde{u}\|_{L^2}^2\|\omega^{(2)}\|_{H^2}^4.
\end{aligned}
$$

Finally, we consider I_6 and I_7, by Hölder's inequality and the Young inequality, and yield

$$
\begin{aligned}
I_6+I_7 &= 2\chi\int\nabla\times\tilde{\omega}\cdot\tilde{u}\mathrm{d}x+2\chi\int\nabla\times\tilde{u}\cdot\tilde{\omega}\mathrm{d}x\\
&= 4\chi\int\nabla\times\tilde{u}\cdot\tilde{\omega}\mathrm{d}x\\
&\le \frac{\mu}{2}\|\nabla_h\tilde{u}\|_{L^2}^2+\frac{16\chi^2}{\mu}\|\tilde{\omega}\|_{L^2}^2+\chi\|\tilde{u}\|_{L^2}^2+4\chi\|\partial_3\tilde{\omega}\|_{L^2}^2.
\end{aligned}
$$

Substituting the above estimates into (2.21), it infers that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}(t)\|_{L^2}^2+\|\tilde{b}(t)\|_{L^2}^2+\|\tilde{\omega}(t)\|_{L^2}^2)+c_1(\|\nabla_h\tilde{u}\|_{L^2}^2+\|\nabla_h\tilde{b}\|_{L^2}^2+\|\partial_3\tilde{\omega}\|_{L^2}^2+\|\tilde{\omega}\|_{L^2}^2)\\
&\quad\le(\|u^{(2)}\|_{H^2}^2+\|b^{(2)}\|_{H^2}^2+\|\omega^{(2)}\|_{H^2}^4)(\|\tilde{u}\|_{L^2}^2+\|\tilde{b}\|_{L^2}^2+\|\tilde{\omega}\|_{L^2}^2),
\end{aligned}
\tag{2.22}
$$

where $c_1=\min\{\frac{\mu}{2},\nu,\gamma-4\chi,4\chi-\frac{16\chi^2}{\mu}\}$. Since $(u^{(2)},b^{(2)},\omega^{(2)})$ is in the regularity class (2.20), we obtain for any $T>0$,

$$
\int_0^T(\|u^{(2)}(t)\|_{H^2}^2+\|b^{(2)}(t)\|_{H^2}^2+\|\omega^{(2)}(t)\|_{H^2}^4)\mathrm{d}t\le C(T)<+\infty. \tag{2.23}
$$

Then, (2.22) together with Gronwall's inequality and (2.23) leads to for any $T>0$,

$$
\|\tilde{u}(t)\|_{L^2}^2+\|\tilde{b}(t)\|_{L^2}^2+\|\tilde{\omega}(t)\|_{L^2}^2\le 0,\quad \forall t\in[0,T],
$$

which implies the uniqueness. Thus, we complete the proof of Theorem 1.1.

□

3. The proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Since the proof is slightly long, we divide it into four propositions for clarity. The strategy is as follows: As preparations, we first establish the H^1 estimates for (u, ω, b) in Proposition 3.1. Secondly, we will show the following preliminary estimate $\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$, $\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \le C(1+t)^{-\frac{5}{6}}$, $\|b(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}}$ and $\|\nabla b(t)\|_{L^2} \le C(1+t)^{-\frac{5}{4}}$ in Propositions 3.2 and 3.5. We give the improve decay estimates $\|\nabla^2 b(t)\|_{L^2} \le C(1+t)^{-\frac{65}{48}+\frac{17}{12}\alpha}$ and $\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \le C(1+t)^{-\frac{157}{64}+\frac{17}{16}\alpha}$ in Proposition 3.7. Finally, we obtain the decay estimate $\|\nabla u(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} \le C(1+t)^{-\frac{471}{512}+\frac{51}{128}\alpha}$ in Proposition 3.8, and thus, the proof of Theorem 1.4 is completed.

Proposition 3.1. *Let the assumptions stated in Theorem 1.4 and (1.13) hold. Then for all $t > 0$, (u, ω, b) obey the following uniform bounds*

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\\&\quad + 2c_2\int_0^t \left(\|\nabla_h u(\tau)\|_{L^2}^2 + \|u(t)\|_{L^2} + \|\nabla b(\tau)\|_{L^2}^2 + \|\partial_3\omega(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2\right) d\tau\\&\le \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2,\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2\\&\quad + c_2\int_0^t (\|\nabla\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^2}^2 + \|\nabla^2 b(\tau)\|_{L^2}^2 + \|\nabla\partial_3\omega(\tau)\|_{L^2}^2 + \|\nabla\omega(\tau)\|_{L^2}^2)d\tau\\&\le \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 + \|\nabla\omega_0\|_{L^2}^2\right) e^{C(\|u_0\|_{H^2}^2+\|b_0\|_{H^2}^2+\|\omega_0\|_{H^2}^2)},\end{aligned}\tag{3.2}$$

where $c_2 = \min\{\frac{\mu}{2}, \frac{\chi}{4}, \nu, \gamma - \frac{16\chi}{3}, 4\chi - \frac{16\chi^2}{\mu}\}$.

Proof. Taking the L^2-inner product of (1.8) with u, ω and b, respectively, and then adding the resulting equations together, we yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + \mu\|\nabla_h u\|_{L^2}^2 + \chi\|u\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2\\&\quad + \gamma\|\partial_3\omega\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\cdot\omega\|_{L^2}^2\\&= 4\chi\int \nabla\times u\cdot\omega \mathrm{d}x.\end{aligned}\tag{3.3}$$

Applying the Young inequality and Hölder's inequality, we obtain

$$4\chi\int \nabla\times u\cdot\omega\mathrm{d}x \le \frac{\mu}{2}\|\nabla_h u\|_{L^2}^2 + \frac{16\chi^2}{\mu}\|\omega\|_{L^2}^2 + \frac{3\chi}{4}\|u\|_{L^2}^2 + \frac{16\chi}{3}\|\partial_3\omega\|_{L^2}^2.$$

Inserting this bound into (3.3) and then integrating in time, we yield the desired bound (3.1).

Now we turn to proof (3.2). Applying ∂_k with $k=1,2,3$ to $(1.8)_1$, $(1.8)_2$ and $(1.8)_3$, dotting the results by $\partial_k u$, $\partial_k b$ and $\partial_k \omega$, respectively, integrating in space domain and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{3}\left(\|\partial_k u(t)\|_{L^2}^2+\|\partial_k b(t)\|_{L^2}^2+\|\partial_k\omega(t)\|_{L^2}^2\right)+\mu\sum_{k=1}^{3}\|\nabla_h\partial_k u\|_{L^2}^2+\chi\sum_{k=1}^{3}\|\partial_k u\|_{L^2}^2\\
&\qquad+\nu\sum_{k=1}^{3}\|\nabla\partial_k b\|_{L^2}^2+\gamma\sum_{k=1}^{3}\|\partial_3\partial_k\omega\|_{L^2}^2+4\chi\sum_{k=1}^{3}\|\partial_k\omega\|_{L^2}^2+\kappa\sum_{k=1}^{3}\|\partial_k\nabla\cdot\omega\|_{L^2}^2\\
&\quad=-\sum_{k=1}^{3}\int\partial_k(u\cdot\nabla u)\cdot\partial_k u\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k(b\cdot\nabla b)\cdot\partial_k u\mathrm{d}x+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k\omega\cdot\partial_k u\mathrm{d}x\\
&\qquad-\sum_{k=1}^{3}\int\partial_k(u\cdot\nabla b)\cdot\partial_k b\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k(b\cdot\nabla u)\cdot\partial_k b\mathrm{d}x-\sum_{k=1}^{3}\int\partial_k(u\cdot\nabla\omega)\cdot\partial_k\omega\mathrm{d}x\\
&\qquad+2\chi\sum_{k=1}^{3}\int\nabla\times\partial_k u\cdot\partial_k\omega\mathrm{d}x\\
&\quad:=A_1+A_2+A_3+A_4+A_5+A_6+A_7.
\end{aligned}\tag{3.4}$$

