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Stability for a nonlinear hyperbolic equation with time-dependent coefficients and boundary damping

Marcelo Moreira Cavalcanti, Valéria Neves Domingos Cavalcanti and André Vicente

Abstract. In this paper, we prove a stability result for a nonlinear wave equation, defined in a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with time-dependent coefficients. The smooth boundary of Ω is $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. On Γ_0 we consider the homogeneous Dirichlet boundary condition and on Γ_1 we consider the Neumann boundary condition with damping term. The presence of time-dependent coefficients and, moreover, of the singularities generated by the condition $\Sigma \neq \emptyset$ brings some technical difficulties. The tools are the combination of appropriate functional with the techniques due to Bey, Loheac, and Moussaoui [2] and new technical arguments.

Mathematics Subject Classification. 35L05, 35L20, 35B35, 35B40.

Keywords. Stability, Hyperbolic equation, Singularity, Boundary damping, Time-dependent coefficients.

1. Introduction

This paper is concerned with the study of the decay rates of the energy associated with the following hyperbolic equation with boundary damping

$$\begin{cases} K(x,t)u_{tt} - A(t)u + F(x,t,u,\nabla u) = 0 \text{ in } \Omega \times (0,\infty) \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu_A} + \beta(x)u_t = 0 \text{ on } \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded open set with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, meas$(\Gamma_0)$ and meas(Γ_1) are positive and such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$. The sets Γ_0 and Γ_1 are specified below;

$$A(t)u = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(a(x,t)\frac{\partial u}{\partial x_j}\right),$$

here $a : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is a known function; ∇ is the gradient operator in the spatial variable;

$$\frac{\partial u}{\partial \nu_A} = \sum_{j=1}^{N} a(x,t)\frac{\partial u}{\partial x_j}\nu_j,$$

is the conormal derivative of u with respect to A, $\nu = (\nu_1, \nu_2, \ldots, \nu_N)$ is the normal unit vector to Γ; $K : \Omega \times (0,\infty) \to \mathbb{R}$, $F : \overline{\Omega} \times [0,\infty) \times \mathbb{R}^{N+1} \to \mathbb{R}$, and $\beta : \Omega \to \mathbb{R}$ are known functions; and u_0 and u_1 are the initial data.

Problems concerning the wave equation with nonconstant coefficient in the principal part have been called the attention of many researchers. We start calling the attention to the important paper Yao [27] where the author studied the boundary exact controllability for the following problem

$$\begin{cases} u_{tt} - \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \text{ in } \Omega \times (0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \\ u = 0 \quad \text{in } \Gamma_1 \times (0,T), \quad y = v \quad \text{in } \Gamma_0 \times (0,T), \end{cases} \tag{1.2}$$

where $v \in L^2(0,T;L^2(\Gamma_0))$ is the control function. The observability inequalities were established by the Riemannian geometry method under some geometric condition for the Dirichlet problem and for the Neumann problem. The Riemannian geometry method was used by Liu, Li, and Yao [25] to prove the decay of the energy associated with a wave equation with variable coefficients in an exterior domain. The damping was considered on a portion the boundary and also in a portion of the interior of the domain. See also Yao [28–30].

When the wave motion holds in an inhomogeneous medium context, the coefficient of u_{tt} is not constant with respect to the spatial variable. A natural way to prove the stability of the problem is use the tools of Microlocal Analysis. A good description of this tools concerning a linear problem can be found in the lecture note due to Burq and Gérard [4]. Nonlinear problems was studied by Cavalcanti *et al.* [1,6–9]. We would like to highlight the work of Cavalcanti *et al.* [5] where was studied the problem

$$\begin{cases} \rho(x)u_{tt} - \operatorname{div}(K(x)\nabla u) + f(u) + a(x)g(u_t) = 0 \text{ in } \Omega \times (0,\infty), \\ u = 0 \text{ on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \ x \in \Omega. \end{cases} \tag{1.3}$$

The use of Microlocal Analysis tools brings us two main assumptions. The first one involves the geometric control condition and the second one involves a unique continuation result for the main operator associated with the problem. Problem in inhomogeneous medium and with dynamical boundary conditions was studied by Coclite, G. Goldstein, and J. Goldstein [11]. Results concerning dynamical boundary conditions can be found in the works of Coclite, G. Goldstein, and J. Goldstein [12–14], Coclite *et al.* [15–17] and references therein. See also the more recent works of Coclite *et al.* [18,19] where the authors studied problems concerning Neumann boundary conditions and discontinuous sources.

When the coefficients are time-dependent the problem becomes more delicate. Indeed, it is well know that the semigroups arguments can not be used. Moreover, the Microlocal Analysis tools also are not appropriate. In [10], using the Faedo–Galerkin method, Cavalcanti, Domingos Cavalcanti, and Soriano proved an existence and uniqueness result to problem (1.1) when the assumption

$$\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset \tag{1.4}$$

is in place. Using an appropriate Lyapunov functional they also proved that the energy decay exponentially.

It is well know that assumption (1.4) allows us to use elliptic results which give us regularity on the solution. When this assumption is not in place, we have some delicate technical difficulties which need to be overcome. In the two- and three- dimensional case the tool to overcome the loss of regularity was introduced by Grisvard [21], see also Grisvard [22,23]. Indeed, he states that a weak solution u of an associate elliptic problem can be split into u_R and u_S, where $u_R \in H^2(\Omega)$ and u_S is given by

$$u_S = \sum_{x \in \Sigma} \rho(r,\theta)\sqrt{r}\sin\left(\frac{\theta}{2}\right),$$

here (r, θ) is a coordinate system with center in $\tilde{x} \in \Sigma$ and ρ and an appropriate smooth function with compact support with $0 \leq \rho \leq 1$. This decomposition allows us to estimate some integrals that are in place due to the presence of singularities.

The ideas of Grisvard was extended to $\mathbb{R}^N$, $N \geq 3$, by Bey, Loheac, and Moussaoui [2]. In fact, they proved a theorem which gives as a decomposition of the solution into two functions u_R and u_S with u_S write as in Grisvard case. Moreover, they give a response to the control of ∇u in a tangential direction. Bey, Loheac, and Moussaoui also proved a stability result to a problem involving the linear wave equation. Problems with singularities also was studied by Cornilleau, Loheac, and Osses [20]. In [20] the authors studied the boundary stabilization of the wave equation by means of a linear or nonlinear Neumann feedback. We highlight that the stability results of [2] and [20] are concerning the wave equation with constant coefficients in the principal operator.

The main goal of the present paper is to study problem (1.1) without assumption (1.4). This work extend the stability results of [2] and [20] to a time-dependent coefficient case. The ideas of Grisvard [21] and, mainly, of Bey, Loheac, and Moussaoui [2] combined with the techniques of Cavalcanti, Domingos Cavalcanti and Soriano [10] are the key to prove our main result.

The difficulties of the present paper are as follows: due to the general assumptions on K and a we do not have control on the derivative of the functional energy. In fact, we do not have the traditional energy identity which is an important tool to prove stability results. This problem combined with the presence of singularities, generated by the change of boundary conditions, brings some technical difficulties which needs to be overcome.

Finally, we also would like to cite the works of Liu and Yao [24], Boiti and Manfrin [3], and Reissig and Smith [26] where the authors studied the wave equation with time-dependent coefficients. In [24] Liu and Yao deal with boundary exact controllability for the dynamics governed by the wave equation subject to Neumann boundary controls. In [3] the authors study the asymptotic behavior of the energy to the Cauchy problem for wave equations with time-dependent propagation speed (i.e., the function which multiply the Laplace operator is time-dependent). $L^p - L^q$ estimates for wave equation with time-dependent propagation speed was studied in [26].

