Z. Angew. Math. Phys. (2022) 73:227

0044-2275/22/060001-32
published online October 16, 2022
https://doi.org/10.1007/s00033-022-01872-z

Zeitschrift für angewandte Mathematik und Physik ZAMP

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Transmission dynamics to a spatially diffusive Tuberculosis model subject to age-since-infection

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Abstract. Distinct Tuberculosis models with one features of age structure and spatial diffusion have been put forward; however, few Tuberculosis models take all two into account. The main objective of this work is to analyse the recent result (the transmission dynamics) on the long-time behaviour of solutions to the model arising from the spreading of Tuberculosis with the fast and slow progression. Such model is traditionally given by ordinary differential equation system. Here, the local diffusion term, which is used to represent the random walk of the population in a connected domain, and age-since-infection, which is employed to describe the contamination process, are introduced. First, one obtains the well-posedness of the model. Second, one denotes the basic reproduction number through the spectral radius of a compact positive linear operator, which determines the dichotomy of disease persistence and extinction. Third, one proves the global dynamics of the model.

Mathematics Subject Classification. 35B35, 35B40, 35K20, 35K57, 35Q92, 37N25, 92D30.

Keywords. A spatially diffusive Tuberculosis model, Age-since-infection, The fast and slow progression, The well-posedness, The basic reproduction number, The global dynamics.

Contents

1. Introduction

1.1. Research background and current situation

Tuberculosis is a bacterial disease caused by Mycobacterium tuberculosis and is usually acquired through airborne infection from active Tuberculosis cases [1–3]. In [3,4], Blower et al. formulate and analyse mathematical models describing the transmission dynamics of untreated tuberculosis epidemics. It was assumed in [3,4] that infected individuals remain non-infectious until they develop disease by one of

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two pathogenic mechanisms: direct progression or endogenous reactivation. Two types of tuberculosis contribute to the incidence rate of tuberculosis disease: one type of tuberculosis develops through direct progression soon after infection which is named fast progression and the other type of tuberculosis develops through endogenous reactivation in the latently infected individuals which is named slow progression.

As everyone knows, Tuberculosis is a familiar and deadly infectious disease, which has developed into a chronic infectious disease threatening human health all over the world. In 2016, an estimated 10 million 400 thousand cases of Tuberculosis were equivalent to 140 cases per 100 thousand population. At the same time, the proportion of people suffering from Tuberculosis and dying of the disease (fatality rate) is 16%. Hence, Tuberculosis has become a social and public health issue of global concern [5].

Overall, half of those affected by Tuberculosis face catastrophic costs (more than 20% of household income) due to Tuberculosis and are distributed in 16 countries, ranging from 19 to 83%. For drug-resistant Tuberculosis patients, this proportion rose to 80%. In most of the countries surveyed, the poorest households are more than 20% more likely to face catastrophic costs. Tuberculosis is a poor disease. The most at risk are often those who have the most problems with access to health services and are most adversely affected by the high out of pocket costs of health care. If there are no strong mitigation measures, including social protection, a higher proportion of Tuberculosis patients and their families will face a risk of catastrophic costs [6].

Many mathematical models have been become the helpful tools attempting to obtain a better understanding of the spread and control of Tuberculosis. For the past few years, many scholars have done a lot of research on the transmission mechanism and prevention strategy of Tuberculosis, such as [7–13].

Now, age is one of the most significant and fashionable variables constituting a population. In short, at the individual level, many internal variables inevitably account on age, since different ages mean different reproductive and survival abilities, as well as different behaviours. Therefore, with a deeper understanding of age, some age-dependent Tuberculosis models have been proposed, analysed and studied, such as [14–17]. Besides, at different age stages, the effects of Tuberculosis transmission are various, which is another important and key factor that needs necessarily to be included in modelling this Tuberculosis transmission process.

Because the population distribute heterogeneously in diverse spatial location in the real life and they will move or diffuse for many reasons, there is increasing testimony that environmental heterogeneity and individual motility have momentous influence on the spread of Tuberculosis.

Owing to the uneven distribution of people in different spatial locations in real life, and the migration or spread of people due to a variety of reasons, more and more evidence in epidemiology shows that environmental heterogeneity and individual initiative have an significant impact on the spread of infectious diseases [18,19]. In recent years, the global behaviour of the spatial diffusion system for Tuberculosis and other diseases has attracted extensive attention and become one of the research hotspots. Spatial diffusion is an intrinsic properties for studying the roles of spatial heterogeneity on Tuberculosis mechanisms and transmission routes and can lead to rich dynamics. Based on this reality, one generalizes (1.1)–(1.7) by taking account of the case that individuals move or diffuse around on the spatial habitat $x \in \Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. However, there are relatively few works on Tuberculosis models with both of the infection age and spatial diffusion.

1.2. Mathematical model

Inspired by the above discussions, in this article, one will do with the following spatially diffusive Tuberculosis model version possessing age-since-infection which is generalization of the model investigated in [5] for the first time to allow for individuals moving around on the spatial habitat $x \in \Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, and is very necessary and reasonable. In order to consider the dynamical structure of Tuberculosis model, the total population is decomposed into four compartments: the susceptible

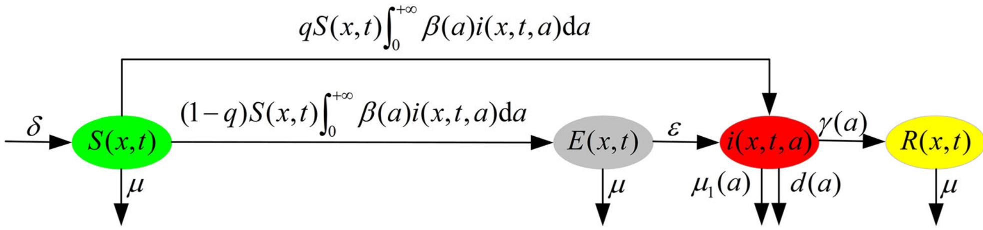

FIG. 1. The transfer diagram of system (1.1)–(1.7)

compartment (S) (individuals not infected but capable of infection), the exposed compartment (E) (individuals infected but not yet infectious, i.e. undetected non-symptomatic (latent) carriers), the infectious compartment (I) (individuals capable of transmitting the disease to susceptibles, i.e. symptomatic infectious individuals), and the removed compartment (R) (individuals who have died or who have recovered with permanent immunity). One assumes that the total population is constant and confined to a bounded spatial domain Ω in R^n with smooth boundary $\partial\Omega$.

Let $i(x,t,a)$ be the density function of the infected population, where x is a point of space in Ω, t is the time elapsed since the beginning of the Tuberculosis, and a is the infective age (i.e. the time elapsed since acquiring the infection), $S(x,t)$, $E(x,t)$, $I(x,t)$, and $R(x,t)$ be the density functions of the compartment (S), (E), (I), and (R), individually, τ be the length of the incubation period and $\hat{\tau}$ the length of the disease duration. Evidently,

$$E(x,t)=\int_0^{\tau} i(x,t,a)\mathrm{d}a, I(x,t)=\int_{\tau}^{\tau+\hat{\tau}} i(x,t,a)\mathrm{d}a.$$

Remark 1.1. These formulas are applicable only to the case where the incubation period is constant. Moreover, the age a in (1.3) is different from that in the above formulas, that is the age a in (1.3) stands for the age from which the individual became infected and the incubation period is not included in a.

If one takes time frames that are comparable with the life span of the infectious individuals into consideration, then one can hypothesize that the length of the disease duration will be infinite. Namely,

$$I(x,t)=\int_0^{+\infty} i(x,t,a)\mathrm{d}a,$$

and the total population $N(x,t)$ is given by

$$N(x,t)=S(x,t)+E(x,t)+\int_0^{+\infty} i(x,t,a)\mathrm{d}a+R(x,t).$$

Obviously , the travel of latent individuals demonstrating no symptoms can spread the Tuberculosis geographically which makes Tuberculosis harder to control.

The transfer diagram of the model is shown in Fig. 1. The transfer diagram leads to the following system of reaction–diffusion equations.

$$\frac{\partial}{\partial t}S(x,t)=D_1\frac{\partial^2}{\partial x^2}S(x,t)+\delta-S(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a-\mu S(x,t), \tag{1.1}$$

$$\frac{\partial}{\partial t}E(x,t) = D_2\frac{\partial^2}{\partial x^2}E(x,t) + \overbrace{(1-q)S(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a}^{\text{slow progression}} - (\mu+\varepsilon)E(x,t), \tag{1.2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i(x,t,a) = D_3\frac{\partial^2}{\partial x^2}i(x,t,a) - \left[\gamma(a)+\mu_1(a)+d(a)\right]i(x,t,a), \tag{1.3}$$

$$i(x,t,0) = \underbrace{qS(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a}_{\text{fast progression}} + \varepsilon E(x,t), \tag{1.4}$$

$$\frac{\partial}{\partial t}R(x,t) = D_4\frac{\partial^2}{\partial x^2}R(x,t) + \int_0^{+\infty}\gamma(a)i(x,t,a)\mathrm{d}a - \mu R(x,t), \tag{1.5}$$

corresponding to initial data

$$\begin{aligned}&[S(x,0),E(x,0),i(x,0,a),R(x,0)]\\ &\quad = [\varphi_1(x),\varphi_2(x),\varphi_3(x,a),\varphi_4(x)]\,, a\geq 0, x\in\Omega,\end{aligned} \tag{1.6}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t) = \frac{\partial}{\partial \mathbf{n}}E(x,t) = \frac{\partial}{\partial \mathbf{n}}i(x,t,a) = \frac{\partial}{\partial \mathbf{n}}R(x,t) = 0, x\in\partial\Omega. \tag{1.7}$$

All the parameters of system (1.1)–(1.7) are positive constants.

▷ D_1 is the diffusion coefficient of susceptible individuals.
▷ δ is the constant recruitment rate of the population (comprising the birth and immigration).
▷ $\beta(a)$ is the transmission coefficient of Tuberculosis which relies upon age-since-infection a, and $\beta(\cdot)\in L^\infty(0,+\infty)$.
▷ μ is the natural death rate independent of age-since-infection a of population.
▷ D_2 is the diffusion coefficient of undetected non-symptomatic (latent) carriers.
▷ q is the proportion of disease by fast progression.
▷ ε is the progression rate from the exposed individuals to the infected individuals.
▷ D_3 is the diffusion coefficient of symptomatic infectious individuals with age-since-infection a.
▷ $\gamma(a)$ is the recovery rate which depends on age-since-infection a, and $\gamma(\cdot)\in L^\infty(0,+\infty)$.
▷ $\mu_1(a)$ is the natural death rate which accounts upon age-since-infection a, and $\mu(\cdot)\in L^\infty(0,+\infty)$.
▷ $d(a)$ is the additional death rate induced by the Tuberculosis which relies on age-since-infection a, and $d(\cdot)\in L^\infty(0,+\infty)$.
▷ $i(x,t,0)$ is used to reflect the resources fluxing into compartment, because the infection occurs at age 0.
▷ $\Omega\in\mathbb{R}^n$ is a spatial habitat with smooth boundary $\partial\Omega$.
▷ $\mathbf{n}$ is the outward normal to $\partial\Omega$, where Ω is bounded and connected.
▷ $\frac{\partial}{\partial\mathbf{n}}$ is the differentiation along the outward unit normal $\mathbf{n}$.

The homogeneous Neumann boundary conditions indicate that there is no population flux across the boundary $\partial\Omega$.

Since the first four equations of system (1.1)–(1.7) are independent of $R(x,t)$, one only needs to consider the following system:

$$\frac{\partial}{\partial t}S(x,t) = D_1\frac{\partial^2}{\partial x^2}S(x,t) + \delta - S(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a - \mu S(x,t), \tag{1.8}$$

$$\frac{\partial}{\partial t}E(x,t) = D_2\frac{\partial^2}{\partial x^2}E(x,t) + \overbrace{(1-q)S(x,t)\int\limits_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a}^{\text{slow progression}} - (\mu+\varepsilon)E(x,t), \tag{1.9}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)i(x,t,a) = D_3\frac{\partial^2}{\partial x^2}i(x,t,a) - \left[\gamma(a) + \mu_1(a) + d(a)\right]i(x,t,a), \tag{1.10}$$

$$i(x,t,0) = \underbrace{qS(x,t)\int\limits_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a}_{\text{fast progression}} + \varepsilon E(x,t), \tag{1.11}$$

associating with initial data

$$\begin{aligned}&[S(x,0), E(x,0), i(x,0,a)]\\ &\quad = \left[\varphi_1(x), \varphi_2(x), \varphi_3(x,a)\right], a \geq 0, x \in \Omega,\end{aligned} \tag{1.12}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t) = \frac{\partial}{\partial \mathbf{n}}E(x,t) = \frac{\partial}{\partial \mathbf{n}}i(x,t,a) = 0, x \in \partial\Omega. \tag{1.13}$$

For systems (1.8)–(1.13), there are two main reduction forms as below:

(1) Spatially diffusive Tuberculosis model.

