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Ill-posedness of the hyperbolic Keller-Segel model in Besov spaces

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Abstract. In this paper, we give a new construction of $u_0 \in B^\sigma_{p,\infty}$ such that the corresponding solution to the hyperbolic Keller-Segel model starting from u_0 is discontinuous at $t = 0$ in the metric of $B^\sigma_{p,\infty}(\mathbb{R}^d)$ with $d \geq 1$ and $1 \leq p \leq \infty$, which implies the ill-posedness for this equation in $B^\sigma_{p,\infty}$. Our result generalizes the recent work in Zhang et al. (J Differ Equ 334:451-489, 2022) where the case $d = 1$ and $p = 2$ was considered.

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1. Introduction

Chemotaxis is the active motion of organisms influenced by chemical gradients. The most prominent model for this process goes back to Patlak, Keller and Segel [6–8,16] which takes the form of

$$\begin{cases} \partial_t u + \nabla \cdot (D_1(u,S)\nabla u - \chi(u,S)\nabla S) = 0, \\ \tau S_t = D_2 \Delta S + k(u,S), \end{cases} \tag{1.1}$$

here $u(x,t)$ represents the cell density at position $x \in \mathbb{R}^d$, time $t > 0$, and $S(x,t)$ is the concentration of a chemical signal. The motility $D_1(u,S)$ and the chemotactic sensitivity $\chi(u,S)$ rely on the cell density and on the signal concentration. The term $k(u,S)$ depicts production and decay or consumption of the signal and D_2 is the diffusion constant for S. The parameter τ illustrates that movement of the species and dynamics of the signal have different characteristic time scales. The Keller-Segel model has been applied to many different problems, ranging from bacteria chemotaxis to cancer growth or the immune response.

Dolak and Schmeiser [5] derived a convection equation with a small diffusion term as higher order correction from a kinetic model for chemotaxis. Inspired by this, Dolak and Schmeiser proposed the following parabolic-type Keller-Segel equations with small diffusivity:

$$\begin{cases} \partial_t u = -\nabla \cdot (u(1-u)\nabla S - \epsilon \nabla u), & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d, \\ -\Delta S = u - S, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d. \end{cases} \tag{1.2}$$

Burger, Dolak and Schmeiser [3] studied the asymptotic behavior of solutions of the chemotaxis model (1.2) in multiple spatial dimensions. Of particular interest is the practically relevant case of small diffusivity, where (as in the one-dimensional case) the cell densities form plateau-like solutions for large time. Some other results related to (1.2) can be found in [18–20].

Nie and Yuan [14] considered the Cauchy problem for multidimensional chemotaxis system

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (uv), \text{ in } & \mathbb{R}^+ \times \mathbb{R}^d, \\ \partial_t v - \nabla u = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ (u,v)|_{t=0} = (u_0, v_0), & \text{in } \mathbb{R}^d, \end{cases} \tag{1.3}$$

they proved that (1.3) is well-posed in $\dot{B}^{\frac{d}{p}-2}_{p,\sigma} \times (\dot{B}^{\frac{d}{p}-1}_{p,\sigma})^d$ when $p < 2d$ and is ill-posed when $p > 2d$. Later, Nie and Yuan [15] also obtaind that (1.3) is ill-posed in $\dot{B}^{\frac{d}{p}-2}_{p,1} \times (\dot{B}^{\frac{d}{p}-1}_{p,1})^d$ when $p = 2d$. Almost in the same time, Xiao and Fei [21] proved that (1.3) is ill-posed in $\dot{B}^{\frac{d}{p}-2}_{p,\sigma} \times (\dot{B}^{\frac{d}{p}-1}_{p,\sigma})^d$ when $p = 2d, \sigma > 2$. Recently, Li, Yu and Zhu [13] proved that (1.3) is ill-posed in $\dot{B}^{\frac{d}{p}-2}_{p,r} \times (\dot{B}^{\frac{d}{p}-1}_{p,r})^d$ when $1 \leq r < d$.