Our paper is organized as follows. In section 2 we present the notations and the assumptions. We also enunciate the existence and uniqueness result. The theorem which gives us the stability also is enunciated in section 2. Finally, in section 3 we prove the stability result, our main result.

2. Preliminaries and existence theorem

Let us denote by $\|\cdot\|_{L^2(\Omega)}$ the usual norm in the Hilbert space $L^2(\Omega)$ endowed with the inner product $(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx$. We also consider the subspace of $H^1(\Omega)$, denoted by V, as the closure of $C^1(\overline{\Omega})$ such that $u_{|\Gamma_0} = 0$ in the strong topology of $H^1(\Omega)$, i.e.,

$$V = \overline{\{u \in C^1(\overline{\Omega});\ u_{|\Gamma_0} = 0\}}^{H^1(\Omega)}.$$

We have that the Poincaré Inequality holds in V, thus there exists a positive constant c_p such that

$$\|\nabla u\|_{L^2(\Omega)} \leq c_p \|u\|_{L^2(\Omega)},$$

for all $u \in V$. Therefore, the space V can be endowed with the norm, $\|\nabla \cdot\|_{L^2(\Omega)}$, induced by the inner product

$$(u,v)_V = (\nabla u, \nabla v)_{L^2(\Omega)},$$

which is equivalent to usual norm of $H^1(\Omega)$.

Let x_0 a fixed point of $\mathbb{R}^N$. We define

$$m(x) = (x - x_0) \cdot \nu,$$

for all $x \in \mathbb{R}^N$. We consider that the boundary Γ of Ω is given by

$$\Gamma_0 = \{x \in \Gamma;\ m \cdot \nu < 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma;\ m \cdot \nu \geq 0\}.$$

Below, we introduce the assumption on the function F. Our prototype of function F is given by $F(x, t, u, \nabla u) = |u|^\gamma u + \vartheta(t) \sum_{j=1}^N \sin\left(\frac{\partial u}{\partial x_j}\right)$, where ϑ is a regular function.

Assumption 1. We suppose that $F : \overline{\Omega} \times [0, \infty) \times \mathbb{R}^{N+1} \to \mathbb{R}$ is continuously differentiable and that there exist positive constants C_0 and C_1 such that

$$\begin{aligned} |F(x, t, \xi, \varsigma)| &\leq C_0(1 + |\xi|^{\gamma+1} + |\varsigma|), \\ |F_t(x, t, \xi, \varsigma)| &\leq C_0(1 + |\xi|^{\gamma+1} + |\varsigma|), \\ |F_\xi(x, t, \xi, \varsigma)| &\leq C_0(1 + |\xi|^\gamma), \\ |F_{\varsigma_j}(x, t, \xi, \varsigma)| &\leq C_1, \text{ for } j = 1, 2, \ldots, N, \end{aligned}$$

for all $(x, t, \xi, \varsigma) \in \overline{\Omega} \times [0, \infty) \times \mathbb{R}^{N+1}$, where $\gamma > 0$, if $N = 2$ and $0 < \gamma \leq \frac{N}{N-2}$, if $N \geq 3$, and $\varsigma = (\varsigma_1, \ldots, \varsigma_N)$. Moreover, we suppose that there exists a function $C \in L^\infty(0, \infty) \cap L^1(0, \infty)$ such that

$$F(x, t, \xi, \varsigma)\eta \geq |\xi|^\gamma \xi \eta - C(t)(1 + |\eta||\varsigma|),$$

for all $(x, t, \xi, \varsigma) \in \overline{\Omega} \times [0, \infty) \times \mathbb{R}^{N+1}$ and for all $\eta \in \mathbb{R}$;

$$F(x, t, \xi, \varsigma) m \cdot \varsigma \geq |\xi|^\gamma \xi m \cdot \varsigma - C(t)(1 + |\varsigma||m \cdot \varsigma|),$$

for all $(x, t, \xi, \varsigma) \in \overline{\Omega} \times [0, \infty) \times \mathbb{R}^{N+1}$. We also suppose that there exist positive constant D_1 and D_2 such that

$$(F(x, t, \xi, \varsigma) - F(x, t, \hat{\xi}, \hat{\varsigma}))(\eta - \hat{\eta}) \geq -D_1(|\xi|^\gamma - |\hat{\xi}|^\gamma)|\xi - \hat{\xi}||\eta - \hat{\eta}| - D_2|\varsigma - \hat{\varsigma}||\eta - \hat{\eta}|,$$

for all $(x, t, \xi, \varsigma), (x, t, \hat{\xi}, \hat{\varsigma}) \in \overline{\Omega} \times [0, \infty) \times \mathbb{R}^{N+1}$ and $\eta, \hat{\eta} \in \mathbb{R}$.

Next, we write the assumptions on the functions K and a.

Assumption 2. We suppose that $K, a : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ satisfies

$$\begin{aligned} K &\in W^{1,\infty}(0, \infty; C^1(\overline{\Omega})), \\ a &\in W^{1,\infty}(0, \infty; C^1(\overline{\Omega})) \cap W^{2,\infty}(0, \infty; L^\infty(\Omega)) \\ K_t, a_t &\in L^1(0, \infty; L^\infty(\Omega)), \end{aligned}$$

Moreover, we suppose that there exist constants K_0 and a_0 such that

$$K > k_0 > 0 \quad \text{and} \quad a > a_0 > 0 \quad \text{in } \Omega \times (0, \infty).$$

Finally, in this paper we consider the case $\beta(x) = m(x)$, for all $x \in \overline{\Omega}$.

Now, we can enunciate an existence and uniqueness theorem. The proof is exactly the same of Theorem 2.1 of [10]. But, we highlight that, since in our case $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, we cannot use elliptic regularity arguments and to conclude that $u(t) \in H^2(\Omega)$ (as it was used in [10]).

Theorem 2.1. (Existence and uniqueness of solution) *Suppose that Assumptions 1 and 2 hold. For each initial data $(u_0, u_1) \in H^2(\Omega) \times H^2(\Omega)$ satisfying $\frac{\partial u_0}{\partial \nu_A} + \beta(x) u_1 = 0$, there exist a unique solution of (1.1) in the class*

$$u \in W^{1,\infty}_{loc}(0, \infty; V) \cap W^{2,\infty}_{loc}(0, \infty, L^2(\Omega)). \tag{2.5}$$

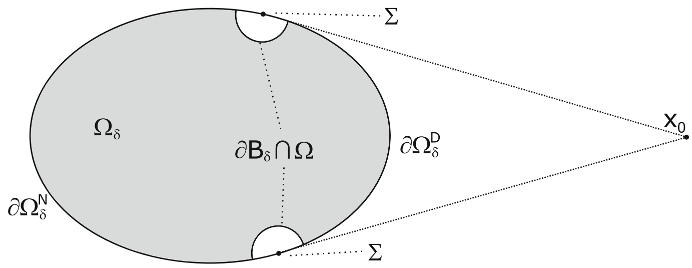

FIG. 1. A prototype of the domain: $\mathbb{R}^2$ case

□

We define the energy associated with problem (1.1) by

$$E(t) = \frac{1}{2}\left(\int_\Omega K u_t^2\,dx + \int_\Omega a|\nabla u|^2\,dx\right) + \frac{1}{\gamma+2}\int_\Omega |u|^{\gamma+2}\,dx. \tag{2.6}$$

Moreover, for each $\varepsilon > 0$, we define the perturbed energy by

$$E_\varepsilon(t) = E(t) + \varepsilon\Psi(t), \tag{2.7}$$

where

$$\Psi(t) = 2\int_\Omega K u_t m\cdot\nabla u\,dx + \theta\int_\Omega K u_t u\,dx, \tag{2.8}$$

here θ is an appropriate positive constant.