If $i(x,t,a) = i(x,t)$ which is not relying upon the age parameter, then systems (1.8)–(1.13) will reduce to the spatially diffusive Tuberculosis model as follows:

$$\frac{\partial}{\partial t}S(x,t) = D_1\frac{\partial^2}{\partial x^2}S(x,t) + \delta - \beta S(x,t)i(x,t) - \mu S(x,t), \tag{1.14}$$

$$\frac{\partial}{\partial t}E(x,t) = D_2\frac{\partial^2}{\partial x^2}E(x,t) + (1-q)\beta S(x,t)i(x,t) - (\mu+\varepsilon)E(x,t), \tag{1.15}$$

$$\frac{\partial}{\partial t}i(x,t) = D_3\frac{\partial^2}{\partial x^2}i(x,t) + q\beta S(x,t)i(x,t) + \varepsilon E(x,t) - (\gamma + \mu + d)\,i(x,t), \tag{1.16}$$

relating to initial data

$$\begin{aligned}&[S(x,0), E(x,0), i(x,0)]\\ &\quad = \left[\varphi_1(x), \varphi_2(x), \varphi_3(x)\right], x \in \Omega,\end{aligned} \tag{1.17}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t) = \frac{\partial}{\partial \mathbf{n}}E(x,t) = \frac{\partial}{\partial \mathbf{n}}i(x,t) = 0, x \in \partial\Omega. \tag{1.18}$$

(2) Age-dependent Tuberculosis model.

If $i(x,t,a) = i(t,a)$ which is not depending upon the spatial variable, then systems (1.8)–(1.13) will reduce to the age-dependent Tuberculosis model as follows:

$$\frac{\partial}{\partial t}S(t) = \delta - S(t)\int\limits_0^{+\infty}\beta(a)i(t,a)\mathrm{d}a - \mu S(t), \tag{1.19}$$

$$\frac{\partial}{\partial t}E(t) = (1-q)S(t)\int\limits_0^{+\infty}\beta(a)i(t,a)\mathrm{d}a - (\mu+\varepsilon)E(t), \tag{1.20}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)i(t,a) = -\left[\gamma(a) + \mu_1(a) + d(a)\right]i(t,a), \tag{1.21}$$

$$i(t,0) = qS(t)\int_0^{+\infty}\beta(a)i(t)\mathrm{d}a + \varepsilon E(t), \tag{1.22}$$

$$S(0) = S_0 \geq 0, E(0) = E_0 \geq 0, i(t,\cdot) = i_0 \in L^1((0,+\infty),\mathbb{R}). \tag{1.23}$$

For systems (1.14)–(1.18) and (1.19)–(1.23), nowdays, there have some good results. Namely, in [20], authors obtain that there exist a traveling wave solutions (TWS) for the model if the threshold number $\Re_0 > 1$ and $c > c^*$, where c^* is the minimum wave speed by using sub-super solution method, Schauders fixed point theorem and Lyapunov functional. In [3], authors show that the transmission dynamics of the disease is fully determined by the basic reproduction number. Besides, authors establish the local stability of a disease-free steady state and an endemic steady state of the model by analysing corresponding characteristic equations. Meanwhile, authors prove the system is uniformly persistent when the basic reproduction number is greater than unity by using the persistence theory for infinite dimensional system. Finally, authors verify that the global dynamics of the system is completely determined by the basic reproduction number by constructing suitable Lyapunov functionals and using LaSalles invariance principle. The major task of this article is to explore the transmission dynamics of system (1.8)–(1.13).

One organizes this manuscript below.

- ▷ One gives the well-posedness of system (1.8)–(1.13) in part 2.
- ▷ One denotes the basic reproduction number of system (1.8)–(1.13) in part 3.
- ▷ One discusses the local dynamics of system (1.8)–(1.13) in part 4.
- ▷ One considers the disease persistence of system (1.8)–(1.13) in part 5.
- ▷ One studies the global dynamics of system (1.8)–(1.13) in part 6.
- ▷ One provides some discussions in part 7.

2. Well-posedness

Consider Banach spaces

$$\mathbb{X}_1 \overset{\text{def}}{=\!=} C\left(\overline{\Omega},\mathbb{R}\right), \mathbb{X}_2 \overset{\text{def}}{=\!=} L^1\left(\mathbb{R}_+,\mathbb{X}_1\right),$$

which are equipped with the norm

$$|\xi|_{\mathbb{X}_1} = \sup_{x\in\overline{\Omega}}|\xi(x)|, ||\eta||_{\mathbb{X}_2} \overset{\text{def}}{=\!=} \int_0^{+\infty}|\eta(\cdot,a)|_{\mathbb{X}_1}\mathrm{d}a, \quad \forall\xi\in\mathbb{X}_1, \eta\in\mathbb{X}_2.$$

Next, the positive cones of $\mathbb{X}_1$, $\mathbb{X}_2$ are defined by $\mathbb{X}_1^+$, $\mathbb{X}_2^+$. Let

$$\chi(a) = e^{-\int_0^a[\gamma(s)+\mu_1(s)+d(s)]\mathrm{d}s},$$

$G : \mathbb{X}_1 \to \mathbb{X}_1$ be the C_0 semigroup generated by $D_3\frac{\partial^2}{\partial x^2}$ subject to the homogeneous Neumann boundary condition. According to [21], one has

$$(G(t)[\xi])(x) = \int_\Omega G(y,x,t)\xi(y)\mathrm{d}y, \quad \forall t > 0, \xi \in \mathbb{X}_1,$$

where $G(y,x,t)$ is the Green function. Due to [22], one knows that G is compact and strongly positive for $\forall t > 0$. Via directly solving the equations (1.10) by the way of the characteristic line $t - a = c$ (as

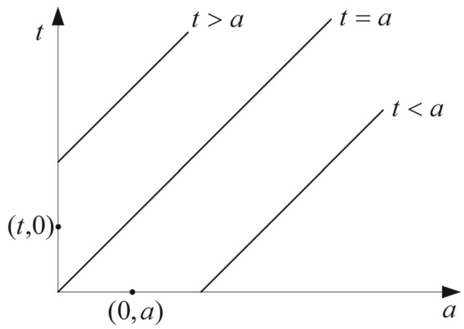

FIG. 2. The characteristic line

shown in Fig. 2), where c is a constant, we have

$$i(x,t,a)=\begin{cases}\chi(a)\int\limits_{\Omega}G(y,a,x)i(y,t-a,0)\mathrm{d}y, & t-a>0, x\in\Omega,\\ \frac{\chi(a)}{\chi(a-t)}\int\limits_{\Omega}G(y,t,x)\varphi_3(y,a-t)\mathrm{d}y, & a-t\geq 0, x\in\Omega.\end{cases} \tag{2.1}$$

Set

$$j(x,t)\xlongequal{\text{def}} i(x,t,0),$$

then substituting (2.1) into systems (1.8)–(1.13) derives

$$\frac{\partial}{\partial t}S(x,t)=D_1\frac{\partial^2}{\partial x^2}S(x,t)+\delta-\frac{1}{q}j(x,t)+\frac{1}{q}\varepsilon E(x,t)-\mu S(x,t), \tag{2.2}$$

$$\frac{\partial}{\partial t}E(x,t)=D_2\frac{\partial^2}{\partial x^2}E(x,t)+\frac{1-q}{q}j(x,t)-\left(\mu+\frac{1}{q}\varepsilon\right)E(x,t), \tag{2.3}$$

$$j(x,t)=qS(x,t)\left[h_1(x,t)+h_2(x,t)\right]+\varepsilon E(x,t), \tag{2.4}$$

associating with initial data

$$[S(x,0),E(x,0),j(x,0)] = \left[\varphi_1(x),\varphi_2(x),q\varphi_1(x)\int\limits_0^{+\infty}\beta(a)\varphi_3(x,a)\mathrm{d}a+\varepsilon\varphi_2(x)\right], a\geq 0, x\in\Omega, \tag{2.5}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t)=\frac{\partial}{\partial \mathbf{n}}E(x,t)=\frac{\partial}{\partial \mathbf{n}}j(x,t)=0, x\in\partial\Omega \tag{2.6}$$

with

$$h_1(x,t)=\int\limits_0^t\beta(a)\chi(a)\int\limits_{\Omega}G(y,x,a)j(y,t-a)\mathrm{d}y\mathrm{d}a,$$

$$h_2(x,t)=\int\limits_t^{+\infty}\beta(a)\frac{\chi(a)}{\chi(a-t)}\int\limits_{\Omega}G(y,x,t)\varphi_3(y,a-t)\mathrm{d}y\mathrm{d}a.$$

Next, one concentrates on the well-posedness of system (2.2)–(2.6).

Theorem 2.1. *For* $\forall(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+$, *system* (2.2)–(2.6) *exists an only solution* (S,E,j) *on* $\overline{\Omega}\times[0,t_{\max}), t_{\max}\le+\infty$.

Proof. Set

$$\mathbb{X}_3 \overset{\text{def}}{=\!=\!=} C\left(\mathbb{X}_1,[0,t_{\max})\right), t_{\max}\le+\infty,$$

which is taken the norm

$$||\eta||_{\mathbb{X}_3} \overset{\text{def}}{=\!=\!=} \sup_{0\le t\le t_{\max}} |\eta(\cdot,t)|_{\mathbb{X}_1}, t_{\max}\le+\infty, \forall\eta\in\mathbb{X}_3.$$

Solving Eqs. (2.2), (2.3) in $(x,t)\in\overline{\Omega}\times[0,t_{\max}), t_{\max}\le+\infty$, one gets

$$S(x,t)=h_3(x,t)+\int_0^t e^{-\mu(t-a)}\int_\Omega \hat{G}(y,x,t-a)\left[\delta-\frac{1}{q}j(y,a)+\frac{1}{q}\varepsilon E(y,a)\right]\mathrm{d}y\mathrm{d}a, \tag{2.7}$$

$$E(x,t)=h_4(x,t)+\frac{1-q}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \tilde{G}(y,x,t-a)j(a,y)\mathrm{d}y\mathrm{d}a, \tag{2.8}$$

$$j(x,t)=qS(x,t)\left[h_1(x,t)+h_2(x,t)\right]+\varepsilon E(x,t), \tag{2.9}$$

where

$$h_3(x,t)=e^{-\mu t}\int_0^\pi \hat{G}(y,x,t)\varphi_1(y)\mathrm{d}y,$$

$$h_4(x,t)=e^{-\left(\mu+\frac{1}{q}\varepsilon\right)t}\int_0^\pi \tilde{G}(y,x,t)\varphi_2(y)\mathrm{d}y$$

with $\hat{G}$, $\tilde{G}$ are the Green function concerning $D_1\frac{\partial^2}{\partial x^2}$, $D_2\frac{\partial^2}{\partial x^2}$ subject to the homogeneous Neumann boundary condition. Substituting Eqs. (2.7), (2.8) into (2.9) allows us to denote $H:\mathbb{X}_3\to\mathbb{X}_3$ as:

$$\begin{aligned} &H[j(x,t)](x,t)\\ &\overset{\text{def}}{=\!=\!=} qh_3(x,t)\left[h_1(x,t)+h_2(x,t)\right]\\ &+\int_0^t e^{-\mu(t-a)}\int_\Omega \hat{G}(y,x,t-a)\left[\delta-\frac{1}{q}j(y,a)+\frac{1}{q}\varepsilon E(y,a)\right]\mathrm{d}y\mathrm{d}a\left[h_1(x,t)+h_2(x,t)\right]\\ &+\varepsilon\left[h_4(x,t)+\frac{1-q}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \tilde{G}(y,x,t-a)j(y,a)\mathrm{d}y\mathrm{d}a\right]. \end{aligned} \tag{2.10}$$

For convenience, one lets

$$h_5(x,t)=\int_0^t e^{-\mu(t-a)}\int_\Omega \hat{G}(y,x,t-a)\left[\delta-\frac{1}{q}j(y,a)+\frac{1}{q}\varepsilon E(y,a)\right]\mathrm{d}y\mathrm{d}a,$$

$$h_6(x,t)=\frac{1-q}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \tilde{G}(y,x,t-a)j(y,a)\mathrm{d}y\mathrm{d}a,$$

then (2.10) becomes

$$H[j(x,t)](x,t)\overset{\text{def}}{=\!=\!=} q\left[h_3(x,t)+h_5(x,t)\right]\left[h_1(x,t)+h_2(x,t)\right]$$

$$+\varepsilon\left[h_4(x,t)+h_6(x,t)\right].$$

Via standard procedures, so as to yield a strict contraction mapping H in $\mathbb{X}_3$, one puts

$$j_1, j_2 \in \mathbb{X}_3, \hat{j} \xlongequal{\text{def}} j_1 - j_2.$$

Then one has

$$\begin{aligned}|H[j_1]-H[j_2]| &= \left|qh_3h_1\left(\hat{j}\right)+\tilde{h}_5\left(\hat{j}\right)h_1(j_2)+h_5(j_1)h_1\left(\hat{j}\right)+\varepsilon h_6\left(\hat{j}\right)\right| \\ &\le \left|(qh_3+h_5)\tilde{h}_1+\tilde{\tilde{h}}_5h_1+\varepsilon\tilde{h}_6\right|\cdot\sup_{0\le s\le t}\left|\hat{j}(s,\cdot)\right|_{\mathbb{X}_1},\end{aligned}$$

where

$$\tilde{h}_5(x,t) = -\frac{1}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \hat{G}(y,x,t-a)\hat{j}(y,a)\mathrm{d}y\mathrm{d}a,$$

$$\tilde{h}_1(x,t) = \int_0^t \beta(a)\chi(a)\int_\Omega G(y,x,a)\mathrm{d}y\mathrm{d}a,$$

$$\tilde{\tilde{h}}_5(x,t) = -\frac{1}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \hat{G}(y,x,t-a)\mathrm{d}y\mathrm{d}a,$$

$$\tilde{h}_6(x,t) = \frac{1-q}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \tilde{G}(y,x,t-a)\mathrm{d}y\mathrm{d}a.$$