In this paper, we consider the Cauchy problem for following hyperbolic Keller-Segel equation:

$$\begin{cases} \partial_t u = -\nabla \cdot (u(1-u)\nabla S), & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ -\Delta S = u - S, & \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{1.4}$$

The unknown scale functions $u(x,t)$ and $S(x,t)$ denote the cell density and the concentration of chemical substance, respectively. Dolak and Schmeiser [4] firstly established the existence and unique of global smooth solution to one dimensional version of (1.2) with suitable conditions on the initial data. On a time scale characteristic for the convective effects, they also proved that the corresponding sequence of solutions u^ϵ converges to the weak entropy solution u to (1.4) as $\epsilon \to 0$. Laterly, Burger, Difrancesco and Dolak [2] obtained the unique local-in-time solution to (1.2) with the initial data belonging to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Perthame and Dalibard [17] proved the existence of an entropy solution to (1.4) by passing to the limit in a sequence of solutions to the parabolic approximation. Lee and Liu [9] proved the sub-threshold for finite time shock formation to solutions of (1.4) in one-dimension.

Recently, Zhou, Zhang and Mu [23] obtained the existence and uniqueness of solution of (1.4) in $B^s_{p,r}(\mathbb{R}^d)$ when $1 \leq p, r \leq \infty, s > 1 + \frac{d}{p}$. Later, Zhang, Mu and Zhou [22] proved that (1.4) is ill-posed in $B^s_{2,\infty}(\mathbb{R})$ with $s > \frac{3}{2}$ and (1.4) is local well-posed in $B^s_{p,1}(\mathbb{R}^d)$ when $1 \leq p < \infty, s = 1 + \frac{d}{p}$. However, their initial data seems to be valid only for $p = 2$ when proving the ill-posedness in $B^s_{p,\infty}$. Motivated by the recent works in [11,12], we aim to extend the ill-posedness result in [22] to more general case, i.e, $1 \leq p \leq \infty$ and $d \geq 1$. The main result of the paper is the following theorem:

Theorem 1.1. *Let $d \geq 1$. Assume that*

$$s > 1 + \frac{d}{p} \quad \textit{with} \quad 1 \leq p \leq \infty,$$

then there exists $u_0 \in B^s_{p,\infty}(\mathbb{R}^d)$ and a positive constant ϵ_0 such that the data-to-solution map $u_0 \mapsto u$ of the Cauchy problem (1.4) satisfies

$$\limsup_{t \to 0^+} \|u(t) - u_0\|_{B^s_{p,\infty}} \geq \epsilon_0.$$

Remark 1.1. Theorem 1.1 demonstrates the ill-posedness of (1.4) in $B^s_{p,\infty}(\mathbb{R}^d)$. Precisely speaking, we can construct $u_0 \in B^s_{p,\infty}(\mathbb{R}^d)$ such that the corresponding solutions of the Keller-Segel equation do not converge to u_0 in the metric of $B^s_{p,\infty}(\mathbb{R}^d)$ as $t \to 0^+$.

Remark 1.2. We should mention that the key decomposition technique and the special initial data used in [22] can not be applied to the present case $p \neq 2$ any more. To overcome these difficulties, we construct a new initial data which is completely different from [22]. In particular, by utilizing the commutator estimate and some basic analysis, we make the proof more simple.

The rest of the paper is organized as follows. In Sect. 2, we introduce some basic definitions and key lemmas. In Sect. 3, we present the proof of Theorem 1.1.

2. Preliminaries

Notation The notation $A \lesssim B$ (resp., $A \gtrsim B$) means that there exists a harmless positive constant c such that $A \leq cB$ (resp., $A \geq cB$). Given a Banach space X, we denote its norm by $\|\cdot\|_X$. For a Banach space X and for any $0 < T \leq \infty$, we use standard notation $L^p(0,T;X)$ to denote the quasi-Banach space of Bochner measurable functions f from $(0,T)$ to X endowed with the norm

$$\|f\|_{L^p_T X} := \begin{cases} \left(\int\limits_0^T \|f(\cdot,t)\|_X^p \, dt \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \sup_{0 \leq t \leq T} \|f(\cdot,t)\|_X, & \text{if } p = \infty. \end{cases}$$

Let us recall that for all $f \in \mathcal{S}'$, the Fourier transform $\widehat{f}$, is defined by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \mathrm{d}x \quad \text{for any}\, \xi \in \mathbb{R}^d.$$

The inverse Fourier transform of any g is given by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi) \mathrm{d}\xi.$$

Next, we recall some facts on the Littlewood-Paley theory which can be found in [1].

