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Wave propagation in a diffusive epidemic model with demography and time-periodic coefficients

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Abstract. In this paper, the periodic traveling wave solution for a reaction–diffusion SIR epidemic model with demography and time-periodic coefficients is investigated. Because the traveling wave system of non-autonomous reaction–diffusion model is a partial differential equation system, some traditional methods using only the theory of ordinary differential equations are no longer applicable. To overcome these difficulties, the traditional methods are extended and improved, and some new techniques are introduced. The research results show that the existence and nonexistence of traveling wave solutions are determined by the basic reproduction number $\mathcal{R}_0$ and the minimal wave speed c^*. Specifically, when $\mathcal{R}_0 > 1$ and the wave speed $c > c^*$ the existence of periodic traveling wave solutions is proved by means of auxiliary system, upper–lower solutions, fixed-point theorems and some limit arguments. Otherwise, when $R_0 < 1$, for any wave speed $c > 0$ the nonexistence of periodic traveling wave solutions is proved. Lastly, the numerical examples are carried out to verify the theoretical results.

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Keywords. Reaction–diffusion SIR epidemic model, Time-periodic coefficient, Periodic traveling wave solution, Upper–lower solutions, Auxiliary system.

1. Introduction

Infectious diseases have always been a huge obstacle to the development of human society. Every outbreak of infectious diseases brings great losses to human life and property. The study of infectious diseases has never stopped. An important research field is the mathematical modeling of infectious diseases, theoretical analysis and applications in the realistic infectious disease to understand the spread dynamics of diseases and formulate effective prevention and control measures.

In recent decades, the infectious disease dynamical models which are described by the differential and difference equations have developed rapidly, and many classical results have played an important role in disease control and prevention. However, we know that with the rapid development of human society and economy, the range of human activities is getting larger and more frequent, so the reaction–diffusion equation has been widely proposed in the modeling of infectious diseases. The existence of endemic equilibrium, calculation of basic reproduction number, stability of equilibria, extinction and persistence of disease, existence of traveling wave solution, bifurcation phenomena, dynamical complexity, etc., are all the focuses in the reaction–diffusion epidemic models (see [1–7] and references therein). Particularly, traveling wave solution is a kind of special formal solution of reaction–diffusion equation, which can well describe the process of disease spreading from a certain location in space to its surroundings at a certain speed. In general, there are two important indicators in the discussion of traveling wave solution, one is asymptotic boundary conditions and the other is minimum wave speed. Specifically, the asymptotic boundary condition can reflect the asymptotic behavior of disease transmission, and the minimum wave speed can describe how quickly transmission takes place in space. As a consequence, the traveling wave solutions have gradually become important mathematical methods and research focus in the disease propagation (see [8–13] and references therein).

It is well known that the spread of disease is influenced by many environmental factors, such as temperature, humidity, season, and so on. These factors have a certain periodicity (especially time-varying periodicity). Therefore, the epidemic models with time-varying periodic coefficients have attracted the attention of many scholars (see [14–16] and references therein). However, the global behavior of the reaction–diffusion system with time periodic coefficients is still a challenging problem. Recently, Zhang et al. [17] discussed the following non-autonomous reaction–diffusion SIR (susceptible–infective–removed) model with bilinear incidence and T-periodic coefficients

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S - \beta(t) S(t,x) I(t,x), \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I + \beta(t) S(t,x) I(t,x) - \gamma(t) I(t,x), \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R + \gamma(t) I(t,x), \end{cases} \tag{1}$$

where $S(t,x)$, $I(t,x)$ and $R(t,x)$ describe the density of susceptible, infective and removed individuals in location x and at time t, respectively; constants $d_i \geq 0$ $(i = 1,2,3)$ reflect the diffusion ability of individuals in space; coefficients $\beta(t)$ and $\gamma(t)$ stand for transmission rate and removed rate, respectively, which are assumed to be positive and continuous T-periodic functions with respect to t. For this model, the authors discussed the existence of positive T-periodic traveling wave solution, which is defined by

$$\begin{pmatrix} S(t,x) \\ I(t,x) \end{pmatrix} = \begin{pmatrix} \phi(t,x+ct) \\ \psi(t,x+ct) \end{pmatrix}, \begin{pmatrix} \phi(t+T,z) \\ \psi(t+T,z) \end{pmatrix} = \begin{pmatrix} \phi(t,z) \\ \psi(t,z) \end{pmatrix}, \tag{2}$$

with asymptotic boundary conditions

$$\begin{pmatrix} \phi(t,-\infty) \\ \psi(t,-\infty) \end{pmatrix} = \begin{pmatrix} S^{\infty} \\ 0 \end{pmatrix}, \begin{pmatrix} \phi(t,+\infty) \\ \psi(t,+\infty) \end{pmatrix} = \begin{pmatrix} S_0 \\ 0 \end{pmatrix} \text{ uniformly for } \ t \in \mathbb{R}, \tag{3}$$

where constant $c > 0$ is the wave speed, $(S_0, 0)$ is the disease-free equilibrium and $(S^{\infty}, 0)$ is another equilibrium after the outbreak. In model (1), the authors proved that the model admits positive T-periodic wave propagation when the basic production number $\mathcal{R}_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} > 1$ and the wave speed c more than the minimum wave speed $c^* = 2\sqrt{\frac{d_2}{T}\int_0^T [\beta(t) - \gamma(t)]dt}$, and there is no such periodic traveling waves when $\mathcal{R}_0 \leq 1$ or $\mathcal{R}_0 > 1$ and $0 < c < c^*$.

Wang et al. in [18] studied the following reaction–diffusion SIR model with standard incidence and T-periodic coefficients

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S - \dfrac{\beta(t) S(t,x) I(t,x)}{S(t,x) + I(t,x)}, \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I + \dfrac{\beta(t) S(t,x) I(t,x)}{S(t,x) + I(t,x)} - \gamma(t) I(t,x), \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R + \gamma(t) I(t,x), \end{cases} \tag{4}$$

where $S(t,x)$, $I(t,x)$, d_i $(i = 1,2,3)$, $\beta(t)$ and $\gamma(t)$ are defined as in (1). The sufficient conditions for the existence of nonnegative T-periodic traveling wave solutions with the asymptotic boundary condition (3) are established by using the semigroup theory, periodic upper and lower solutions, fixed-point theorem and some limit techniques.

It should be pointed out that in models (1) and (4) many vital factors, such as the natural mortality of individuals, the recruitment rate of susceptible individuals, and the disease-related mortality of infected individuals, are ignored. This means that the infectious diseases described by models (1) and (4) break out so quickly that population dynamics can be ignored. However, many outbreaks of infectious disease continue for some time and always be accompanied by population dynamics. Assume that population dynamics are taken into account, then new infections may occur where the outbreak of disease has passed, which may lead to a secondary outbreak of disease and may induce some spatiotemporal oscillations (see [19]). Therefore, a more interesting and challenging problem is to study the traveling wave solution of reaction–diffusion epidemic model with the influence of population dynamics factors in the time-periodic environments. Driven by this problem, in this paper we consider the following reaction–diffusion SIR epidemic model with demography and T-periodic coefficients

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S + \Lambda(t) - \beta(t)S(t,x)I(t,x) - \mu(t)S(t,x), \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I + \beta(t)S(t,x)I(t,x) - (\gamma(t) + \alpha(t) + \mu(t))I(t,x), \\ \dfrac{\partial R}{\partial t} = d_3 \Delta R + \gamma(t)I(t,x) - \mu(t)R(t,x), \end{cases} \tag{5}$$

where variables $S(t,x)$, $I(t,x)$, $R(t,x)$ and coefficients $d_i > 0$ $(i = 1,2,3)$, $\beta(t)$, $\gamma(t)$ are defined like in model (1); $\Lambda(t)$ stands for the supplement rate of susceptible; $\mu(t)$ represents the natural death rate; $\alpha(t)$ is the death rate due to disease. All functions $\Lambda(t)$, $\beta(t)$, $\gamma(t)$ and $\mu(t)$ are positive and T-periodic continuous functions for time t.

It should be emphasized that compared with models (1) and (4), model (5) takes into account the input of susceptible individual, natural death, disease death, which makes the model (5) more general and the study of traveling wave solutions more complex and challenging. On the other hand, since the infected term in model (5) is not integrable, some traditional methods, such as monotone iterative method, Lebesgue's dominated convergence theorem and Laplace transform in models (1) and (4) are no longer applicable. Thus, we study the existence of traveling waves through fixed-point theorems on a convex and closed set by constructing suitable solution maps. Besides, the corresponding non-diffusion periodic kinetic system of model (5) admits a disease-free T-periodic steady state $(S^0(t), 0)$ and a positive T-periodic steady state $(S^*(t), I^*(t))$ when the basic reproduction number $\mathcal{R}_0 > 1$. Therefore, the traveling waves connecting two steady states $(S^0(t), 0)$ and $(S^*(t), I^*(t))$ will be explored by the numerical examples.

This paper is organized as follows. In Sect. 2, some conclusions about non-diffusion periodic kinetic system and some important lemmas are given. In Sect. 3, upper–lower solutions are established. In Sect. 4, the periodic traveling waves are obtained by the fixed-point theorem on closed convex sets consisting of proper periodic functions. In Sect. 5, some numerical examples are given. In Sect. 6, some brief conclusion and discussion are presented.

2. Preliminary

Firstly, for a second-order continuously differentiable function $f(\xi)$ with $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n$, for the convenience we denote the partial derivative $\frac{\partial f(\xi)}{\partial \xi_i}$ by $f_{\xi_i}(\xi)$ and $\partial_{\xi_i} f(\xi)$, and $\frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j}$ by $f_{\xi_i \xi_j}(\xi)$ and $\partial_{\xi_i \xi_j} f(\xi)$.

Since variable R does not appear in the first two equations of model (5), it is sufficient to investigate the following reduced model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \Delta S + \Lambda(t) - \beta(t)S(t,x)I(t,x) - \mu(t)S(t,x), \\ \dfrac{\partial I}{\partial t} = d_2 \Delta I + \beta(t)S(t,x)I(t,x) - (\gamma(t) + \alpha(t) + \mu(t))I(t,x), \end{cases} \tag{6}$$

where $x \in \mathbb{R}$ and $t > 0$. The corresponding non-diffusion T-periodic kinetic system for model (6) is

$$\begin{cases} \dfrac{dS}{dt} = \Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t), \\ \dfrac{dI}{dt} = \beta(t)S(t)I(t) - (\gamma(t) + \alpha(t) + \mu(t))I(t). \end{cases} \tag{7}$$

If $I(t) \equiv 0$, then system (7) becomes the following susceptible T-periodic linear equation:

$$\frac{dS}{dt} = \Lambda(t) - \mu(t)S(t). \tag{8}$$

We have the following lemma.

Lemma 1. *(see [20]) There exists a globally uniformly attractive T-periodic solution $S^0(t)$ for equation (8). Furthermore,*

$$S^0(t) = \int_{-\infty}^{t} e^{-\int_s^t \mu(\tau)d\tau} \Lambda(s) ds. \tag{9}$$

For system (7), we define the basic reproduction number

$$\mathcal{R}_0 := \frac{\int_0^T \beta(t)S^0(t)dt}{\int_0^T (\gamma(t) + \alpha(t) + \mu(t))dt},$$

where $S^0(t)$ is defined by (9). We further have the following lemma.

Lemma 2. *(see [20]) The following conclusions are equivalent for system (7).*

(i) Infected I is permanent;
(ii) Infected I is strong persistent;
(iii) System (7) has a positive T-periodic solution;
(iv) The basic reproduction number $\mathcal{R}_0 > 1$.

Remark 1. From Lemma 1, we know that system (7) admits a unique disease-free T-periodic solution $(S^0(t), 0)$. When $\mathcal{R}_0 > 1$, then from Lemma 2 there is a positive T-periodic solution $(S^*(t), I^*(t))$ for system (7).

Remark 2. For the reaction–diffusion T-periodic model (6), it is clear that steady state $(S^0(t), 0)$ always is a disease-free T-periodic solution, and when $\mathcal{R}_0 > 1$, steady state $(S^*(t), I^*(t))$ is a positive T-periodic solution.

Remark 3. When all coefficients in system (7) degenerate as constants, then system (7) will become to an autonomous system. For autonomous system (7), it is clear that $E_0(\frac{\lambda}{\mu}, 0)$ is the disease-free equilibrium, the basic reproduction number $\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\mu+\alpha+\gamma)}$, and when $\mathcal{R}_0 > 1$ there exists a unique endemic equilibrium $E^*(S^*, I^*)$. We have a lot of conclusions about persistence and stability (see [20–25]). Specifically, in [21], the results show that if $\mathcal{R}_0 \le 1$, then equilibrium E_0 is globally asymptotically stable, and if $\mathcal{R}_0 > 1$, then equilibrium E^* is locally asymptotically stable. In [20], the authors discussed the relationship between persistence, stability and basic reproduction number $\mathcal{R}_0$. More conclusions can be found in [22–25].