Let

$$C(t_{\max}) \xlongequal{\text{def}} \sup_{0\le t\le t_{\max}}\left|(qh_3+h_5)\tilde{h}_1+\tilde{\tilde{h}}_5h_1+\varepsilon\tilde{h}_6\right|_{\mathbb{X}_1}, t_{\max}\le+\infty,$$

then

$$|H[j_1]-H[j_2]|_{\mathbb{X}_3} \le C(t_{\max})\,|j_1-j_2|_{\mathbb{X}_3}, t_{\max}\le+\infty.$$

Select $0 < t_{\max} \le 1$ small enough such that $C(t_{\max}) < 1$. Thus, H is a strict contraction in $\mathbb{X}_3$. According to the theorem 9.23 (i.e. contraction mapping theorem) of [23], one finishes the proof of this theorem. □

Theorem 2.2. *For $\forall(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+$, the solution (S,E,j) of system (2.2)–(2.6) meets*

$$S>0, E\ge0, j\ge0 \text{ on } \overline{\Omega}\times[0,t_{\max}), t_{\max}\le+\infty.$$

Proof. For $\forall\eta\in\mathbb{X}_2$, define

$$A(\eta)(x,t) \xlongequal{\text{def}} \int_0^t \beta(a)\chi(a)\int_\Omega G(y,x,a)\alpha(y,t-a)\mathrm{d}y\mathrm{d}a.$$

And then, $A:\mathbb{X}_2\to\mathbb{X}_2$ is the positive linear operator in the sense that $A\left(\mathbb{X}_2^+\right)\subset\mathbb{X}_2^+$. Noting that $h_1(x,t)$ can be expressed by $A(\eta)$. Then for $(x,t)\in\overline{\Omega}\times[0,t_{\max}), t_{\max}\le+\infty$, one gains

$$\begin{aligned}\frac{\partial}{\partial t}S(x,t) &> D_1\frac{\partial^2}{\partial x^2}S(x,t)-S(x,t)\left[A(\eta)+h_2(x,t)+\mu\right],\\ \frac{\partial}{\partial t}E(x,t) &= D_2\frac{\partial^2}{\partial x^2}E(x,t)+(1-q)S(x,t)\left[A(\eta)+h_2(x,t)\right]-(\varepsilon+\mu)E(x,t),\\ j(x,t) &= qS(x,t)\left[A(\eta)+h_2(x,t)\right]+\varepsilon E(x,t),\end{aligned}$$

relating with initial data

$$[S(x,0),E(x,0),j(x,0)] \\ =\left[\varphi_1(x),\varphi_2(x),q\varphi_1(x)\int\limits_0^{+\infty}\beta(a)\varphi_3(x,a)\mathrm{d}a+\varepsilon\varphi_2(x)\right],a\geq 0,x\in\Omega,$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t)=\frac{\partial}{\partial \mathbf{n}}E(x,t)=\frac{\partial}{\partial \mathbf{n}}j(x,t)=0,x\in\partial\Omega.$$

Owing to the continuity and boundedness of $A(\eta)+h_2(x,t)+\mu$, one attains

$$S(x,t)>0 \text{ on } \overline{\Omega}\times[0,t_{\max}),t_{\max}\leq+\infty.$$

Proceed to the next step, for $(x,t)\in\overline{\Omega}\times[0,t_{\max}),t_{\max}\leq+\infty$, one obtains

$$\begin{aligned}\frac{\partial}{\partial t}S(x,t)&>D_1\frac{\partial^2}{\partial x^2}S(x,t)-S(x,t)\left[A(\eta)+h_2(x,t)+\mu\right],\\ \frac{\partial}{\partial t}E(x,t)&>D_2\frac{\partial^2}{\partial x^2}E(x,t)-(\varepsilon+\mu)E(x,t),\\ j(x,t)&=qS(x,t)\left[A(\eta)+h_2(x,t)\right]+\varepsilon E(x,t),\end{aligned}$$

associating with initial data

$$[S(x,0),E(x,0),j(x,0)] \\ =\left[\varphi_1(x),\varphi_2(x),q\varphi_1(x)\int\limits_0^{+\infty}\beta(a)\varphi_3(x,a)\mathrm{d}a+\varepsilon\varphi_2(x)\right],a\geq 0,x\in\Omega,$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}S(x,t)=\frac{\partial}{\partial \mathbf{n}}E(x,t)=\frac{\partial}{\partial \mathbf{n}}j(x,t)=0,x\in\partial\Omega.$$

Consequently,

$$E(x,t)\geq 0,J(x,t)\geq 0 \text{ on } \overline{\Omega}\times[0,t_{\max}),t_{\max}\leq+\infty.$$

One finishes the proof of this theorem. □

Theorem 2.3. *For* $\forall(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+$, *the solution* (S,E,j) *of system* (2.2)–(2.6) *is bounded in* $\overline{\Omega}\times[0,t_{\max}),t_{\max}\leq+\infty$.

Proof. According to system (1.8)–(1.13), one obtains

$$\begin{aligned}&\frac{\partial}{\partial t}S(x,t)+\frac{\partial}{\partial t}E(x,t)+\int\limits_0^{+\infty}\frac{\partial}{\partial t}i(x,t,a)\mathrm{d}a\\ &=D_1\frac{\partial^2}{\partial x^2}S(x,t)+\delta-S(x,t)\int\limits_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a\\ &\quad-\mu S(x,t)+D_2\frac{\partial^2}{\partial x^2}E(x,t)+(1-q)S(x,t)\int\limits_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a\end{aligned}$$

$$
\begin{aligned}
&- (\mu+\varepsilon)E(x,t) + D_3\int_0^{+\infty}\frac{\partial^2}{\partial x^2}i(x,t,a)\mathrm{d}a \\
&- \int_0^{+\infty}\left[\gamma(a)+\mu_1(a)+d(a)\right]i(x,t,a)\mathrm{d}a - \int_0^{+\infty}\frac{\partial}{\partial a}i(x,t,a)\mathrm{d}a \\
=\ & D_1\frac{\partial^2}{\partial x^2}S(x,t) + D_2\frac{\partial^2}{\partial x^2}E(x,t) + D_3\int_0^{+\infty}\frac{\partial^2}{\partial x^2}i(x,t,a)\mathrm{d}a \\
&- qS(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a + \delta - \mu S(x,t) - (\mu+\varepsilon)E(x,t) \\
&- \int_0^{+\infty}\left[\gamma(a)+\mu_1(a)+d(a)\right]i(x,t,a)\mathrm{d}a - \int_0^{+\infty}\frac{\partial}{\partial a}i(x,t,a)\mathrm{d}a \\
<\ & D_1\frac{\partial^2}{\partial x^2}S(x,t) + D_2\frac{\partial^2}{\partial x^2}E(x,t) + D_3\int_0^{+\infty}\frac{\partial^2}{\partial x^2}i(x,t,a)\mathrm{d}a \\
&+ \delta - \mu\left[S(x,t)+E(x,t)+\int_0^{+\infty}i(x,t,a)\mathrm{d}a\right].
\end{aligned}
$$

Noting the homogeneous Neumann boundary conditions of system (1.8)–(1.13) and employing the Gauss formula, one derives

$$
\int_\Omega D_1\frac{\partial^2}{\partial x^2}S(x,t)\mathrm{d}x = \int_\Omega D_2\frac{\partial^2}{\partial x^2}E(x,t)\mathrm{d}x = \int_\Omega D_3\int_0^{+\infty}\frac{\partial^2}{\partial x^2}i(x,t,a)\mathrm{d}a\mathrm{d}x = 0.
$$

It follows that

$$
\begin{aligned}
\frac{\mathrm{d}\tilde{N}(t)}{\mathrm{d}t} &= \int_\Omega\left[\frac{\partial}{\partial t}S(x,t)+\frac{\partial}{\partial t}E(x,t)+\int_0^{+\infty}\frac{\partial}{\partial t}i(x,t,a)\mathrm{d}a\right]\mathrm{d}x \\
&< \int_\Omega\left\{\delta-\mu\left[S(x,t)+E(x,t)+\int_0^{+\infty}i(x,t,a)\mathrm{d}a\right]\right\}\mathrm{d}x \\
&= \delta\left|\Omega\right| - \mu\tilde{N}(t).
\end{aligned}
$$

Hence, if

$$
\tilde{N}(t) > \frac{\delta}{\mu}\left|\Omega\right|,
$$

then

$$
\frac{\mathrm{d}\tilde{N}(t)}{\mathrm{d}t} < 0.
$$

In addition, one finds the ODE

$$
\frac{\mathrm{d}\tilde{N}(t)}{\mathrm{d}t} = \delta\left|\Omega\right| - \mu\tilde{N}(t)
$$

with general solution

$$\tilde{N}(t) = \frac{\delta}{\mu}|\Omega| + \left[\tilde{N}(0) - \frac{\delta}{\mu}|\Omega|\right] e^{-\mu t},$$

where $\tilde{N}(0)$ shows the initial value of total population in region Ω. Via utilizing the standard comparison theorem, one finds out for all $t \geq 0$,

$$\tilde{N}(t) \leq \frac{\delta}{\mu}|\Omega|,$$

if

$$\tilde{N}(0) \leq \frac{\delta}{\mu}|\Omega|.$$

Therefore,

$$\Phi = \left\{(S, E, i)|\tilde{N} \leq \frac{\delta}{\mu}|\Omega|\right\}$$

is positive invariant for system (2.2)–(2.6), where $|\Omega|$ is the volume of Ω. One finishes the proof of this theorem. □

3. Basic reproduction number

Apparently, systems (1.8)–(1.13) admit the disease-free equilibrium $P_0 = (S_0, 0, 0)$ with $S_0 = \frac{\delta}{\mu}$. Linearizing system (1.8)–(1.13) near such equilibrium in disease invasion phase, one has

$$\frac{\partial}{\partial t}E(x,t) = D_2\frac{\partial^2}{\partial x^2}E(x,t) + (1-q)S(x,t)\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a - (\mu+\varepsilon)E(x,t), \tag{3.1}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)i(x,t,a) = D_3\frac{\partial^2}{\partial x^2}i(x,t,a) - \left[\gamma(a) + \mu_1(a) + d(a)\right]i(x,t,a), \tag{3.2}$$

$$i(x,t,0) = q\frac{\delta}{\mu}\int_0^{+\infty}\beta(a)i(x,t,a)\mathrm{d}a + \varepsilon E(x,t), \tag{3.3}$$

coresponding with initial data

$$\begin{aligned}&[E(x,0), i(x,0,a)]\\ &\quad= [\varphi_2(x), \varphi_3(x,0)], a \geq 0, x \in \Omega,\end{aligned} \tag{3.4}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}E(x,t) = \frac{\partial}{\partial \mathbf{n}}i(x,t,a) = 0, x \in \partial\Omega. \tag{3.5}$$

According to the above results, one finds

$$\begin{aligned}j(x,t) = {}& \frac{q\delta}{\mu}\int_0^t \beta(a)\chi(a)\int_\Omega G(y,x,a)j(y,t-a)\mathrm{d}y\mathrm{d}a\\ &+ \frac{\varepsilon(1-q)}{q}\int_0^t e^{-\mu(t-a)}\int_\Omega \tilde{G}(y,x,t-a)j(y,a)\mathrm{d}y\mathrm{d}a.\end{aligned} \tag{3.6}$$

Making the Laplace transformation to (3.6), one gets

$$\begin{aligned} L[j] &\xlongequal{\text{def}} \int_0^{+\infty} e^{-\lambda t} j \mathrm{d}t \\ &= \frac{q\delta}{\mu} \int_0^{+\infty} e^{-\lambda t} \int_0^t \beta(a)\chi(a) \int_\Omega G(y,x,a) j(y,t-a) \mathrm{d}y\mathrm{d}a\mathrm{d}t \\ &+ \frac{\varepsilon(1-q)}{q} \int_0^{+\infty} e^{-\lambda t} \int_0^t e^{-\mu(t-a)} \int_\Omega \tilde{G}(y,x,t-a) j(y,a) \mathrm{d}y\mathrm{d}a\mathrm{d}t. \end{aligned}$$

Hence, after multiple interchanging the order of integration, one derives

$$\begin{aligned} L[j] = \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a) e^{-\lambda a} \int_\Omega G(y,x,a) \int_0^t j(y,t) \mathrm{d}t\mathrm{d}y\mathrm{d}a \\ + \frac{\varepsilon(1-q)}{q} \int_0^{+\infty} e^{\mu a} j(y,a) \int_\Omega \tilde{G}(y,x,t-a) \int_0^t e^{-(\lambda+\mu)t} \mathrm{d}t\mathrm{d}y\mathrm{d}a. \end{aligned}$$

Setting $\lambda = 0$, one figures out

$$\begin{aligned} \int_0^{+\infty} j(t,x)\mathrm{d}t = \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a) \int_\Omega G(a,x,y) \int_0^t j(t,y) \mathrm{d}t\mathrm{d}y\mathrm{d}a \\ + \frac{\varepsilon(1-q)}{q} \int_0^{+\infty} e^{\mu a} j(a,y) \int_\Omega \tilde{G}(t-a,x,y) \int_0^t e^{-\mu t} \mathrm{d}t\mathrm{d}y\mathrm{d}a. \end{aligned}$$

So, due to the discussions of [24], the operator $\mathcal{Q} : \mathbb{X}_1 \to \mathbb{X}_1$ can be regarded as the next-generation operator,

$$\begin{aligned} \mathcal{Q}[\eta](x) &\xlongequal{\text{def}} \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a) \int_\Omega G(y,x,a)\eta(y) \mathrm{d}y\mathrm{d}a \\ &+ \frac{\varepsilon(1-q)}{q} \int_0^{+\infty} e^{\mu a} j(y,a) \int_\Omega \tilde{G}(y,x,t-a) \int_0^t e^{-\mu t} \mathrm{d}t\mathrm{d}y\mathrm{d}a \\ &= \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a) \int_\Omega G(y,x,a)\eta(y) \mathrm{d}y\mathrm{d}a \\ &+ \frac{\varepsilon(1-q)}{q} \int_0^{+\infty} \int_a^{+\infty} e^{-\mu a} \int_\Omega \tilde{G}(y,x,a)\eta(y) \mathrm{d}y\mathrm{d}t\mathrm{d}a \\ &= \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a) \int_\Omega G(y,x,a)\eta(y) \mathrm{d}y\mathrm{d}a \end{aligned}$$

$$+\frac{\varepsilon(1-q)}{q}\int_0^{+\infty} e^{-\mu a}\int_\Omega \tilde{G}(y,x,a)\eta(y)\mathrm{d}y\mathrm{d}a. \tag{3.7}$$

For operator $\mathcal{Q}$, one obtains the following result.