Using the divergence free condition $\nabla\cdot u=0$, together with Lemma 2.2 and the Young inequality, one infers

$$\begin{aligned}
A_1&=-\sum_{k=1}^{3}\sum_{j=1}^{3}\int\partial_k u_j\partial_j u\cdot\partial_k u\mathrm{d}x\\
&=-\sum_{k=1}^{2}\sum_{j=1}^{3}\int\partial_k u_j\partial_j u\cdot\partial_k u\mathrm{d}x-\sum_{j=1}^{2}\int\partial_3 u_j\partial_j u\cdot\partial_3 u\mathrm{d}x-\int\partial_3 u_3\partial_3 u\cdot\partial_3 u\mathrm{d}x\\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^2}^{\frac12}\|\partial_1\partial_k u_j\|_{L^2}^{\frac12}\|\partial_j u\|_{L^2}^{\frac12}\|\partial_2\partial_j u\|_{L^2}^{\frac12}\|\partial_k u\|_{L^2}^{\frac12}\|\partial_3\partial_k u\|_{L^2}^{\frac12}\\
&\quad+\sum_{j=1}^{2}\|\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_j u\|_{L^2}^{\frac12}\|\partial_3\partial_j u\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\\
&\quad+\|\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3\partial_h u_h\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\\
&\le C\|\nabla u\|_{L^2}^2\|\nabla_h u\|_{H^2}^2+\frac{c_2}{32}\|\nabla\nabla_h u\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_6&=-\sum_{k=1}^{3}\int\partial_k(u\cdot\nabla\omega)\cdot\partial_k\omega\mathrm{d}x\\
&=-\sum_{k=1}^{2}\sum_{j=1}^{3}\int\partial_k u_j\partial_j\omega\cdot\partial_k\omega\mathrm{d}x-\sum_{j=1}^{3}\int\partial_3 u_j\partial_j\omega\cdot\partial_3\omega\mathrm{d}x\\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^2}^{\frac12}\|\partial_1\partial_k u_j\|_{L^2}^{\frac12}\|\partial_j\omega\|_{L^2}^{\frac12}\|\partial_2\partial_j\omega\|_{L^2}^{\frac12}\|\partial_k\omega\|_{L^2}^{\frac12}\|\partial_3\partial_k\omega\|_{L^2}^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{3}\|\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_1\partial_3 u_j\|_{L^2}^{\frac12}\|\partial_j\omega\|_{L^2}^{\frac12}\|\partial_3\partial_j\omega\|_{L^2}^{\frac12}\|\partial_3\omega\|_{L^2}^{\frac12}\|\partial_2\partial_3\omega\|_{L^2}^{\frac12}\\
&\le C(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)(\|\nabla_h u\|_{H^2}^2+\|\partial_3\omega\|_{H^2}^2+\|\omega\|_{H^2}^2)+\frac{c_2}{32}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla\partial_3\omega\|_{L^2}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
A_4&=-\sum_{k=1}^{3}\int\partial_k(u\cdot\nabla b)\cdot\partial_k b\mathrm{d}x\\
&\le C(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\|\nabla b\|_{H^2}^2+\frac{c_2}{32}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2).
\end{aligned}
$$

Due to

$$
\sum_{k=1}^{3}\int b\cdot\nabla\partial_k b\cdot\partial_k u\mathrm{d}x+\sum_{k=1}^{3}\int b\cdot\nabla\partial_k u\cdot\partial_k b\mathrm{d}x=0,
$$

then, we have

$$
\begin{aligned}
A_2+A_5&=\sum_{k=1}^{3}\int\partial_k b\cdot\nabla b\cdot\partial_k u\mathrm{d}x+\sum_{k=1}^{3}\int\partial_k b\cdot\nabla u\cdot\partial_k b\mathrm{d}x\\
&\le C(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\|\nabla b\|_{H^2}^2+\frac{c_2}{32}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2).
\end{aligned}
$$

By the Young inequality, we have

$$
A_3+A_7\le\frac{\mu}{2}\sum_{k=1}^{3}\|\nabla_h\partial_k u\|_{L^2}^2+\frac{16\chi^2}{\mu}\sum_{k=1}^{3}\|\partial_k\omega\|_{L^2}^2+\frac{3\chi}{4}\sum_{k=1}^{3}\|\partial_k u\|_{L^2}^2+\frac{16\chi}{3}\sum_{k=1}^{3}\|\partial_3\partial_k\omega\|_{L^2}^2.
$$

Inserting the above estimates into (3.4), we infer that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\nabla\omega(t)\|_{L^2}^2\right)+\frac{\mu}{2}\|\nabla_h\nabla u\|_{L^2}^2+\frac{\chi}{4}\|\nabla u\|_{L^2}^2\\
&\quad+\nu\|\nabla^2 b\|_{L^2}^2+(\gamma-\frac{16\chi}{3})\|\partial_3\nabla\omega\|_{L^2}^2+\left(4\chi-\frac{16\chi^2}{\mu}\right)\|\nabla\omega\|_{L^2}^2+\kappa\|\nabla\nabla\cdot\omega\|_{L^2}^2\\
&\le\frac{c_2}{2}(\|\nabla\nabla_h u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2+\|\nabla\partial_3\omega\|_{L^2}^2)\\
&\quad+C(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\nabla\omega(t)\|_{L^2}^2)(\|\nabla_h u\|_{H^2}^2+\|\nabla b\|_{H^2}^2+\|\partial_3\omega\|_{H^2}^2+\|\omega\|_{H^2}^2),
\end{aligned}
$$

where $c_2=\min\{\frac{\mu}{2},\frac{\chi}{4},\nu,\gamma-\frac{16\chi}{3},4\chi-\frac{16\chi^2}{\mu}\}$. Then applying Gronwall's inequality, we obtain

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\nabla\omega(t)\|_{L^2}^2+c_2\int_0^t\left(\|\nabla_h\nabla u(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^2}^2\right.\\
&\quad\left.+\|\nabla^2 b(\tau)\|_{L^2}^2+\|\partial_3\nabla\omega(\tau)\|_{L^2}^2+\|\nabla\omega(\tau)\|_{L^2}^2\right)d\tau\\
&\le\left(\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2\right)\\
&\quad\times\exp\left\{\int_0^t(\|\nabla_h u(\tau)\|_{H^2}^2+\|\nabla b(\tau)\|_{H^2}^2+\|\nabla_3\omega(\tau)\|_{H^2}^2+\|\omega(\tau)\|_{H^2}^2)d\tau\right\}\\
&\le\left(\|\nabla u_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2\right)e^{C(\|u_0\|_{H^2}^2+\|b_0\|_{H^2}^2+\|\omega_0\|_{H^2}^2)},
\end{aligned}
\tag{3.5}
$$

where we used (1.11) in the last step. □

With Proposition 3.1 at our disposal, we now start to prove our decay estimates.