Due to the presence of singularities, initially it is necessary to work away of these points. Therefore, first we define

$$\Sigma = \overline{\Gamma}_0 \cap \overline{\Gamma}_1.$$

Now, let $\delta > 0$ a small and fixed number. We consider

$$B_\delta = \bigcup_{x\in\Sigma} B(x,\delta),$$

where $B(x,\delta) = \{y\in\Omega;\ \|x-y\| < \delta\}$. The boundary of B_δ is denoted by ∂B_δ. We work in the following subset of Ω:

$$\Omega_\delta = \Omega \setminus B_\delta.$$

Its boundary $\partial\Omega_\delta$ is denoted by

$$\partial\Omega_\delta = \partial\Omega_\delta^D \cup \partial\Omega_\delta^N \cup (\partial B_\delta \cap \Omega),$$

where

$$\partial\Omega_\delta^D = \partial\Omega_\delta \cap \Gamma_0 \quad \text{and} \quad \partial\Omega_\delta^N = \partial\Omega_\delta \cap \Gamma_1.$$

See Figures 1 and 2.

We define the energy associated with problem (1.1) and to Ω_δ by

$$E_{\Omega_\delta}(t) = \frac{1}{2}\left(\int_{\Omega_\delta} K u_t^2\,dx + \int_{\Omega_\delta} a|\nabla u|^2\,dx\right) + \frac{1}{\gamma+2}\int_{\Omega_\delta} |u|^{\gamma+2}\,dx. \tag{2.9}$$

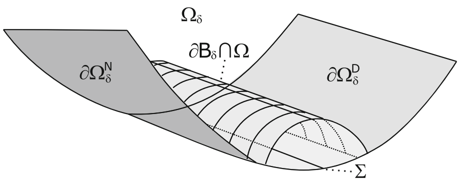

FIG. 2. The sets $\partial\Omega_\delta^N$, $\partial\Omega_\delta^D$, $\partial B_\delta \cap \Omega$, and Σ in the $\mathbb{R}^3$ case

Finally, for each $\varepsilon > 0$, we define the perturbed energy associated with Ω_δ by

$$E_{\delta,\varepsilon}(t) = E_{\Omega_\delta}(t) + \varepsilon\Psi_\delta(t), \tag{2.10}$$

where

$$\Psi_\delta(t) = 2\int_{\Omega_\delta} K u_t m \cdot \nabla u \, dx + \theta \int_{\Omega_\delta} K u_t u \, dx. \tag{2.11}$$

We have the following lemma connecting $E(t)$ with $E_{\Omega_\delta}(t)$.

Lemma 2.1. *It holds*

$$E_{\Omega_\delta}(t) \to E(t) \quad \textit{and} \quad \Psi_\delta(t) \to \Psi(t),$$

as $\delta \to 0$. *Therefore,*

$$E_{\delta,\varepsilon}(t) \to E_\varepsilon(t), \tag{2.12}$$

as $\delta \to 0$.

Proof. It follows Lebesgue converge theorem. □

To prove the stability result, it is necessary the following assumption (this assumption also was used by Bey, Loheac, and Moussaoui [2] and Grisvard [21]).

Assumption 3. We denote by τ the unit tangent vector to Γ and normal to Σ pointing towards the exterior of Γ_1, from Γ_1 to Γ_0. We suppose that

$$m(x) \cdot \tau(x) < 0,$$

for all $x \in \Sigma$.

See Figure 3.

Theorem 2.2. *Assume that assumptions* 1, 2, *and* 3 *hold and let* $E(t)$ *the energy associated with* (1.1). *Assume that there exist positive constants* α, r, ϵ, *and* θ_0 *such that for all* t *sufficiently large, it holds*

$$\int_0^t e^{\epsilon\theta_0 s}\varphi(s)\, ds \le \alpha t^r, \tag{2.13}$$

where

$$\begin{aligned}\varphi(t) = {}& \frac{r_1}{a_0}\|a_t(t)\|_{L^\infty(\Omega)} + \left(\frac{1}{k_0} + C_2\varepsilon\right) r_1 \|K_t(t)\|_{L^\infty(\Omega)} + \varepsilon\left(C_3\|\nabla a(t)\|_{L^\infty(\Omega)} + C_5\|\nabla K(t)\|_{L^\infty(\Omega)}\right)\\ &+ \left\{[1+\varepsilon(2+\theta)]meas(\Omega) + \frac{4r_1}{\sqrt{k_0 a_0}} + \varepsilon(C_4 + C_7)r_1\right\} C(t),\end{aligned} \tag{2.14}$$

where $C_2, \ldots, C_7$ *and* r_1 *are known constants, then the energy decay exponentially, i.e., there exist positive constants* β_1 *and* β_2 *such that*

$$E(t) \le \beta_1\left(E(0) + \alpha t^r\right) e^{-\beta_2 t} \tag{2.15}$$

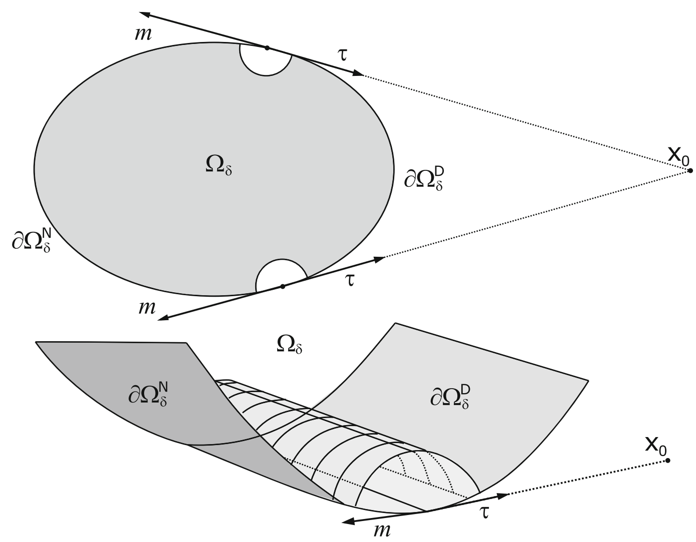

FIG. 3. Examples of domain Ω and a x_0 satisfying Assumption 3

It is possible to verify that the energy $E_{\Omega_\delta}(t)$ and the perturbed energy $E_{\delta,\varepsilon}(t)$ are equivalent. Precisely, there exists a positive constant r_0 such that

$$|E_{\delta,\varepsilon}(t) - E_{\Omega_\delta}(t)| \leq \varepsilon r_0 E_{\Omega_\delta}(t), \tag{2.16}$$

for all $t \geq 0$ and for all $\varepsilon > 0$.

Moreover, there exists a positive constant r_1 such that

$$E_{\Omega_\delta}(t) \leq r_1, \tag{2.17}$$

for all $t \geq 0$.

Next lemma gives us a kind of inequality of energy. We observe that this lemma does not allow us to conclude that the energy decay. It holds because the assumptions of K and a are very general.