Theorem 3.1. *The operator $\mathcal{Q}$ is strictly positive and compact.*

Proof. The positivity of the operator $\mathcal{Q}$ is apparent. In order to yield the compactness of the operator $\mathcal{Q}$, one needs the two steps as below.

First step The operator $\mathcal{Q}$ is uniformly bounded.

Choosing a bounded sequence $\{\eta_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_1$ with $\{\eta_n\}_{\mathbb{X}_1}\leq\mathbb{K}$ for some $\mathbb{K}>0$. Denoting a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ by

$$\zeta_n \overset{\text{def}}{=\!=} \mathbb{K}\eta_n.$$

Thus, for all $n\in\mathbb{N}$ and $x\in\Omega$, one has

$$\begin{aligned}\zeta_n(x)\leq \frac{q\delta\mathbb{K}}{\mu}\int_0^{+\infty}\beta(a)\chi(a)\int_\Omega G(y,x,a)\mathrm{d}y\mathrm{d}a\\ +\frac{\varepsilon(1-q)\mathbb{K}}{q}\int_0^{+\infty}e^{-\mu a}\int_\Omega\tilde{G}(y,x,a)\mathrm{d}y\mathrm{d}a.\end{aligned}$$

Go a step further, one finds out that $\{\zeta_n\}_{n\in\mathbb{N}}$ is uniformly bounded.

Second step The sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is equi-continuous.

For $\forall x,\hat{x}\in\Omega$, one yields

$$\begin{aligned}|\zeta_n(x)-\zeta_n(\hat{x})| &= |\mathcal{Q}\eta_n(x)-\mathcal{Q}\eta_n(\hat{x})|\\ &\leq \frac{q\delta}{\mu}\int_0^{+\infty}\beta(a)\chi(a)\int_\Omega |G(y,x,a)-G(y,\hat{x},a)|\,\eta_n(y)\mathrm{d}y\mathrm{d}a\\ &+\frac{\varepsilon(1-q)}{q}\int_0^{+\infty}e^{-\mu a}\int_\Omega\left|\tilde{G}(y,x,a)-\tilde{G}(y,\hat{x},a)\right|\eta_n(y)\mathrm{d}y\mathrm{d}a\\ &\leq\frac{q\delta\mathbb{K}}{\mu}\text{ess.sup}_{a\geq0}\beta(a)\int_0^{+\infty}\chi(a)\int_\Omega|G(y,x,a)-G(y,\hat{x},a)|\,\mathrm{d}y\mathrm{d}a\\ &+\frac{\varepsilon(1-q)\mathbb{K}}{q}\int_0^{+\infty}e^{-\mu a}\int_\Omega\left|\tilde{G}(y,x,a)-\tilde{G}(y,\hat{x},a)\right|\mathrm{d}y\mathrm{d}a.\end{aligned}$$

Owing to the compactness of the operator $\frac{\partial^2}{\partial x^2}$ and the uniform continuity of $G(y,x,a)$ and $\tilde{G}(y,x,a)$, there exists $\hat{\delta}>0$ such that

$$\begin{aligned}|G(y,x,a)-G(y,\hat{x},a)|&\leq\frac{\mu\hat{\varepsilon}}{2\delta\text{ess.sup}_{a\geq0}\beta(a)\mathbb{K}\int_0^{+\infty}\chi(a)\mathrm{d}a\,|\Omega|},\\ \left|\tilde{G}(y,x,a)-\tilde{G}(y,\hat{x},a)\right|&\leq\frac{\mu\hat{\varepsilon}}{2\delta\mathbb{K}\int_0^{+\infty}e^{-\mu a}\mathrm{d}a},\end{aligned}$$

$\forall \hat{\varepsilon} > 0$, $|x - \hat{x}| < \hat{\delta}$, $y \in \Omega$. For this $\hat{\delta}$, $\hat{\varepsilon}$,

$$|\zeta_n(x) - \zeta_n(\hat{x})| < \hat{\varepsilon}, \text{ for all } |x - \hat{x}| < \hat{\delta},$$

namely, the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is equi-continuous. One finishes the proof of this theorem. □

According to [24], one denotes that the basic reproduction number of (2.2)–(2.6) is

$$R_0 = r(\mathcal{Q}),$$

where $r(\mathcal{Q})$ is the spectral radius of $\mathcal{Q}$. Theorem 3.1 together with Krein–Rutman theorem [25] indicates that the basic reproduction number R_0 is the unique positive eigenvalue of $\mathcal{Q}$ associating with a positive eigenvector. Without losing generality, substituting $\eta(x) \equiv 1 > 0$ into (3.7) and applying

$$\int_\Omega G(y, x, \cdot)\mathrm{d}y = 1, \int_\Omega \tilde{G}(y, x, \cdot)\mathrm{d}y = 1,$$

one has

$$\begin{aligned}
\mathcal{Q}[1] &= \frac{q\delta[1]}{\mu} \int_0^{+\infty} \beta(a)\chi(a) \int_\Omega G(y, x, a)\mathrm{d}y\mathrm{d}a \\
&\quad + \frac{\varepsilon(1-q)[1]}{q} \int_0^{+\infty} e^{-\mu a} \int_\Omega \tilde{G}(y, x, a)\eta(y)\mathrm{d}y\mathrm{d}a \\
&= \frac{q\delta[1]}{\mu} \int_0^{+\infty} \beta(a)\chi(a)\mathrm{d}a + \frac{\varepsilon(1-q)[1]}{q} \int_0^{+\infty} e^{-\mu a}\mathrm{d}a. \qquad (3.8)
\end{aligned}$$

Thereby, $R_0 = r(\mathcal{Q})$ can be explicitly expressed by:

$$R_0 = \frac{q\delta}{\mu} \int_0^{+\infty} \beta(a)\chi(a)\mathrm{d}a + \frac{\varepsilon(1-q)}{\mu q}.$$

Nowadays, one can easily figures out that R_0 is a threshold value for the existence of a positive space independent endemic steady state $P_* = [S_*, E_*, i_*(a)]$ of the original system (1.8)–(1.13), which is a solution of the following equations:

$$0 = \delta - S_* \int_0^{+\infty} \beta(a)i_*(a)\mathrm{d}a - \mu S_*, \qquad (3.9)$$

$$0 = (1-q)S_* \int_0^{+\infty} \beta(a)i_*(a)\mathrm{d}a - (\mu + \varepsilon)E_*, \qquad (3.10)$$

$$\frac{\mathrm{d}i_*(a)}{\mathrm{d}a} = -[\gamma(a) + \mu_1(a) + d(a)]\, i_*(a), \qquad (3.11)$$

$$i_*(0) = qS_* \int_0^{+\infty} \beta(a)i_*(a)\mathrm{d}a + \varepsilon E_*. \qquad (3.12)$$

Theorem 3.2. *Assume that $R_0 > 1$, then the original system (1.8)–(1.13) has a space independent endemic steady state $P_* = [S_*, E_*, i_*(a)]$, which is a solution of (3.9)–(3.12), where*

$$S_* = \frac{\mu+\varepsilon}{(\varepsilon+q\mu)\int_0^{+\infty}\beta(a)\chi(a)\mathrm{d}a},$$

$$E_* = \frac{\delta(\varepsilon+q\mu)\int_0^{+\infty}\beta(a)\chi(a)\mathrm{d}a-\mu(\mu+\varepsilon)}{\varepsilon(\mu+\varepsilon)} - \frac{q\delta(\varepsilon+q\mu)\int_0^{+\infty}\beta(a)\chi(a)\mathrm{d}a-\mu(\mu+\varepsilon)}{\varepsilon(\varepsilon+q\mu)},$$

$$i_*(a) = \frac{\chi(a)\left[\delta(\varepsilon+q\mu)\int_0^{+\infty}\beta(a)\chi(a)\mathrm{d}a-\mu(\mu+\varepsilon)\right]}{\mu+\varepsilon}.$$

4. Local dynamics

This part will prove that P_0 is locally asymptotically stable if $R_0 < 1$ and P_* are locally asymptotically stable if $R_0 > 1$.

Theorem 4.1. (1) *The disease-free equilibrium $P_0 = (S_0, 0, 0)$ with $S_0 = \frac{\delta}{\mu}$ is locally asymptotically stable if $R_0 < 1$;*

(2) *The space independent endemic steady state $P_* = [S_*, E_*, i_*(a)]$ is locally asymptotically stable if $R_0 > 1$.*

Proof. **First step** One proves (1). Set

$$\tilde{S}(x,t) = S(x,t) - S_0, \tilde{E}(x,t) = E(x,t), \tilde{i}(x,t,a) = i(x,t,a),$$

then the linearized equation of system (1.8)–(1.13) near $P_0 = (S_0, 0, 0)$ reads as:

$$\frac{\partial}{\partial t}\tilde{S}(x,t) = D_1\frac{\partial^2}{\partial x^2}\tilde{S}(x,t) - S_0\int_0^{+\infty}\beta(a)\tilde{i}(x,t,a)\mathrm{d}a - \mu\tilde{S}(x,t), \tag{4.1}$$

$$\frac{\partial}{\partial t}\tilde{E}(x,t) = D_2\frac{\partial^2}{\partial x^2}\tilde{E}(x,t) + (1-q)S_0\int_0^{+\infty}\beta(a)\tilde{i}(x,t,a)\mathrm{d}a - (\mu+\varepsilon)\tilde{E}(x,t), \tag{4.2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\tilde{i}(x,t,a) = D_3\frac{\partial^2}{\partial x^2}\tilde{i}(x,t,a) - [\gamma(a)+\mu_1(a)+d(a)]\,\tilde{i}(x,t,a), \tag{4.3}$$

$$\tilde{i}(x,t,0) = qS_0\int_0^{+\infty}\beta(a)\tilde{i}(x,t,a)\mathrm{d}a + \varepsilon\tilde{E}(x,t), \tag{4.4}$$

relating with initial data

$$\left[\tilde{S}(x,0), \tilde{E}(x,0), \tilde{i}(x,0,a)\right] = \left[\varphi_1(x) - S_0, \varphi_2(x), \varphi_3(x,a)\right], a \ge 0, x \in \Omega, \tag{4.5}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}\tilde{S}(x,t)=\frac{\partial}{\partial \mathbf{n}}\tilde{E}(x,t)=\frac{\partial}{\partial \mathbf{n}}\tilde{i}(x,t,a)=0, x\in\partial\Omega. \tag{4.6}$$

Because the linear system contains $\frac{\partial^2}{\partial x^2}$, one introduces the related theory from [19]. Define $\theta_k, k=1,2,\cdots$ be the eigenvalues of operator $-\frac{\partial^2}{\partial x^2}$ on a bounded set Ω with boundary condition (1.13), that is to say,

$$\frac{\partial^2}{\partial x^2}\nu(x)=-\theta_k\nu(x), k=1,2,\cdots.$$

Thus,

$$0=\theta_0<\theta_1<\theta_2<\cdots,$$

associating to which, there is the space of eigenfunctions in $C^1(\Omega)$, defined by $E(\theta_k), k=1,2,\cdots$. Define

$$\{\omega_{km}|m=1,2,\cdots,\dim E(\theta_k), k=1,2,\cdots\}$$

be the orthogonal basis of $E(\theta_k), k=1,2,\cdots$. Make further efforts, put

$$\mathbb{Y}_{km}=\left\{c\omega_{km}|c\in\mathbb{R}^3, m=1,2,\cdots,\dim E(\theta_k), k=1,2,\cdots\right\},$$

then

$$\tilde{\mathbb{Y}}=\bigoplus_{k=0}^{+\infty}\mathbb{Y}_k, \mathbb{Y}_k=\bigoplus_{m=1}^{\dim E(\theta_k)}\mathbb{Y}_{km}, m=1,2,\cdots,\dim E(\theta_k), k=1,2,\cdots.$$