Now, we investigate the traveling waves defined in (2) for periodic reaction–diffusion model (6). It is clear that such solution $(\phi(t,z), \psi(t,z))$ satisfies

$$\begin{cases} \phi_t = d_1\phi_{zz} - c\phi_z + \Lambda(t) - \beta(t)\phi\psi - \mu(t)\phi, \\ \psi_t = d_2\psi_{zz} - c\psi_z + \beta(t)\phi\psi - (\gamma(t) + \alpha(t) + \mu(t))\psi, \\ \phi(t+T,z) = \phi(t,z),\ \psi(t+T,z) = \psi(t,z). \end{cases} \tag{10}$$

In the present work, we consider the traveling waves of model (6) satisfying the following conditions:

$$\lim_{z\to-\infty}(\phi(t,z), \psi(t,z)) = (S^0(t), 0) \text{ uniformly for } \ t \in \mathbb{R}, \tag{11}$$

and

$$\liminf_{z\to+\infty} \inf_{t\in[0,T]} \phi^*(t,z) > 0,\ \liminf_{z\to+\infty} \inf_{t\in[0,T]} \psi^*(t,z) > 0. \tag{12}$$

Linearizing system (10) at steady state $(S^0(t), 0)$, from the second equation we obtain

$$J_t = d_2J_{zz} - cJ_z + [\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))]J(t,z). \tag{13}$$

Define

$$\Phi_c(\lambda) = d_2\lambda^2 - c\lambda + \rho,\ c \in \mathbb{R},\ \lambda \in \mathbb{R},$$

where $\rho := \frac{1}{T}\int_0^T \left[\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))\right]dt$. Obviously, $\rho > 0 \Leftrightarrow \mathcal{R}_0 = \frac{\int_0^T \beta(t)S^0(t)dt}{\int_0^T(\gamma(t)+\alpha(t)+\mu(t))dt} > 1$. If $c > c^* := 2\sqrt{d_2\rho}$, let

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4d_2\rho}}{2d_2},\ \ \lambda_2 = \frac{c + \sqrt{c^2 - 4d_2\rho}}{2d_2}.$$

Then, we have $\Phi_c(\lambda_1) = \Phi_c(\lambda_2) = 0$ and $\Phi_c(\lambda) < 0$ for all $\lambda \in (\lambda_1, \lambda_2)$.

In the following, the proper upper and lower solutions for system (10) are constructed. Unfortunately, it is very troublesome to construct the upper–lower solutions directly to system (10). To overcome this difficulty, we introduce the following auxiliary system:

$$\begin{cases} \phi_t = d_1\phi_{zz} - c\phi_z + \Lambda(t) - \beta(t)\phi\psi - \mu(t)\phi, \\ \psi_t = d_2\psi_{zz} - c\psi_z + \beta(t)\phi\psi - (\gamma(t) + \alpha(t) + \mu(t))\psi(t,z) - \varepsilon\psi^2, \end{cases} \tag{14}$$

where $\varepsilon > 0$ is a constant.

3. Upper–lower solutions

In this section, we always assume $\mathcal{R}_0 > 1$ and fix $c > c^*$. To construct the appropriate upper and lower solutions, we first define a periodic function

$$K(t) = \exp\left(\int_0^t \left[d_2\lambda_1^2 - c\lambda_1 + \left(\beta(\tau)S^0(\tau) - (\gamma(\tau) + \alpha(\tau) + \mu(\tau))\right)\right]d\tau\right).$$

Construct the four T-periodic nonnegative functions as follows

$$\begin{aligned} &\phi^+ := S^0(t), && \phi^- := \max\{S^0(t)(1 - M_1e^{\varepsilon_1 z}), 0\}, \\ &\psi^+ := \min\{K(t)e^{\lambda_1 z}, K_\varepsilon\}, && \psi^- := \max\{K(t)e^{\lambda_1 z}(1 - M_2e^{\varepsilon_2 z}), 0\}, \end{aligned}$$

where $t \in \mathbb{R}, z \in \mathbb{R}$ and $K_\varepsilon = \max_{t\in[0,T]}\left\{\frac{\beta(t)S^0(t)-(\gamma(t)+\alpha(t)+\mu(t))}{\varepsilon}, 1\right\}$. Parameters $M_i > 0$ and $\varepsilon_i > 0$ $(i = 1, 2)$ will be chosen later. We have the following lemmas to indicate that (ϕ^+, ψ^+) is the upper solution and (ϕ^-, ψ^-) is the lower solution for system (14).

Lemma 3. *Function* $\phi^+ = S^0(t)$ *satisfies*

$$\phi_t^+ \geq d_1\phi_{zz}^+ - c\phi_z^+ + \Lambda(t) - \beta(t)\phi^+\psi^- - \mu(t)\phi^+.$$

Proof. It follows from $\phi_t^+ = S_t^0(t) = \Lambda(t) - \mu(t)S^0(t)$ and $\psi^- \geq 0$ that

$$\begin{aligned} &d_1\phi_{zz}^+ - c\phi_z^+ + \Lambda(t) - \beta(t)\phi^+\psi^- - \mu(t)\phi^+ \\ &\quad = \Lambda(t) - \beta(t)\phi^+\psi^- - \mu(t)\phi^+ \leq \Lambda(t) - \mu(t)\phi^+ = \phi_t^+. \end{aligned}$$

Therefore, the conclusion in Lemma 3 holds. □

Lemma 4. *Function* $\psi^+ = \min\{K(t)e^{\lambda_1 z}, K_\varepsilon\}$ *satisfies*

$$\psi_t^+ \geq d_2\psi_{zz}^+ - c\psi_z^+ + \beta(t)\phi^+\psi^+ - (\gamma(t) + \alpha(t) + \mu(t))\psi^+ - \varepsilon{\psi^+}^2,$$

for any $z \neq z_1 := \frac{1}{\varepsilon_1}\ln K_\varepsilon$.

Proof. If $z > z_1$, then $\psi^+ = K_\varepsilon$. Thus,

$$\begin{aligned} &d_2\psi_{zz}^+ - c\psi_z^+ + \beta(t)\phi^+\psi^+ - (\gamma(t) + \alpha(t) + \mu(t))\psi^+ - \varepsilon{\psi^+}^2 \\ &= \beta(t)S^0(t)K_\varepsilon - (\gamma(t) + \alpha(t) + \mu(t))K_\varepsilon - \varepsilon K_\varepsilon^2 \\ &= K_\varepsilon[\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t)) - \varepsilon K_\varepsilon] \leq 0 = \psi_t^+. \end{aligned}$$

If $z < z_1$, then $\psi^+ = K(t)e^{\lambda_1 z}$. Thus,

$$\begin{aligned} &d_2\psi_{zz}^+ - c\psi_z^+ + \beta(t)\phi^+\psi^+ - (\gamma(t) + \alpha(t) + \mu(t))\psi^+ - \varepsilon{\psi^+}^2 \\ &\leq d_2\lambda_1^2K(t)e^{\lambda_1 z} - c\lambda_1K(t)e^{\lambda_1 z} + \beta(t)S^0(t)K(t)e^{\lambda_1 z} - (\gamma(t) + \alpha(t) + \mu(t))K(t)e^{\lambda_1 z} \\ &= [d_2\lambda_1^2 - c\lambda_1 + \beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))]K(t)e^{\lambda_1 z} = \psi_t^+. \end{aligned}$$

The proof is completed. □

Lemma 5. *Suppose* $0 < \varepsilon_1 < \min\{\lambda_1, \frac{c}{d_1}\}$ *and* $M_1 > 1$ *large enough. Then,* $\phi^- = \max\{S^0(t)(1 - M_1e^{\varepsilon_1 z}), 0\}$ *satisfies*

$$\phi_t^- \leq d_1\phi_{zz}^- - c\phi_z^- + \Lambda(t) - \beta(t)\phi^-\psi^+ - \mu(t)\phi^-, \tag{15}$$

for any $z \neq z_2 := \frac{1}{\varepsilon_1}\ln\frac{1}{M_1}$.

Proof. If $z > z_2$, then $\phi^- = 0$. Clearly, (15) holds. If $z < z_2$, then $\phi^- = S^0(t)(1 - M_1e^{\varepsilon_1 z})$ and $\psi^+ = K(t)e^{\lambda_1 z}$ since $z_2 < 0 < z_1$. Therefore,

$$\begin{aligned} &d_1\phi_{zz}^- - c\phi_z^- + \Lambda(t) - \beta(t)\phi^-\psi^+ - \mu(t)\phi^- \\ =&(-d_1\varepsilon_1^2 + c\varepsilon_1)S^0(t)M_1e^{\varepsilon_1 z} + \Lambda(t) - \beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} - \mu(t)S^0(t)(1 - M_1e^{\varepsilon_1 z}) \\ =&(c\varepsilon_1 - d_1\varepsilon_1^2)S^0(t)M_1e^{\varepsilon_1 z} + \Lambda(t) - \mu(t)S^0(t) - \beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} + \mu(t)S^0(t)M_1e^{\varepsilon_1 z} \\ \geq&\Lambda(t) - \mu(t)S^0(t) - \beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} + \mu(t)S^0(t)M_1e^{\varepsilon_1 z} \\ =&S_t^0(t) - \beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} + \mu(t)S^0(t)M_1e^{\varepsilon_1 z}. \end{aligned}$$

To obtain inequality (15), we only need to verify

$$S_t^0(t) - \beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} + \mu(t)S^0(t)M_1e^{\varepsilon_1 z} \geq \phi_t^-(t,z) = S_t^0(t)(1 - M_1e^{\varepsilon_1 z}),$$

that is

$$\beta(t)S^0(t)(1 - M_1e^{\varepsilon_1 z})K(t)e^{\lambda_1 z} \leq S_t^0(t)M_1e^{\varepsilon_1 z} + \mu(t)S^0(t)M_1e^{\varepsilon_1 z} = \Lambda(t)M_1e^{\varepsilon_1 z}.$$

It is sufficient to verify

$$\beta(t)S^0(t)e^{(\lambda_1 - \varepsilon_1)z}K(t) \leq \Lambda(t)M_1. \tag{16}$$

Since $z < z_2 = \frac{1}{\varepsilon_1}\ln\frac{1}{M_1} < 0$ and $0 < \varepsilon_1 < \{\lambda_1, \frac{c}{d_1}\}$, we know that $e^{(\lambda_1-\varepsilon_1)z} < 1$. Combining the positivity of $\Lambda(t)$, taking $M_1 > \max_{t\in[0,T]}\{\frac{\beta(t)S^0(t)K(t)}{\Lambda(t)}, 1\}$, then inequality (16) is true. Therefore, (15) holds for all $z < z_2$. This completes the proof. □

Lemma 6. *Suppose* $0 < \varepsilon_2 < \min\{\varepsilon_1, \lambda_2 - \lambda_1\}$ *and* $M_2 > 1$ *large enough. Then,* $\psi^- := \max\{K(t)e^{\lambda_1 z}(1 - M_2 e^{\varepsilon_2 z}), 0\}$ *satisfies the inequality*

$$\psi_t^- \le d_2\psi_{zz}^- - c\psi_z^- + \beta(t)\phi^-\psi^- - (\gamma(t)+\alpha(t)+\mu(t))\phi^- - \varepsilon{\psi^-}^2, \tag{17}$$

for any $z \ne z_3 := \frac{1}{\varepsilon_2}\ln\frac{1}{M_2}$.

Proof. If $z > z_3$, then $\psi^- = 0$, which implies that (17) holds. If $z < z_3$, we let $M_2 > 1$ large enough, such that $z_3 = \frac{1}{\varepsilon_2}\ln\frac{1}{M_2} < z_2$. Then, $\psi^- = K(t)e^{\lambda_1 z}(1 - M_2e^{\varepsilon_2 z})$ and $\phi^- = S^0(t)(1 - M_1e^{\varepsilon_1 z})$ since $z_3 < z_2$. Rewriting inequality (17), we have

$$\psi_t^- - d_2\psi_{zz}^- + c\psi_z^- - \big(\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t))\big)\psi^- \le \beta(t)\big(\phi^- - S^0(t)\big)\psi^- - \varepsilon{\psi^-}^2. \tag{18}$$

By direct calculations, we can obtain that

$$\begin{aligned}
&\psi_t^- - d_2\psi_{zz}^- + c\psi_z^- - \big(\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t))\big)\psi^-(t,z)\\
&= K'(t)e^{\lambda_1 z}(1 - M_2e^{\varepsilon_2 z}) - d_2K(t)\big(\lambda_1^2e^{\lambda_1 z} - (\lambda_1+\varepsilon_2)^2M_2e^{(\lambda_1+\varepsilon_2)z}\big)\\
&+ cK(t)\big(\lambda_1e^{\lambda_1 z} - (\lambda_1+\varepsilon_2)e^{(\lambda_1+\varepsilon_2)z}\big) - \big(\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t))\big)K(t)e^{\lambda_1 z}(1 - M_2e^{\varepsilon_2 z})\\
&= e^{\lambda_1 z}\big[K'(t) - d_2\lambda_1^2K(t) + c\lambda_1K(t) - \big(\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t))\big)K(t)\big] - M_2e^{(\lambda_1+\varepsilon_2)z}\\
&\times\big[K'(t) - d_2(\lambda_1+\varepsilon_2)^2K(t) + c(\lambda_1+\varepsilon_2)K(t) - \big(\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t))\big)K(t)\big]\\
&= -M_2e^{(\lambda_1+\varepsilon_2)z}K(t)\big[(d_2\lambda_1^2 - c\lambda_1) - \big(d_2(\lambda_1+\varepsilon_2)^2 - c(\lambda_1+\varepsilon_2)\big)\big]\\
&= M_2e^{(\lambda_1+\varepsilon_2)z}K(t)\Phi_c(\lambda_1+\varepsilon_2).
\end{aligned}$$

Clearly, inequality (18) is equivalent to

$$M_2e^{(\lambda_1+\varepsilon_2)z}K(t)\Phi_c(\lambda_1+\varepsilon_2) \le \beta(t)\big(\phi^- - S^0(t)\big)\psi^- + \varepsilon{\psi^-}^2, \tag{19}$$

that is,

$$\begin{aligned}
&- M_2e^{(\lambda_1+\varepsilon_2)z}K(t)\Phi_c(\lambda_1+\varepsilon_2)\\
\ge&\beta(t)S^0(t)M_1K(t)e^{(\lambda_1+\varepsilon_1)z}(1 - M_2e^{\varepsilon_2 z}) - \varepsilon e^{2\lambda_1 z}(1 - M_2e^{\varepsilon_2 z})^2K^2(t).
\end{aligned}$$