Proposition 3.2. *Let the assumptions stated in Theorem 1.4 hold. Then*

$$\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}. \tag{3.6}$$

Proof. Using (3.5) and (1.11), for $0 < s < t$, we have

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 \\ &\quad \leq \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla b(s)\|_{L^2}^2 + \|\nabla \omega(s)\|_{L^2}^2\right) e^{C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2)}.\end{aligned} \tag{3.7}$$

By (3.1) and (3.2), we have

$$\int_0^\infty \|\nabla b(\tau)\|_{L^2}^2 d\tau \leq C(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2), \tag{3.8}$$

$$\int_0^\infty \|\nabla \omega(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq C(\|u_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2 + \|b_0\|_{H^2}^2). \tag{3.9}$$

Integrating (3.7) in $(\frac{t}{2}, t)$ with respect to s, together with (3.8)–(3.9), we obtain

$$\begin{aligned}&t(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2) \\ &\quad \leq 2e^{C(\|u_0\|_{H^2}^2 + \|b_0\|_{H^2}^2 + \|\omega_0\|_{H^2}^2)} \int_{\frac{t}{2}}^{t} \left(\|\nabla u(s)\|_{L^2}^2 + \|\nabla b(s)\|_{L^2}^2 + \|\nabla \omega(s)\|_{L^2}^2\right) \mathrm{d}s \\ &\leq C.\end{aligned}$$

Therefore, for $t \geq 1$, we have

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 \leq Ct^{-1} \leq C(1+t)^{-1}. \tag{3.10}$$

For $0 < t < 1$, it follows from (3.2); we have

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 \leq C \leq C(1+t)^{-1}. \tag{3.11}$$

Then (3.10) and (3.11) yield (3.6).

□

To give the decay estimates for (u, ω, b) and to improve the decay estimate for ∇b, we recall the following estimate for the heat operator (see, e.g., [48]).

Lemma 3.3. *Let $m \geq 0$, $a > 0$ and $1 \leq p \leq q \leq +\infty$. Then for any $t > 0$,*

$$\|\nabla^m e^{a\Delta t} f\|_{L^q(\mathbb{R}^3)} \leq Ct^{-\frac{m}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^3)}, \tag{3.12}$$

where

$$e^{a\Delta t} f(x) = (4\pi a t)^{-1} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4at}} f(y) dy.$$

And the following calculus inequalities (see, e.g., [49,50]) involving fractional differential operators Λ^s with $s > 0$ and

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi), \quad \hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) \mathrm{d}x.$$

Lemma 3.4. *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $p_2, q_1 \in (1, \infty)$ and $p_1, q_2 \in [1, +\infty]$. Then*

$$\|\Lambda(fg)\|_{L^r} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}),$$

where C is constant depending on the indices s, r, p_1, q_1, p_2 and q_2.

Now, we can start to establish the desired decay estimates.

Proposition 3.5. *Let the assumptions stated in Theorem 1.4 hold. Then*

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{6}}, \tag{3.13}$$

$$\|b(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{4}}, \quad \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-\frac{5}{4}}. \tag{3.14}$$

Proof. Taking the L^2-inner product to the first and the third equations of (1.8) with u and ω, respectively, and adding the resulting equations together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + \mu\|\nabla_h u\|_{L^2}^2 + \chi\|u\|_{L^2}^2 + \gamma\|\partial_3\omega\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \kappa\|\nabla\cdot\omega\|_{L^2}^2 \\
&\quad = \int b\cdot\nabla b\cdot u\mathrm{d}x + 4\chi\int \nabla\times u\cdot\omega\mathrm{d}x \\
&\quad \leq C\|\nabla b\|_{L^2}^2\|u\|_{L^3} + \frac{\mu}{2}\|\nabla_h u\|_{L^2}^2 + \frac{16\chi^2}{\mu}\|\omega\|_{L^2}^2 + \frac{3\chi}{4}\|u\|_{L^2}^2 + \frac{16\chi}{3}\|\partial_3\omega\|_{L^2}^2,
\end{aligned} \tag{3.15}$$

where we have used Sobolev's inequality. Set $c = \min\{\frac{\chi}{2}, 8\chi - \frac{32\chi^2}{\mu}\}$.Then integrating (3.15) in time, we yield

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \\
&\quad \leq e^{-ct}(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2) + C\int_0^t e^{-c(t-s)}\|\nabla b(s)\|_{L^2}^2\|u(s)\|_{L^2}^{\frac{1}{2}}\|\nabla u(s)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s \\
&\quad \leq e^{-ct}(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2) + C(Q_1 + Q_2),
\end{aligned} \tag{3.16}$$

where

$$Q_1 = \int_0^{\frac{t}{2}} e^{-c(t-s)}\|\nabla b(s)\|_{L^2}^2\|u(s)\|_{L^2}^{\frac{1}{2}}\|\nabla u(s)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s,$$

$$Q_2 = \int_{\frac{t}{2}}^{t} e^{-c(t-s)}\|\nabla b(s)\|_{L^2}^2\|u(s)\|_{L^2}^{\frac{1}{2}}\|\nabla u(s)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s.$$

By (3.1) and (3.2), we get

$$Q_1 \leq Ce^{-\frac{ct}{2}}\int_0^{\frac{t}{2}}\|\nabla b(s)\|_{L^2}^2\mathrm{d}s \leq Ce^{-\frac{ct}{2}}. \tag{3.17}$$

Set

$$\mathcal{M}(t) = \sup_{0\leq s\leq t}\{(1+s)^{\frac{1}{2}}(\|\nabla u(s)\|_{L^2} + \|\nabla\omega(s)\|_{L^2} + \|\nabla b(s)\|_{L^2}\}.$$

Then

$$Q_2 \le C\mathcal{M}^{\frac{5}{2}}(t)\int_{\frac{t}{2}}^{t} e^{-c(t-s)}(1+s)^{-\frac{5}{4}}\|u(s)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s. \tag{3.18}$$

Set

$$\mathcal{N}(t) = \sup_{0\le s\le t}\{(1+s)^{\frac{5}{6}}(\|u(s)\|_{L^2} + \|\omega(s)\|_{L^2})\}.$$

Inserting (3.17), (3.18) into (3.16), we obtain

$$\mathcal{N}^2(t) \le C(1+t)^{-\frac{5}{3}}e^{-\frac{ct}{2}} + C\mathcal{M}^{\frac{5}{2}}(t)\mathcal{N}^{\frac{1}{2}}(t).$$

Then Young's inequality and $\mathcal{M}(t) \le C$ lead to the desired result.