Note that the parabolic equation

$$\begin{aligned}&\frac{\partial}{\partial t}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t),\\&\frac{\partial}{\partial \mathbf{n}}u(x,t)=0\end{aligned}$$

exists the exponential solution

$$u(x,t)=e^{\varsigma t}\nu(x), \nu(x)\in\mathbb{Y}_k, k=1,2,\cdots.$$

Substituting

$$\left[\tilde{S}(x,t),\tilde{E}(x,t),\tilde{i}(x,t,a)\right]=e^{\varsigma t}\left[\rho_1^0(x),\rho_2^0(x),\rho_3^0(x,a)\right]$$

into system (4.1)–(4.6), one has

$$\varsigma\rho_1^0(x)=-D_1\theta_k\rho_1^0(x)-S_0\int_0^{+\infty}\beta(a)\rho_3^0(x,a)\mathrm{d}a-\mu\rho_1^0(x), \tag{4.7}$$

$$\varsigma\rho_2^0(x)=-D_2\theta_k\rho_2^0(x)+(1-q)S_0\int_0^{+\infty}\beta(a)\rho_3^0(x,a)\mathrm{d}a-(\mu+\varepsilon)\rho_2^0(x), \tag{4.8}$$

$$\varsigma\rho_3^0(x,a)+\frac{\partial\rho_3^0(x,a)}{\partial a}=-D_3\theta_k\rho_3^0(x,a)-\left[\gamma(a)+\mu_1(a)+d(a)\right]\rho_3^0(x,a), \tag{4.9}$$

$$\rho_3^0(x,0)=qS_0\int_0^{+\infty}\beta(a)\rho_3^0(x,a)\mathrm{d}a+\varepsilon\rho_2^0(x), \tag{4.10}$$

associating with initial data

$$\left[\tilde{S}(x,0),\tilde{E}(x,0),\tilde{i}(x,0,a)\right]$$

$$= \left[\varphi_1(x) - S_0, \varphi_2(x), \varphi_3(x,a)\right], a \geq 0, x \in \Omega, \tag{4.11}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}\tilde{S}(x,t) = \frac{\partial}{\partial \mathbf{n}}\tilde{E}(x,t) = \frac{\partial}{\partial \mathbf{n}}\tilde{i}(x,t,a) = 0, x \in \partial\Omega. \tag{4.12}$$

Solving equation (4.9), one yields

$$\rho_3^0(x,a) = \rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}, \tilde{\chi}(a) = \chi(a)e^{-D_3\theta_k a}, k = 1, 2, \cdots . \tag{4.13}$$

Substituting (4.13), (4.8) into (4.10), one derives

$$\begin{aligned}
\rho_3^0(x,0) = {} & qS_0 \int_0^{+\infty} \beta(a)\rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a - D_2\theta_k\rho_2^0(x) \\
& + (1-q)S_0 \int_0^{+\infty} \beta(a)\rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a - (\mu+\varsigma)\rho_2^0(x) \\
& = S_0 \int_0^{+\infty} \beta(a)\rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a - D_2\theta_k\rho_2^0(x) - (\mu+\varsigma)\rho_2^0(x) \\
& < S_0 \int_0^{+\infty} \beta(a)\rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a \\
& \overset{\text{def}}{=\!=\!=} \mathcal{A}(\varsigma).
\end{aligned} \tag{4.14}$$

Clearly,

$$\begin{aligned}
\mathcal{A}'(\varsigma) = {} & -\varsigma S_0 \int_0^{+\infty} \beta(a)\rho_3^0(x,0)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a \\
& < 0.
\end{aligned}$$

If (4.14) has a unique real positive root ς, one gets

$$\begin{aligned}
1 = {} & \frac{\delta}{\mu}\int_0^{+\infty} \beta(a)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a \\
& < \frac{q\delta}{\mu}\int_0^{+\infty} \beta(a)\chi(a)\mathrm{d}a + \frac{\varepsilon(1-q)}{\mu q} \\
& = R_0,
\end{aligned}$$

which leads to a contradiction. Thus, all roots of (4.14) are negative. If (4.14) has complex roots in the form of $\varsigma = x' \pm y'i$ with $x' > 0$, one obtains

$$\begin{aligned}
1 = {} & \frac{\delta}{\mu}\int_0^{+\infty} \beta(a)\tilde{\chi}(a)e^{-x'a}\cos(y'a)\mathrm{d}a \\
& \leq R_0,
\end{aligned}$$

which creats a contradiction. Thereform, the disease-free equilibrium $P_0 = (S_0, 0, 0)$ with $S_0 = \frac{\delta}{\mu}$ is locally asymptotically stable if $R_0 < 1$.

Second step One proves (2). Let

$$\widehat{S}(x,t)=S(x,t)-S_*,\widehat{E}(x,t)=E(x,t)-E_*,\widehat{i}(x,t,a)=i(x,t,a)-i_*(a),$$

then the linearized equation of system (1.8)–(1.13) around $P_*=[S_*,E_*,i_*(a)]$ reads as:

$$\frac{\partial}{\partial t}\widehat{S}(x,t)=D_1\frac{\partial^2}{\partial x^2}\widehat{S}(x,t)-S_*\int_0^{+\infty}\beta(a)\widehat{i}(x,t,a)\mathrm{d}a-\mu\widehat{S}(x,t), \tag{4.15}$$

$$\frac{\partial}{\partial t}\widehat{E}(x,t)=D_2\frac{\partial^2}{\partial x^2}\widehat{E}(x,t)+(1-q)S_*\int_0^{+\infty}\beta(a)\widehat{i}(x,t,a)\mathrm{d}a-(\mu+\varepsilon)\widehat{E}(x,t), \tag{4.16}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\widehat{i}(x,t,a)=D_3\frac{\partial^2}{\partial x^2}\widehat{i}(x,t,a)-[\gamma(a)+\mu_1(a)+d(a)]\,\widehat{i}(x,t,a), \tag{4.17}$$

$$\widehat{i}(x,t,0)=qS_*\int_0^{+\infty}\beta(a)\widehat{i}(x,t,a)\mathrm{d}a+\varepsilon\widehat{E}(x,t), \tag{4.18}$$

conresponding with initial data

$$\begin{aligned}&\left[\widehat{S}(x,0),\widehat{E}(x,0),\widehat{i}(x,0,a)\right]\\&\quad=\left[\varphi_1(x)-S_*,\varphi_2(x)-E_*,\varphi_3(x,a)-i_*(a)\right],a\geq 0,x\in\Omega,\end{aligned} \tag{4.19}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}\widehat{S}(x,t)=\frac{\partial}{\partial \mathbf{n}}\widehat{E}(x,t)=\frac{\partial}{\partial \mathbf{n}}\widehat{i}(x,t,a)=0,x\in\partial\Omega. \tag{4.20}$$

Homoplastically, substituting

$$\left[\widehat{S}(x,t),\widehat{E}(x,t),\widehat{i}(x,t,a)\right]=e^{\varsigma t}\left[\rho_1(x),\rho_2(x),\rho_3(x,a)\right]$$

into system (4.15)–(4.20), one gets

$$\varsigma\rho_1(x)=-D_1\theta_k\rho_1(x)-S_*\int_0^{+\infty}\beta(a)\rho_3(x,a)\mathrm{d}a-\mu\rho_1(x), \tag{4.21}$$

$$\varsigma\rho_2(x)=-D_2\theta_k\rho_2(x)+(1-q)S_*\int_0^{+\infty}\beta(a)\rho_3(x,a)\mathrm{d}a-(\mu+\varepsilon)\rho_2(x), \tag{4.22}$$

$$\varsigma\rho_3(x,a)+\frac{\partial\rho_3(x,a)}{\partial a}=-D_3\theta_k\rho_3(x,a)-[\gamma(a)+\mu_1(a)+d(a)]\,\rho_3^0(x,a), \tag{4.23}$$

$$\rho_3(x,0)=qS_*\int_0^{+\infty}\beta(a)\rho_3(x,a)\mathrm{d}a+\varepsilon\rho_2(x), \tag{4.24}$$

associating with initial data

$$\begin{aligned}&\left[\widehat{S}(x,0),\widehat{E}(x,0),\widehat{i}(x,0,a)\right]\\&\quad=\left[\varphi_1(x)-S_*,\varphi_2(x)-E_*,\varphi_3(x,a)-i_*(a)\right],a\geq 0,x\in\Omega,\end{aligned} \tag{4.25}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial \mathbf{n}}\widehat{S}(x,t)=\frac{\partial}{\partial \mathbf{n}}\widehat{E}(x,t)=\frac{\partial}{\partial \mathbf{n}}\widehat{i}(x,t,a)=0,x\in\partial\Omega. \tag{4.26}$$

Solving the equation (4.23), one yields

$$\rho_3(x,a) = \rho_3(x,0)\tilde{\chi}(a)e^{-\varsigma a}, \tilde{\chi}(a) = \chi(a)e^{-D_3\theta_k a}, k = 1,2,\cdots. \tag{4.27}$$

Plugging (4.27) into systems (4.21)–(4.26), one gets that the characteristic equation of system (1.8)–(1.13) at $P_* = [S_*, E_*, i_*(a)]$ is

$$\mu\left(R_0 - 1\right)\Xi(\varsigma) - \left(D_1\theta_k + \mu R_0 + \varsigma\right)\left[\Xi(\varsigma) - 1\right] = 0, k = 1,2,\cdots,$$

where

$$\Xi(\varsigma) = S_* \int_0^{+\infty} \beta(a)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a.$$

Hence,

$$\left(\varsigma + D_1\theta_k + \mu\right)\Xi(\varsigma) - \left(\varsigma + D_1\theta_k + \mu R_0\right) = 0, k = 1,2,\cdots. \tag{4.28}$$

Assume that (4.28) has a positive real root ς, due to $R_0 > 1$, then one yields

$$\begin{aligned}\Xi(\varsigma) =& \frac{\varsigma + D_1\theta_k + \mu R_0}{\varsigma + D_1\theta_k + \mu}\\ &>1, k = 1,2,\cdots.\end{aligned} \tag{4.29}$$

Evidently, $\Xi'(\varsigma) < 0$. This manifests

$$\begin{aligned}\Xi(\varsigma) <&\Xi(0)\\ &= S_* \int_0^{+\infty} \beta(a)\tilde{\chi}(a)e^{-\varsigma a}\mathrm{d}a\\ &= 1,\end{aligned}$$

which creates a contradiction with (4.29). Therefore, the total real roots of (4.28) are negative.

Suppose that (4.28) has complex roots $\varsigma = x'' \pm y''i$ with $x'' > 0$, then

$$\left(x'' \pm y''i + D_1\theta_k + \mu\right)\Xi\left(x'' \pm y''i\right) - \left(x'' \pm y''i + D_1\theta_k + \mu R_0\right) = 0, k = 1,2,\cdots.$$

Furthermore, one obtains

$$\begin{aligned}\mathrm{Re}\left[\Xi\left(x'' \pm y''i\right)\right] =& \frac{\left(x'' + D_1\theta_k + \mu R_0\right)\left(x'' + D_1\theta_k + \mu\right) + y''^2}{\left(x'' + D_1\theta_k + \mu\right)^2 + y''^2}\\ &>1, k = 1,2,\cdots.\end{aligned} \tag{4.30}$$

On the other side,

$$\begin{aligned}\mathrm{Re}\left[\Xi\left(x'' \pm y''i\right)\right] \leq& \Xi(0)\\ &<1,\end{aligned}$$

which leads to a contradiction with (4.30). Consequently, the space independent endemic steady state $P_* = [S_*, E_*, i_*(a)]$ is locally asymptotically stable if $R_0 > 1$. One finishes the proof of this theorem. □

5. Disease persistence

In this part, one will demonstrate the disease persistence when $R_0 > 1$. First, one rewrites

$$i(x,t,a)=\begin{cases}\frac{\chi(a)}{\chi(a-t)}\int\limits_{\Omega}G(y,t,x)\varphi_3(y,a-t)\mathrm{d}y, & a-t\geq 0, x\in\Omega,\\ \chi(a)\int\limits_{\Omega}G(y,a,x)j(y,t-a,0)\mathrm{d}y, & t-a>0, x\in\Omega.\end{cases} \tag{5.1}$$

Next, one gives the first result of this portion.