It is obvious that

$$-M_2\Phi_c(\lambda_1+\varepsilon_2) \ge \beta(t)S^0(t)M_1e^{(\varepsilon_1-\varepsilon_2)z}(1 - M_2e^{\varepsilon_2 z}) - \varepsilon e^{(\lambda_1-\varepsilon_2)z}(1 - M_2e^{\varepsilon_2 z})^2K(t). \tag{20}$$

Noting $0 < \varepsilon_2 < \min\{\varepsilon_1, \lambda_2 - \lambda_1\}$ and $0 < \varepsilon_1 < \lambda_1$, we have $\lambda_1 - \varepsilon_2 > 0$, $\varepsilon_1 - \varepsilon_2 > 0, z < 0$ and $0 < 1 - M_2e^{\varepsilon_2 z} < 1$. Hence,

$$\beta(t)S^0(t)M_1e^{(\varepsilon_1-\varepsilon_2)z}(1 - M_2e^{\varepsilon_2 z}) + \varepsilon e^{(\lambda_1-\varepsilon_2)z}(1 - M_2e^{\varepsilon_2 z})^2K(t) \le \beta(t)S^0(t)M_1. \tag{21}$$

Combining (20) and (21), to obtain inequality (18), it is sufficient to verify

$$\beta(t)S^0(t)M_1 \le -M_2\Phi_c(\lambda_1+\varepsilon_2). \tag{22}$$

Obviously, the inequality (22) is true as long as $M_2 > \max_{t\in[0,T]}\{\frac{\beta(t)S^0(t)M_1}{-\Phi_c(\lambda_1+\varepsilon_2)}, \frac{1}{e^{\varepsilon_2 z_2}}, 1\}$ large enough. This completes the proof. □

4. Periodic traveling wave solution

In this section, the traveling waves of system (14) will be investigated by some integral transformations and fixed-point theorems. Firstly, we need to construct an appropriate bounded closed convex set by the upper–lower solutions obtained in the previous section.

For any given constants $\eta > 0$ and $T > 0$, denote by $B_\eta([0,T]\times\mathbb{R},\mathbb{R}^2)$ the Banach space of bounded and uniformly continuous functions $v = (v_1, v_2) : [0,T]\times\mathbb{R} \to \mathbb{R}^2$ satisfying $v(0,\xi) = v(T,\xi)$ for all $\xi \in \mathbb{R}$ with the norm defined by

$$\|v\|_\eta := \max\left\{\sup_{t\in[0,T],\xi\in\mathbb{R}} e^{-\eta|\xi|}|v_1(t,\xi)|, \sup_{t\in[0,T],\xi\in\mathbb{R}} e^{-\eta|\xi|}|v_2(t,\xi)|\right\}.$$

Let

$$\mathcal{D} := \left\{(\widetilde{\phi},\widetilde{\psi}) \in B_\mu([0,T]\times\mathbb{R},\mathbb{R}^2) : \phi^- \le \widetilde{\phi} \le \phi^+, \psi^- \le \widetilde{\psi} \le \psi^+\right\}.$$

Choose two constants α_i $(i = 1,2)$ satisfying $\alpha_1 > \max_{t\in[0,T]}\{\beta(t)S^0(t) + \mu(t), \beta(t)K_\varepsilon + \mu(t)\}$ and $\alpha_2 > \max_{t\in[0,T]}\{\gamma(t) + \alpha(t) + \mu(t)\} + 2\varepsilon K_\varepsilon$. Define

$$f_1[\phi,\psi](t,z) = \alpha_1\phi(t,z) + \Lambda(t) - \beta(t)\phi(t,z)\psi(t,z) - \mu(t)\phi(t,z),$$
$$f_2[\phi,\psi](t,z) = \alpha_2\psi(t,z) + \beta(t)\phi(t,z)\psi(t,z) - (\gamma(t) + \alpha(t) + \mu(t))\psi(t,z) - \varepsilon\psi^2(t,z).$$

For $(\widetilde{\phi},\widetilde{\psi}) \in \mathcal{D}$, consider the following initial value problem:

$$\begin{cases} \phi_t - d_1\phi_{zz} + c\phi_z + \alpha_1\phi = f_1[\widetilde{\phi},\widetilde{\psi}], \\ \psi_t - d_2\psi_{zz} + c\psi_z + \alpha_2\psi = f_2[\widetilde{\phi},\widetilde{\psi}], \\ \phi(0,z) = \phi_0(z), \psi(0,z) = \psi_0(z), z \in \mathbb{R}. \end{cases} \tag{23}$$

By means of analytic semigroups, the problem (23) is written as the integral equations

$$\begin{cases} \phi(t,z) = T_1(t)\phi_0(z) + \int_0^t T_1(t-s)f_1[\widetilde{\phi},\widetilde{\psi}](s,z)ds, \\ \psi(t,z) = T_2(t)\psi_0(z) + \int_0^t T_2(t-s)f_2[\widetilde{\phi},\widetilde{\psi}](s,z)ds, \end{cases} \tag{24}$$

where $T_i(t)$ is the analytic semigroup (see [28]) generated by the operator $\mathcal{A}_i : D(\mathcal{A}_i) \to C(\mathbb{R})$ defined by $\mathcal{A}_i\nu = d_i\nu_{zz} - c\nu_z - \alpha_i\nu$ $(i = 1,2)$, and

$$D(\mathcal{A}_i) := \left\{\nu \in \bigcap_{1\le p<\infty} W^{2,p}_{loc}(\mathbb{R}) : \mathcal{A}_i\nu \in C(\mathbb{R})\right\}, i = 1,2,$$

where the definition of $W^{2,p}_{loc}(\mathbb{R})$ can be found in [26]. Further, we have

$$(T_i(t)\nu)(x) = e^{-\alpha_i t}\frac{1}{\sqrt{4\pi d_i t}}\int_{\mathbb{R}} e^{-\frac{(x-ct-\xi)^2}{4d_i t}}\nu(\xi)d\xi,\ t > 0,\ x \in \mathbb{R},\ \nu(\cdot) \in \Gamma, \tag{25}$$

where Γ is the Banach space of bounded and uniformly continuous functions $\nu : \mathbb{R} \to \mathbb{R}$ with the supremum norm.

It follows from the Definition 4.1.4 in [28] that the solution of equations (24) is the mild solution of problem (23). Define the Banach space

$$\widetilde{B}_\eta(\mathbb{R},\mathbb{R}^2) := \left\{ \nu = (\nu_1,\nu_2) : \nu_i \in \Gamma,\ \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}|\nu_i(\xi)| < +\infty,\ i = 1,2 \right\},$$

with the norm $\|\nu\|_\eta := \max\{\sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}|\nu_1(\xi)|,\ \sup_{\xi\in\mathbb{R}} e^{-\eta|\xi|}|\nu_2(\xi)|\}$.

From [27], we introduce the following interesting property which will be used in the proof of main results of this paper.

Proposition 1. *(see [27]) Let the set $\Omega \subset Y^2 := Y \times Y$ and $Y = C([0,T]\times\mathbb{R},\mathbb{R})$. If for any bounded interval $I \subset \mathbb{R}$, the restriction of Ω on $[0,T]\times I$ is precompact in the sense of the supremum norm, then Ω is precompact in the sense of the norm $\|\cdot\|_\eta$.*

Moreover, we need to introduce the following important lemma which will be used in the proof of Lemma 9.

Lemma 7. *(see [28]) Let $\rho : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function such that $t \mapsto \rho(t,\cdot)$ belongs to $C([0,T],C(\mathbb{R})$, and let $T(t) : C(\mathbb{R}) \to C(\mathbb{R})$ is a compact semigroup for $t \geq 0$. Set*

$$\mu(t,x) = (T(t)\mu_0)(x) + \int_0^t T(t-\tau)\rho(\tau,x)d\tau,\ \ \mu_0 \in C(\mathbb{R}). \tag{26}$$

Then, the function $\mu \in C([0,T]\times\mathbb{R}) \cap C^{\theta,2\theta}([\epsilon,T]\times\mathbb{R})$ for any $0 < \epsilon < T$ and some $0 < \theta < 1$. Moreover, there exist constants $P > 0$ and $M(\epsilon,\theta) > 0$ such that

$$\|\mu\|_\infty \leq P(\|\mu_0\|_\infty + \|\rho\|_\infty), \tag{27}$$

and

$$\|\mu\|_{C^{\theta,2\theta}([\epsilon,T]\times\mathbb{R})} \leq M(\epsilon,\theta)(\epsilon^{-\theta}\|\mu_0\|_\infty + \|\rho\|_\infty). \tag{28}$$

With the help of the above results, we will discuss the existence of time-periodic solutions of system (10) by using the fixed-point theory. Let

$$\widetilde{\mathcal{D}} := \left\{ (\phi_0(\cdot),\psi_0(\cdot)) \in \widetilde{B}_\eta(\mathbb{R},\mathbb{R}^2) : \begin{array}{l} \phi^-(0,z) \leq \phi_0(z) \leq \phi^+(0,z) \\ \psi^-(0,z) \leq \psi_0(z) \leq \psi^+(0,z) \end{array} \right\}.$$

Now we need to prove the fact that for any $(\widetilde{\phi},\widetilde{\psi}) \in \mathcal{D}$, system (24) admits a unique solution $(\phi^*,\psi^*) \in \mathcal{D}$. To obtain this, we need the invariance of integral equations (24) and introduce the following useful inequalities.

Lemma 8. *Functions ϕ^+, ψ^+, ϕ^- and ψ^- satisfy*

$$\phi^+(t,\cdot) \geq T_1(t)\phi^+(0,\cdot) + \int_0^t T_1(t-s)f_1[\phi^+,\psi^-](s,\cdot)ds. \tag{29}$$

$$\psi^+(t,\cdot) \geq T_2(t)\psi^+(0,\cdot) + \int_0^t T_2(t-s)f_2[\phi^+,\psi^+](s,\cdot)ds. \tag{30}$$

$$\phi^-(t,\cdot) \leq T_1(t)\phi^-(0,\cdot) + \int_0^t T_1(t-s)f_1[\phi^-,\psi^+](s,\cdot)ds. \tag{31}$$

and

$$\psi^-(t,\cdot) \le T_2(t)\psi^-(0,\cdot) + \int_0^t T_2(t-s) f_2[\phi^-,\psi^-](s,\cdot)ds. \tag{32}$$

Proof. It follows from Lemma 3 and $\mathcal{A}_1\phi = d_1\phi_{zz} - c\phi_z - \alpha_1\phi$ that

$$\phi_t^+ = \Lambda(t) - \mu(t)\phi^+ = \mathcal{A}_1\phi^+ + \alpha_1\phi^+ + \Lambda(t) - \mu(t)\phi^+.$$

Combining the positivity of $T_1(t)$ yields

$$\begin{aligned}\phi^+(t,\cdot) =& T_1(t)\phi^+(0,\cdot) + \int_0^t T_1(t-s)[\alpha_1\phi^+(s,\cdot) + \Lambda(s) - \mu(s)\phi^+(s,\cdot)]ds \\ \ge& T_1(t)\phi^+(0,\cdot) + \int_0^t T_1(t-s) f_1[\phi^+,\psi^-](s,\cdot)ds.\end{aligned}$$

This implies that the inequality (29) holds.

Combining the definition of $\psi^+(t,z)$, Lemma 4 and $\mathcal{A}_2\psi = d_2\psi_{zz} - c\psi_z - \alpha_2\psi$. If $\psi^+(t,z) = K_\varepsilon$, then $\psi_t^+ = 0 = \mathcal{A}_2\psi^+ + \alpha_2\psi^+(t)$. Combining the fact $\beta(t)S^0(t)K_\varepsilon - (\gamma(t)+\alpha(t)+\mu(t))K_\varepsilon - \varepsilon {K_\varepsilon}^2 \le 0$, one has

$$\begin{aligned}\psi^+(t,\cdot) =& T_2(t)\psi^+(0,\cdot) + \int_0^t T_2(t-s)[\alpha_2\psi^+(s,\cdot)]ds \\ \ge& T_2(t)\psi^+(0,\cdot) + \int_0^t T_2(t-s) f_2[\phi^+,\psi^+](s,\cdot)ds.\end{aligned}$$

If $\psi^+ = K(t)e^{\lambda_1 z}$, then $\psi_t^+ = \mathcal{A}_2\psi^+ + \alpha_2\psi^+ + (\beta(t)S^0(t) - (\gamma(t)+\alpha(t)+\mu(t)))\psi^+$. Therefore,

$$\begin{aligned}\psi^+(t,\cdot) =& T_2(t)\psi^+(0,\cdot) + \int_0^t T_2(t-s)[\alpha_2\psi^+(s,\cdot) + \big(\beta(s)S^0(s) - (\gamma(s)+\alpha(s)+\mu(s))\big)\psi^+(s,\cdot)]ds \\ \ge& T_2(t)\psi^+(0,\cdot) + \int_0^t T_2(t-s) f_2[\phi^+,\psi^+](s,\cdot)ds.\end{aligned}$$

This shows that the inequality (30) holds.