To get the decay estimate of b, we write the second equation of (1.8) into integral form.

$$\begin{aligned} b(t) &= e^{\nu\Delta t}b_0 + \int_0^t e^{\nu\Delta(t-s)}(b\cdot\nabla u - u\cdot\nabla b)(s)\mathrm{d}s \\ &= e^{\nu\Delta t}b_0 + \int_0^{\frac{t}{2}} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s + \int_{\frac{t}{2}}^{t} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s, \end{aligned} \tag{3.19}$$

where $f\otimes g = (f_i g_j)$ defines the tensor product. By Lemma 3.3, for $0 < t < 1$, we have

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le C\|b_0\|_{L^2},$$

and for $t \ge 1$,

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le Ct^{-\frac{3}{4}}\|b_0\|_{L^1}.$$

Therefore, for any $t > 0$,

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le C(t+1)^{-\frac{3}{4}}. \tag{3.20}$$

Again applying Lemma 3.3, together with (3.13), we obtain for $t \ge 1$,

$$\begin{aligned} &\left\|\int_0^{\frac{t}{2}} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s\right\|_{L^2} \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{3}{4}}\|(b\otimes u - u\otimes b)(s)\|_{L^1}\mathrm{d}s \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-\frac{5}{4}}\|b(s)\|_{L^2}\|u(s)\|_{L^2}\mathrm{d}s \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-\frac{5}{4}}(1+s)^{-\frac{5}{6}}\mathrm{d}s. \end{aligned} \tag{3.21}$$

Using Lemma 3.3 and the Gagliardo–Nirenberg inequality, together with (3.6) and (3.13), for any $t > 0$, we yield

$$
\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s \right\|_{L^2} \\
&\quad\le C\int_{\frac{t}{2}}^{t}(t-s)^{\frac{1}{4}-\frac{3}{2p}}\|(b\otimes u-u\otimes b)(s)\|_{L^p}\mathrm{d}s \\
&\quad\le C\int_{\frac{t}{2}}^{t}(t-s)^{\frac{1}{4}-\frac{3}{2p}}\|b(s)\|_{L^{2p}}\|u(s)\|_{L^{2p}}\mathrm{d}s \\
&\quad\le C\int_{\frac{t}{2}}^{t}(t-s)^{\frac{1}{4}-\frac{3}{2p}}\|u(s)\|_{L^2}^{\frac{3-p}{2p}}\|\nabla u(s)\|_{L^2}^{\frac{3p-3}{2p}}\|b(s)\|_{L^2}^{\frac{3-p}{2p}}\|\nabla b(s)\|_{L^2}^{\frac{3p-3}{2p}}\mathrm{d}s \\
&\quad\le C\mathcal{M}^{\frac{3p-3}{p}}(t)\int_{\frac{t}{2}}^{t}(t-s)^{\frac{1}{4}-\frac{3}{2p}}(1+s)^{-\frac{5}{6}\times\frac{3-p}{2p}-\frac{3p-3}{2p}}\mathrm{d}s \\
&\quad\le C(1+t)^{-\frac{15-2p}{12p}},
\end{aligned}
\tag{3.22}
$$

with $\frac{12}{10} < p \le \frac{15}{11}$. Taking the L^2-norm for space to (3.19), together with (3.20)–(3.22), we obtain for any $t \ge 1$,

$$
\begin{aligned}
\|b(t)\|_{L^2} &\le C(t+1)^{-\frac{3}{4}} + C(1+t)^{-\frac{13}{12}} + C(1+t)^{-\frac{15-2p}{12p}} \\
&\le C(t+1)^{-\frac{3}{4}}.
\end{aligned}
\tag{3.23}
$$

Note that for $0 < t < 1$, (3.1) implies

$$\|b(t)\|_{L^2} \le C,$$

then we immediately obtain the first decay estimate (3.14).

Now we turn to the decay estimate of ∇b. Applying ∇ to (3.19),

$$
\begin{aligned}
\nabla b(t) = \nabla e^{\nu\Delta t}b_0 &+ \int_0^{\frac{t}{2}} \nabla^2 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s \\
&+ \int_{\frac{t}{2}}^{t} \nabla^2 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s.
\end{aligned}
\tag{3.24}
$$

By Lemma 3.3, for $0 < t < 1$, we have

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2} \le C\|\nabla b_0\|_{L^2},$$

and for $t \ge 1$,

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2} \le Ct^{-\frac{5}{4}}\|b_0\|_{L^1}.$$

Therefore, for any $t > 0$,

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2} \le C(t+1)^{-\frac{5}{4}}. \tag{3.25}$$

Using Lemma 3.3, together with (3.13) and (3.23), we have for any $t \geq 1$,

$$\begin{aligned} &\left\| \int_0^{\frac{t}{2}} \nabla^2 e^{\nu\Delta(t-s)}(b \otimes u - u \otimes b)(s)\mathrm{d}s \right\|_{L^2} \\ &\leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{2}{2}-\frac{3}{4}} \|(b \otimes u - u \otimes b)(s)\|_{L^1}\mathrm{d}s \\ &\leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{7}{4}} \|b(s)\|_{L^2}\|u(s)\|_{L^2}\mathrm{d}s \\ &\leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{7}{4}} (1+s)^{-\frac{19}{12}}\mathrm{d}s. \\ &\leq C(1+t)^{-\frac{7}{4}}. \end{aligned} \tag{3.26}$$

Applying Lemmas 3.3 and 3.4, together with (3.6), (3.13) and (3.23), for any $t > 0$, we yield

$$\begin{aligned} &\left\| \int_{\frac{t}{2}}^{t} \nabla^2 e^{\nu\Delta(t-s)}(b \otimes u - u \otimes b)(s)\mathrm{d}s \right\|_{L^2} \\ &= \left\| \int_{\frac{t}{2}}^{t} \Lambda^{-\beta}\nabla^2 e^{\nu\Delta(t-s)}\Lambda^{\beta}(b \otimes u - u \otimes b)(s)\mathrm{d}s \right\|_{L^2} \\ &\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} \|\Lambda^{\beta}(b \otimes u - u \otimes b)(s)\|_{L^2}\mathrm{d}s \\ &\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} (\|\Lambda^{\beta}b(s)\|_{L^4}\|u(s)\|_{L^4} + \|\Lambda^{\beta}u(s)\|_{L^4}\|b(s)\|_{L^4})\mathrm{d}s \\ &\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} \Big(\|u(s)\|_{L^2}^{\frac{1}{4}} \|\nabla u(s)\|_{L^2}^{\frac{3}{4}} \|b(s)\|_{L^2}^{\frac{1}{4}-\beta} \|\nabla b(s)\|_{L^2}^{\frac{3}{4}+\beta} \\ &\quad + \|b(s)\|_{L^2}^{\frac{1}{4}} \|\nabla b(s)\|_{L^2}^{\frac{3}{4}} \|u(s)\|_{L^2}^{\frac{1}{4}-\beta} \|\nabla u(s)\|_{L^2}^{\frac{3}{4}+\beta} \Big) \mathrm{d}s \\ &\leq C\mathcal{M}^{\frac{3}{2}+\beta}(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} \Big((1+s)^{-(\frac{55}{48}-\frac{\beta}{4})} + (1+s)^{-(\frac{55}{48}-\frac{\beta}{3})} \Big) \mathrm{d}s \\ &\leq C(1+t)^{\frac{5}{6}\beta-\frac{55}{48}}, \end{aligned} \tag{3.27}$$

where we set $0 < \beta \leq \frac{7}{40}$; then,

$$C(1+t)^{\frac{5}{6}\beta - \frac{55}{48}} \leq C(1+t)^{-1}.$$

Taking the L^2-inner for space to (3.24), together with (3.25) - (3.27), for $t \geq 1$, we obtain

$$\|\nabla b(t)\|_{L^2} \leq C(t+1)^{-\frac{5}{4}} + C(1+t)^{-\frac{7}{4}} + C(1+t)^{\frac{5}{6}\beta - \frac{55}{48}} \leq C(1+t)^{-1}. \tag{3.28}$$