Lemma 2.2. *Let $E_{\Omega_\delta}(t)$ the energy of (1.1) associated with δ. The following inequality holds*

$$\begin{aligned} E'_{\Omega_\delta}(t) \leq & -\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2 \, d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial \nu_A} u_t \, d\Gamma \\ & + C(t)\int_{\Omega_\delta} (1+|u_t||\nabla u|)\, dx + \frac{1}{2}\int_{\Omega_\delta} K_t u_t^2 \, dx + \frac{1}{2}\int_{\Omega_\delta} a_t |\nabla u|^2 \, dx. \end{aligned} \tag{2.18}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω_δ, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_\delta} K u_t^2 \, dx + \int_{\Omega_\delta} a|\nabla u|^2 \, dx\right) + \int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2 \, d\Gamma + \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial \nu_A} u_t \, d\Gamma \\ & + \int_{\Omega_\delta} F(u) u_t \, dx - \frac{1}{2}\int_{\Omega_\delta} K_t u_t^2 \, dx - \frac{1}{2}\int_{\Omega_\delta} a_t |\nabla u|^2 \, dx = 0. \end{aligned} \tag{2.19}$$

From this and observing Assumption 1, we obtain (2.18). □

3. Stability theorem proof

Proof of Theorem 2.2. Differentiating $\Psi_\delta(t)$ and observing (1.1), we have

$$\Psi'_\delta(t) = 2\int_{\Omega_\delta} K_t u_t m\cdot\nabla u\,dx - 2\int_{\Omega_\delta} A(t)u\,m\cdot\nabla u\,dx - 2\int_{\Omega_\delta} F(x,t,u,\nabla u)m\cdot\nabla u\,dx + 2\int_{\Omega_\delta} K u_t m\cdot\nabla u_t\,dx$$
$$+\theta\int_{\Omega_\delta} K_t u_t u\,dx + \theta\int_{\Omega_\delta} K u_t^2\,dx - \theta\int_{\Omega_\delta} uA(t)u\,dx - \theta\int_{\Omega_\delta} F(x,t,u,\nabla u)u\,dx.$$

From this and using Assumption 1, we infer

$$\Psi'_\delta(t) \le 2\int_{\Omega_\delta} K_t u_t m\cdot\nabla u\,dx - 2\int_{\Omega_\delta} A(t)u\,m\cdot\nabla u\,dx - 2\int_{\Omega_\delta} |u|^\gamma u\,m\cdot\nabla u\,dx - 2C(t)\int_{\Omega_\delta}(1+|\nabla u|)|m\cdot\nabla u|\,dx$$
$$+2\int_{\Omega_\delta} K u_t m\cdot\nabla u_t\,dx + \theta\int_{\Omega_\delta} K_t u_t u\,dx + \theta\int_{\Omega_\delta} K u_t^2\,dx$$
$$-\theta\int_{\Omega_\delta} uA(t)u\,dx - \theta\int_{\Omega_\delta} |u|^{\gamma+2}\,dx + \theta C(t)\int_{\Omega_\delta}(1+|u||\nabla u|)\,dx. \tag{3.20}$$

Now, we are going to estimate the right-hand side of (3.20).

Estimate for $-2\int_{\Omega_\delta} A(t)u\,m\cdot\nabla u\,dx$. Using Gauss theorem, we have

$$-2\int_{\Omega_\delta} A(t)u\,m\cdot\nabla u\,dx = -\int_{\Omega_\delta} a\,m\cdot\nabla(|\nabla u|^2)\,dx - 2\int_{\Omega_\delta} a|\nabla u|^2\,dx + 2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma. \tag{3.21}$$

Using Gauss theorem one more time, we obtain

$$\int_{\Omega_\delta} a\,m\cdot\nabla(|\nabla u|^2)\,dx = -\int_{\Omega_\delta}\nabla a\cdot m|\nabla u|^2\,dx - n\int_{\Omega_\delta} a|\nabla u|^2\,dx + \int_{\partial\Omega_\delta} a\,m\cdot\nu|\nabla u|^2\,d\Gamma. \tag{3.22}$$

Combining (3.21) with (3.22), we have

$$-2\int_{\Omega_\delta} A(t)u\,m\cdot\nabla u\,dx = (n-2)\int_{\Omega_\delta} a|\nabla u|^2\,dx + \int_{\Omega_\delta}\nabla a\cdot m|\nabla u|^2\,dx$$
$$+2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta} a\,m\cdot\nu|\nabla u|^2\,d\Gamma. \tag{3.23}$$

Estimate for $-2\int_{\Omega_\delta}|u|^\gamma u\,m\cdot\nabla u\,dx$. We have that

$$-2\int_{\Omega_\delta}|u|^\gamma u\,m\cdot\nabla u\,dx = -\frac{2}{\gamma+2}\int_{\Omega_\delta}\nabla(|u|^{\gamma+2})\cdot m\,dx$$
$$= \frac{2n}{\gamma+2}\int_{\Omega_\delta}|u|^{\gamma+2}\,dx - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta} m\cdot\nu|u|^{\gamma+2}\,d\Gamma. \tag{3.24}$$

Estimate for $2\int_{\Omega_\delta} K u_t m\cdot\nabla u_t\,dx$. We observe that

$$2\int_{\Omega_\delta} K u_t m\cdot\nabla u_t\,dx = \int_{\Omega_\delta} K m\cdot\nabla u_t^2\,dx$$
$$= -\int_{\Omega_\delta}(\nabla K\cdot m)u_t^2\,dx - n\int_{\Omega_\delta} K u_t^2\,dx + 2\int_{\partial\Omega_\delta} m\cdot\nu K u_t^2\,d\Gamma. \quad (3.25)$$

Estimate for $-\theta\int_{\Omega_\delta} uA(t)u\,dx$. Using Gauss theorem, we obtain

$$-\theta\int_{\Omega_\delta} uA(t)u\,dx = -\theta\int_{\Omega_\delta} a|\nabla u|^2\,dx + \theta\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}u\,d\Gamma. \quad (3.26)$$

Substituting (3.21)–(3.26) into (3.20), we have

$$\begin{aligned}\Psi_\delta'(t) \le\ & 2\int_{\Omega_\delta} K_t u_t m\cdot\nabla u\,dx - [\theta-(n-2)]\int_{\Omega_\delta} a|\nabla u|^2\,dx + \int_{\Omega_\delta}\nabla a\cdot m|\nabla u|^2\,dx\\ & -\left(\theta-\frac{2n}{\gamma+2}\right)\int_{\Omega_\delta}|u|^{\gamma+2}\,dx - 2C(t)\int_{\Omega_\delta}(1+|\nabla u|)|m\cdot\nabla u|\,dx - \int_{\Omega_\delta}(\nabla K\cdot m)u_t^2\,dx\\ & -(n-\theta)\int_{\Omega_\delta} K u_t^2\,dx + \theta\int_{\Omega_\delta} K_t u_t u\,dx + \theta C(t)\int_{\Omega_\delta}(1+|u||\nabla u|)\,dx\\ & +2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta} a\,m\cdot\nu|\nabla u|^2\,d\Gamma - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta} m\cdot\nu|u|^{\gamma+2}\,d\Gamma\\ & +2\int_{\partial\Omega_\delta} m\cdot\nu K u_t^2\,d\Gamma + \theta\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}u\,d\Gamma. \end{aligned} \quad (3.27)$$

Now, we are going to estimate the integrals over $\partial\Omega_\delta$. Observing that $u=0$ on Γ_0, we have

$$m\cdot\nabla u = m\cdot\frac{\partial u}{\partial\nu} \quad\text{and}\quad |\nabla u|^2 = \left(\frac{\partial u}{\partial\nu}\right)^2 \quad\text{on }\Gamma_0.$$

From this and observing the boundary condition on Γ_1, we infer

$$2\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma = 2\int_{\partial\Omega_\delta^D} am\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma$$
$$-2\int_{\partial\Omega_\delta^N} m\cdot\nu u_t\, m\cdot\nabla u\,d\Gamma + 2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma. \quad (3.28)$$

For the second integral on $\partial\Omega_\delta$, we have

$$-\int_{\partial\Omega_\delta} a\,m\cdot\nu|\nabla u|^2\,d\Gamma = -\int_{\partial\Omega_\delta^D} a\,m\cdot\nu\left(\frac{\partial u}{\partial\nu}\right)^2 d\Gamma$$
$$-\int_{\partial\Omega_\delta^N} a\,m\cdot\nu|\nabla u|^2\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\,m\cdot\nu|\nabla u|^2\,d\Gamma. \quad (3.29)$$