Let $\hat{\phi}^-(t,z) = \phi^-(t,z+ct)$ and $\hat{\psi}^+(t,z) = \psi^+(t,z+ct)$. Combining Lemma 5 yields

$$\hat{\phi}_t^- - d_1\hat{\phi}_{zz}^- + \alpha_1\hat{\phi}^- - f_1[\hat{\phi}^-,\hat{\psi}^+] \le 0,$$

for $z \ne \hat{z}_1(t) := -\varepsilon_1^{-1}\ln M_1 - ct$. A straightforward computation shows

$$\frac{\partial\hat{\phi}^-(t,\hat{z}_1(t)-0)}{\partial z} = \lim_{z\to\hat{z}_1(t)-0}\left\{-S^0(t)M_1\varepsilon_1 e^{\varepsilon_1(z+ct)}\right\} = -\varepsilon_1 M_1 S^0(t) < 0.$$

Let $\Gamma(t,z) := -\hat{\phi}_t^- + d_1\hat{\phi}_{zz}^- - \alpha_1\hat{\phi}^- + f_1[\hat{\phi}^-,\hat{\psi}^+] \ge 0$ and

$$H(\hat{\phi}^-)(t,z,s) := \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy. \tag{33}$$

Then, the derivative of $H(\hat{\phi}^-)$ respect to s satisfies

$$
\begin{aligned}
&\frac{\partial}{\partial s}H(\hat{\phi}^-)(t,z,s)\\
=&\frac{\alpha_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy+\frac{e^{-\alpha_1(t-s)}}{2(t-s)\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy\\
&-\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}}\frac{(z-y)^2}{4d_1(t-s)^2}e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy\\
&+\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\frac{\partial^2\hat{\phi}^-(s,y)}{\partial y^2}dy-\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\alpha_1\hat{\phi}^-(s,y)dy\\
&+\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\Big[f_1[\hat{\phi}^-,\hat{\psi}^+](s,y)-\Gamma(s,y)\Big]dy.
\end{aligned}
$$

Combining $\hat{\phi}^-(t,z)\equiv 0$ when $z>\hat{z}_1(t)$, and using the integral formula of parts, we obtain

$$
\begin{aligned}
&\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\frac{\partial^2\hat{\phi}^-(s,y)}{\partial y^2}dy=\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{-\infty}^{\hat{z}_1(s)} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\frac{\partial^2\hat{\phi}^-(s,y)}{\partial y^2}dy\\
=&\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}}\frac{\partial\hat{\phi}^-(s,\hat{z}_1(s)-0)}{\partial z}-\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{-\infty}^{\hat{z}_1(s)}\frac{e^{-\frac{(z-y)^2}{4d_1(t-s)}}}{2(t-s)}\hat{\phi}^-(s,y)dy\\
&+\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{-\infty}^{\hat{z}_1(s)}\frac{(z-y)^2}{4d_1(t-s)^2}e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy.
\end{aligned}
$$

Noting $\frac{\partial\hat{\phi}^-(t,\hat{z}_1(t)-0)}{\partial z}=-\varepsilon_1 M_1 S^0(t)$, thus,

$$
\begin{aligned}
\frac{\partial}{\partial s}H(\hat{\phi}^-)(t,z,s)=&-\varepsilon_1 M_1 S^0(t)\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}}\\
&+\frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\Big[f_1[\hat{\phi}^-,\hat{\psi}^+](s,y)-\Gamma(s,y)\Big]dy.
\end{aligned}
\tag{34}
$$

Furthermore, from Chapter I in [26], we know

$$
\hat{\phi}^-(t,z)=\lim_{s\to t-0}T(t-s)\hat{\phi}^-(t,z)=\lim_{s\to t-0}\frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int\limits_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\hat{\phi}^-(s,y)dy. \tag{35}
$$

Therefore, from (34), (35) and Newton–Leibniz formula, we have

$$\begin{aligned}\hat{\phi}^-(t,z) &= \lim_{\xi\to 0^+} H(\hat{\phi}^-)(t,z,t-\xi) = H(\hat{\phi}^-)(t,z,0) + \lim_{\xi\to 0^+}\int_0^{t-\xi}\frac{\partial}{\partial s}H(\hat{\phi}^-)(t,z,s)ds\\ &= \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}}\hat{\phi}^-(0,y)dy - \varepsilon_1 M_1 S^0(t)\int_0^t \frac{d_1 e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}e^{-\frac{(z-\hat{z}_1(s))^2}{4d_1(t-s)}}ds\\ &\quad + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}}\Big[f_1[\hat{\phi}^-,\hat{\psi}^+](s,y)-\Gamma(s,y)\Big]dyds.\end{aligned}$$

Due to $\Gamma(s,y)\geq 0$, we have

$$\begin{aligned}\hat{\phi}^-(t,z) \leq& \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}}\hat{\phi}^-(0,y)dy\\ &+ \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} f_1[\hat{\phi}^-,\hat{\psi}^+](s,y)dyds.\end{aligned}$$

Using an integral transformation, one has

$$\begin{aligned}&\phi^-(t,z+ct)\\ \leq& \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}}\phi^-(0,y)dy + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1(t-s)}} f_1[\phi^-,\psi^+](s,y+cs)dyds\\ =& \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{4d_1 t}}\phi^-(0,y)dy + \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(z-y+cs)^2}{4d_1(t-s)}} f_1[\phi^-,\psi^+](s,y)dyds.\end{aligned}$$

Let $\hat{z}=z+ct$, then

$$\begin{aligned}\phi^-(t,\hat{z}) \leq& \frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(\hat{z}-y-ct)^2}{4d_1 t}}\phi^-(0,y)dy\\ &+ \int_0^t \frac{e^{-\alpha_1(t-s)}}{\sqrt{4\pi d_1(t-s)}}\int_{\mathbb{R}} e^{-\frac{(\hat{z}-y-c(t-s))^2}{4d_1(t-s)}} f_1[\phi^-,\psi^+](s,y)dyds\\ =& T_1(t)\phi^-(0,\cdot) + \int_0^t T_1(t-s)f_1[\phi^-,\psi^+](s,\cdot)ds.\end{aligned}$$

Therefore, the inequality (31) holds.

Lastly, the inequality (32) can be obtained by similar method. This completes the proof. □

Lemma 9. *Let* $\big(\phi(t,z;\phi_0,\psi_0),\psi(t,z;\phi_0,\psi_0)\big)$ *be the solution of integral equations (24) with the initial value* $(\phi_0,\psi_0)\in\widetilde{\mathcal{D}}$. *Then,*

$$\phi^- \leq \phi(t,z;\phi_0,\psi_0)\leq \phi^+,\ \ \psi^-\leq \psi(t,z;\phi_0,\psi_0)\leq\psi^+,\ \forall\ (t,z)\in[0,T]\times\mathbb{R}.$$

Proof. Due to $(\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}$ and $(\phi_0,\psi_0)\in\widetilde{\mathcal{D}}$, thus $\phi^-\leq\widetilde{\phi}\leq\phi^+,\ \psi^-\leq\widetilde{\psi}\leq\psi^+$, and

$$\phi^-(0,z)\leq\phi_0(z)\leq\phi^+(0,z),\ \psi^-(0,z)\leq\psi_0(z)\leq\psi^+(0,z).$$

Since $\phi^+ = S^0(t)$, one has $\phi_t^+ = \mathcal{A}_1\phi^+ + \alpha_1\phi^+ + \Lambda(t) - \mu(t)\phi^+$, thus

$$\phi^+(t,\cdot) = T_1(t)\phi^+(0,\cdot) + \int_0^t T_1(t-s)[\alpha_1\phi^+(s,\cdot) + \Lambda(s) - \mu(s)\phi^+(s,\cdot)]ds. \tag{36}$$

Since $\alpha_1 > \max_{t\in[0,T]}\{\beta(t)S^0(t)+\mu(t), \beta(t)K_\varepsilon+\mu(t)\}$, we have $\alpha_1\phi^+ + \Lambda(t) - \mu(t)\phi^+ \geq \alpha_1\tilde{\phi} + \Lambda(t) - \mu(t)\tilde{\phi} \geq \alpha_1\tilde{\phi} + \Lambda(t) - \beta(t)\tilde{\phi}\tilde{\psi} - \mu(t)\tilde{\phi} = f_1[\widetilde{\phi}, \widetilde{\psi}]$. Noting the positivity of semigroup $T_1(t)$, from (36) we can deduce

$$\begin{aligned}\int_0^t T_1(t-s)f_1[\widetilde{\phi},\widetilde{\psi}](s,\cdot)ds \leq & \int_0^t T_1(t-s)[\alpha_1\phi^+(s,\cdot) + \Lambda(s) - \mu(s)\phi^+(s,\cdot)]ds \\ = & \phi^+(t,\cdot) - T_1(t)\phi^+(0,\cdot) \leq \phi^+(t,\cdot) - T_1(t)\phi_0(\cdot),\ t\in(0,T].\end{aligned}$$

Consequently, $\phi(t,z;\phi_0,\psi_0) \leq \phi^+(t,z)$.

Let $\Phi(t,z) = \phi(t,z;\phi_0,\psi_0) - \phi^-(t,z)$. Combining the inequality (31) in Lemma 8, we can deduce

$$\begin{aligned}\Phi(t,\cdot) \geq & T_1(t)[\phi_0 - \phi^-(0,\cdot)] \\ & + \int_0^t T_1(t-s)\left(\alpha_1[\widetilde{\phi} - \phi^-](s,\cdot) - \beta(s)[\widetilde{\phi}\widetilde{\psi} - \phi^-\psi^+](s,\cdot) - \mu(s)[\widetilde{\phi} - \phi^-](s,\cdot)\right)ds \\ \geq & T_1(t)[\phi_0 - \phi^-(0,\cdot)] + \int_0^t T_1(t-s)\left(\alpha_1[\widetilde{\phi} - \phi^-](s,\cdot) - (\beta(s)K_\varepsilon + \mu(s))[\widetilde{\phi} - \phi^-](s,\cdot)\right)ds.\end{aligned}$$

Noting $\alpha_1 > \max_{t\in[0,T]}\{\beta(t)K_\varepsilon + \mu(t), \beta(t)S^0(t) + \mu(t)\}$ and $\widetilde{\phi} \geq \phi^-$, one has $\Phi(t,z) \geq 0$, that is $\phi(t,z;\phi_0,\psi_0) \geq \phi^-(t,z)$.

Recalling $\alpha_2 > \max_{t\in[0,T]}\{\gamma(t) + \alpha(t) + \mu(t)\} + 2\varepsilon K_\varepsilon$, we obtain

$$\begin{aligned}&\int_0^t T_2(t-s)f_2[\widetilde{\phi},\widetilde{\psi}](s,\cdot)ds \\ =& \int_0^t T_2(t-s)\left[\alpha_2\widetilde{\psi}(s,\cdot) + \beta(s)\widetilde{\phi}(s,\cdot)\widetilde{\psi}(s,\cdot) - (\gamma(s)+\alpha(s)+\mu(s))\widetilde{\psi}(s,\cdot) - \varepsilon\widetilde{\psi}^2(s,\cdot)\right]ds \\ \leq& \int_0^t T_2(t-s)\left[\alpha_2\psi^+(s,\cdot) + \big(\beta(s)p(s) - (\gamma(s)+\alpha(s)+\mu(s))\big)\psi^+(s,\cdot) - \varepsilon{\psi^+}^2(s,\cdot)\right](s,\cdot)ds \\ =& \int_0^t T_2(t-s)f_2[\phi^+,\psi^+](s,\cdot)ds,\end{aligned}$$

for $t\in(0,T]$. Combining inequality (30), one has

$$\int_0^t T_2(t-s)f_2[\widetilde{\phi},\widetilde{\psi}](s,\cdot)ds \leq \psi^+(t,\cdot) - T_2(t)\psi^+(0,\cdot) \leq \psi^+(t,\cdot) - T_2(t)\psi_0(\cdot),\ t\in(0,T].$$

According to $\psi(0,z) \leq \psi^+(0,z)$, it further obtains that $\psi(t,z;\phi_0,\psi_0) \leq \psi^+(t,z)$.

From $\alpha_2 > \max_{t\in[0,T]}\{\gamma(t)+\alpha(t)+\mu(t)\}+2\varepsilon K_\varepsilon$ and inequality (32), we deduce that

$$\begin{aligned}\int_0^t T_2(t-s)f_2[\widetilde{\phi},\widetilde{\psi}](s,\cdot)ds &\geq \int_0^t T_2(t-s)f_2[\phi^-,\psi^-](s,\cdot)ds\\ &\geq \psi^-(t,\cdot)-T_2(t)\psi^-(0,\cdot)\geq \psi^-(t,\cdot)-T_2(t)\psi_0(\cdot),\end{aligned}$$

which implies $T_2(t)\psi_0(z)+\int_0^t T_2(t-s)f_2[\widetilde{\phi},\widetilde{\psi}](s,z)ds \geq \psi^-(t,z)$. Since $\psi(0,z)=\psi_0(z)\geq\psi^-(0,z)$, we further obtain $\psi(t,z;\phi_0,\psi_0)\geq\psi^-(t,z)$. This completes the proof. □

For any $(\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}$, define the map $\Gamma_{(\widetilde{\phi},\widetilde{\psi})}$ for system (24) as follows:

$$\Gamma_{(\widetilde{\phi},\widetilde{\psi})}(\phi_0(\cdot),\psi_0(\cdot))=(\phi(T,\cdot;\phi_0,\psi_0),\psi(T,\cdot;\phi_0,\psi_0)),\ \ (\phi_0(\cdot),\psi_0(\cdot))\in\widetilde{\mathcal{D}}.$$

Theorem 1. *For any given $(\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}$, map $\Gamma_{(\widetilde{\phi},\widetilde{\psi})}$ admits a unique fixed point in $\widetilde{\mathcal{D}}$.*