To improve the decay rates of $\|\nabla b\|_{L^2}$, when $t \geq 1$, we insert (3.28) into (3.27). Set

$$\mathcal{M}_1(t) = \sup_{0 \leq s \leq t} \{(1+s)\|\nabla b(t)\|_{L^2}\}.$$

Then, we have

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \nabla^2 e^{\nu\Delta(t-s)}(b \otimes u - u \otimes b)(s)\mathrm{d}s \right\|_{L^2} \\
&\leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} \Big(\|u(s)\|_{L^2}^{\frac{1}{4}} \|\nabla u(s)\|_{L^2}^{\frac{3}{4}} \|b(s)\|_{L^2}^{\frac{1}{4}-\beta} \|\nabla b(s)\|_{L^2}^{\frac{3}{4}+\beta} \\
&\quad + \|b(s)\|_{L^2}^{\frac{1}{4}} \|\nabla b(s)\|_{L^2}^{\frac{3}{4}} \|u(s)\|_{L^2}^{\frac{1}{4}-\beta} \|\nabla u(s)\|_{L^2}^{\frac{3}{4}+\beta} \Big) \mathrm{d}s \\
&\leq C\mathcal{M}^{\frac{3}{4}}(t)\mathcal{M}_1^{\frac{3}{4}+\beta}(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} (1+s)^{-(\frac{73}{48}+\frac{\beta}{4})}\mathrm{d}s \\
&\quad + C\mathcal{M}_1^{\frac{3}{4}}(t)\mathcal{M}^{\frac{3}{4}+\beta}(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\beta}{2}} (1+s)^{\frac{\beta}{3}-\frac{73}{48}}\mathrm{d}s \\
&\leq C(1+t)^{\frac{5}{6}\beta - \frac{73}{48}},
\end{aligned} \tag{3.29}$$

where $0 < \beta \leq \frac{1}{4}$. Together with (3.25), (3.26) and (3.29), for $t \geq 1$, we have

$$\|\nabla b(t)\|_{L^2} \leq C(t+1)^{-\frac{5}{4}} + C(1+t)^{-\frac{7}{4}} + C(1+t)^{\frac{5}{6}\beta - \frac{73}{48}} \leq C(1+t)^{-\frac{5}{4}}.$$

Note that for $0 < t < 1$, (3.2) implies

$$\|\nabla b(t)\|_{L^2} \leq C.$$

Thus, we obtain the second decay estimate in (3.14).

□

Proposition 3.6. *Let the assumption stated in Theorem 1.4 hold. Then*

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} \leq C(1+t)^{-\frac{11}{6}}. \tag{3.30}$$

Proof. By the Young inequality, we have

$$C\|\nabla b\|_{L^2}^2 \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \leq \frac{\chi}{8}\|u\|_{L^2}^2 + C\|\nabla b\|_{L^2}^{\frac{8}{3}} \|\nabla u\|_{L^2}^{\frac{2}{3}}.$$

Inserting it into (3.15),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right)+\mu\|\nabla_h u\|_{L^2}^2+\frac{\chi}{8}\|u\|_{L^2}^2+(4\chi-\frac{16\chi^2}{\mu})\|\omega\|_{L^2}^2$$
$$\le C\|\nabla b\|_{L^2}^{\frac{8}{3}}\|\nabla u\|_{L^2}^{\frac{2}{3}}$$
$$\le C(1+t)^{-\frac{11}{3}}. \tag{3.31}$$

Set $c_3=\min\{\frac{\chi}{4},8\chi-\frac{32\chi^2}{\mu}\}$. Integrating (3.31) in time, we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\\
&\le e^{-c_3 t}(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2)+C\int_0^t e^{-c_3(t-s)}(1+s)^{-\frac{11}{3}}\mathrm{d}s\\
&= e^{-c_3 t}(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2)+C\int_0^{\frac{t}{2}} e^{-c_3(t-s)}(1+s)^{-\frac{11}{3}}\mathrm{d}s\\
&\quad+C\int_{\frac{t}{2}}^t e^{-c_3(t-s)}(1+s)^{-\frac{11}{3}}\mathrm{d}s\\
&\le Ce^{-c_3 t}+Ce^{-\frac{c_3 t}{2}}+C(1+t)^{-\frac{11}{3}}\\
&\le C(1+t)^{-\frac{11}{3}},
\end{aligned} \tag{3.32}$$

which immediately implies the desired bound. Thus, the proof of Proposition 3.6 is completed. □

Finally, with Propositions 3.2, 3.5 and 3.6 at our disposal, we can improve the L^2 decay for (u,ω) and $(\nabla u,\nabla\omega)$, and obtain the decay for $\nabla^2 b$ in L^2.

Proposition 3.7. *Let the assumptions stated in Theorem 1.4 hold. Then for any* $0<\alpha<\frac{1}{4}$, *we have*

$$\|\nabla^2 b(t)\|_{L^2}\le C(1+t)^{-\frac{65}{48}+\frac{17}{12}\alpha}, \tag{3.33}$$

$$\|u(t)\|_{L^2}+\|\omega(t)\|_{L^2}\le Ct^{-\frac{157}{64}+\frac{17}{16}\alpha}. \tag{3.34}$$

Proof. Firstly, we establish the decay estimate of $\nabla^2 b$. Applying ∇^2 to (3.19), we have

$$\begin{aligned}
\nabla^2 b(t)=\nabla^2 e^{\nu\Delta t}b_0&+\int_0^{\frac{t}{2}}\nabla^3 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s\\
&+\int_{\frac{t}{2}}^t\nabla^3 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s.
\end{aligned} \tag{3.35}$$

By Lemma 3.3, for $0 < t < 1$, we have

$$\|\nabla^2 e^{\nu\Delta t} b_0\|_{L^2} \leq C\|\nabla^2 b_0\|_{L^2},$$

and for $t \geq 1$,

$$\|\nabla^2 e^{\nu\Delta t} b_0\|_{L^2} \leq Ct^{-\frac{7}{4}}\|b_0\|_{L^1}.$$

Therefore, for any $t > 0$,

$$\|\nabla^2 e^{\nu\Delta t} b_0\|_{L^2} \leq C(t+1)^{-\frac{7}{4}}. \tag{3.36}$$