Theorem 5.1. *For* $\forall(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+$*, system* (2.2)–(2.6) *denotes a continuous semiflow*

$$\Phi(\varphi,t)\overset{def}{=\!=}\left[S\left(\varphi_1,\cdot,t\right),E\left(\varphi_2,\cdot,t\right),i\left(\varphi_3,\cdot,t\right)\right]\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+,\forall t\geq 0.$$

Proof. For $\forall\pi\geq 0$, $t\geq 0$, $a\geq 0$, $x\in\Omega$, set

$$S_\pi(x,t)=S(x,\pi+t), j_\pi(x,t)=j(x,\pi+t), i_\pi(x,t,a)=i(x,\pi+t,a),$$

then

$$\frac{\partial}{\partial t}S_\pi(x,t)=D_1\frac{\partial^2}{\partial x^2}S_\pi(x,t)+\delta-\frac{1}{q}j_\pi(x,t)+\frac{1}{q}\varepsilon E_\pi(x,t)-\mu S_\pi(x,t), \tag{5.2}$$

$$\frac{\partial}{\partial t}E_\pi(x,t)=D_2\frac{\partial^2}{\partial x^2}E_\pi(x,t)+\frac{1-q}{q}j_\pi(x,t)-\left(\mu+\frac{1}{q}\varepsilon\right)E_\pi(x,t), \tag{5.3}$$

$$j_\pi(x,t)=qS_\pi(x,t)\int\limits_0^{+\infty}\beta(a)i_\pi(x,t,a)\mathrm{d}a+\varepsilon E_\pi(x,t), \tag{5.4}$$

associating with initial data

$$\begin{aligned}&[S_\pi(x,0),E_\pi(x,0),j_\pi(x,0)]\\ &=\left[\varphi_{1\pi}(x),\varphi_{2\pi}(x),q\varphi_{1\pi}(x)\int\limits_0^{+\infty}\beta(a)\varphi_{3\pi}(x,a)\mathrm{d}a+\varepsilon\varphi_{2\pi}(x)\right],a\geq 0,x\in\Omega,\end{aligned} \tag{5.5}$$

and the homogeneous Neumann boundary condition

$$\frac{\partial}{\partial\mathbf{n}}S_\pi(x,t)=\frac{\partial}{\partial\mathbf{n}}E_\pi(x,t)=\frac{\partial}{\partial\mathbf{n}}j_\pi(x,t)=0, x\in\partial\Omega. \tag{5.6}$$

And then, (5.1) becomes

$$i_\pi(x,t,a)=\begin{cases}\frac{\chi(a)}{\chi(a-t-\pi)}\int\limits_{\Omega}G(y,\pi+t,x)\varphi_3(y,a-t-\pi)\mathrm{d}y, & a-t\geq\pi, x\in\Omega,\\ \chi(a)\int\limits_{\Omega}G(y,a,x)j_\pi(y,t-a,0)\mathrm{d}y, & t-a>\pi, x\in\Omega.\end{cases} \tag{5.7}$$

In addition, for $\forall\pi\geq 0$, $a>t\geq 0$, $x\in\Omega$, one gets

$$i_\pi(x,0,a-t)=\begin{cases}\frac{\chi(a-t)}{\chi(a-t-\pi)}\int\limits_{\Omega}G(y,\pi,x)\varphi_3(y,a-t-\pi)\mathrm{d}y, & a-t\geq\pi, x\in\Omega,\\ \chi(a-t)\int\limits_{\Omega}G(y,a-t,x)j_\pi(y,t-a,0)\mathrm{d}y, & \pi>a-t\geq 0, x\in\Omega.\end{cases} \tag{5.8}$$

According to the property of G, one derives

$$\frac{\chi(a)}{\chi(a-t)}\int\limits_{\Omega}G(y,t,x)i_\pi(y,a-t,0)\mathrm{d}y$$

$$= \begin{cases} \frac{\chi(a)}{\chi(a-t-\pi)} \int\limits_{\Omega} G(y,\pi+t,x)\varphi_3(y,a-t-\pi)\mathrm{d}y, & a-t\geq\pi, x\in\Omega, \\ \chi(a)\int\limits_{\Omega} G(y,a,x)j_\pi(y,t-a,0)\mathrm{d}y, & \pi>a-t\geq 0, x\in\Omega. \end{cases} \quad (5.9)$$

Combining with (5.7) and (5.9), one can yield

$$i_\pi(x,t,a) = \begin{cases} \frac{\chi(a)}{\chi(a-t)} \int\limits_{\Omega} G(y,t,x)\varphi_3(y,a-t,0)\mathrm{d}y, & a-t\geq 0, x\in\Omega, \\ \chi(a)\int\limits_{\Omega} G(y,a,x)j_\pi(y,t-a)\mathrm{d}y, & t-a>0, x\in\Omega. \end{cases} \quad (5.10)$$

Thus, from (5.2)–(5.6) and (5.10), one has

$$\begin{aligned} \Phi\left[S(\cdot,\pi),E(\cdot,\pi),i(\cdot,\cdot,\pi),t\right] &= \left[S_\pi(t),E_\pi(t),i_\pi(\cdot,\cdot,t)\right] \\ &= \Phi\left(\varphi_1,\varphi_2,\varphi_3,\pi+t\right), \forall\pi\geq 0, t\geq 0. \end{aligned}$$

Therefore, the time continuity of Φ follows from theorem 2.1–2.3. One finishes the proof of this theorem. □

Theorem 5.2. *If $\varphi\in\Psi$, $R_0>1$, then there exists $\omega>0$, such that*

$$\limsup_{t\to+\infty} |j(\cdot,t)|_{\mathbb{X}_1} > \omega,$$

where

$$\Psi \overset{def}{=\!=} \left\{(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+ \middle| q\varphi_1(\cdot)\int_0^{+\infty}\beta(\cdot)\varphi_2(\cdot,a)\mathrm{d}a+\varepsilon\varphi_3(\cdot)>0 \text{ for some } x\in\Omega\right\}.$$

Proof. According to the expression of R_0, opting $\omega>0$, such that

$$\frac{q\delta-\omega}{\mu}\int_0^{+\infty}\beta(a)\chi(a)\mathrm{d}a+\frac{\varepsilon(1-q)}{\mu q}>1. \quad (5.11)$$

Presume

$$j(x,t)\leq\omega, \forall x\in\Omega, t\geq t_1>0,$$

then due to (5.11), one obtains that there exist enough small $\lambda>0$ and sufficiently large $t_2>t_1$, such that

$$\begin{aligned} \tilde{R} \overset{\text{def}}{=\!=} & \frac{q\delta-\omega}{\mu}\left[1-e^{-\mu(t_2-t_1)}\right]\int_0^{+\infty}\beta(a)\chi(a)e^{-\lambda a}\mathrm{d}a+\frac{\varepsilon(1-q)}{\mu q} \\ & >1. \end{aligned} \quad (5.12)$$

Therefore, one has

$$\begin{aligned} & \frac{\partial}{\partial t}\left[S(x,t)+E(x,t)\right] \\ & \geq D_1\frac{\partial^2}{\partial x^2}\left[S(x,t)+E(x,t)\right]+q\delta-\omega-\mu\left[S(x,t)+E(x,t)\right] \text{ on } [t_2,+\infty)\times\Omega. \end{aligned} \quad (5.13)$$

Solving (5.13) and employing comparison principle, one obtains

$$\left[S(x,t)+E(x,t)\right]\geq\frac{q\delta-\omega}{\mu}\left[1-e^{-\mu(t_2-t_1)}\right] \text{ on } [t_2,+\infty)\times\Omega. \quad (5.14)$$

Combining theorem 5.1 and (5.14), and letting $S(x,t_2)$, $E(x,t_2)$, $j(x,t_2)$ with $t_2=0$ be initial conditions, we have

$$j(x,t) \geq \frac{q\delta-\omega}{\mu}\left[1-e^{-\mu(t_2-t_1)}\right]\int_0^t \beta(a)\chi(a)\int_\Omega G(y,x,a)j(y,t-a)\mathrm{d}y\mathrm{d}a \text{ on } [t_2,+\infty)\times\Omega. \tag{5.15}$$

Clearly,

$$L[j] \xlongequal{\text{def}} \int_0^{+\infty} e^{-\lambda t} j(x,t)\mathrm{d}t < +\infty, \forall x\in\Omega.$$

Denote

$$L[j]\,(\hat{x}) \xlongequal{\text{def}} \min_{x\in\Omega} L[j].$$

According to (5.15), one gets

$$L[j]\,(\hat{x}) \geq \frac{q\delta-\omega}{\mu}\left[1-e^{-\mu(t_2-t_1)}\right]\int_0^{+\infty} e^{-\lambda t}\int_0^t \beta(a)\chi(a)\int_\Omega G(y,x,a)j(y,t-a)\mathrm{d}y\mathrm{d}a\mathrm{d}t \text{ on } [t_2,+\infty)\times\Omega.$$

Thus, after multiple interchanging the order of integration, one obtains

$$\begin{aligned} L[j]\,(\hat{x}) &\geq \frac{q\delta-\omega}{\mu}\left[1-e^{-\mu(t_2-t_1)}\right]\int_0^{+\infty} \beta(a)\chi(a)e^{-\lambda a}\int_\Omega G(y,x,a)\int_0^{+\infty} e^{-\lambda t}j(y,t)\mathrm{d}t\mathrm{d}y\mathrm{d}a \\ &\geq \tilde{R}L[j]\,(\hat{x}) \text{ on } [t_2,+\infty)\times\Omega, \end{aligned}$$

which creates a contradiction with (5.12). The second claim directly follows from (5.14). One finishes the proof of this theorem. □

According to theorem 5.2, one will prove the strong $|\cdot|_{\mathbb{X}_1}$-persistence.

Theorem 5.3. *If $\varphi\in\Psi$, $R_0>1$, then there exists $\omega'>0$, such that*

$$\limsup_{t\to+\infty} |j(\cdot,t)|_{\mathbb{X}_1} > \omega',$$

where

$$\Psi \xlongequal{\text{def}} \left\{(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{X}_1^+\times\mathbb{X}_2^+\times\mathbb{X}_2^+ \middle| q\varphi_1(\cdot)\int_0^{+\infty}\beta(\cdot)\varphi_2(\cdot,a)\mathrm{d}a+\varepsilon\varphi_3(\cdot)>0, \textit{for some } x\in\Omega\right\}.$$

Proof. Hypothesize that

$$\limsup_{t\to+\infty} |j(\cdot,t)|_{\mathbb{X}_1} \leq \omega', \text{ for some } \omega'>0.$$

And then, owing to theorem 5.2, one indicates that there exist increasing sequences $\{t_{1l}\}_{l=1}^{+\infty}$, $\{t_{2l}\}_{l=1}^{+\infty}$, $\{t_{3l}\}_{l=1}^{+\infty}$ with $t_{1l}>t_{12}>t_{13}$, $l=1,2,\cdots,+\infty$ and decreasing sequence $\{t_{4l}\}_{l=1}^{+\infty}$ with $\lim\limits_{l\to+\infty} t_{4l}=0$. And they also meet

$$|j(\cdot,t_{3l})|_{\mathbb{X}_1} > \omega, t=t_{3l}, l=1,2,\cdots,+\infty, \tag{5.16}$$

$$|j(\cdot,t_{2l})|_{\mathbb{X}_1} = \omega, t = t_{2l}, l = 1,2,\cdots,+\infty, \tag{5.17}$$

$$|j(\cdot,t_{1l})|_{\mathbb{X}_1} < t_{4l} < \omega, t = t_{2l}, l = 1,2,\cdots,+\infty, \tag{5.18}$$

$$|j(\cdot,t)|_{\mathbb{X}_1} < \omega, t \in (t_{2l},t_{1l}), l = 1,2,\cdots,+\infty, \tag{5.19}$$

and set $\{S_l\}_{l=1}^{+\infty}$, $\{E_l\}_{l=1}^{+\infty}$, $\{j_l\}_{l=1}^{+\infty}$ denote

$$S_l \xlongequal{\text{def}} S(\cdot,t_{2l}) \in \mathbb{X}_1, E_l \xlongequal{\text{def}} E(\cdot,t_{2l}) \in \mathbb{X}_1, j_l \xlongequal{\text{def}} j(\cdot,t_{2l}) \in \mathbb{X}_1, l = 1,2,\cdots,+\infty.$$

Due to (2.7)–(2.9), (2.10) and using the Arzela–Ascoli theorem, one gets that there exists $(S_*,E_*,j_*) \in \mathbb{X}_1^+ \times \mathbb{X}_1^+ \times \mathbb{X}_1^+$, such that

$$\liminf_{l\to+\infty} S_l = S_*, \liminf_{l\to+\infty} E_l = E_*, \liminf_{l\to+\infty} j_l = j_*.$$

Put $\left(\widehat{S},\widehat{E},\widehat{j}\right)$ be a solution of (2.2)–(2.6) with

$$\varphi_1(x) = S_*(x), \varphi_2(x) = E_*(x), \varphi_3(x,a) = \chi(a)\int_\Omega G(y,x,a)j_*(y)\mathrm{d}y, \forall a \geq 0, x \in \Omega.$$

Because φ_3 accounts upon (5.1) and due to theorem 5.2, one gets that there exists $t' > 0$, $\omega'' > 0$, such that

$$\left|\hat{j}(\cdot,t')\right|_{\mathbb{X}_1} > \omega, t = t', \tag{5.20}$$

$$\left|\hat{j}(\cdot,t)\right|_{\mathbb{X}_1} > \omega'', 0 < t < t'. \tag{5.21}$$

Set

$$\hat{j}_{1l}(\cdot,t) \xlongequal{\text{def}} \hat{j}(\cdot,t_{2l}+t), l = 1,2,\cdots,+\infty,$$

then it follows from the semiflow property that

$$\left|\hat{j}_{1l}(\cdot,t')\right|_{\mathbb{X}_1} > \omega, t = t', \tag{5.22}$$

$$\left|\hat{j}_{1l}(\cdot,t)\right|_{\mathbb{X}_1} > \omega'' > t_{4l}, 0 < t < t', \tag{5.23}$$

for enough large l. In contrast, for

$$\hat{t}_l \xlongequal{\text{def}} t_{1l} - t_{2l}, l = 1,2,\cdots,+\infty,$$

it follows from (5.16)–(5.19) that

$$\left|\hat{j}_{1l}(\cdot,t')\right|_{\mathbb{X}_1} > \omega, t = t', \tag{5.24}$$

$$\left|\hat{j}_{1l}(\cdot,t)\right|_{\mathbb{X}_1} > \omega'' > t_{4l}, 0 < t < t', l = 1,2,\cdots,+\infty. \tag{5.25}$$

Obviously, it leads to a contradiction between (5.22)–(5.23) and (5.24)–(5.25). Here, one completes the proof of $\limsup_{t\to+\infty} |j(\cdot,t)|_{\mathbb{X}_1} > \omega'$, for some constant $\omega' > 0$. One finishes the proof of this theorem. □

6. Global dynamics

The main aim of this portion is to prove that P_0 is globally asymptotically stable if $R_0 \leq 1$ and P_* is globally asymptotically stable if $R_0 > 1$.