Proof. It follows from Lemma 9 that $\Gamma_{(\widetilde{\phi},\widetilde{\psi})}(\widetilde{\mathcal{D}})\subset\widetilde{\mathcal{D}}$. Let $(\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}$, then $f_i[\widetilde{\phi},\widetilde{\psi}](t,\cdot)\in C([0,T],C(\mathbb{R}))$, and $f_i[\widetilde{\phi},\widetilde{\psi}]$ $(i=1,2)$ are uniformly bounded. Combining Lemma 7, we can deduce that (ϕ,ψ) defined by system (24) belongs to $C([0,T]\times\mathbb{R},\mathbb{R})\cap C^{\theta,2\theta}([\epsilon,T]\times\mathbb{R},\mathbb{R})$ for any $0<\epsilon<T$ and some constant $0<\theta<1$. Moreover, there exist $C_1,\ C_2>0$ satisfying

$$\|\phi(T,\cdot)\|_{C^{2\theta}(\mathbb{R})}\leq C_1\big(\epsilon^{-\theta}\|\phi_0\|_\infty+\|f_1[\widetilde{\phi},\widetilde{\psi}]\|_\infty\big) \tag{37}$$

and

$$\|\psi(T,\cdot)\|_{C^{2\theta}(\mathbb{R})}\leq C_2\big(\epsilon^{-\theta}\|\psi_0\|_\infty+\|f_2[\widetilde{\phi},\widetilde{\psi}]\|_\infty\big). \tag{38}$$

From (37) and (38), we conclude that $\{(\phi(T,\cdot;\phi_0,\psi_0),\psi(T,\cdot;\phi_0,\psi_0)),(\phi_0,\psi_0)\in\widetilde{\mathcal{D}}\}$ is compact on $C(\mathbb{R},\mathbb{R}^2)$. On the other hand, for any given sequences $\{(\phi_{0n},\psi_{0n})\}\subset\widetilde{\mathcal{D}}$, let $\Gamma(\phi_{0n},\psi_{0n})=\big(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n})\big),\ n=1,2,\cdots$. Because $\phi_n(T,\cdot;\phi_{0n},\psi_{0n})$ and $\psi_n(T,\cdot;\phi_{0n},\psi_{0n})$ satisfy the inequalities (37) and (38), there exists a subsequence of $\{(\phi_n(T,\cdot;\phi_{0n},\ \psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\ \psi_{0n}))\}$, still labeled by $\{(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\ \ \psi_n(T,\cdot;\ \ \phi_{0n},\psi_{0n}))\}$, which satisfies $(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\ \ \psi_n(T,\cdot;\phi_{0n},\psi_{0n}))\rightarrow(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))$ as $n\to\infty$ in $C_{loc}([0,T]\times\mathbb{R},\mathbb{R}^2)$, that is, for any $Z>0$,

$$\begin{aligned}\lim_{n\to\infty}\|&(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n}))\\ &-(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))\|_{C([0,T]\times[-Z,Z],\mathbb{R}^2)}=0.\end{aligned} \tag{39}$$

Now, we show that

$$\lim_{n\to\infty}\|(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n})-(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*})\|_\mu=0.$$

From the boundedness of $\widetilde{\mathcal{D}}$, we can find

$$\|(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n}))-(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))\|_\mu$$

uniformly bounded for $n\in\mathbb{N}_+$. On the other hand, for any $\delta>0$, there exists some constant $Z^*>0$ such that

$$e^{-\mu|z|}|(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n}))-(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))|<\delta \tag{40}$$

for all $|z|>Z^*$ and $n\in\mathbb{N}_+$. Furthermore, for $|z|\leq Z^*$, we can extract a $N^*\in\mathbb{N}_+$ such that for all $n>N^*$, the following inequality holds:

$$e^{-\mu|z|}|(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n}))-(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))|<\delta. \tag{41}$$

Therefore, inequalities (40) and (41) mean that $(\phi_n(T,\cdot;\phi_{0n},\psi_{0n}),\psi_n(T,\cdot;\phi_{0n},\psi_{0n}))\to(\phi_*(T,\cdot;\phi_{0*},\psi_{0*}),\psi_*(T,\cdot;\phi_{0*},\psi_{0*}))$ as $n\to\infty$. Consequently, the Schauder's fixed-point theorem implies that $\Gamma_{(\widetilde{\phi},\widetilde{\psi})}$ admits a fixed point $(\phi_0^*,\psi_0^*)\in\widetilde{\mathcal{D}}$ satisfying $\phi(T,z;\phi_0^*,\psi_0^*)=\phi_0^*(z)$ and $\psi(T,z;\phi_0^*,\psi_0^*)=\psi_0^*(z)$ for $z\in\mathbb{R}$.

The uniqueness of fixed point will be proved by the contradiction. Suppose that there exists $(\phi_0^{**},\psi_0^{**})\in\widetilde{\mathcal{D}}$ such that $\phi(T,z;\phi_0^{**},\psi_0^{**})=\phi_0^{**}(z)$, $\psi(T,z;\phi_0^{**},\psi_0^{**})=\psi_0^{**}(z)$ for $z\in\mathbb{R}$. If $\phi_0^*(\cdot)\neq\phi_0^{**}(\cdot)$, then

$$|\phi(T,z;\phi_0^*,\psi_0^*)-\phi(T,z;\phi_0^{**},\psi_0^{**})|\le e^{-\alpha_1T}\int_{\mathbb{R}}\frac{e^{-\frac{(z-x-cT)^2}{4d_1T}}}{\sqrt{4\pi d_1T}}|\phi_0^*(x)-\phi_0^{**}(x)|dx$$

$$\le\|\phi_0^*(\cdot)-\phi_0^{**}(\cdot)\|_{L^\infty}e^{-\alpha_1T}\int_{\mathbb{R}}\frac{e^{-\frac{(z-x-cT)^2}{4d_1T}}}{\sqrt{4\pi d_1T}}dx=e^{-\alpha_1T}\|\phi_0^*(\cdot)-\phi_0^{**}(\cdot)\|_{L^\infty},$$

which implies $e^{-\alpha_1T}\ge 1$. This leads to a contradiction. Therefore, the uniqueness of $\phi_0^*(\cdot)$ is achieved. By the similar arguments, we also have $\psi_0^*(\cdot)=\psi_0^{**}(\cdot)$. The proof is completed. □

Remark 4. The unique fixed point $(\phi^*,\psi^*)\in\widetilde{\mathcal{D}}$ of map $\Gamma_{(\widetilde{\phi},\widetilde{\psi})}$ is corresponding to the unique positive solution $(\phi(t,z;\phi^*,\psi^*),\psi(t,z;\phi^*,\psi^*))$ of system (24) defined for $(t,z)\in[0,T]\times\mathbb{R}$ satisfying $(\phi^*,\psi^*)=(\phi(T,\cdot;\phi^*,\psi^*),\psi(T,\cdot;\phi^*,\psi^*))$.

Based on the above results, we can further define the solution map $\mathcal{F}:\mathcal{D}\to\mathcal{D}$ as follows:

$$\mathcal{F}(\widetilde{\phi},\widetilde{\psi})=(\phi,\psi),\ \ (\widetilde{\phi},\widetilde{\psi})\in\mathcal{D},$$

where $(\phi(t,z),\psi(t,z))$ is the unique positive solution of system (24) defined on $(t,z)\in[0,T]\times\mathbb{R}$ satisfying $(\phi(0,\cdot),\psi(0,\cdot))=(\phi(T,\cdot),\psi(T,\cdot))$.

Theorem 2. *Map $\mathcal{F}:\mathcal{D}\to\mathcal{D}$ admits a fixed point $(\widetilde{\phi}^*,\widetilde{\psi}^*)\in\mathcal{D}$.*

Proof. Firstly, for any $(\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}$, let $\mathcal{F}(\widetilde{\phi},\widetilde{\psi})=(\phi,\psi)$. Since $(\phi(t,z),\psi(t,z))$ is the positive solution of system (24) defined on $(t,z)\in[0,T]\times\mathbb{R}$ satisfying $(\phi(0,\cdot),\psi(0,\cdot))=(\phi(T,\cdot),\psi(T,\cdot))$, we also have $(\phi,\psi)\in\mathcal{D}$. Therefore, $\mathcal{F}$ maps the set $\mathcal{D}$ to oneself.

Similar to Lemma 2.7 in [18], we can easily obtain the continuity of map $\mathcal{F}:\mathcal{D}\to\mathcal{D}$ with respect to norm $\|\cdot\|_\eta$ of space $B_\eta([0,T]\times\mathbb{R},\mathbb{R}^2)$. Now, we consider the compactness of map $\mathcal{F}$ in $B_\eta([0,T]\times\mathbb{R},\mathbb{R}^2)$. By the periodicity of system (24), for $t\in[T,2T]$ we let

$$(\phi^*(t,z;\widetilde{\phi},\widetilde{\psi}),\psi^*(t,z;\widetilde{\phi},\widetilde{\psi}))=(\phi^*(t-T,z;\widetilde{\phi},\widetilde{\psi}),\psi^*(t-T,z;\widetilde{\phi},\widetilde{\psi})).\tag{42}$$

Then, $(\phi^*(t,z;\widetilde{\phi},\widetilde{\psi}),\psi^*(t,z;\widetilde{\phi},\widetilde{\psi}))$ is well defined for $t\in[0,2T]$ and satisfies system (24). Hence, by Lemma 7, we can obtain that the set

$$\{(\phi^*(t,z;\widetilde{\phi},\widetilde{\psi}),\psi^*(t,z;\widetilde{\phi},\widetilde{\psi})),\ t\in[\epsilon,2T],\ z\in\mathbb{R},\ (\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}\}$$

is uniformly bounded on $C^{\theta,2\theta}([\epsilon,2T]\times\mathbb{R})$ for any $0<\epsilon<T$ and some $0<\theta<1$. The Ascoli–Arzela theorem ensures that $\{(\phi^*(t,z;\widetilde{\phi},\widetilde{\psi}),z\in I,\ t\in[T,2T],\ (\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}\}$ is precompact in the sense of the supremum norm for any bounded interval $I\subset\mathbb{R}$. From Proposition 1, we know that $\{(\phi^*(t,z;\widetilde{\phi},\widetilde{\psi}),z\in\mathbb{R},\ t\in[T,2T],\ (\widetilde{\phi},\widetilde{\psi})\in\mathcal{D}\}$ is precompact in the sense of $\|\cdot\|_\eta$. Combining (42), we finally get that the map $\mathcal{F}$ is precompact in $B_\eta([0,T]\times\mathbb{R},\mathbb{R}^2)$.

Therefore, the existence of fixed point of $\mathcal{F}$ can be obtained by Schauder's fixed-point theorem. The proof is completed. □

Remark 5. The fixed point $(\widetilde{\phi}^*,\widetilde{\psi}^*)\in\mathcal{D}$ of map $\mathcal{F}$ is a positive solution of auxiliary system (14) defined on $(t,z)\in[0,T]\times\mathbb{R}$ satisfying $(\widetilde{\phi}^*(0,\cdot),\widetilde{\psi}^*(0,\cdot))=(\widetilde{\phi}^*(T,\cdot),\widetilde{\psi}^*(T,\cdot))$.

Return to the original system (10), we can conclude the following results.

Theorem 3. *System (10) admits a positive solution* $(\phi^*(t,z),\psi^*(t,z))$ *satisfying*

$$(\phi^-(t,z),\psi^-(t,z)) \le (\phi^*(t,z),\psi^*(t,z)) \le (\phi^+(t,z),K(t)e^{\lambda_1 z}),$$

$0<\phi^*(t,z)<S^0(t)$ *and* $\psi^*(t,z)>0$ *for any* $(t,z)\in[0,T]\times\mathbb{R}$, *and* $(\phi^*(0,z),\psi^*(0,z))=(\phi^*(T,z),\psi^*(T,z))$ *for* $z\in\mathbb{R}$.

Proof. Let sequence $\{\varepsilon_n\}$ satisfying $0<\varepsilon_{n+1}<\varepsilon_n<1,\ n=1,2,\cdots$ and $\varepsilon_n\to 0$ as $n\to\infty$. Theorem 2 and Remark 5 ensure that there exists a positive function sequence $\{(\phi_n(t,z),\psi_n(t,z))\}$ with $\varepsilon=\varepsilon_n$ $(n=1,2,\cdots)$ for $(t,z)\in[0,T]\times\mathbb{R}$, and $(\phi_n(0,z),\psi_n(0,z))=(\phi_n(T,z),\psi_n(T,z))$, satisfying integral system (24) and system (14), that is

$$\begin{cases} \partial_t\phi_n = d_1\partial_{zz}\phi_n - c\partial_z\phi_n + \Lambda(t) - \beta(t)\phi_n\psi_n - \mu(t)\phi_n, \\ \partial_t\psi_n = d_2\partial_{zz}\psi_n - c\partial_z\psi_n + \beta(t)\phi_n\psi_n - (\gamma(t)+\alpha(t)+\mu(t))\psi_n - \varepsilon_n\psi_n^2, \end{cases} \tag{43}$$

and

$$(\phi^-,\psi^-) \le (\phi_n,\psi_n) \le (\phi^+,\psi^+),\ t\in[0,T],\ z\in\mathbb{R}, \tag{44}$$

where $\psi_n^+(t,z):=\min\{K(t)e^{\lambda_1 z},K_{\varepsilon_n}\}$. For each interval $[-k,k],\ k=1,2,\cdots$, there exists $n_k\in\mathbb{N}_+$ such that $\psi^+(t,z)=K(t)e^{\lambda_1 z}<K_{\varepsilon_n}$ for all $n>n_k$ and $t\in[0,T]$. Using Arzelà–Ascoli theorem, we can obtain a uniformly convergent subsequence $\{(\phi_{k,m},\ \psi_{k,m})\}$ of $\{(\phi_n,\psi_n)\}$ for $(t,z)\in[0,T]\times[-k,k]$ as $m\to\infty$.