Using Lemma 3.3, together with (3.23) and (3.30), we have for any $t \geq 1$,

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \nabla^3 e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s \right\|_{L^2} \\
&\leq C\int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}-\frac{3}{4}}\|(b\otimes u - u\otimes b)(s)\|_{L^1}\mathrm{d}s \\
&\leq C\int_0^{\frac{t}{2}} (t-s)^{-\frac{9}{4}}\|b(s)\|_{L^2}\|u(s)\|_{L^2}\mathrm{d}s \\
&\leq C\int_0^{\frac{t}{2}} (t-s)^{-\frac{9}{4}}(1+s)^{-\frac{31}{12}}\mathrm{d}s. \\
&\leq C(1+t)^{-\frac{9}{4}}.
\end{aligned} \tag{3.37}$$

Applying Lemmas 3.3 and 3.4, for any $t > 0$, we yield

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \nabla^3 e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s \right\|_{L^2} \\
&= \left\| \int_{\frac{t}{2}}^{t} \Lambda^{-\alpha}\nabla^2 e^{\nu\Delta(t-s)}\Lambda^{\alpha}\nabla(b\otimes u - u\otimes b)(s)\mathrm{d}s \right\|_{L^2} \\
&\leq C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}}\|\Lambda^{1+\alpha}(b\otimes u - u\otimes b)(s)\|_{L^2}\mathrm{d}s \\
&\leq C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}}(\|\Lambda^{1+\alpha}u(s)\|_{L^4}\|b(s)\|_{L^4} + \|\Lambda^{1+\alpha}b(s)\|_{L^4}\|u(s)\|_{L^4})\mathrm{d}s \\
&\leq F_1 + F_2,
\end{aligned} \tag{3.38}$$

where we set $0 < \alpha < \frac{1}{4}$. Then by using Gagliardo–Nirenberg inequality, (1.11), (3.14) and (3.30), we have

$$\begin{aligned}
F_1 &= C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}\|\Lambda^{1+\alpha}u(s)\|_{L^4}\|b(s)\|_{L^4}\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}(\|u(s)\|_{L^2}^{\frac{1}{8}-\frac{\alpha}{2}}\|\nabla^2u(s)\|_{L^2}^{\frac{7}{8}+\frac{\alpha}{2}}\|b(s)\|_{L^2}^{\frac{1}{4}}\|\nabla b(s)\|_{L^2}^{\frac{3}{4}})\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}\|u(s)\|_{H^2}(1+s)^{-\frac{11}{6}(\frac{1}{8}-\frac{\alpha}{2})-\frac{1}{4}\times\frac{3}{4}-\frac{3}{4}\times\frac{5}{4}}\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}(1+s)^{\frac{11}{12}\alpha-\frac{65}{48}}\mathrm{d}s\\
&\le C(1+t)^{\frac{17}{12}\alpha-\frac{65}{48}}.
\end{aligned}$$

Next, we consider F_2, set

$$\mathcal{M}_2(t) = \sup_{0\le s\le t}\{(1+s)^{-\frac{17}{12}\alpha+\frac{65}{48}}\|\nabla^2 b(s)\|_{L^2}\},$$

then we have

$$\begin{aligned}
F_2 &= C\int_{\frac{t}{2}}^{t}(t-s)^{-1+\frac{\alpha}{2}}\|\Lambda^{1+\alpha}b(s)\|_{L^4}\|u(s)\|_{L^4}\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}(\|b(s)\|_{L^2}^{\frac{1}{8}-\frac{\alpha}{2}}\|\nabla^2b(s)\|_{L^2}^{\frac{7}{8}+\frac{\alpha}{2}}\|u(s)\|_{L^2}^{\frac{1}{4}}\|\nabla u(s)\|_{L^2}^{\frac{3}{4}})\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}\|\nabla^2b(s)\|_{L^2}^{\frac{7}{8}+\frac{\alpha}{2}}(1+s)^{-\frac{3}{4}(\frac{1}{8}-\frac{\alpha}{2})-\frac{1}{4}\times\frac{11}{6}-\frac{3}{4}\times\frac{1}{2}}\mathrm{d}s\\
&\le C\mathcal{M}_2^{\frac{7}{8}+\frac{\alpha}{2}}(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{2-\alpha}{2}}(1+s)^{-(-\frac{17}{12}\alpha+\frac{65}{48})(\frac{7}{8}+\frac{\alpha}{2})-\frac{89}{96}+\frac{3}{8}\alpha}\mathrm{d}s\\
&\le C\mathcal{M}_2^{\frac{7}{8}+\frac{\alpha}{2}}(t)(1+t)^{-(-\frac{17}{12}\alpha+\frac{65}{48})(\frac{7}{8}+\frac{\alpha}{2})-\frac{89}{96}+\frac{7}{8}\alpha}.
\end{aligned}$$

Inserting F_1 and F_2 into (3.38), and combining (3.36), (3.37) and (3.38), when $t \ge 1$, we have

$$\begin{aligned}
\mathcal{M}_2(t) &\le C(1+t)^{-\frac{7}{4}-\frac{17}{12}\alpha+\frac{65}{48}} + C(1+t)^{-\frac{9}{4}-\frac{17}{12}\alpha+\frac{65}{48}}\\
&\quad + C + C\mathcal{M}_2^{\frac{7}{8}+\frac{\alpha}{2}}(1+t)^{-(-\frac{17}{12}\alpha+\frac{65}{48})(\frac{7}{8}+\frac{\alpha}{2})-\frac{89}{96}+\frac{7}{8}\alpha-\frac{17}{12}\alpha+\frac{65}{48}},
\end{aligned}$$

where $0<\alpha<\frac{1}{4}$. Through calculation, it can be seen that when $0<\alpha<\frac{1}{4}$,

$$\begin{aligned}&-\left(-\frac{17}{12}\alpha+\frac{65}{48}\right)\left(\frac{7}{8}+\frac{\alpha}{2}\right)-\frac{89}{96}+\frac{7}{8}\alpha-\frac{17}{12}\alpha+\frac{65}{48}\\&=\frac{17}{24}\alpha^2+\frac{1}{48}\alpha-\frac{291}{384}<0,\end{aligned}$$

$$-\frac{7}{4}-\frac{17}{12}\alpha+\frac{65}{48}<0,\quad\text{and}\quad-\frac{9}{4}-\frac{17}{12}\alpha+\frac{65}{48}<0.$$

Then for the $t\geq 1$, by the Young inequality we yield

$$\mathcal{M}_2(t)\leq C.$$

Note that for $0<t<1$, (1.11) implies

$$\|\nabla^2 b(t)\|_{L^2}\leq C.$$

Thus, we obtain (3.33).