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\chi \in C_c^\infty(\mathbb{R}^d)$ be a radial positive function such that

$$\begin{aligned}
&\text{supp } \varphi \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, \quad \text{supp } \chi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\}, \\
&\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R}^d, \\
&|i-j| \geq 2 \Rightarrow \text{supp } \varphi(2^{-i}\cdot) \cap \text{supp } \varphi(2^{-j}\cdot) = \varnothing, \\
&j \geq 1 \Rightarrow \text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \chi(x) = \varnothing, \\
&\varphi(\xi) \equiv 1 \quad \text{for} \quad \frac{4}{3} \leq |\xi| \leq \frac{3}{2}.
\end{aligned}$$

We can define the nonhomogeneous localization operators as follows.

$$\Delta_j u = 0, \;\; j \leq -1; \quad \Delta_j u = \chi(D)u, \;\; j = -1; \quad \Delta_j u = \varphi(2^{-j}D)u, \;\; j \geq 0,$$

where the pseudo-differential operator $f(D) : u \to \mathcal{F}^{-1}(f\mathcal{F}u)$.

Let us now define the Besov spaces as follows.

Definition 2.1. ([1]) Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ is defined by

$$B^s_{p,r}(\mathbb{R}) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \; \|f\|_{B^s_{p,r}(\mathbb{R}^d)} < \infty \Big\},$$

where

$$\|f\|_{B^s_{p,r}(\mathbb{R}^d)} = \begin{cases} \left(\sum_{j \geq -1} 2^{sjr} \|\Delta_j f\|^r_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\ \sup_{j \geq -1} 2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}, & \text{if } r = \infty. \end{cases}$$

Remark 2.1. It should be emphasized that $B^s_{p,\infty}(\mathbb{R}^d)$ with $s > \frac{d}{p}$ is a Banach algebra and $B^s_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^t_{p,\infty}(\mathbb{R}^d)$ with $s > t$. These facts will be often used implicity.

Finally, we recall some lemmas which be used later.

Lemma 2.1. *(Bernstein's inequality, [1]) Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ be a ball. There exists a constant C such that for any nonnegative integer k, any couple $(p,q)\in[1,\infty]^2$ with $1\le p\le q$, and any L^p function u we have*

$$\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q(\mathbb{R}^d)}\le C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p(\mathbb{R}^d)},\ \operatorname{supp}\widehat{u}\subset\lambda\mathcal{B},$$
$$C^{-(k+1)}\lambda^k\|u\|_{L^p(\mathbb{R}^d)}\le\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p(\mathbb{R}^d)}\le C^{(k+1)}\lambda^k\|u\|_{L^p(\mathbb{R}^d)},\ \operatorname{supp}\widehat{u}\subset\lambda\mathcal{C}.$$

Lemma 2.2. *([1]) A smooth function $f:\mathbb{R}^d\to\mathbb{R}$ is said to be an S^m-multiplier: if$\forall\alpha\in\mathbb{N}^d$, there exists a constant $C_\alpha>0$ such that*

$$|\partial^\alpha f(\xi)|\le C_\alpha(1+|\xi|)^{m-\alpha},\ \xi\in\mathbb{R}^d.$$

If f is a S^m-multiplier, then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$ for all $s\in\mathbb{R}$ and $1\le p,r\le\infty$.