Moreover, since $m \cdot \nu \geq 0$ on Γ_1, we obtain

$$-\frac{2}{\gamma+2}\int_{\partial\Omega_\delta} m\cdot\nu|u|^{\gamma+2}\,d\Gamma \leq -\frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma. \tag{3.30}$$

We also have,

$$2\int_{\partial\Omega_\delta} m\cdot\nu K u_t^2\,d\Gamma = 2\int_{\partial\Omega_\delta^N} m\cdot\nu K u_t^2\,d\Gamma + 2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma \tag{3.31}$$

and

$$\theta\int_{\partial\Omega_\delta}\frac{\partial u}{\partial\nu_A}u\,d\Gamma = -\theta\int_{\partial\Omega_\delta^N} m\cdot\nu u_t u\,d\Gamma + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}u\,d\Gamma. \tag{3.32}$$

Combining (3.27)–(3.32), we infer

$$\begin{aligned}
\Psi_\delta'(t) \leq\ & 2\int_{\Omega_\delta} K_t u_t m\cdot\nabla u\,dx - [\theta-(n-2)]\int_{\Omega_\delta} a|\nabla u|^2\,dx + \int_{\Omega_\delta}\nabla a\cdot m|\nabla u|^2\,dx\\
& -\left(\theta-\frac{2n}{\gamma+2}\right)\int_{\Omega_\delta}|u|^{\gamma+2}\,dx - 2C(t)\int_{\Omega_\delta}(1+|\nabla u|)|m\cdot\nabla u|\,dx - \int_{\Omega_\delta}(\nabla K\cdot m)u_t^2\,dx\\
& -(n-\theta)\int_{\Omega_\delta} K u_t^2\,dx + \theta\int_{\Omega_\delta} K_t u_t u\,dx + \theta C(t)\int_{\Omega_\delta}(1+|u||\nabla u|)\,dx\\
& -2\int_{\partial\Omega_\delta^N} m\cdot\nu u_t\, m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta^N} a\, m\cdot\nu|\nabla u|^2\,d\Gamma + 2\int_{\partial\Omega_\delta^N} m\cdot\nu K u_t^2\,d\Gamma - \theta\int_{\partial\Omega_\delta^N} m\cdot\nu u_t u\,d\Gamma\\
& +2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\, m\cdot\nu|\nabla u|^2\,d\Gamma - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma\\
& +2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}u\,d\Gamma.
\end{aligned} \tag{3.33}$$

Recovering the energy on Ω_δ. We observe that

$$\begin{aligned}
&-[\theta-(n-2)]\int_{\Omega_\delta} a|\nabla u|^2\,dx - \left(\theta-\frac{2n}{\gamma+2}\right)\int_{\Omega_\delta}|u|^{\gamma+2}\,dx - (n-\theta)\int_{\Omega_\delta} K u_t^2\,dx\\
&\quad\leq -\min\{2[\theta-(n-2)], 2(n-\theta), 2n-(\gamma+2)\theta\}\, E_{\Omega_\delta}(t) := -C_1 E_{\Omega_\delta}(t).
\end{aligned} \tag{3.34}$$

Moreover,

$$2\int_{\Omega_\delta} K_t u_t m\cdot\nabla u\,dx \leq \frac{4}{\sqrt{k_0 a_0}}\max_{x\in\overline{\Omega}}\{|m(x)|\}\,\|k_t(t)\|_{L^\infty(\Omega)} E_{\Omega_\delta}(t) := C_2\|K_t(t)\|_{L^\infty(\Omega)} E_{\Omega_\delta}(t), \tag{3.35}$$

$$\int_{\Omega_\delta}\nabla a\cdot m|\nabla u|^2\,dx \leq \frac{2}{a_0}\max_{x\in\overline{\Omega}}\{|m(x)|\}\,\|\nabla a(t)\|_{L^\infty(\Omega)} E_{\Omega_\delta}(t) := C_3\|\nabla a(t)\|_{L^\infty(\Omega)} E_{\Omega_\delta}(t),$$

$$-2C(t)\int_{\Omega_\delta}(1+|\nabla u|)|m\cdot\nabla u|\,dx \leq 2\mathrm{meas}(\Omega)C(t) + 4\max_{x\in\overline{\Omega}}\{|m(x)|\}\,C(t)E_{\Omega_\delta}(t) \tag{3.36}$$

$$:= 2\mathrm{meas}(\Omega)C(t) + C_4 C(t)E_{\Omega_\delta}(t), \tag{3.37}$$

$$-\int_{\Omega_\delta}(\nabla K\cdot m)u_t^2\,dx \le \frac{2}{k_0}\max_{x\in\overline{\Omega}}\{|m(x)|\}\,\|\nabla K(t)\|_{L^\infty(\Omega)}E_{\Omega_\delta}(t) := C_5\|\nabla K(t)\|_{L^\infty(\Omega)}E_{\Omega_\delta}(t), \quad (3.38)$$

$$\theta\int_{\Omega_\delta}K_t u_t u\,dx \le 4\theta\sqrt{\frac{c_p}{a_0k_0}}\|K_t(t)\|_{L^\infty(\Omega)}E_{\Omega_\delta}(t) := \theta C_6\|K_t(t)\|_{L^\infty(\Omega)}E_{\Omega_\delta}(t),$$

$$\theta C(t)\int_{\Omega_\delta}(1+|u||\nabla u|)\,dx \le \theta\text{meas}(\Omega)C(t) + 4\theta\frac{\sqrt{c_p}}{a_0}C(t)E_{\Omega_\delta}(t) \quad (3.39)$$

$$:= \theta\text{meas}(\Omega)C(t) + C_7C(t)E_{\Omega_\delta}(t). \quad (3.40)$$

Using estimates (3.34)–(3.40) in (3.33), we obtain

$$\begin{aligned}\Psi_\delta'(t) \le{}& -C_1E_{\Omega_\delta}(t) + \lambda(t)E_{\Omega_\delta}(t) + (2+\theta)\text{meas}(\Omega)C(t)\\ &-2\int_{\partial\Omega_\delta^N} m\cdot\nu u_t\, m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta^N} a\, m\cdot\nu|\nabla u|^2\,d\Gamma + 2\int_{\partial\Omega_\delta^N} m\cdot\nu K u_t^2\,d\Gamma - \theta\int_{\partial\Omega_\delta^N} m\cdot\nu u_t u\,d\Gamma\\ &+2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\, m\cdot\nu|\nabla u|^2\,d\Gamma - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma\\ &+2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}u\,d\Gamma, \end{aligned} \quad (3.41)$$

where

$$\lambda(t) = (C_2+\theta C_6)\|K_t(t)\|_{L^\infty(\Omega)} + C_3\|\nabla a(t)\|_{L^\infty(\Omega)} + (C_4+C_7)C(t) + C_5\|\nabla K(t)\|_{L^\infty(\Omega)}.$$

Estimate for the integrals on $\partial\Omega_\delta^N$. We have

$$-2\int_{\partial\Omega_\delta^N} m\cdot\nu u_t\, m\cdot\nabla u\,d\Gamma \le \frac{1}{a_0}\left(\max_{x\in\overline{\Omega}}\{|m(x)|\}\right)^2\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma + \int_{\partial\Omega_\delta^N} m\cdot\nu a|\nabla u|^2\,d\Gamma, \quad (3.42)$$

$$-\theta\int_{\partial\Omega_\delta^N} m\cdot\nu u_t u\,d\Gamma \le \frac{\theta^2}{2a_0\sigma}\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma + \sigma E_{\Omega_\delta}(t), \quad (3.43)$$

for all $\sigma > 0$ constant. We also obtain that

$$2\int_{\partial\Omega_\delta^N} m\cdot\nu K u_t^2\,d\Gamma \le 2\|K\|_{L^\infty(\Omega\times(0,\infty))}\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma. \quad (3.44)$$