Finally, with (3.33) at our disposal, we can turn to establish the decay estimate (3.34). Again applying Hölder's inequality, we obtain

$$\begin{aligned}\int b\cdot\nabla b\cdot u&\leq\|b\|_{L^6}\|\nabla b\|_{L^2}\|u\|_{L^3}\\&\leq C\|\nabla b\|_{L^2}^2\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}},\end{aligned}\tag{3.39}$$

and

$$\begin{aligned}\int b\cdot\nabla b\cdot u&\leq\|b\|_{L^3}\|\nabla b\|_{L^6}\|u\|_{L^2}\\&\leq C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}\|u\|_{L^2}.\end{aligned}\tag{3.40}$$

Inserting (3.39) and (3.40) into (3.15) and integrating in time yields

$$\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\leq Ce^{-ct}+Z_1+Z_2,\tag{3.41}$$

where

$$Z_1=C\int_0^{\frac{t}{2}}e^{-c(t-s)}\|\nabla b(s)\|_{L^2}^2\|u(s)\|_{L^2}^{\frac{1}{2}}\|\nabla u(s)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s,$$

$$Z_2=C\int_{\frac{t}{2}}^{t}e^{-c(t-s)}\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 b\|_{L^2}\|u\|_{L^2}\mathrm{d}s.$$

Applying Hölder's inequality, together with (3.1) and (3.2), we obtain

$$Z_1\leq Ce^{-\frac{ct}{2}}.\tag{3.42}$$

Set

$$\mathcal{N}_1(t)=\sup_{0\leq s\leq t}\{(1+s)^{(\frac{157}{64}-\frac{17}{16}\alpha)}(\|u(s)\|_{L^2}+\|\omega(s)\|_{L^2})\}.$$

Then (3.14), (3.30) and (3.33) yield

$$Z_2\leq\mathcal{N}_1(t)\int_{\frac{t}{2}}^{t}e^{-c(t-s)}(1+s)^{-\frac{157}{32}+\frac{17}{8}\alpha}\mathrm{d}s.\tag{3.43}$$

Inserting (3.42) and (3.43) into (3.41), we get

$$\mathcal{N}_1^2(t) \le C + C\mathcal{N}_1(t),$$

since for any $t > 0$, $(1+t)^{(\frac{157}{64}-\frac{17}{16}\alpha)}e^{-\frac{ct}{2}} \le C$. Then the Young inequality yields

$$\mathcal{N}_1(t) \le C,$$

which implies (3.34). Thus, the proof of Proposition 3.7 is completed. □

Proposition 3.8. *Let the assumptions stated in Theorem 1.4 hold. Then for any $0 < \alpha < \frac{1}{4}$, we have*

$$\|\nabla u(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} \le C(1+t)^{-\frac{471}{512}+\frac{51}{128}\alpha}. \tag{3.44}$$

Proof. Applying ∂_k with $k = 1, 2, 3$ to $(1.8)_1$ and $(1.8)_3$, dotting the results by $\partial_k u$ and $\partial_k \omega$, respectively, integrating in space domain and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{3}\left(\|\partial_k u(t)\|_{L^2}^2 + \|\partial_k \omega(t)\|_{L^2}^2\right) + \mu\sum_{k=1}^{3}\|\nabla_h\partial_k u\|_{L^2}^2 + \chi\sum_{k=1}^{3}\|\partial_k u\|_{L^2}^2 \\
&\quad + \gamma\sum_{k=1}^{3}\|\partial_3\partial_k\omega\|_{L^2}^2 + 4\chi\sum_{k=1}^{3}\|\partial_k\omega\|_{L^2}^2 + \kappa\sum_{k=1}^{3}\|\partial_k\nabla\cdot\omega\|_{L^2}^2 \\
&= -\sum_{k=1}^{3}\int \partial_k(u\cdot\nabla u)\cdot\partial_k u\mathrm{d}x + \sum_{k=1}^{3}\int \partial_k(b\cdot\nabla b)\cdot\partial_k u\mathrm{d}x + 2\chi\sum_{k=1}^{3}\int \nabla\times\partial_k\omega\cdot\partial_k u\mathrm{d}x \\
&\quad - \sum_{k=1}^{3}\int \partial_k(u\cdot\nabla\omega)\cdot\partial_k\omega\mathrm{d}x + 2\chi\sum_{k=1}^{3}\int \nabla\times\partial_k u\cdot\partial_k\omega\mathrm{d}x \\
&:= D_1 + D_2 + D_3 + D_4 + D_5.
\end{aligned} \tag{3.45}$$

Using the divergence free condition $\nabla\cdot u = 0$, together with Lemma 2.2 and the Young inequality, we have

$$\begin{aligned}
D_1 &= -\sum_{k=1}^{3}\sum_{j=1}^{3}\int \partial_k u_j\partial_j u\cdot\partial_k u\mathrm{d}x \\
&= -\sum_{k=1}^{2}\sum_{j=1}^{3}\int \partial_k u_j\partial_j u\cdot\partial_k u\mathrm{d}x - \sum_{j=1}^{2}\int \partial_3 u_j\partial_j u\cdot\partial_3 u\mathrm{d}x - \int \partial_3 u_3\partial_3 u\cdot\partial_3 u\mathrm{d}x \\
&\le C\sum_{k=1}^{2}\sum_{j=1}^{3}\|\partial_k u_j\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_k u_j\|_{L^2}^{\frac{1}{2}}\|\partial_j u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_j u\|_{L^2}^{\frac{1}{2}}\|\partial_k u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_k u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{2}\|\partial_3 u_j\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 u_j\|_{L^2}^{\frac{1}{2}}\|\partial_j u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_j u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_h u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_h u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{2}} \\
&\le C(\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|\nabla u\|_{L^2}\|\nabla_h u\|_{L^2}^{\frac{1}{2}})\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{2}}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
D_2 &= \sum_{k=1}^{3} \int \partial_k (b \cdot \nabla b) \cdot \partial_k u \mathrm{d}x \\
&\leq C \|\nabla^2 b\|_{L^2} (\|\nabla b\|_{L^2} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} + \|b\|_{L^\infty} \|\nabla u\|_{L^2}), \\
D_4 &= -\sum_{k=1}^{3} \int \partial_k (u \cdot \nabla \omega) \cdot \partial_k \omega \mathrm{d}x \\
&\leq C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \omega\|_{L^2}^{\frac{1}{2}} (\|\nabla^2 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} + \|\partial_3 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 \omega\|_{L^2}^{\frac{1}{2}}).
\end{aligned}
$$

By the Young inequality, we have

$$
D_3 + D_5 \leq \frac{\mu}{2} \sum_{k=1}^{3} \|\nabla_h \partial_k u\|_{L^2}^2 + \frac{16\chi^2}{\mu} \sum_{k=1}^{3} \|\partial_k \omega\|_{L^2}^2 + \frac{3\chi}{4} \sum_{k=1}^{3} \|\partial_k u\|_{L^2}^2 + \frac{16\chi}{3} \sum_{k=1}^{3} \|\partial_3 \partial_k \omega\|_{L^2}^2.
$$

Inserting the above estimates into (3.45), we infer that

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 \right) + \frac{\mu}{2} \|\nabla_h \nabla u\|_{L^2}^2 + \frac{\chi}{4} \|\nabla u\|_{L^2}^2 \\
&\quad + (\gamma - \frac{16\chi}{3}) \|\partial_3 \nabla \omega\|_{L^2}^2 + \left(4\chi - \frac{16\chi^2}{\mu} \right) \|\nabla \omega\|_{L^2}^2 + \kappa \|\nabla \nabla \cdot \omega\|_{L^2}^2 \\
&\leq C \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{3}{2}} \\
&\quad + C \|\nabla^2 b\|_{L^2} (\|\nabla b\|_{L^2} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} + \|b\|_{L^\infty} \|\nabla u\|_{L^2}) \\
&\quad + C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2} \|\partial_3 \nabla \omega\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \omega\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Then integrating the above inequality in time, set $c = \min\{\frac{\chi}{2}, 8\chi - \frac{32\chi^2}{\mu}\}$, we yield