Lemma 2.3. *([1]) For $1\le p\le\infty$ and $s>0$, there exists a constant C,depending continuously on p and s, we have*

$$\left\|2^{js}\|[\Delta_j,v]\cdot\nabla f\|_{L^p}\right\|_{\ell^\infty}\le C(\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,\infty}}+\|\nabla f\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,\infty}}),$$

where $[\Delta_j,v]\cdot\nabla f=\Delta_j(v\cdot\nabla f)-v\cdot\Delta_j\nabla f$.

3. Proof of Theorem 1.1

For convenience of computation, we rewrite (1.4) as follows

$$\begin{cases}\partial_t u+(1-2u)\nabla S\cdot\nabla u+u(1-u)\Delta S=0, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^d,\\ S=(1-\Delta)^{-1}u, & \text{in}\quad \mathbb{R}^+\times\mathbb{R}^d,\\ u(x,0)=u_0(x), & \text{in}\quad \mathbb{R}^d.\end{cases}\tag{3.1}$$

Let $\widehat{\phi}\in C_0^\infty(\mathbb{R})$ be an even, real-valued and nonnegative function which satisfies

$$\widehat{\phi}(\xi)=\begin{cases}1, & \text{if } |\xi|\le\frac{1}{4^d},\\ 0, & \text{if } |\xi|\ge\frac{1}{2^d}.\end{cases}$$

Remark 3.1. By the Fourier-Plancherel formula, we have $\phi(x)=\mathcal{F}^{-1}(\widehat{\phi}(\xi))$. It is easy to check that

$$\phi(0)=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\phi}(\xi)\mathrm{d}\xi>0\quad\text{and}\quad\phi'(0)=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\phi}'(\xi)\mathrm{d}\xi=0.$$

Lemma 3.1. *Define the function $f_n(x)$ by*

$$f_n(x)=\phi(x_1)\sin\left(\frac{17}{12}2^nx_1\right)\phi(x_2)\cdots\phi(x_d),\quad n\ge3.$$

Then

$$\Delta_j(f_n)=\begin{cases}f_j, & \text{if } j=n,\\ 0, & \text{if } j\ne n.\end{cases}$$

Proof. Notice that

$$\operatorname{supp}\widehat{f}_n\subset\Big\{\xi\in\mathbb{R}^d:\ \frac{17}{12}2^n-\frac{1}{2}\le|\xi|\le\frac{17}{12}2^n+\frac{1}{2}\Big\},$$

using the definition of Δ_j enables us to get the desired result. For more details see [10]. □

Proposition 3.1. *Define the initial data* $u_0(x)$ *as*

$$S_0(x) := \sum_{n=3}^{\infty} 2^{-n(s+2)} f_n(x),$$
$$u_0(x) := (1-\Delta)S_0(x).$$

If $s > 1 + \frac{d}{p}$*, we have*

$$\|u_0\|_{B^s_{p,\infty}} \leq C.$$

Proof. By Lemma 3.1, we have

$$\Delta_j S_0 = 2^{-j(s+2)} f_j(x). \tag{3.2}$$

Combining Lemma 2.2 and (3.2) yields

$$\|u_0\|_{B^s_{p,\infty}} \leq C\|S_0\|_{B^{s+2}_{p,\infty}} = \sup_{j\geq 0} 2^{(s+2)j}\|\Delta_j S_0\|_{L^p} \leq C.$$

We complete the proof of Proposition 3.1. □

Using Proposition 3.1 and Theorem 1.1 in [23], we can obtain that there exists a short time $T > 0$ that (3.1) has a unique solution $u \in L^\infty([0,T); B^s_{p,\infty}) \cap Lip([0,T); B^{s-1}_{p,\infty})$ for $s > 1 + \frac{d}{p}$. Moreover, it holds

$$\|u(t)\|_{L^\infty_T(B^s_{p,\infty})} \leq C\|u_0\|_{B^s_{p,\infty}}. \tag{3.3}$$