Therefore,

$$\begin{aligned}\Psi_\delta'(t) \le{}& -(C_1-\sigma)E_{\Omega_\delta}(t) + \lambda(t)E_{\Omega_\delta}(t) + (2+\theta)\text{meas}(\Omega)C(t) + C_8\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma\\ &+2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\, m\cdot\nu|\nabla u|^2\,d\Gamma - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma\\ &+2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A}u\,d\Gamma, \end{aligned} \quad (3.45)$$

where

$$C_8 = \frac{1}{a_0}\left(\max_{x\in\overline{\Omega}}\{|m(x)|\}\right)^2 + \frac{\theta^2}{2a_0\sigma} + 2\|K\|_{L^\infty(\Omega\times(0,\infty))}.$$

Observing Lemma 2.2, (3.45) and the definition of $E_{\delta,\varepsilon}(t)$, we have

$$\begin{aligned} E'_{\delta,\varepsilon}(t) \le & -\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\, d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial\nu_A} u_t\, d\Gamma \\ & + C(t)\int_{\Omega_\delta}(1+|u_t||\nabla u|)\,dx + \frac{1}{2}\int_{\Omega_\delta} K_t u_t^2\,dx + \frac{1}{2}\int_{\Omega_\delta} a_t|\nabla u|^2\,dx \\ & + \varepsilon\Bigg[-(C_1-\sigma)E_{\Omega_\delta}(t) + \lambda(t)E_{\Omega_\delta}(t) + (2+\theta)\mathrm{meas}(\Omega)C(t) + C_8\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma \\ & + 2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\, m\cdot\nu|\nabla u|^2\,d\Gamma - \frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma \\ & + 2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} u\,d\Gamma\Bigg]. \end{aligned} \tag{3.46}$$

We have that

$$\begin{aligned} & C(t)\int_{\Omega_\delta}(1+|u_t||\nabla u|)\,dx + \frac{1}{2}\int_{\Omega_\delta} K_t u_t^2\,dx + \frac{1}{2}\int_{\Omega_\delta} a_t|\nabla u|^2\,dx \\ & \quad \le J(t)E_{\Omega_\delta}(t) + C(t)\mathrm{meas}(\Omega), \end{aligned}$$

where

$$J(t) = \frac{\|a_t(t)\|_{L^\infty(\Omega)}}{a_0} + \frac{\|K_t(t)\|_{L^\infty(\Omega)}}{k_0} + \frac{4}{\sqrt{k_0 a_0}}C(t).$$

Thus,

$$\begin{aligned} E'_{\delta,\varepsilon}(t) \le & -\varepsilon(C_1-\sigma)E_{\Omega_\delta}(t) + (J(t)+\varepsilon\lambda(t))E_{\Omega_\delta}(t) + [1+\varepsilon(2+\theta)]\mathrm{meas}(\Omega)C(t) \\ & -(1-\varepsilon C_8)\int_{\partial\Omega_\delta^N} m\cdot\nu u_t^2\,d\Gamma + \Lambda_\delta(t) + \Xi_\delta(t), \end{aligned}$$

where

$$\Lambda_\delta(t) = \varepsilon\left[2\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} m\cdot\nabla u\,d\Gamma - \int_{\partial\Omega_\delta\cap\Omega} a\, m\cdot\nu|\nabla u|^2\,d\Gamma\right] \tag{3.47}$$

and

$$\begin{aligned} \Xi_\delta(t) = \varepsilon & \Bigg[-\frac{2}{\gamma+2}\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u|^{\gamma+2}\,d\Gamma + 2\int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K u_t^2\,d\Gamma \\ & + \theta\int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} u\,d\Gamma\Bigg] - \int_{\partial\Omega_\delta\cap\Omega}\frac{\partial u}{\partial\nu_A} u_t\,d\Gamma. \end{aligned} \tag{3.48}$$

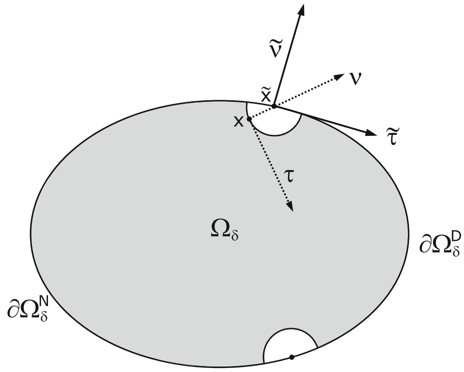

FIG. 4. The vectors ν, τ, $\tilde{\nu}$, and $\tilde{\tau}$ in the $\mathbb{R}^2$ case

Choosing $n-2<\theta<n$ and, after this, $\sigma = \frac{C_1}{4}$ and $\varepsilon < \frac{1}{C_8}$, we infer

$$E'_{\delta,\varepsilon}(t) \le -\frac{\varepsilon C_1}{4}E_{\Omega_\delta}(t) + (J(t)+\varepsilon\lambda(t))E_{\Omega_\delta}(t) + [1+\varepsilon(2+\theta)]\text{meas}(\Omega)C(t) + \Lambda_\delta(t) + \Xi_\delta(t). \tag{3.49}$$

From this, using (2.16), and as

$$\varphi(t) = (J(t)+\varepsilon\lambda(t))r_1 + [1+\varepsilon(2+\theta)]\text{meas}(\Omega)C(t)$$

(see (2.14)) we obtain

$$E'_{\delta,\varepsilon}(t) \le -\frac{\varepsilon C_1}{8}E_{\delta,\varepsilon}(t) + \varphi(t) + \Lambda_\delta(t) + \Xi_\delta(t), \tag{3.50}$$

for $\varepsilon > 0$ small enough. Thus,

$$\frac{d}{dt}\left(E_{\delta,\varepsilon}(t)e^{\frac{\varepsilon C_1 t}{8}}\right) \le e^{\frac{\varepsilon C_1 t}{8}}\left(\varphi(t) + \Lambda_\delta(t) + \Xi_\delta(t)\right). \tag{3.51}$$

Integrating from 0 to t, we obtain

$$E_{\delta,\varepsilon}(t) \le E_{\delta,\varepsilon}(0)e^{-\frac{\varepsilon C_1 t}{8}} + \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}}\left(\varphi(s) + \Lambda_\delta(s) + \Xi_\delta(s)\right)\, ds\right)e^{-\frac{\varepsilon C_1 t}{8}}. \tag{3.52}$$

From this and using Assumption 2.13, we conclude that

$$E_{\delta,\varepsilon}(t) \le E_{\delta,\varepsilon}(0)e^{-\frac{\varepsilon C_1 t}{8}} + \alpha t^r e^{-\frac{\varepsilon C_1 t}{8}} + \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}}[\Lambda_\delta(s) + \Xi_\delta(s)]\, ds\right)e^{-\frac{\varepsilon C_1 t}{8}}. \tag{3.53}$$

Estimate for $\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}}\Lambda_\delta(s)\, ds\right)e^{-\frac{\varepsilon C_1 t}{8}}$. First, we consider the two-dimensional case. For each $\tilde{x} \in \Sigma$, we denote by $\tilde{\nu} = \nu(\tilde{x})$ and $\tilde{\tau} = \tau(\tilde{x})$ the unit normal vector pointing towards the exterior and the tangent vector of Σ, respectively. We consider $\tilde{\tau}$ from Γ_1 to Γ_0. Using coordinate system $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$, it is possible to verify that

$$2\frac{\partial u}{\partial \nu_A}m\cdot\nabla u - a\, m\cdot\nu|\nabla u|^2 = \frac{a}{4\delta}\tilde{m}\cdot\tilde{\tau} - \frac{a}{4}\nu\cdot\tilde{\tau}.$$

See Figures 4 and 5.