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 \\
&\quad \leq e^{-ct} (\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2) + C(J_1 + J_2 + J_3),
\end{aligned} \tag{3.46}
$$

where

$$
\begin{aligned}
J_1 &= \int_0^t e^{-c(t-s)} \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{3}{2}} \mathrm{d}s, \\
J_2 &= \int_0^t e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} (\|\nabla b(s)\|_{L^2} \|\nabla \nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} + \|b(s)\|_{L^\infty} \|\nabla u(s)\|_{L^2}) \mathrm{d}s, \\
J_3 &= \int_0^t e^{-c(t-s)} \|\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla \omega(s)\|_{L^2} \|\partial_3 \nabla \omega(s)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \omega(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s.
\end{aligned}
$$

Finally, we need consider J_1 - J_3, respectively. Applying the Gagliardo– Nirenberg inequality, (3.2) and (3.34), we have,

$$\begin{aligned}
J_1 &= \int_0^{\frac{t}{2}} e^{-c(t-s)} \|\nabla u(s)\|_{L^2}^{\frac{3}{2}} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{3}{2}} \mathrm{d}s + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla u(s)\|_{L^2}^{\frac{3}{2}} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{3}{2}} \mathrm{d}s \\
&\le Ce^{-\frac{ct}{2}} \int_0^{\frac{t}{2}} \|u(s)\|_{H^2} (\|\nabla u(s)\|_{L^2}^2 + \|\nabla\nabla_h u(s)\|_{L^2}^2) \mathrm{d}s \\
&\quad + C\int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|u(s)\|_{L^2}^{\frac{3}{4}} \|\nabla^2 u(s)\|_{L^2}^{\frac{9}{4}} \mathrm{d}s \\
&\le Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{471}{256}+\frac{51}{64}\alpha},
\end{aligned}$$

where C is dependent on μ, χ, ν, κ and the initial data $\|(u_0, b_0, \omega_0)\|_{H^2}$. For J_2,

$$\begin{aligned}
J_2 &= \int_0^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|\nabla b(s)\|_{L^2} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\quad + \int_0^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|b(s)\|_{L^\infty} \|\nabla u(s)\|_{L^2} \mathrm{d}s \\
&:= J_{21} + J_{22}.
\end{aligned}$$

Using the Young inequality, (3.1), (3.2), (3.14) and (3.34), we obtain

$$\begin{aligned}
J_{21} &= \int_0^{\frac{t}{2}} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|\nabla b(s)\|_{L^2} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\quad + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|\nabla b(s)\|_{L^2} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\le \int_0^{\frac{t}{2}} e^{-c(t-s)} \|b(s)\|_{H^2} (\|\nabla b(s)\|_{L^2}^2 + \|\nabla\nabla_h u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2) \mathrm{d}s \\
&\quad + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|\nabla b(s)\|_{L^2} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\le Ce^{-\frac{ct}{2}} + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|\nabla b(s)\|_{L^2} \|\nabla^2 u(s)\|_{L^2}^{\frac{3}{4}} \|u(s)\|_{L^2}^{\frac{1}{4}} \mathrm{d}s \\
&\le Ce^{-\frac{ct}{2}} + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} (1+s)^{-\frac{5}{4}-\frac{1}{4}(\frac{157}{64}-\frac{17}{16}\alpha)} \mathrm{d}s \\
&\le Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{477}{256}+\frac{17}{64}\alpha},
\end{aligned}$$

and

$$
\begin{aligned}
J_{22} &= \int_0^{\frac{t}{2}} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|b(s)\|_{L^\infty} \|\nabla u(s)\|_{L^2} \mathrm{d}s \\
&\quad + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla^2 b(s)\|_{L^2} \|b(s)\|_{L^\infty} \|\nabla u(s)\|_{L^2} \mathrm{d}s \\
&\leq \int_0^{\frac{t}{2}} e^{-c(t-s)} \|b(s)\|_{H^2} (\|\nabla^2 b(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2) \mathrm{d}s \\
&\quad + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|b(s)\|_{L^2}^{\frac{1}{4}} \|\nabla^2 b(s)\|_{L^2}^{\frac{7}{4}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 u(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\leq Ce^{-\frac{ct}{2}} + C \int_{\frac{t}{2}}^{t} e^{-c(t-s)} (1+s)^{-\frac{1}{4}\times\frac{3}{4} + \frac{7}{4}(\frac{17}{12}\alpha - \frac{65}{48}) - \frac{1}{2}(\frac{157}{64} - \frac{17}{16}\alpha)} \mathrm{d}s \\
&\leq Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{23248}{6144} + \frac{289}{96}\alpha}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
J_2 &\leq Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{477}{256} + \frac{17}{64}\alpha} + C(1+t)^{-\frac{23248}{6144} + \frac{289}{96}\alpha} \\
&\leq Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{477}{256} + \frac{17}{64}\alpha},
\end{aligned}
$$

where $0 < \alpha < \frac{1}{4}$. Similarly,

$$
\begin{aligned}
J_3 &= \int_0^{\frac{t}{2}} e^{-c(t-s)} \|\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla \omega(s)\|_{L^2} \|\partial_3 \nabla \omega(s)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \omega(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\quad + \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla \omega(s)\|_{L^2} \|\partial_3 \nabla \omega(s)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \omega(s)\|_{L^2}^{\frac{1}{2}} \mathrm{d}s \\
&\leq \int_0^{\frac{t}{2}} e^{-c(t-s)} \|u(s)\|_{H^2}^{\frac{1}{2}} \|\omega(s)\|_{H^2}^{\frac{1}{2}} (\|\nabla_h u(s)\|_{L^2}^2 + \|\partial_3 \nabla \omega(s)\|_{L^2}^2 + \|\nabla^2 \omega(s)\|_{L^2}^2) \mathrm{d}s \\
&\quad + C \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla\nabla_h u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla \omega(s)\|_{L^2} \|\nabla^2 \omega(s)\|_{L^2} \mathrm{d}s \\
&\leq Ce^{-\frac{ct}{2}} + C \int_{\frac{t}{2}}^{t} e^{-c(t-s)} \|\omega(s)\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \omega(s)\|_{L^2}^{\frac{3}{2}} \|u(s)\|_{L^2}^{\frac{1}{4}} \|\nabla^2 u(s)\|_{L^2}^{\frac{3}{4}} \mathrm{d}s \\
&\leq Ce^{-\frac{ct}{2}} + C \int_{\frac{t}{2}}^{t} e^{-c(t-s)} (1+s)^{-\frac{3}{4}(\frac{157}{64} - \frac{17}{16}\alpha)} \mathrm{d}s
\end{aligned}
$$

$$\leq Ce^{-\frac{ct}{2}} + C(1+t)^{-(\frac{471}{256}-\frac{51}{64}\alpha)}.$$

Inserting J_1–J_3 into (3.46), we obtain

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2\\&\quad\leq Ce^{-\frac{ct}{2}} + C(1+t)^{-\frac{477}{256}+\frac{17}{64}\alpha} + C(1+t)^{-\frac{471}{256}+\frac{51}{64}\alpha} \leq C(1+t)^{-\frac{471}{256}+\frac{51}{64}\alpha}.\end{aligned}$$

which implies (3.44). Thus, the proof of Proposition 3.8 is completed. □

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