By the diagonal method, we can choose a subsequence $\{(\phi_{m,m},\ \psi_{m,m})\}$ which is uniformly convergent for $(t,z)\in[0,T]\times[-k,k]$ as $m\to\infty$ for any $k=1,2,\cdots$, and each $(\phi_{m,m},\ \psi_{m,m})$ satisfies

$$\begin{cases} \partial_t\phi_{m,m} = d_1\partial_{zz}\phi_{m,m} - c\partial_z\phi_{m,m} + \Lambda(t) - \beta(t)\phi_{m,m}\psi_{m,m} - \mu(t)\phi_{m,m}, \\ \partial_t\psi_{m,m} = d_2\partial_{zz}\psi_{m,m} - c\partial_z\psi_{m,m} + \beta(t)\phi_{m,m}\psi_{m,m} - (\gamma(t)+\alpha(t)+\mu(t))\psi_{m,m} - \varepsilon_{m,m}\psi_{m,m}^2. \end{cases} \tag{45}$$

Let $\lim_{m\to\infty}(\phi_{m,m}(t,z),\psi_{m,m}(t,z))=(\phi^*(t,z),\psi^*(t,z))$. Then, $(\phi^*(t,z),\psi^*(t,z))$ is defined for $(t,z)\in[0,T]\times\mathbb{R}$ and satisfies $(\phi^*(0,z),\psi^*(0,z))=(\phi^*(T,z),\psi^*(T,z))$. Since $\lim_{m\to\infty}\varepsilon_{m,m}=0$, take $m\to\infty$ in system (45), and then we can easily obtain that

$$\begin{cases} \partial_t\phi^* = d_1\partial_{zz}\phi^* - c\partial_z\phi^* + \Lambda(t) - \beta(t)\phi^*\psi^* - \mu(t)\phi^*, \\ \partial_t\psi^* = d_2\partial_{zz}\psi^* - c\partial_z\psi^* + \beta(t)\phi^*\psi^* - (\gamma(t)+\alpha(t)+\mu(t))\psi^*. \end{cases}$$

That is, $(\phi^*(t,z),\psi^*(t,z))$ is a solution of system (10). Furthermore, from (44) we can obtain

$$(\phi^-,\psi^-) \le (\phi^*,\psi^*) \le (\phi^+,K(t)e^{\lambda_1 z}),\ t\in[0,T],\ z\in\mathbb{R}.$$

Finally, we prove $0<\phi^*<S^0(t)$ and $\psi^*>0$ for $t\in[0,T]$ and $z\in\mathbb{R}$. From $\psi^-\le\psi^*\le\psi^+$ and $\psi^-=\max\{K(t)e^{\lambda_1 z}(1-M_2e^{\varepsilon_2 z}),0\}$, we deduce that

$$\psi^*(t,z)\ge 0 \ \text{ for all } (t,z)\in[0,T]\times\mathbb{R}$$

and

$$\psi^*(t,z)>0 \ \text{ for all } \ t\in[0,T],\ z<z_3=-{\varepsilon_2}^{-1}\ln M_2. \tag{46}$$

Therefore, we have

$$\begin{cases} \psi_t^* \ge d_2\psi_{zz}^* - c\psi_z^* - (\gamma(t)+\alpha(t)+\mu(t))\psi^*, \\ \psi^*(t,z)\ge 0,\ \psi^*(0,z)\ge 0, \end{cases}$$

for $(t,z)\in(0,T]\times\mathbb{R}$. Combining (46), we know that $\psi^*(0,z)\not\equiv 0$. By the maximum principle (see Lemma 2.1.8 in [26]), we obtain $\psi^*(t,z)>0$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}$.

Furthermore, from the first equation of system (10) and $\psi^*(t,z)>0$, one has $\phi^*(t,z)<S^0(t)$ on $\mathbb{R}\times\mathbb{R}$. Lastly, we prove $\phi^*(t,z)>0$ on $\mathbb{R}\times\mathbb{R}$. For a contrary, suppose that there exists (t^*,z^*) such that $\phi^*(t^*,z^*)=\phi_t^*(t^*,z^*)=\phi_z^*(t^*,z^*)=0$ and $\phi_{zz}^*(t^*,z^*)\ge 0$ since $\phi^*(t,z)\ge 0$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}$.

Then, combining the first equation of system (10), one has $0 = d_1\phi^*_{zz}(t^*, z^*) + \Lambda(t^*) > 0$, which is a contradiction. Thus, all the claims of this theorem are proved. This completes the proof. □

Theorem 4. *Suppose $\mathcal{R}_0 > 1$. Then, for every $c > c^*$ model (6) has a nontrivial T-periodic traveling wave solution $(\phi^*(t,z), \psi^*(t,z))$ defined on $\mathbb{R}\times\mathbb{R}$ satisfying*

$$\lim_{z\to-\infty}(\phi^*(t,z), \psi^*(t,z)) = (S^0(t), 0) \ \text{ uniformly for } \ t \in \mathbb{R},$$

and

$$0 < \phi^*(t,z) < S^0(t), \ \psi^*(t,z) > 0 \ \text{ for all } \ (t,z) \in \mathbb{R}\times\mathbb{R}.$$

Moreover,

$$\liminf_{z\to+\infty}\inf_{t\in[0,T]}\phi^*(t,z) > 0, \ \liminf_{z\to+\infty}\inf_{t\in[0,T]}\psi^*(t,z) > 0. \tag{47}$$

Proof. It follows from the discussion in Sect. 4 that the solution $(\phi^*(t,z), \psi^*(t,z))$ of model (6) satisfying

$$(\phi^-, \psi^-) \le (\phi^*, \psi^*) \le (\phi^+, K(t)e^{\lambda_1 z}), \ t \in [0,T], \ z \in \mathbb{R},$$

and $0 < \phi^*(t,z) < S^0(t)$ and $\psi^*(t,z) > 0$ for all $t \in [0,T]$ and $z \in \mathbb{R}$. Combining the definition of ϕ^-, ψ^-, ϕ^+ and $K(t)e^{\lambda_1 z}$, one has $\lim_{z\to-\infty}(\phi^*(t,z), \psi^*(t,z)) = (S^0(t), 0)$ uniformly for $t \in \mathbb{R}$.

Theorem 2 implies that, for each $c > c^*$, map $\mathcal{F}$ admits a fixed point $(\phi^*, \psi^*) \in \mathcal{D}$ satisfying $(\phi^*(T,\cdot), \psi^*(T,\cdot)) = (\phi^*(0,\cdot), \psi^*(0,\cdot))$. Furthermore, we also have

$$\begin{cases} \phi^*(t,\cdot) = T_1(t)\phi^*(0,\cdot) + \int_0^t T_1(t-s)f_1[\phi^*, \psi^*](s,\cdot)ds, \\ \psi^*(t,\cdot) = T_2(t)\psi^*(0,\cdot) + \int_0^t T_2(t-s)f_2[\phi^*, \psi^*](s,\cdot)ds. \end{cases} \tag{48}$$

Denote $\hat{\phi}^*(t,z) = \phi^*(t-nT, z)$ and $\hat{\psi}^*(t,z) = \psi^*(t-nT, z)$, where $n \in \mathbb{N}_+$ can be taken such that $nT \le t < (n+1)T$. Obviously, $\hat{\phi}^*$ and $\hat{\psi}^*$ are T-periodic functions. From $(\phi^*, \psi^*) \in C^{\theta,2\theta}([0,T]\times\mathbb{R}, \mathbb{R}^2)$, we know that $(\hat{\phi}^*, \hat{\psi}^*) \in C^{\theta,2\theta}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)$. Furthermore,

$$\begin{cases} \hat{\phi}^*(t,\cdot) = T_1(t)\hat{\phi}^*(0,\cdot) + \int_0^t T_1(t-s)f_1[\hat{\phi}^*, \hat{\psi}^*](s,\cdot)ds, \\ \hat{\psi}^*(t,\cdot) = T_2(t)\hat{\psi}^*(0,\cdot) + \int_0^t T_2(t-s)f_2[\hat{\phi}^*, \hat{\psi}^*](s,\cdot)ds, \end{cases} \tag{49}$$

for $t \in \mathbb{R}$. Without loss of generality, $(\hat{\phi}^*, \hat{\psi}^*)$ still is labeled by (ϕ^*, ψ^*). From Theorems 5.1.2, 5.1.3 and 5.1.4 in [28], one can see that $(\phi^*, \psi^*) \in C^{1,2+2\theta}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)$ satisfies

$$\begin{cases} \phi^*_t = d_1\phi^*_{zz} - c\phi^*_z + \Lambda(t) - \beta(t)\phi^*\psi^* - \mu(t)\phi^*, \\ \psi^*_t = d_2\psi^*_{zz} - c\psi^*_z + \beta(t)\phi^*\psi^* - (\gamma(t) + \alpha(t) + \mu(t))\psi^*. \end{cases} \tag{50}$$

Moreover,

$$\|\phi^*\|_{C^{1,2+2\theta}} + \|\psi^*\|_{C^{1,2+2\theta}} < +\infty \ \text{ for some } \theta \in (0,1). \tag{51}$$

Now, we show that conclusions (47) hold. Let

$$\phi_* = \liminf_{z\to+\infty}\inf_{t\in[0,T]}\phi^*(t,z), \ \psi_* = \liminf_{z\to+\infty}\inf_{t\in[0,T]}\psi^*(t,z).$$

We first show $\phi_* > 0$. By contradiction, suppose $\phi_* = 0$. Then, we can choose a sequence $\{(t_n, z_n)\}$ satisfying $t_n \in [0, T]$ and $z_n \to +\infty$ as $n \to +\infty$ such that $\phi^*(t_n, z_n) \to 0$ as $n \to +\infty$. Let

$$\phi_n^*(t,z) = \phi^*(t_n + t, z_n + z),\ \psi_n^*(t,z) = \psi^*(t_n + t, z_n + z),\ (t,z) \in \mathbb{R} \times \mathbb{R}.$$

Due to the estimation (51), we deduce that there exist nonnegative functions $\phi_\infty^*(t,z)$ and $\psi_\infty^*(t,z)$, without loss of generality, satisfying

$$\phi_n^*(t,z) \to \phi_\infty^*(t,z),\ \psi_n^*(t,z) \to \psi_\infty^*(t,z),\ \text{as } n \to +\infty,\ \text{in } C_{loc}^{\theta,2\theta}(\mathbb{R} \times \mathbb{R}).$$

Clearly, $\phi_\infty^*(0,0) = 0$. Furthermore,

$$\partial_t \phi_\infty^*(0,0) = \partial_z^* \phi_\infty(0,0) = 0 \text{ and } \partial_{zz}^* \phi_\infty(0,0) \geq 0, \tag{52}$$

since $\phi_\infty^*(t,z) \geq 0$. Due to $t_n \in [0,T]$, we can assume that $t_n \to t^* \in [0,T]$ as $n \to +\infty$. Combining the first equation of system (10), one has

$$\partial_t \phi_\infty^* = d_1 \partial_{zz} \phi_\infty^* - c\partial_z \phi_\infty^* + \Lambda(t^* + t) - \beta(t^* + t)\phi_\infty^* \psi_\infty^* - \mu(t^* + t)\phi_\infty^*.$$

Let $t = 0$ and $z = 0$, then from (52) it follows that $0 = d_1 \partial_{zz} \phi_\infty^*(0,0) + \Lambda(t^*) > 0$, which is a contradiction. Therefore, we have $\phi_* > 0$, that is, $\liminf_{z \to +\infty} \inf_{t \in [0,T]} \phi^*(t,z) > 0$.

Next, we intend to show the second asymptotic boundary condition in (47). On the one hand, because all coefficients in model (6) are the time-periodic functions, and the lack of conclusions on the stability of the time-periodic coefficient reaction–diffusion model, some classical methods are no longer applicable (see [12,29–31]). On the other hand, the population dynamics behavior is considered in model (6), thus making the infectious part integrable with respect to time, so the boundary conditions cannot be discussed using the integral transformation methods proposed in the literature works [17,18]. The asymptotic boundary conditions in this paper concerning the infectious disease part are difficult to obtain by conventional methods [32–37]. Therefore, in the following, we draw on the methods of authors such as Ambrosio, Ducrot and Ruan [38], which derives its core idea from the uniform persistence theory of infectious disease models.

From the time periodicity of ψ^*, we obtain

$$\inf_{t \in \mathbb{R}} \psi^*(t,0) > 0. \tag{53}$$

We firstly show that for each wave speed $c > c^*$, the following inequality holds:

$$\liminf_{t \to +\infty} \inf_{\tau \in \mathbb{R}} \psi^*(t - \tau, ct) > 0. \tag{54}$$

Define

$$(u,v)(t,z;\tau) = (\phi^*, \psi^*)(t - \tau, z + ct),\ (t,z) \in \mathbb{R} \times \mathbb{R}, \tau \in \mathbb{R}. \tag{55}$$

Obviously, from system (10), (u,v) satisfies the following equations:

$$\begin{cases} u_t = d_1 u_{zz} - cu_z + \Lambda(t-\tau) - \beta(t-\tau)uv - \mu(t-\tau)u, \\ v_t = d_2 v_{zz} - cv_z + \beta(t-\tau)uv - (\gamma(t-\tau) + \alpha(t-\tau) + \mu(t-\tau))v \end{cases}$$

for $t \in \mathbb{R}$, $z \in \mathbb{R}$ and $\tau \in \mathbb{R}$. Define a set

$$\mathcal{T} = \{(u,v)(t+h, z; \tau) : h \in \mathbb{R},\ \tau \in \mathbb{R}\}.$$

It follows from the parabolic regularity that the set $\mathcal{T}$ is relatively compact with respect to the open compact topology on $\mathbb{R} \times \mathbb{R}$. The closure of $\mathcal{T}$ is labeled by $\overline{\mathcal{T}}$. Then, $(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}}$ if and only if there exist two sequences $\{h_n\}$ and $\{\tau_n\}$ such that

$$\lim_{n \to +\infty} (u,v)(t + h_n, z; \tau_n) = (\tilde{u}, \tilde{v})(t,z) \ \text{ in } C_{loc}^0(\mathbb{R} \times \mathbb{R})^2.$$

Claim 1. There exists a constant $\varepsilon > 0$ such that $\limsup_{t \to +\infty} \tilde{v}(t,0) \geq \varepsilon$ for all $(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}}$ with $\tilde{v} \not\equiv 0$. The proof of Claim 1 is similar to Lemma 4.3 given in [38], we here omit it.