Proposition 3.2. *Let* $s - 1 > \frac{d}{p}$ *and* $\|u_0\|_{B^s_{p,\infty}} \lesssim 1$*. Assume that* $u \in L^\infty(0,T; B^s_{p,\infty}(\mathbb{R}^d))$ *be the solution of (1.4), then we have*

$$\|u(t) - u_0\|_{B^{s-1}_{p,\infty}} \lesssim t.$$

Proof. Using the Newton-Leibniz formula, Minkowski's inequality, Remark 2.1, Lemma 2.2 and Proposition 3.1, we have

$$\begin{aligned}
\|u(t) - u_0\|_{B^{s-1}_{p,\infty}} &\leq \int_0^t \|(1-2u)\nabla S \cdot \nabla u\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau + \int_0^t \|u(1-u)\Delta S\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \\
&\lesssim t\|u\|_{L^\infty_t B^{s-2}_{p,\infty}}\|u\|_{L^\infty_t B^s_{p,\infty}} + t\|u\|_{L^\infty_t B^{s-2}_{p,\infty}}\|u\|_{L^\infty_t B^{s-1}_{p,\infty}}\|u\|_{L^\infty_t B^s_{p,\infty}} \\
&\quad + t\|u\|^3_{L^\infty_t B^{s-1}_{p,\infty}} + t\|u\|^2_{L^\infty_t B^{s-1}_{p,\infty}} \\
&\lesssim t\left(\|u\|^3_{L^\infty_t B^s_{p,\infty}} + \|u\|^2_{L^\infty_t B^s_{p,\infty}}\right) \\
&\lesssim t\left(\|u_0\|^3_{B^s_{p,\infty}} + \|u_0\|^2_{B^s_{p,\infty}}\right) \\
&\lesssim t,
\end{aligned}$$

where we have used (3.3).

We complete the proof of Proposition 3.2. □

Proposition 3.3. *Let* $s - 1 > \frac{d}{p}$ *and* $\|u_0\|_{B^s_{p,\infty}} \lesssim 1$*. Assume that* $u \in L^\infty(0,T; B^s_{p,\infty}(\mathbb{R}^d))$ *be the solution of (1.4), then we have*

$$\|h(t,u_0)\|_{B^{s-2}_{p,\infty}} \lesssim t^2,$$

where we denote

$$h(t,u_0) := u - u_0 + tv_0$$

and

$$v_0 := \nabla \cdot (u_0(1-u_0)\nabla S_0) = (1-2u_0)\nabla S_0 \cdot \nabla u_0 + u_0(1-u_0)\Delta S_0.$$

Proof. Using the Newton-Leibniz formula, Minkowski's inequality, Remark 2.1, Lemma 2.2 and (3.3), we have

$$\begin{aligned}
\|h(t,u_0)\|_{B^{s-2}_{p,\infty}} &\le \int_0^t \|\partial_\tau u + v_0\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau \\
&\le \int_0^t \|\nabla\cdot(u_0(1-u_0)\nabla S_0) - \nabla\cdot(u(1-u)\nabla S)\|_{B^{s-2}_{p,\infty}} \mathrm{d}\tau \\
&\lesssim \int_0^t \|u_0(1-u_0)\nabla S_0 - u(1-u)\nabla S\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \\
&\lesssim \int_0^t \|u(\tau)-u_0\|_{B^{s-1}_{p,\infty}} \mathrm{d}\tau \\
&\lesssim t^2,
\end{aligned}$$

where we have used Proposition 3.2 in the last step.

We complete the proof of Proposition 3.3. □

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1.. Notice that $u(t)-u_0 = h(t,u_0) - tv_0$, then

$$\begin{aligned}
\|u(t)-u_0\|_{B^s_{p,\infty}} &\ge 2^{js}\|\Delta_j(h(t,u_0)-tv_0)\|_{L^p} \\
&\ge 2^{js}t\|\Delta_j v_0\|_{L^p} - 2^{js}\|\Delta_j h(t,u_0)\|_{L^p} \\
&\ge 2^{js}t\|\Delta_j((1-2u_0)\nabla S_0\cdot\nabla u_0)\|_{L^p} - 2^{js}t\|\Delta_j(u_0^2\Delta S_0)\|_{L^p} \\
&\quad - 2^{js}t\|\Delta_j(u_0\partial_x^2 S_0)\|_{L^p} - 2^{js}\|\Delta_j h(t,u_0)\|_{L^p}.
\end{aligned} \tag{3.4}$$