Theorem 6.1. (1) *The disease-free equilibrium* $P_0 = (S_0, 0, 0)$ *with* $S_0 = \frac{\delta}{\mu}$ *is globally asymptotically stable if* $R_0 \leq 1$*;*

(2) *The space-independent endemic steady state* $P_* = [S_*, E_*, i_*(a)]$ *is globally asymptotically stable if* $R_0 > 1$*.*

Proof. **First step** One proves (1). Denote the following Lyapunov functional

$$V_{P_0}(t) \overset{\text{def}}{=\!=} \int\limits_{\Omega} [V_S(x,t) + V_i(x,t)]\,\mathrm{d}x$$

with

$$\begin{aligned} V_S(x,t) &\overset{\text{def}}{=\!=} \Delta\left[S(x,t), S_0\right](x,t) \\ &= S(x,t) - S_0 - S_0 \ln \frac{S(x,t)}{S_0} \\ &\geq 0, \\ &\forall S(x,t), S_0 \in \mathbb{X}_1^+, \Delta\left[S(x,t), S(x,t)\right](x,t) = 0, \end{aligned}$$

$$V_i(x,t) \overset{\text{def}}{=\!=} \int\limits_0^{+\infty} \Gamma(a) i(x,t,a)\mathrm{d}a,$$

$$\Gamma(a) = \frac{1}{\chi(a)} \int\limits_a^{+\infty} S_0 \beta(\pi)\chi(\pi)\mathrm{d}\pi,$$

clearly, $\Gamma(a)$ meets

$$\begin{aligned} S_0\beta(a) + \frac{\mathrm{d}\Gamma(a)}{\mathrm{d}a} - \left[\gamma(a) + \mu_1(a) + d(a)\right]\Gamma(a) = 0, \\ \Gamma(0) = R_0, \end{aligned}$$

then

$$\begin{aligned} \frac{\partial}{\partial t} V_S(x,t) = D_1 \frac{S(x,t) - S_0}{S(x,t)} \frac{\partial^2}{\partial x^2} S(x,t) - \frac{\mu\left[S(x,t) - S_0\right]^2}{S(x,t)} \\ - \frac{1}{q} j(x,t) + \frac{1}{q}\varepsilon \frac{S(x,t) - S_0}{S(x,t)} E(x,t) + \frac{1}{q} S_0 \int\limits_0^{+\infty} \beta(a) i(x,t,a)\mathrm{d}a. \end{aligned}$$

Moreover,

$$\begin{aligned} V_i(x,t) = &\int\limits_0^{+\infty} \Gamma(t+a) \frac{\chi(t+a)}{\chi(a)} \int\limits_{\Omega} G(y,t,x)\varphi_3(y,a)\mathrm{d}y\mathrm{d}a \\ &+ \int\limits_0^t \Gamma(t-a)\chi(t-a) \int\limits_{\Omega} G(y,t-a,x) j(y,a)\mathrm{d}y\mathrm{d}a. \end{aligned} \tag{6.1}$$

Thus,

$$\frac{\partial}{\partial t} V_i(x,t) = \Gamma(0) \int\limits_{\Omega} G(y,0,x) j(y,t)\mathrm{d}y$$

$$
\begin{aligned}
&+\int_0^t \frac{\mathrm{d}\Gamma(t-a)}{\mathrm{d}t}\chi(t-a)\int_\Omega G(y,t-a,x)j(y,a)\mathrm{d}y\mathrm{d}a\\
&+\int_0^t \Gamma(t-a)\chi(t-a)\int_\Omega \frac{\partial G(y,t-a,x)}{\partial t}j(y,a)\mathrm{d}y\mathrm{d}a\\
&-\int_0^t \left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right]\Gamma(t-a)\chi(t-a)\int_\Omega G(y,t-a,x)j(y,a)\mathrm{d}y\mathrm{d}a\\
&+\int_0^{+\infty} \frac{\mathrm{d}\Gamma(t+a)}{\mathrm{d}t}\frac{\chi(t+a)}{\chi(a)}\int_\Omega G(y,t,x)\varphi_3(y,a)\mathrm{d}y\mathrm{d}a\\
&+\int_0^{+\infty} \Gamma(t+a)\frac{\chi(t+a)}{\chi(a)}\int_\Omega \frac{\partial G(y,t,x)}{\partial t}\varphi_3(y,a)\mathrm{d}y\mathrm{d}a\\
&-\int_0^{+\infty} \left[\gamma(t+a)+\mu_1(t+a)+d(t+a)\right]\Gamma(t+a)\frac{\chi(t+a)}{\chi(a)}\int_\Omega G(y,t,x)\varphi_3(y,a)\mathrm{d}y\mathrm{d}a\\
=\ &\Gamma(0)j(x,t)+\int_0^{+\infty}\left\{\frac{\mathrm{d}\Gamma(a)}{\mathrm{d}a}-\left[\gamma(a)+\mu_1(a)+d(a)-D_3\frac{\partial^2}{\partial x^2}\right]\Gamma(a)\right\}i(x,t,a)\mathrm{d}a.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\mathrm{d}V_{P_0}(t)}{\mathrm{d}t} &= -D_1\int_\Omega \frac{|\nabla S(x,t)|^2}{S(x,t)^2}\mathrm{d}x-\int_\Omega \frac{\mu}{S(x,t)}\left[S(x,t)-S_0\right]^2\mathrm{d}x-\int_\Omega\left[1-\Gamma(0)\right]j(x,t)\mathrm{d}x\\
&+\int_\Omega\int_0^{+\infty}\left\{S_0\beta(a)+\frac{\mathrm{d}\Gamma(a)}{\mathrm{d}a}-\left[\gamma(a)+\mu_1(a)+d(a)-D_3\frac{\partial^2}{\partial x^2}\right]\Gamma(a)\right\}i(x,t,a)\mathrm{d}a\mathrm{d}x. \qquad (6.2)
\end{aligned}
$$

Accordingly, one has

$$
\begin{aligned}
\frac{\mathrm{d}V_{P_0}(t)}{\mathrm{d}t} &= -D_1\int_\Omega \frac{|\nabla S(x,t)|^2}{S(x,t)^2}\mathrm{d}x-\int_\Omega \frac{\mu}{S(x,t)}\left[S(x,t)-S_0\right]^2\mathrm{d}x-\int_\Omega(1-R_0)j(x,t)\mathrm{d}x\\
&\le 0,
\end{aligned}
$$

if $R_0\le 1$. In summary, $\{P_0\}$ is the largest invariant set such that $\frac{\mathrm{d}V_{P_0}(t)}{\mathrm{d}t}=0$, and according to the invariance principle in [26], one knows that P_0 is globally attractive.

Second step One proves (2). Define the Lyapunov functional as follows

$$
V_{P_*}(t)\overset{\text{def}}{=\!=}\int_\Omega\left[\tilde{V}_S(x,t)+\tilde{V}_i(x,t)\right]\mathrm{d}x
$$

with

$$
\begin{aligned}
\tilde{V}_S(x,t)&\overset{\text{def}}{=\!=}\Delta\left[S(x,t),S_*\right](x,t)\\
&=S(x,t)-S_*-S_*\ln\frac{S(x,t)}{S_*}\\
&\ge 0,
\end{aligned}
$$

$$\forall S(x,t), S_* \in \mathbb{X}_1^+, \Delta\left[S(x,t), S(x,t)\right](x,t) = 0,$$
$$\tilde{V}_i(x,t) \overset{\text{def}}{=\!=} \int_0^{+\infty} \Xi(a)\Delta\left[i(x,t,a), i_*(a)\right](x,t)\mathrm{d}a,$$
$$\Xi(a) = \frac{1}{\chi(a)} \int_a^{+\infty} S_*\beta(\pi)\chi(\pi)\mathrm{d}\pi,$$

then

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}_S(x,t) = D_1 \frac{S(x,t)-S_*}{S(x,t)}\frac{\partial^2}{\partial x^2}S(x,t) - \frac{\mu\left[S(x,t)-S_*\right]^2}{S(x,t)} \\ + \frac{1}{q}\varepsilon\frac{S(x,t)-S_*}{S(x,t)}E(x,t) + \frac{i(x,t,0)S_*}{qS(x,t)} \\ + \frac{i_*(0)}{q} - \frac{i(x,t,0)}{q} - \frac{i_*(0)S_*}{qS(x,t)}.\end{aligned} \tag{6.3}$$

Besides,

$$\tilde{V}_i(x,t) = \int_0^t \Xi(t-a)\Delta\left[u_1, v_1\right]\mathrm{d}y\mathrm{d}a + \int_0^{+\infty} \Xi(t+a)\Delta\left[u_2, v_2\right]\mathrm{d}y\mathrm{d}a$$

with

$$u_1 = \chi(t-a)\int_\Omega G(y,t-a,x)i(y,a,0)\mathrm{d}y, v_1 = i_*(t-a),$$
$$u_2 = \frac{\chi(t+a)}{\chi(a)}\int_\Omega G(y,t,x)\varphi_3(y,a)\mathrm{d}y, v_2 = i_*(t+a).$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}_i(x,t) = \Xi(0)\Delta\left[\int_\Omega G(y,0,x)i(y,a,0)\mathrm{d}y, i_*(0)\right] + \int_0^t \frac{\mathrm{d}\Xi(t-a)}{\mathrm{d}t}\Delta\left[u_1,v_1\right]\mathrm{d}a \\ + \int_0^{+\infty} \frac{\mathrm{d}\Xi(t+a)}{\mathrm{d}t}\Delta\left[u_2,v_2\right]\mathrm{d}a \\ + \int_0^t \Xi(t-a)\chi(t-a)\int_\Omega \frac{\partial G(y,t-a,x)}{\partial t}i(y,a,0)\mathrm{d}y\frac{\partial\Delta\left[u_1,v_1\right]}{\partial u_1}\mathrm{d}a \\ - \int_0^t \Xi(t-a)\left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right]u_1\frac{\partial\Delta\left[u_1,v_1\right]}{\partial u_1}\mathrm{d}a \\ - \int_0^t \Xi(t-a)\left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right]i_*(t-a)\frac{\partial\Delta\left[u_1,v_1\right]}{\partial v_1}\mathrm{d}a \\ + \int_0^{+\infty} \Xi(t+a)\frac{\chi(t+a)}{\chi(a)}\int_\Omega \frac{\partial G(y,t,x)}{\partial t}\varphi_3(y,a)\mathrm{d}y\frac{\partial\Delta\left[u_2,v_2\right]}{\partial u_2}\mathrm{d}a\end{aligned}$$

$$-\int_0^{+\infty} \Xi(t+a)\left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right] u_2 \frac{\partial \Delta\left[u_2, v_2\right]}{\partial u_2} \mathrm{d}a$$
$$-\int_0^{+\infty} \Xi(t+a)\left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right] i_*(t+a) \frac{\partial \Delta\left[u_2, v_2\right]}{\partial v_2} \mathrm{d}a.$$

Owing to

$$u \frac{\partial \Delta[u, v]}{\partial u}+v \frac{\partial \Delta[u, v]}{\partial v}=\Delta[u, v],$$

one has

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{V}_i(x, t)= & \Xi(0) \Delta\left[\int_{\Omega} G(y, 0, x) i(y, a, 0) \mathrm{d}y, i_*(0)\right] \\
& +\int_0^{+\infty}\left\{\frac{\mathrm{d}\Xi(a)}{\mathrm{d}a}-\left[\gamma(t-a)+\mu_1(t-a)+d(t-a)\right] \Xi(a)\right\} \Delta\left[i(x, t, a), i_*(a)\right] \mathrm{d}a \\
& +\int_0^t \Xi(t-a) \chi(t-a) \int_{\Omega} \frac{\partial G(y, t-a, x)}{\partial t} i(y, a, 0) \mathrm{d}y \frac{\partial \Delta\left[u_1, v_1\right]}{\partial u_1} \mathrm{d}a \\
& +\int_0^{+\infty} \Xi(t+a) \frac{\chi(t+a)}{\chi(a)} \int_{\Omega} \frac{\partial G(y, t, x)}{\partial t} \varphi_3(y, a) \mathrm{d}y \frac{\partial \Delta\left[u_2, v_2\right]}{\partial v_2} \mathrm{d}a.
\end{aligned}$$

Denote the unit semigroup

$$(T(0)[\varphi])(x) \stackrel{\text{def}}{=\!=} \int_{\Omega} G(y, 0, x) \varphi_3(y) \mathrm{d}y,$$

then

$$\begin{aligned}
\frac{\partial G}{\partial t} & =D_2 \frac{\partial^2}{\partial x^2} G, \\
\frac{\partial \Delta[u, v]}{\partial u} & =1-\frac{v}{u}.
\end{aligned}$$