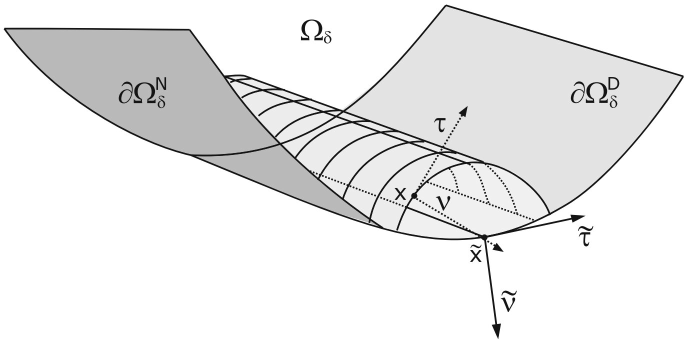

Fig. 5. The vectors ν, τ, $\tilde{\nu}$, and $\tilde{\tau}$ in the $\mathbb{R}^3$ case

Observing that

$$\frac{1}{\pi\delta}\int_{\partial\Omega_\delta\cap\Omega} a\, d\Gamma \to a(\tilde{x}) := \tilde{a},$$

as $\delta \to \infty$, we infer

$$\int_{\partial\Omega_\delta\cap\Omega}\left(2\frac{\partial u}{\partial\nu_A} m\cdot\nabla u - a\, m\cdot\nu|\nabla u|^2\right) d\Gamma \to \frac{\pi}{4}\tilde{a}\,\tilde{m}\cdot\tilde{\tau}, \tag{3.54}$$

as $\delta \to 0$. As Assumption 3 is in place and since $a \geq 0$, we conclude that the integral converges to a negative number.

Now, we consider the case in $\mathbb{R}^N$, with $N \geq 3$. For each $x \in \partial\Omega_\delta \cap \Omega$, there exists $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$ such that x is into the plane defined by $(\tilde{x}, \tilde{\nu}, \tilde{\tau})$. Moreover, there exists an arc of circumference $\gamma(\tilde{x}, \delta)$ into this plane such that $x \in \gamma(\tilde{x}, \delta)$. Thus, writing

$$\nabla u = \nabla_2 u + \nabla_T u, \tag{3.55}$$

where $\nabla_2 u$ is into the plan describe above and $\nabla_T u$ is orthogonal to $\nabla_2 u$, we have

$$\nabla_T u\cdot\nabla_2 u = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla_2 u|^2 + |\nabla_T u|^2.$$

Therefore,

$$\begin{aligned}
&\int_{\partial\Omega_\delta\cap\Omega}\left(2\frac{\partial u}{\partial\nu_A} m\cdot\nabla u - a\, m\cdot\nu|\nabla u|^2\right) d\Gamma\\
&= \int_{\partial\Omega_\delta\cap\Omega}\left[2a(\nabla_2 u + \nabla_T u)\cdot\nu\, m\cdot(\nabla_2 u + \nabla_T u) - a\, m\cdot\nu(|\nabla_2 u|^2 + |\nabla_T u|^2)\right] d\Gamma\\
&= \int_{\partial\Omega_\delta\cap\Omega}\left(2a\nabla_T u\cdot\nu\, m\cdot\nabla_T u - a\, m\cdot\nu|\nabla_T u|^2\right) d\Gamma\\
&\quad + \int_{\partial\Omega_\delta\cap\Omega}\left(2a\nabla_2 u\cdot\nu\, m\cdot\nabla_2 u - a\, m\cdot\nu|\nabla_2 u|^2\right) d\Gamma\\
&\quad +2\int_{\partial\Omega_\delta\cap\Omega} a\left(\nabla_T u\cdot\nu\, m\cdot\nabla_2 u + \nabla_2 u\cdot\nu\, m\cdot\nabla_T u\right) d\Gamma.
\end{aligned} \tag{3.56}$$

Using the first part of Theorem 4 of [2], we have

$$\int_{\partial\Omega_\delta\cap\Omega} \left(2a\nabla_T u\cdot\nu\, m\cdot\nabla_T u - a\, m\cdot\nu|\nabla_T u|^2\right)\, d\Gamma \to 0, \tag{3.57}$$

as $\delta\to 0$.

Next step is to estimate the second integral of the right side of (3.56). From Theorem 4 of [2], we have that u can be written locally as a sum of regular part with a singular part

$$\Phi(\tilde{x})u_S(x-\tilde{x}),$$

where Φ is locally in $H^{\frac{1}{2}}(\Sigma)$ and u_S is given by

$$u_S(r,w,t) = c(t)r^{\frac{1}{2}}\varrho\sin\left(\frac{w}{2}\right), \tag{3.58}$$

where $\varrho\in C^\infty$ with compact support and such that $\varrho=1$ into a neighborhood of zero and supp$(\varrho)\subset[-\varrho_0,\varrho_0]\subset(-1,1)$, with $\varrho_0>0$ small enough. As in the two-dimensional case, using the coordinate system $\tilde{x},\tilde{\nu},\tilde{\tau}$ we obtain

$$2a\nabla_2 u_S\cdot\nu\, m\cdot\nabla_2 u_S - a\, m\cdot\nu|\nabla_2 u_S|^2 = \frac{a}{4\delta}\tilde{m}\cdot\tilde{\tau} - \frac{a}{4}\nu\cdot\tilde{\tau}.$$

Integrating over the arc $\gamma(\tilde{x},\delta)$ and taking the limit as $\delta\to 0$, as in (3.54), we infer

$$\int_{\gamma(\tilde{x},\delta)} \left(2a\nabla_2 u_S\cdot\nu\, m\cdot\nabla_2 u_S - a\, m\cdot\nu|\nabla_2 u_S|^2\right)\, d\Gamma \to \frac{\pi}{4}\tilde{a}\,\tilde{m}\cdot\tilde{\tau}, \tag{3.59}$$

as $\delta\to 0$. Using Assumption 3, we infer that this integral converges to a negative number. Therefore, using Fubini's theorem, we conclude that the second integral of the right-hand side of (3.56) converges to a negative number.

Finally, we are going to estimate the last integral of the right-hand side of (3.56). Using Hölder's inequality, we have

$$2\int_{\gamma(\tilde{x},\delta)} a\left(\nabla_T u\cdot\nu\, m\cdot\nabla_2 u + \nabla_2 u\cdot\nu\, m\cdot\nabla_T u\right)\, d\Gamma \le C\left(\int_{\gamma(\tilde{x},\delta)} |\nabla_T u|^2\, d\Gamma\right)^{\frac{1}{2}}\left(\int_{\gamma(\tilde{x},\delta)} |\nabla_2 u|^2\, d\Gamma\right)^{\frac{1}{2}}.$$

From Theorem 4 of [2], we have

$$\left(\int_{\gamma(\tilde{x},\delta)} |\nabla_T u|^2\, d\Gamma\right)^{\frac{1}{2}} \to 0,$$

as $\delta\to 0$, using the decomposition described above, we infer

$$\left(\int_{\gamma(\tilde{x},\delta)} |\nabla_2 u|^2\, d\Gamma\right)^{\frac{1}{2}} \le C\int_0^{2\pi} \frac{1}{\sqrt{\delta}}\, ds = C\sqrt{\delta}\to 0,$$

as $\delta\to 0$. Therefore,

$$2\int_{\gamma(\tilde{x},\delta)} a\left(\nabla_T u\cdot\nu\, m\cdot\nabla_2 u + \nabla_2 u\cdot\nu\, m\cdot\nabla_T u\right)\, d\Gamma \to 0, \tag{3.60}$$

as $\delta\to 0$. Thus, (3.60) and Fubini's theorem allow us to conclude that the third integral of the right-hand side of (3.56) converges to zero.