It is not difficult to deduce that

$$\begin{aligned}
&2^{js}t\|\Delta_j(u_0^2\Delta S_0)\|_{L^p} \lesssim t\|u_0^2\Delta S_0\|_{B^s_{p,\infty}} \lesssim t\|u_0\|^3_{B^s_{p,\infty}} \lesssim t, \\
&2^{js}t\|\Delta_j(u_0\Delta S_0)\|_{L^p} \lesssim t\|u_0\Delta S_0\|_{B^s_{p,\infty}} \lesssim t\|u_0\|^2_{B^s_{p,\infty}} \lesssim t, \\
&2^{js}\|\Delta_j h(t,u_0)\|_{L^p} \le 2^{2j}\|h(t,u_0)\|_{B^{s-2}_{p,\infty}} \lesssim t^2 2^{2j}.
\end{aligned}$$

Gathering the above estimates together with (3.4) yields

$$\begin{aligned}
\|u(t)-u_0\|_{B^s_{p,\infty}} &\ge 2^{js}\|\Delta_j(h+tv_0)\|_{L^2} \\
&\ge 2^{js}t\|\Delta_j((1-2u_0)\nabla S_0\cdot\nabla u_0)\|_{L^p} - Ct - Ct^2 2^{2j} \\
&\ge 2^{js}t\|(1-2u_0)\nabla S_0\cdot\Delta_j\nabla u_0\|_{L^p} \\
&\quad - 2^{js}t\|[\Delta_j,(1-2u_0)\nabla S_0]\cdot\nabla u_0\|_{L^p} - Ct - Ct^2 2^{2j}.
\end{aligned} \tag{3.5}$$

On the one hand, by Lemma 2.3, we deduce

$$2^{js}\|[\Delta_j,(1-2u_0)\nabla S_0]\cdot\nabla u_0\|_{L^p} \le C. \tag{3.6}$$

On the other hand, we have

$$2^{js}\|(1-2u_0)(\nabla S_0\cdot\Delta_j\nabla u_0)\|_{L^p} = 2^{js}\left\|(1-2u_0)\sum_{i=1}^d \partial_{x_i}S_0\Delta_j\partial_{x_i}u_0\right\|_{L^p} \ge J-K, \tag{3.7}$$

where

$$J := 2^{js} \|(1-2u_0)\partial_{x_1} S_0 \Delta_j \partial_{x_1} u_0\|_{L^p},$$
$$K := 2^{js} \sum_{i=2}^{d} \|(1-2u_0)\partial_{x_i} S_0 \Delta_j \partial_{x_i} u_0\|_{L^p}.$$

By Lemma 3.1, we infer

$$J = 2^{-2j} \|(1-2u_0)\partial_{x_1} S_0 \partial_{x_1}(1-\Delta) f_j\|_{L^p} \geq 2^{-2j}(J_1 - J_2 - J_3), \tag{3.8}$$

where

$$J_1 := \left\|(1-2u_0)\partial_{x_1} S_0 \partial_{x_1}^3 f_j\right\|_{L^p},$$
$$J_2 := \sum_{i=2}^{d} \left\|(1-2u_0)\partial_{x_1} S_0 \partial_{x_1}\partial_{x_i}^2 f_j\right\|_{L^p},$$
$$J_3 := \|(1-2u_0)\partial_{x_1} S_0 \partial_{x_1} f_j\|_{L^p}.$$