Let $\sigma(t) = (\Lambda(t), \beta(t), \alpha(t), \gamma(t), \mu(t), S^*(t))$. Then, $\sigma \in L^\infty(\mathbb{R})^5 \times BUC(\mathbb{R})$, where $BUC(\mathbb{R})$ is the Banach space of all bounded and uniformly continuous functions defined on $\mathbb{R}$ with the supremum norm. Define the hull of function σ, denoted by $\Sigma = \mathbf{cl}\{\sigma(\cdot + h),\ h \in \mathbb{R}\}$, where **cl** is the closure with respect to the open compact topology for $L^\infty(\mathbb{R})^5$ weak-$\star$ topology. Then, $\tilde{\sigma} = (\tilde{\Lambda}, \tilde{\beta}, \tilde{\alpha}, \tilde{\gamma}, \tilde{\mu}, \tilde{S^*}) \in \Sigma$ if there exists a sequence $\{h_n\}$ satisfying $S^*(t + h_n) \to \tilde{S^*}(t)$ in $C^0_{loc}(\mathbb{R})$, and $(\Lambda, \beta, \alpha, \gamma, \mu)(t + h_n) \to (\tilde{\Lambda}, \tilde{\beta}, \tilde{\alpha}, \tilde{\gamma}, \tilde{\mu})(t)$ for $L^\infty(\mathbb{R})^5$ weak $-\star$ topology. Furthermore, define

$$\mathcal{S} = \{((u, v)(t + h, z; \tau), \sigma(t + h - \tau)), (h, \tau) \in \mathbb{R} \times \mathbb{R}\}.$$

$\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ with respect to the product topology of $\overline{\mathcal{T}} \times \Sigma$.

For given sequences $\{h_n\}$ and $\{\tau_n\}$, function sequences $(u_n, v_n)(t, z) := (u, v)(t + h_n, z; \tau_n)$ satisfy

$$\begin{cases} \partial_t u_n = d_1\partial_{zz}u_n - c\partial_z u_n + \Lambda(t + h_n - \tau_n) - \beta(t + h_n - \tau_n)u_n v_n - \mu(t + h_n - \tau_n)u_n, \\ \partial_t v_n = d_2\partial_{zz}v_n - c\partial_z v_n + \beta(t + h_n - \tau_n)u_n v_n - (\gamma + \alpha + \mu)(t + h_n - \tau_n)v_n. \end{cases}$$

For every $(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}}$, there exists a $\tilde{\sigma} \in \Sigma$ satisfying $((\tilde{u}, \tilde{v}), \tilde{\sigma}) \in \overline{\mathcal{S}}$ when $(\tilde{u}, \tilde{v})$ satisfies

$$(\mathcal{P}_{\tilde{\sigma}}) \begin{cases} [\partial_t - d_1\partial_{zz} + c\partial_z]\tilde{u} = \tilde{\Lambda} - \tilde{\beta}\tilde{u}\tilde{v} - \tilde{\mu}\tilde{u}, \\ [\partial_t - d_2\partial_{zz} + c\partial_z]\tilde{v} = \tilde{v}[\tilde{\beta}\tilde{u}\tilde{v} - (\tilde{\gamma} + \tilde{\alpha} + \tilde{\mu})], \end{cases} \tag{56}$$

where $\tilde{\sigma} = (\tilde{\Lambda}, \tilde{\beta}, \tilde{\alpha}, \tilde{\gamma}, \tilde{\mu}, , \tilde{S^*})$. It follows from the strong comparison principle and the fact $\tilde{v} \geq 0$ that

$$(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}} \text{ and } \tilde{v} \not\equiv 0 \ \Leftrightarrow \tilde{v} > 0,$$

and

$$(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}} \text{ and } \tilde{v} \equiv 0 \ \Leftrightarrow (\tilde{u}, \tilde{v}) \equiv (\tilde{S^*}(t), 0).$$

Claim 2. The function $v = v(t, z; \tau)$ defined in (55) satisfies $\inf_{t \geq 0, \tau \in \mathbb{R}} v(t, 0; \tau) > 0$.

In fact, suppose that there exist two sequences $\{l_n\}$ and $\{\tau_n\}$ such that $l_n \geq 0$ and $v(l_n, 0; \tau_n) \leq \frac{1}{n+1}$ for all $n \in \mathbb{N}$. From the definition of v in (55) and (53), one has

$$\inf_{\tau \in \mathbb{R}} v(0, 0; \tau) = \inf_{\tau \in \mathbb{R}} \psi^*(-\tau, 0) = \inf_{\tau \in \mathbb{R}} \psi^*(\tau, 0) := \sigma > 0.$$

Hence, for $n \in \mathbb{N}$ large enough, there exists $s_n \in [0, l_n)$ satisfying

$$\begin{cases} v(s_n, 0; \tau_n) = \sigma_0 := \dfrac{1}{2}\min\{\sigma, \varepsilon\}, \\ v(t + s_n, 0; \tau_n) \leq \sigma_0 \text{ for } t \in [0, l_n - s_n], \\ v(l_n, 0; \tau_n) \leq \dfrac{1}{n + 1}, \end{cases}$$

where $\varepsilon > 0$ provided by Claim 1.

Now, we check that $l_n - s_n \to +\infty$ as $n \to +\infty$. Consider the sequence of functions $\{(u_n, v_n)(t, z)\} := \{(u, v)(s_n + t, z; \tau_n)\}$. Clearly, each $(u_n, v_n)(t, z)$ is a bounded solution of equations (56) with some suitable $\sigma_n \in \Sigma$. We have

$$v_n(0, 0) = \sigma_0 \ \text{ and } \ v_n(l_n - s_n, 0) \leq \frac{1}{n + 1} \ \text{ for } n \text{ large enough.}$$

Therefore, from the parabolic regularity, one may assume that there exists a pair of function (u_∞, v_∞) such that $(u_n, v_n) \to (u_\infty, v_\infty)$ locally uniformly as $n \to +\infty$. Furthermore, (u_∞, v_∞) is a bounded solution of equations (56) for a suitable $\sigma_\infty \in \Sigma$, while the function v_∞ satisfies $v_\infty(0, 0) = \sigma_0 > 0$. Hence, if the sequence $\{l_n - s_n\}$ was bounded, then function v_∞ furthermore satisfies $v_\infty(\iota, 0) = 0$, where $\iota \geq 0$ denotes a limit point of the bounded sequence $\{l_n - s_n\}$. This is a contradiction with the maximum principle. Therefore, we finally have that $l_n - s_n \to +\infty$ as $n \to +\infty$.

Consider once again the sequence of functions $\{(u_n, v_n)\}$ defined above. Then, possibly along a subsequence, we have

$$(u_n, v_n)(t,z) \to (\tilde{u}, \tilde{v})(t,z) \text{ in } C^0_{loc}(\mathbb{R})^2,$$

where $(\tilde{u}, \tilde{v})$ is a bounded solution of equations (56) for some $\tilde{\sigma} \in \Sigma$ with $\tilde{v}(0,0) = \sigma_0 > 0$. Since $l_n - s_n \to +\infty$, the function $\tilde{v}$ satisfies $\tilde{v}(t,0) \le \sigma_0 < \varepsilon$ for all $t \ge 0$. Thus $(\tilde{u}, \tilde{v}) \in \overline{\mathcal{T}}$ and $\tilde{v} \not\equiv 0$. It follows from Claim 1 that $\limsup_{t\to+\infty} \tilde{v}(t,0) \ge \varepsilon$, which leads to a contradiction with $\tilde{v}(t,0) \le \sigma_0 < \varepsilon$ for $t \ge 0$. Therefore, Claim 2 is proved. It follows directly from Claim 2 that (54) holds, since $v(t,0;\tau) = \psi^*(t-\tau, ct)$.

Now, we go back to the proof of conclusion (47). Letting $\tau = t - s$, then from (54), we obtain that there exist $\varepsilon > 0$ and $X_0 > 0$ large enough such that $\psi^*(s, ct) \ge \varepsilon$ for all $s \in \mathbb{R}$ and $t \ge X_0$. Since $c > c^* > 0$, there exist $\varepsilon > 0$ and $Z > 0$ large enough such that $\psi^*(s,z) \ge \varepsilon$ for all $s \in \mathbb{R}$ and $z \ge Z$, which implies that the second conclusion in (47) is valid. Therefore, Theorem 4 is proved. □

Remark 6. Theorem 4 gives a weaker boundary condition $\liminf_{z\to+\infty} \inf_{t\in[0,T]} \phi^*(t,z) > 0$ and $\liminf_{z\to+\infty} \inf_{t\in[0,T]} \psi^*(t,z) > 0$. However, from Remark 2, we know that the periodic model (6) admits a positive T-periodic solution $(S^*(t), I^*(t))$ when $\mathcal{R}_0 > 1$. Therefore, an interesting and challenging open problem is to explore the traveling waves connecting $(S^0(t), 0)$ and $(S^*(t), I^*(t))$, that is, satisfying strong asymptotic boundary condition $\lim_{z\to+\infty}(\phi^*(t,z), \psi^*(t,z)) = (S^*(t), I^*(t))$ uniformly for $t \in \mathbb{R}$.

In the rest of this section, we give the nonexistence of time periodic traveling wave when $\mathcal{R}_0 < 1$ and $c \ge 0$ for model (6).

Theorem 5. *Suppose that $\mathcal{R}_0 < 1$ and $c \ge 0$. Then, there is no T-periodic traveling wave solution satisfying (11) and (12) for model (6).*

Proof. Suppose that model (6) admits a T-periodic traveling wave solution $(\phi^*(t,z), \psi^*(t,z))$ satisfying (11) and (12). Denote $\bar{\phi}^* = \sup_{(t,z)\in[0,T]\times\mathbb{R}} \phi^*(t,z)$. Let $S(t, \bar{\phi}^*)$ be the solution of equation (8) with initial condition $S(0) = \bar{\phi}^*$. Since $S(t, \bar{\phi}^*)$ also is the solution of equation

$$\phi_t = d_1\phi_{zz} - c\phi_z + \Lambda(t) - \mu(t)\phi,$$

and

$$\phi^*_t(t,z) \le d_1\phi^*_{zz}(t,z) - c\phi^*_z(t,z) + \Lambda(t) - \mu(t)\phi^*(t,z),$$

by the comparison principle we have $\phi^*(t,z) \le S(t, \bar{\phi}^*)$ for all $(t,z) \in [0,T] \times \mathbb{R}$. From Lemma 1, we know that $\lim_{t\to\infty} S(t, \bar{\phi}^*) = S^0(t)$. By the time periodicity of $\phi^*(t,z)$, we finally have $\phi^*(t,z) \le S^0(t)$ for all $(t,z) \in [0,T] \times \mathbb{R}$.

Since

$$\begin{aligned} \psi^*_t =& d_2\psi^*_{zz} - c\psi^*_z + \beta(t)\phi^*\psi^* - (\gamma(t) + \alpha(t) + \mu(t))\psi^* \\ \le& d_2\psi^*_{zz} - c\psi^*_z + [\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))]\psi^*, \ (t,z) \in (0,\infty) \times \mathbb{R}. \end{aligned} \tag{57}$$

Let $\sigma := \sup_{z\in\mathbb{R}} \psi^*(0,z) < +\infty$, then $\psi^*(0,z) \le \sigma$ for all $z \in \mathbb{R}$. It follows from the comparison principle that

$$\psi^*(t,z) \le \omega(t;\sigma), \text{ for all } t > 0, z \in \mathbb{R},$$

where $\omega(t;\sigma)$ is the solution of the following problem:

$$\begin{cases} \omega'(t) = [\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))]\omega(t), \ t > 0, \\ \omega(0) = \sigma. \end{cases}$$

Since $\mathcal{R}_0 < 1$, we obtain $\frac{1}{T}\int\limits_0^T [\beta(t)S^0(t) - (\gamma(t) + \alpha(t) + \mu(t))]dt < 0$. Consequently, from Theorem 3.1.2 in [39] we get $\lim_{t\to+\infty} \omega(t;\sigma) = 0$, and then $\lim_{t\to+\infty} \psi^*(t,z) = 0$ for $z \in \mathbb{R}$. This contradicts the T-periodicity of function $\psi^*(t,z)$ in t. □

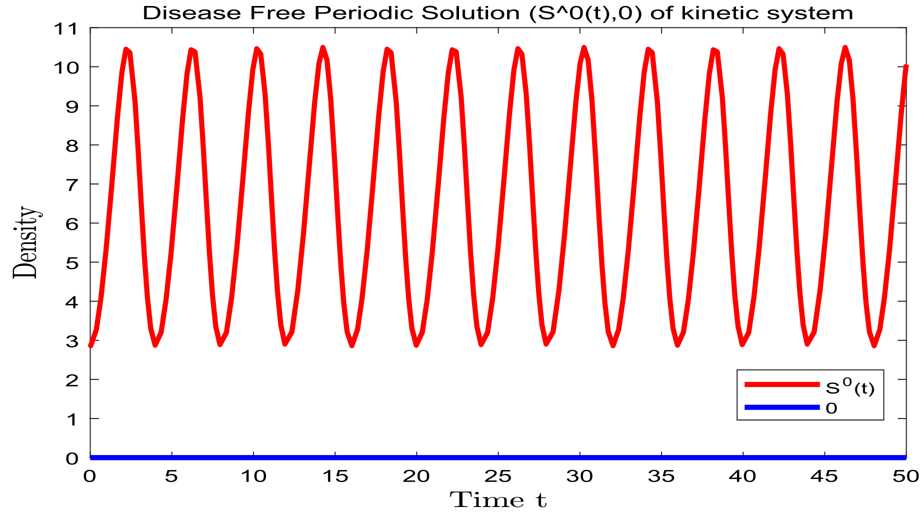

FIG. 1. Disease-free periodic steady state $(S^0(t), 0)$ of the corresponding dynamic system (7)