Therefore, we infer

$$\Lambda_\delta(t) = \int\limits_{\partial\Omega_\delta\cap\Omega} \left(2\frac{\partial u}{\partial \nu_A} m\cdot\nabla u - a\, m\cdot\nu|\nabla u|^2\right)\, d\Gamma \to \alpha,$$

as $\delta \to 0$, where $\alpha \leq 0$ is a real number. At this moment, we consider two cases $\alpha < 0$ and $\alpha = 0$. If $\alpha < 0$, then there exists $\delta_1 > 0$ such that

$$\Lambda_\delta(t) < 0, \tag{3.61}$$

for all $\delta < \delta_1$. Therefore,

$$\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}}\Lambda_\delta(s)\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} < 0, \tag{3.62}$$

for all $\delta < \delta_1$. We conclude that term (3.62) can be removed of (3.53), when $\delta \to 0$.

If $\alpha = 0$, then we consider two subcases. First, if there exists a positive δ_2 such that

$$\Lambda_\delta(t) < 0, \tag{3.63}$$

for all $\delta < \delta_2$, then we take the same way of the case $\alpha < 0$. On the other hand, if there exists a positive r_1 such that

$$\Lambda_\delta(t) > 0, \tag{3.64}$$

for all $\delta < r_2$, then

$$0 \leq \int\limits_0^t \left(e^{\frac{\varepsilon C_1 s}{8}}\Lambda_\delta(s)\right)\, ds e^{-\frac{\varepsilon C_1 t}{8}} \leq \int\limits_0^t \Lambda_\delta(s)\, ds \to 0, \tag{3.65}$$

as $\delta \to 0$. Therefore, in this case the term also goes to 0.

Thus, we conclude that

$$\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}}\Lambda_\delta(s)\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.66}$$

as $\delta \to 0$.

Estimate for $\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}}\Xi_\delta(s)\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}}$. Using the decomposition described above as

$$u = u_R + u_S, \tag{3.67}$$

where u_R is the regular part of u and u_S is the singular one, we have

$$\begin{aligned} &\left|2\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}} \int\limits_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K(x,s)|u_t(x,s)|^2\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}}\right| \\ &\quad\leq C\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}} \int\limits_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu|(u_R)_t^2\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \\ &\quad\quad + C\left(\int\limits_0^t e^{\frac{\varepsilon C_1 s}{8}} \int\limits_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu|(u_S)_t^2\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}}. \end{aligned} \tag{3.68}$$

From Lebesgue convergence theorem, we have

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu|(u_R)_t^2\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.69}$$

as $\delta \to 0$. Now, observing (3.58), we have

$$\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\gamma(\tilde{x},\delta)} |m\cdot\nu|(u_S)_t^2\, d\Gamma\, ds e^{-\frac{\varepsilon C_1 t}{8}} \le C\int_0^{2\pi} \sqrt{\delta}\, ds \int_0^t e^{\frac{\varepsilon C_1 s}{8}}\, ds\, e^{-\frac{\varepsilon C_1 t}{8}} \le C\delta^{\frac{3}{2}} \to 0, \tag{3.70}$$

as $\delta \to 0$. From this and using the Fubini's theorem, we have that the last integral of (3.68) goes to zero, as $\delta \to 0$. Summarizing, (3.68)–(3.70) give us that

$$2\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu K(x,s)|u_t(x,s)|^2\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.71}$$

as $\delta \to 0$.

On the other hand, using decomposition (3.67), we have

$$\begin{aligned}
&\left|\frac{2}{\gamma+2}\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u(x,s)|^{\gamma+2}\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}}\right| \\
&\le C\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu||u_R|^{\gamma+2}\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \\
&\quad + C\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu||u_S|^{\gamma+2}\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}}.
\end{aligned} \tag{3.72}$$

From Lebesgue convergence theorem, we have

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} |m\cdot\nu||u_R|^{\gamma+2}\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.73}$$

as $\delta \to 0$. Now, observing (3.58), we have

$$\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\gamma(\tilde{x},\delta)} |m\cdot\nu||u_S|^{\gamma+2}\, d\Gamma\, ds e^{-\frac{\varepsilon C_1 t}{8}} \le C\int_0^{2\pi} \delta^{\frac{\gamma+2}{2}}\, ds \int_0^t e^{\frac{\varepsilon C_1 s}{8}}\, ds\, e^{-\frac{\varepsilon C_1 t}{8}} \le C\delta^{\frac{\gamma+4}{2}} \to 0, \tag{3.74}$$

as $\delta \to 0$. From this and using the Fubini's theorem, we have that the last integral of (3.68) goes to zero, as $\delta \to 0$. Summarizing, (3.72)–(3.74) give us that

$$\frac{2}{\gamma+2}\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} m\cdot\nu|u(x,s)|^{\gamma+2}\, d\Gamma\, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.75}$$

as $\delta \to 0$.

On the other hand, using the decomposition of u into a regular and a singular part and (3.55), we have that

$$\begin{aligned}
&\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial \nu_A} u \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \\
&= \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_T u \cdot \nu u_R \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} + \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_2 u \cdot \nu u_S \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \\
&\quad + \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_2 u \cdot \nu u_R \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} + \left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_T u \cdot \nu u_S \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \\
&\leq C\sqrt{\delta}.
\end{aligned} \tag{3.76}$$

From Lebesgue convergence theorem, we have

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_T u \cdot \nu u_R \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.77}$$

as $\delta \to 0$. Making calculus analogous to the preview ones, we infer

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_2 u \cdot \nu u_S \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \leq C\delta \to 0, \tag{3.78}$$

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_2 u \cdot \nu u_R \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \leq C\sqrt{\delta} \to 0, \tag{3.79}$$

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} a\nabla_T u \cdot \nu u_S \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \leq C\delta^{\frac{3}{2}} \to 0, \tag{3.80}$$

as $\delta \to 0$.

Therefore,

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial \nu_A} u \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.81}$$

as $\delta \to 0$.

Analogously,

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \int_{\partial\Omega_\delta\cap\Omega} \frac{\partial u}{\partial \nu_A} u_t \, d\Gamma \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.82}$$

as $\delta \to 0$.

From (3.71), (3.75), (3.81), and (3.82), we conclude that

$$\left(\int_0^t e^{\frac{\varepsilon C_1 s}{8}} \Xi_\delta(s) \, ds\right) e^{-\frac{\varepsilon C_1 t}{8}} \to 0, \tag{3.83}$$

as $\delta \to 0$.

Returning to the estimate of $E_{\delta,\varepsilon}(t)$. Observing Lemma 2.2, (3.53), (3.66), and (3.83), we infer

$$E_\varepsilon(t) \leq (E_\varepsilon(0) + \alpha t^r)\, e^{-\frac{\varepsilon C_1 t}{8}}, \tag{3.84}$$

for all $t \geq 0$. From (3.84) and using (2.16), we conclude that (2.15) holds. □

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Marcelo Moreira Cavalcanti and Valéria Neves Domingos Cavalcanti
Department of Mathematics
Universidade Estadual de Maringá
Maringá PR87020-900
Brazil
e-mail: mmcavalcanti@uem.br

Valéria Neves Domingos Cavalcanti
e-mail: vndcavalcanti@uem.br

André Vicente
Center of Exact and Technological Sciences
Universidade Estadual do Oeste do Paraná
Cascavel PR
Brazil
e-mail: andre.vicente@unioeste.br

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