We have

$$\partial_{x_1}^3 f_j(x) = -\left(\frac{17}{12}\right)^3 2^{3j}\phi(x_1)\cos\left(\frac{17}{12}2^j x_1\right)\phi(x_2)\cdots\phi(x_d) + R,$$

where

$$\begin{aligned} R = &\frac{17}{4}2^j\phi''(x_1)\cos\left(\frac{17}{12}2^j x_1\right)\phi(x_2)\cdots\phi(x_d) \\ &- 3\left(\frac{17}{12}\right)^2 2^{2j}\phi'(x_1)\sin\left(\frac{17}{12}2^j x_1\right)\phi(x_2)\cdots\phi(x_d) \\ &+ \phi'''(x_1)\sin\left(\frac{17}{12}2^j x_1\right)\phi(x_2)\cdots\phi(x_d). \end{aligned}$$

Obviously, $\|(1-2u_0)\partial_{x_1} S_0 R\|_{L^p} \leq C2^{2j}$, then

$$J_1 \geq \left(\frac{17}{12}\right)^3 2^{3j}\left\|(1-2u_0)\partial_{x_1} S_0 \phi(x_1)\cos\left(\frac{17}{12}2^j x_1\right)\phi(x_2)\cdots\phi(x_d)\right\|_{L^p} - C2^{2j}.$$

By the construction of f_n, it is not difficult to deduce that

$$S_0(0) = \sum_{n=3}^{\infty} 2^{-n(s+2)} f_n(0) = 0.$$

By easy computations, we have

$$\begin{aligned} \Delta S_0(x) &= \partial_{x_1}^2 S_0(x) + \sum_{i=2}^{d} \partial_{x_i}^2 S_0(x) \\ &= \sum_{n=3}^{\infty} 2^{-n(s+2)}\partial_{x_1}^2 f_n + \sum_{i=2}^{d}\sum_{n=3}^{\infty} 2^{-n(s+2)}\partial_{x_i}^2 f_n. \end{aligned}$$

Noticing that the construction of f_n again and using the fact $\phi'(0) = 0$ from Remark 3.1, then we obtain

$$\Delta S_0(0) = 0,$$

which implies that

$$u_0(0) = S_0(0) - \Delta S_0(0) = 0.$$

Since $(1-2u_0)\partial_{x_1}S_0\phi(x_1)\phi(x_2)\cdots\phi(x_d)$ is a real-valued and continuous function on $\mathbb{R}$, then there exists some $\delta>0$ such that for any $x\in B_\delta(0):=\{x\in\mathbb{R}^d: |x|\le\delta\}$

$$\begin{aligned}
&|[(1-2u_0)\partial_{x_1}S_0\phi(x_1)\phi(x_2)\cdots\phi(x_d)](x)|\\
\ge&\frac{1}{2}\,|[(1-2u_0)\partial_{x_1}S_0\phi(x_1)\phi(x_2)\cdots\phi(x_d)](0)|\\
=&\frac{1}{2}\phi^d(0)|\partial_{x_1}S_0(0)|\\
=&\frac{17}{24}\phi^{2d}(0)\sum_{n=3}^{\infty}2^{-n(s+1)}=:c_0>0.
\end{aligned}$$

Thus we have for j large enough

$$J_1\ge c_0 2^{3j}\left\|\cos\left(\frac{17}{12}2^jx_1\right)\right\|_{L^p(B_\delta(0))}-C2^{2j}\ge\tilde{c}_0 2^{3j}.$$

By direct computations, we can verify that

$$J_2+J_3\le C2^j.$$

Thus, we have

$$J\ge C2^j. \tag{3.9}$$

Similarly, we also have

$$K\le C. \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$2^{js}t\|(1-2u_0)\nabla S_0\cdot\Delta_j\nabla u_0\|_{L^p}\ge C2^jt. \tag{3.11}$$

Inserting (3.11) and (3.6) into (3.5), we deduce that for large j

$$\|u(t)-u_0\|_{B^s_{p,\infty}}\ge C2^jt-Ct-C2^{2j}t^2\ge C2^jt-C2^{2j}t^2.$$

Thus, picking $t2^j\approx\epsilon_0$ with small ϵ_0, we have

$$\|u(t)-u_0\|_{B^s_{p,\infty}}\ge C\epsilon_0-C\epsilon_0^2\ge c_1\epsilon_0.$$

This completes the proof of Theorem 1.1. □

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Declarations

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