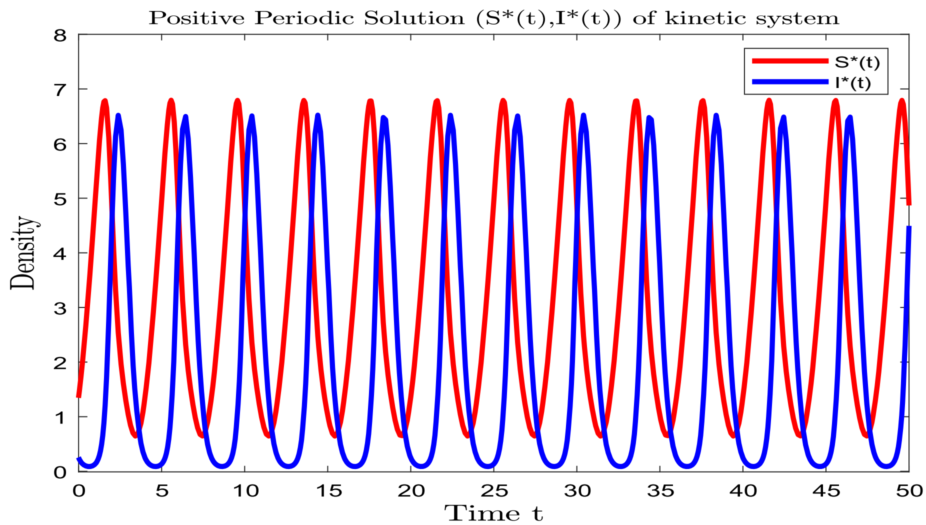

FIG. 2. Positive periodic steady state $(S^*(t), I^*(t))$ of the corresponding dynamic system (7)

Remark 7. In model (6), when coefficients $\Lambda(t), \beta(t), \mu(t), \alpha(t)$ and $\gamma(t)$ are degraded as constants, then model (6) becomes an autonomous model. Then, we have $S^0(t) \equiv \frac{\Lambda}{\mu}$, $\mathcal{R}_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma+\alpha)}$ and $c^* = 2\sqrt{d_2[\frac{\beta\Lambda}{\mu} - (\mu+\gamma+\alpha)]}$. Obviously, Theorems 4 and 5 in this paper are still valid. In addition, there is a stronger asymptotic boundary condition (see [12,29]). That is, when $\mathcal{R}_0 > 1$ and wave speed $c > c^*$, then the autonomous model has a nontrivial, nonnegative traveling waves $(\phi^*(s), \psi^*(s))$ with $s = x + ct$ connecting equilibria $E_0(\frac{\Lambda}{\mu}, 0)$ and $E^*(S^*, I^*)$, that is,

$$\lim_{s\to-\infty}(\phi^*(s), \psi^*(s)) = (\frac{\Lambda}{\mu}, 0), \quad \lim_{s\to+\infty}(\phi^*(s), \psi^*(s)) = (S^*, I^*).$$

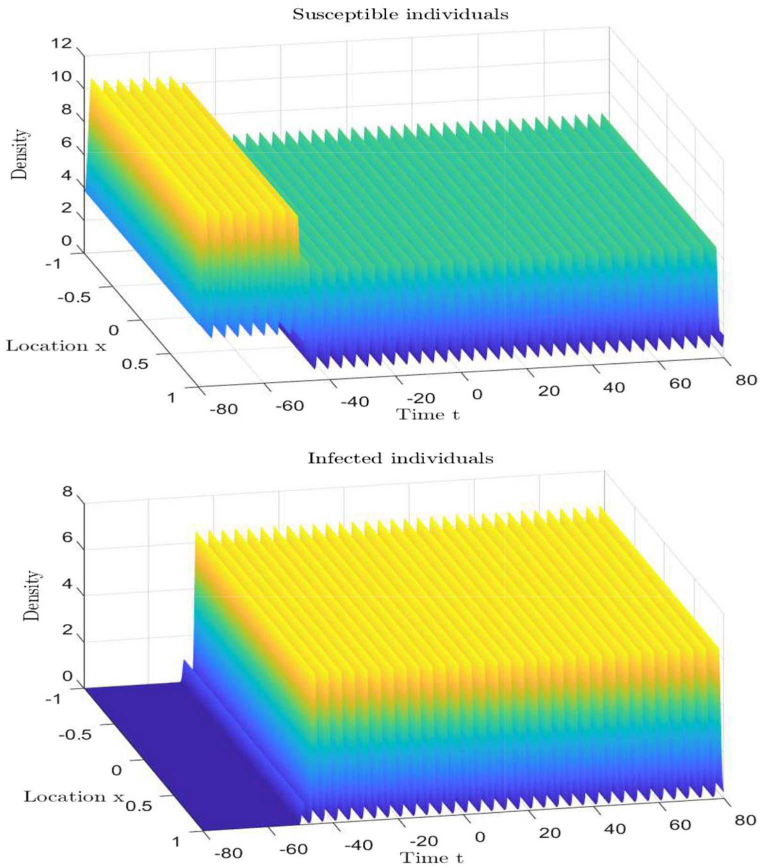

FIG. 3. Periodic traveling wave solution connecting the steady states $(S^0(t), 0)$ and $(S^*(t), I^*(t))$ of the corresponding dynamic system

Unfortunately, for non-autonomous model (6), we can not obtain that the traveling wave solution satisfies $(\phi^*(t,z), \psi^*(t,z)) \to (S^*(t), I^*(t))$ uniformly for $t \in \mathbb{R}$ as $z \to +\infty$. This is a very interesting issue that needs to be studied further.

5. Numerical examples

In this section, we give the numerical examples to illustrate the theoretical results above. In system (6), Let $\Lambda(t) = 5+4.9\sin(0.5\pi t)$, $\beta(t) = 0.35(\sin(0.5\pi t)+1.5)$, $\mu(t) = 1+0.8\cos(0.5\pi t)$, $\alpha(t) = 0.5+0.3\cos(0.5\pi t)$, $\gamma(t) = 0.5 + 0.4\cos(0.5\pi t)$, $d_1 = 0.08$, $d_2 = 0.08$ and $d_3 = 0.17$. By simple calculation, we obtain $\mathcal{R}_0 \approx 1.8375 > 1$ and $c^* \approx 0.7321$.

Firstly, we observe the periodic steady states $(S^0(t), 0)$ and $(S^*(t), I^*(t))$ of the corresponding kinetic system (7) (see Figs. 1 and 2, respectively). The space dynamical behavior of the periodic traveling wave solution for $-80 < t < 80$ is shown in Fig. 3.

In addition, wave speed c has some influence on the spread of disease. So, we study the effects of wave speed on the asymptotic behavior of traveling waves. For the convenience of observation, let the initial conditions $S(0, x) = 3.5 + 0.2\sin(2\pi x)$ and $I(0, x) = 0.5 + 0.3\sin(2\pi x)$, and we take wave speed $c = 1$ and $c = 5$, respectively. To observe the change of spatial density of individuals, equipotential graphs were

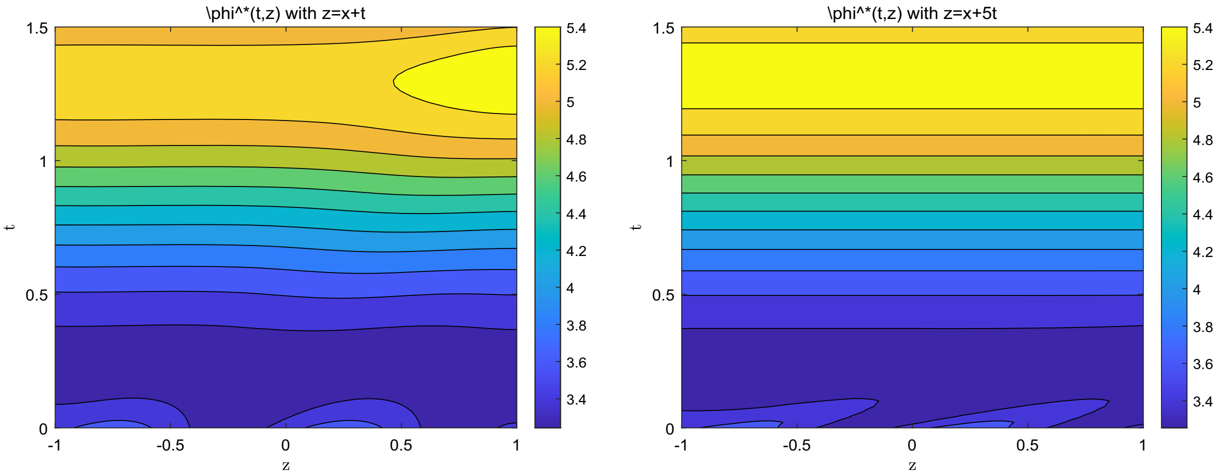

FIG. 4. Density of the susceptible individuals when the wave speed $c = 1$(Left) and $c = 5$(Right), respectively. Figures show that when the wave speed increases, the spatial density tends to be stable more quickly

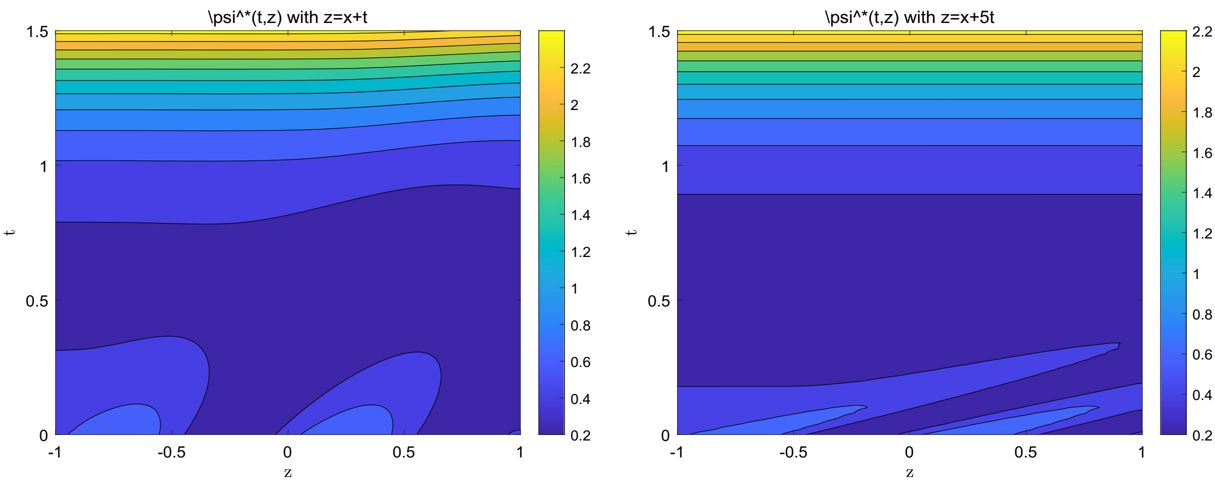

FIG. 5. Density of infected individuals when the wave speed $c = 1$(Left) and $c = 5$(Right), respectively. Figures show that when the wave speed increases, the spatial density tends to be stable more quickly

used to show the changing of the spatial density of individuals. The numerical simulation results in region $(t, z) \in (0, 1.5) \times (-1, 1)$ with $z = x + ct$ are shown in Figs. 4 and 5. The simulations illustrate that when the wave speed increases the trend of disease spread in space will reach a steady state more quickly.

6. Conclusion

In this paper, we investigated the existence and nonexistence of periodic traveling wave solutions satisfying asymptotic boundary conditions (11) and (12) for reaction–diffusion SIR epidemic model (5) with demography and time-periodic coefficients. The traveling wave solutions of model (5) are different from those of the autonomous reaction–diffusion models with homogeneous space. The traveling wave solutions in those models can be transformed as a solution of ordinary differential equation and solved by ODE theory. However, the periodic traveling wave solutions in model (5) still satisfied a partial differential equation. Therefore, some traditional methods are no longer applicable (see [9,29–31]). Moreover, since the upper solution of the infected individual is difficult to construct directly for model (5), we have adopted auxiliary systems and some limit techniques. We obtain the existence of periodic traveling waves satisfying (11) and (12) when wave speed $c > c^*$ and $\mathcal{R}_0 > 1$. By numerical examples, we found that the wave speed can affect the disease propagation in space, and the greater the wave speed, the shorter the time it takes to reach the stable state.

Finally, there are some open problems with periodic traveling waves. For example, the existence and nonexistence of T-periodic traveling wave solutions connecting the steady states $(S^0(t), 0)$ and $(S^*(t), I^*(t))$ are still challenging problems. Furthermore, the methods proposed in this paper can be used further in many similar models. In fact, we can consider reaction–diffusion SIR epidemic model with nonlinear incidence and more general time-space periodic environment and investigate the existence of traveling wave solutions and their related properties, such as uniqueness, asymptotic behavior, and stability. We will leave these issues for further investigation.

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Availability of data and material Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Code Availability Statement Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest/competing interests.

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