Furthermore, one gets

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{V}_i(x, t)= & \Xi(0) \Delta\left[i(x, t, 0), i_*(0)\right]+\int_0^{+\infty} \Xi(a) D_2 \frac{\partial^2}{\partial x^2} i(x, t, a)\left[1-\frac{i_*(a)}{i(x, t, a)}\right] \mathrm{d}a \\
& +\int_0^{+\infty}\left\{\frac{\mathrm{d}\Xi(a)}{\mathrm{d}a}-\left[\gamma(a)+\mu_1(a)+d(a)\right] \Xi(a)\right\} \Delta\left[i(x, t, a), i_*(a)\right] \mathrm{d}a.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
& \left[\gamma(a)+\mu_1(a)+d(a)\right] \Xi(a)-\frac{\mathrm{d}\Xi(a)}{\mathrm{d}a}=S_* \beta(a), \\
& \Xi(0)=\int_0^{+\infty} S_* \beta(\pi) \chi(\pi) \mathrm{d}\pi
\end{aligned}$$

$$= S_* \int_0^{+\infty} \beta(\pi)\chi(\pi)\mathrm{d}\pi.$$

Consequently, one derives

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}_i(x,t) = S_* &\int_0^{+\infty} \beta(\pi)\chi(\pi)\mathrm{d}\pi\Delta\left[i(x,t,0), i_*(0)\right] \\ &+ \int_0^{+\infty} \Xi(a)D_2\frac{\partial^2}{\partial x^2}i(x,t,a)\left[1-\frac{i_*(a)}{i(x,t,a)}\right]\mathrm{d}a \\ &- \int_0^{+\infty} S_*\beta(a)\Delta\left[i(x,t,a), i_*(a)\right]\mathrm{d}a. \end{aligned} \tag{6.4}$$

Put

$$\tilde{V}(x,t) = \tilde{V}_S(x,t) + \tilde{V}_i(x,t),$$

then uniting with (6.3) and (6.4), one yields

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}(x,t) = D_1&\frac{S(x,t)-S_*}{S(x,t)}\frac{\partial^2}{\partial x^2}S(x,t) - \frac{\mu\left[S(x,t)-S_*\right]^2}{S(x,t)} \\ &+ \frac{i(x,t,0)S_*}{qS(x,t)} - \frac{i_*(0)S_*}{qS(x,t)} + i_*(0) - i(x,t,0) \\ &+ S_*\int_0^{+\infty} \beta(\pi)\chi(\pi)\mathrm{d}\pi\Delta\left[i(x,t,0), i_*(0)\right] \\ &+ \int_0^{+\infty} \Xi(a)D_2\frac{\partial^2}{\partial x^2}i(x,t,a)\left[1-\frac{i_*(a)}{i(x,t,a)}\right]\mathrm{d}a \\ &- \int_0^{+\infty} S_*\beta(a)\Delta\left[i(x,t,a), i_*(a)\right]\mathrm{d}a. \end{aligned}$$

Proceed to the next step, one has

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}(x,t) = D_1&\frac{S(x,t)-S_*}{S(x,t)}\frac{\partial^2}{\partial x^2}S(x,t) - \frac{\mu\left[S(x,t)-S_*\right]^2}{S(x,t)} + \frac{i(x,t,0)S_*}{qS(x,t)} \\ &- \frac{i_*(0)S_*}{qS(x,t)} + i_*(0) - i(x,t,0) \\ &+ S_*\int_0^{+\infty} \beta(\pi)\chi(\pi)\mathrm{d}\pi\left[i(x,t,0) - i_*(0) - i_*(0)\ln\frac{i(x,t,0)}{i_*(0)}\right] \\ &+ \int_0^{+\infty} \Xi(a)D_2\frac{\partial^2}{\partial x^2}i(x,t,a)\left[1-\frac{i_*(a)}{i(x,t,a)}\right]\mathrm{d}a \end{aligned}$$

$$-\int_0^{+\infty} S_*\beta(a)\left[i(x,t,a)-i_*(a)-i_*(a)\ln\frac{i(x,t,a)}{i_*(a)}\right]\mathrm{d}a. \tag{6.5}$$

By computations, one can get

$$S_*\int_0^{+\infty}\beta(a)i_*(a)\left[1-\frac{Si_*(0)i(x,t,a)}{S_*i(x,t,0)i_*(a)}\right]\mathrm{d}a=0.$$

And then, (6.5) can be transformed into

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{V}(x,t)&=D_1\frac{S(x,t)-S_*}{S(x,t)}\frac{\partial^2}{\partial x^2}S(x,t)-\frac{\mu\left[S(x,t)-S_*\right]^2}{S(x,t)}\\&+\frac{i(x,t,0)S_*}{qS(x,t)}-\frac{i_*(0)S_*}{qS(x,t)}+i_*(0)-i(x,t,0)\\&+S_*\int_0^{+\infty}\beta(\pi)\chi(\pi)\mathrm{d}\pi\left[i(x,t,0)-i_*(0)-i_*(0)\ln\frac{i(x,t,0)}{i_*(0)}\right]\\&+\int_0^{+\infty}\Xi(a)D_2\frac{\partial^2}{\partial x^2}i(x,t,a)\left[1-\frac{i_*(a)}{i(x,t,a)}\right]\mathrm{d}a\\&-\int_0^{+\infty}S_*\beta(a)\left[2-\frac{S_*}{S(x,t)}+\ln\frac{S_*}{S(x,t)}+\frac{Si_*(0)i(x,t,a)}{S_*i(x,t,0)i_*(a)}+\ln\frac{Si_*(0)i(x,t,a)}{S_*i(x,t,0)i_*(a)}\right]\mathrm{d}a.\end{aligned} \tag{6.6}$$

Therefore, integrating (6.6) over Ω, one has

$$\begin{aligned}\frac{\mathrm{d}V_{P_*}(t)}{\mathrm{d}t}&=-D_1S_*\int_\Omega\frac{|\nabla S(x,t)|^2}{S^2(x,t)}\mathrm{d}x-D_3\int_\Omega\int_0^{+\infty}\Xi(a)i_*(a)\frac{|\nabla i(x,t,a)|^2}{i^2(x,t,a)}\mathrm{d}a\mathrm{d}x\\&-\int_\Omega\frac{\mu}{S(x,t)}\left[S(x,t)-S_*\right]^2\mathrm{d}x\\&+S_*\int_\Omega\int_0^{+\infty}\beta(\pi)\chi(\pi)\mathrm{d}\pi\left[i(x,t,0)-i_*(0)-i_*(0)\ln\frac{i(x,t,0)}{i_*(0)}\right]\mathrm{d}x\\&-\int_\Omega\int_0^{+\infty}S_*\beta(a)\left\{h\left[\frac{S_*}{S(x,t)}\right]+h\left[\frac{Si_*(0)i(x,t,a)}{S_*i(x,t,0)i_*(a)}\right]\right\}\mathrm{d}a\mathrm{d}x\\&\le 0,\end{aligned}$$

where $h[s]=1-s+\ln s, s\in\mathbb{R}_+$ has the properties that $h(s)\le 0$ while $s>0$ and $h(s)$ reaches the global minimum 0 at $s=1$. In summary, $\{P_*\}$ is the largest invariant set such that $\frac{\mathrm{d}V_{P_*}(t)}{\mathrm{d}t}=0$, and due to the invariance principle in [26], one gets that P_* is globally attractive. One finishes the proof of this theorem. □

7. Discussions

In this study, one considers the Tuberculosis model with the fast and slow progression, where age structure and spatial diffusion are introduced. Although the basic reproduction number is independent of the diffusion coefficients D_1, D_2 and D_3 and the dynamics of this model is similar to the model (1.19)–(1.23) without spatial diffusions, the process and method of solving and proving are different from models (1.19)–(1.23) without spatial diffusions. The difficulty in obtaining the dynamics of this model is the construction of Lyapunov functional when proving global asymptotic stability. It is a pity that this spatial diffusion is the Laplace (local) diffusion describing the random walk of the population in a connected domain. Nevertheless, in modern time many infectious diseases are spread geographically by long-distance travel such as air travel [27]. It seems that the Laplace (local) diffusion is unsuitable to model the spatial spread of infectious diseases through this phnomenon [28,29]. Hence, some scholars propose convolution (nonlocal) diffusion to describe the long-distance dispersal [30,31]. And then, one discusses the new type of Tuberculosis model, namely age-dependent Tuberculosis model with convolution (nonlocal) diffusion. One puts these in the future.

Acknowledgements

One is grateful for the detailed and constructive comments and suggestions from an editor and two reviewers.

References

[1] Bjune, G.: Tuberculosis in the 21st century: an emerging pandemic? Norsk Epidemiol. **15**, 133–139 (2005)
[2] Brewer, T.F., Heymann, S.J.: To control and beyond: moving towards eliminating the global Tuberculosis threat. J. Epidemiol. Community Health **58**, 822–825 (2004)
[3] Xu, R., Yang, J.Y., Tian, X.H., Lin, J.Z.: Global dynamics of a Tuberculosis model with fast and slow progression and age-dependent latency and infection. J. Biol. Dyn. **13**(1), 675–705 (2019)
[4] Blower, S.M., McLean, A.R., Porco, T.C., Small, P.M., Hopwell, P.C., Sanchez, M.A., Moss, A.R.: The intrinsic transmission dynamics of Tuberculosis epidemics. Nat. Med. **1**, 815–821 (1995)
[5] Zhang, Y.D., Huo, H.F., Xiang, H.: Dynamics of Tuberculosis with fast and slow progression and media coverage. Math. Biosci. Eng. **16**(3), 1150–1170 (2019)
[6] World health statistics 2021: Monitoring health for the SDGs, sustainable development goals. World Health Organization, Geneva (2021). Licence: CC BY-NC-SA 3.0 IGO
[7] Feng, Z., Iannelli, M., Milner, F.A.: A two-strain Tuberculosis model with age of infection. SIAM J. Appl. Math. **62**(5), 1634–1656 (2002)
[8] Xiang, H., Zou, M.X., Huo, H.F.: Modeling the effects of health education and early therapy on Tuberculosis transmission dynamics. Int. J. Nonlinear Sci. Numer. **20**(3–4), 243–255 (2019)
[9] Wallace, D., Wallace, R.: Problems with the WHO TB model. Math. Biosci. **313**, 71–80 (2019)
[10] Renardy, M., Kirschner, D.E.: A framework for network-based epidemiological modeling of Tuberculosis dynamics using synthetic datasets. Bull. Math. Biol. **82**(6), 78 (2020)
[11] Chinnathambi, R., Rihan, F.A., Alsakaji, H.J.: A fractional-order model with time delay for Tuberculosis with endogenous reactivation and exogenous reinfections. Math. Methods Appl. Sci. **44**(10), 8011–8025 (2021)
[12] Yao, M.R., Zhang, Y.X., Wang, W.D.: Bifurcation analysis for an in-host mycobacterium Tuberculosis model. Discrete Contin. Dyn. Syst. Ser. B **26**(4), 2299–2322 (2021)
[13] Fernandez-Peralta, R., Gomez-Corral, A.: A structured Markov chain model to investigate the effects of pre-exposure vaccines in Tuberculosis control. J. Theor. Biol. **509**, 110490 (2021)
[14] Das, D., Kar, T.K.: Dynamical analysis of an age-structured Tuberculosis mathematical model with LTBI detectivity. J. Math. Anal. Appl. **492**, 124407 (2020)
[15] Yan, D.X., Cao, H., Zhang, S.X.: Global dynamics for a TB transmission model with age-structure and delay. Int. J. Biomath. **13**(7), 2050055 (2020)
[16] Lee, S., Park, H.Y., Ryu, H., Kwon, J.W.: Age-specific mathematical model for Tuberculosis transmission dynamics in South Korea. Mathematics **9**(8), 804 (2021)
[17] Guo, Z.K., Xiang, H., Huo, H.F.: Analysis of an age-structured Tuberculosis model with treatment and relapse. J. Math. Biol. **82**(5), 45 (2021)
[18] Wu, J.H.: Theory and Applications of Partial Functional Differential Equations. Springer, New York (1996)

[19] Cantrell, R.S., Cosner, C.: Spatial Ecology via Reaction–Diffusion Equations. Wiley, Chichester (2003)
[20] Zhang, R., Liu, L.L., Feng, X.M., Jin, Z.: Existence of traveling wave solutions for a diffusive Tuberculosis model with fast and slow progression. Appl. Math. Lett. **112**, 106848 (2021)
[21] Ren, X.Z., Tian, Y.N., Liu, L.L., Liu, X.N.: A reaction–diffusion within-host HIV model with cell-to-cell transmission. J. Math. Biol. **76**, 1831–1872 (2018)
[22] Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. American Mathematical Society, Providence (1995)
[23] Rudin, W.: Principles of Mathematical Analysis. McGraw-Hill, New York (1976)
[24] Diekmann, O., Heesterbeek, J.A.P., Metz, J.A.J.: On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. J. Math. Biol. **28**(4), 365–382 (1990)
[25] Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**, 620–709 (1976)
[26] Walker, J.A.: Dynamical Systems and Evolution Equations: Theory and Applications. Mathematical Concepts and Methods in Science and Engineering, vol. 20. Plenum Press, New York (1980)
[27] Findlater, A., Bogoch, I.I.: Human mobility and the global spread of infectious diseases: a focus on air travel. Trends Parasitol. **34**, 772–783 (2018)
[28] Blyuss, K.B.: On a model of spatial spread of epidemics with long-distance travel. Phys. Lett. A **345**, 129–136 (2005)
[29] Ruan, S.G.: Spatial-Temporal Dynamics in Nonlocal Epidemiological Models. Springer, Berlin (2007)
[30] Yang, F.Y., Li, Y., Li, W.T., Wang, Z.C.: Traveling waves in a nonlocal dispersal Kermack–Mckendrick epidemic model. Discrete Contin. Dyn. Syst. Ser. B **18**, 1969–1993 (2013)
[31] Xu, W.B., Li, W.T., Ruan, S.G.: Spatial propagation in an epidemic model with nonlocal diffusion: the influences of initial data and dispersals. Sci. China Math. **63**, 2177–2206 (2020)

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(Received: September 14, 2021; revised: September 13, 2022; accepted: September 28, 2022)