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Threshold dynamics of a reaction–diffusion equation model for cholera transmission with waning vaccine-induced immunity and seasonality

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Abstract. Cholera is an acute intestinal infectious disease caused by the bacterium *Vibrio cholerae*. To explore the multiple effects of spatial mobility, spatial heterogeneity and the seasonality on the transmission of cholera, we propose a time periodic reaction–diffusion equation model with latent period. Based on the basic reproduction number $\mathscr{R}_0$, we establish a threshold-type result. And in the case where all the parameters are constants and $\mathscr{R}_0 > 1$, we show the global attractivity of the endemic steady state by constructing Lyapunov functionals. Finally, we perform some numerical simulations. Our simulations show that (i) increasing the vaccination rate of susceptible individuals and vaccine protective efficacy can reduce the transmission risk $\mathscr{R}_0$; (ii) decreasing the transmission coefficient of contact with infected individuals, the transmission coefficient of contact with hyperinfectious vibrios and the transmission coefficient of contact with hypoinfectious vibrios can reduce the transmission risk $\mathscr{R}_0$; (iii) it is possible to underestimate the transmission risk $\mathscr{R}_0$ in the periodic system if the spatial averaged system is used, based on some experimental data.

Mathematics Subject Classification. 35K57, 37C75, 35C07.

Keywords. Cholera transmission model, Seasonality, Reaction–diffusion equation model, Basic reproduction number, Threshold dynamics.

1. Introduction

Cholera is an acute intestinal infectious disease caused by a bacterium *Vibrio cholerae*. It can be transmitted by many ways, such as direct person-to-person contact or indirect transmission through the environment [21,31]. It remains as one of the major public health threats in underdeveloped countries even though tremendous efforts have been made towards the prevention, intervention and control of the spread of the disease. According to the World Health Organization (WHO), there are approximately 1.3–4.0 million cholera cases occur worldwide annually, leading to the related deaths of 21,000–143,000 yearly [55]. It is widely accepted that mathematical models of the cholera dynamics play an important role in providing deep and useful insights into the understanding of the multiple transmission pathways of cholera (see, e.g. [1,7,8,10,11,17,19,21,31,32,34,37,43,44,49,51]). Most of these models are governed by ordinary differential equations (ODEs), where model parameters are assumed to be independent of time and space so that detailed mathematical results on stabilities and bifurcations can be achieved.

It is now well known that the most effective measures for prevention, intervention and control on the spread of the cholera transmission in the long term are improvements in personal hygiene, drinking water and basic sanitation system. Unfortunately, these measures cannot be applied properly in some cholera endemic countries. Thus, vaccination becomes a useful measure, for example in 2010, WHO recommended that oral vaccines should be used in some cholera endemic countries [55]. Since then researchers have been

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Table 1. *Biological considerations of parameters for model* (1.1)

Parameters	Description
A	The natural human birth rate
μ	The natural human death rate
d	The disease-induced human death rate
β_h	The transmission coefficient of contact with infected individuals
β_H	The transmission coefficient of contact with hyperinfectious vibrios
β_L	The transmission coefficient of contact with hypoinfectious vibrios
k_H	The concentration of hyperinfectious vibrios in contaminated water that yields 50% chance of catching cholera
k_L	The concentration of hypoinfectious vibrios in contaminated water that yields 50% chance of catching cholera
ϖ	The vaccination rate of susceptible individuals
η	The rate at which the vaccine wears off
σ	The reduction of vaccine efficacy (less than one)
Υ	The recovery rate of infected individuals
ξ	The contribution of each infected individual to the concentration of hyperinfectious vibrios
χ	The decay rate from hyperinfectious state to reduced infectiousness
δ_L	The net death rate of hypoinfectious vibrios

working on mathematical models to investigate the effect of vaccination on the cholera transmission, see [24,35,41,60] and the references therein.

To better understand the multiple transmission pathways of cholera, very recently, Bai et al. [5] proposed a mathematical model, by incorporating the combined effects of hyperinfectious and hypoinfectious vibrios, both human-to-human and environment-to-human transmission pathways and waning vaccine-induced immunity. Let $S(t)$, $V(t)$, $I(t)$, $R(t)$, $B_H(t)$ and $B_L(t)$, respectively, represent the densities of susceptible individuals, vaccinated individuals, infected individuals, recovered individuals, hyperinfectious vibrios and hypoinfectious vibrios at time t. Then, the model takes a form of

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = A - \left(\beta_h I(t) + \dfrac{\beta_H B_H(t)}{k_H + B_H(t)} + \dfrac{\beta_L B_L(t)}{k_L + B_L(t)}\right) S(t) - (\varpi + \mu) S(t) + \eta V(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} = \varpi S(t) - \sigma \left(\beta_h I(t) + \dfrac{\beta_H B_H(t)}{k_H + B_H(t)} + \dfrac{\beta_L B_L(t)}{k_L + B_L(t)}\right) V(t) - (\eta + \mu) V(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \left(\beta_h I(t) + \dfrac{\beta_H B_H(t)}{k_H + B_H(t)} + \dfrac{\beta_L B_L(t)}{k_L + B_L(t)}\right)(S(t) + \sigma V(t)) - (\Upsilon + d + \mu) I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \Upsilon I(t) - \mu R(t), \\ \dfrac{\mathrm{d}B_H(t)}{\mathrm{d}t} = \xi I(t) - \chi B_H(t), \\ \dfrac{\mathrm{d}B_L(t)}{\mathrm{d}t} = \chi B_H(t) - \delta_L B_L(t). \end{cases} \tag{1.1}$$

The biological meanings of parameters in model (1.1) are summarized in Table 1. The global dynamics and the control measures of cholera of model (1.1) were discussed.

In biology, spatial and temporal heterogeneity involving differences in ecological and geographic environments, demographic characteristics and socio-economic structures lead to differences in disease exposure rates, levels of human activity, pathogen growth and mortality. This, in turn, leads to a strong impact on cholera dynamics, resulting some natural questions regarding the spread of cholera transmission:

(i) How does the spatial mobility of hosts and pathogens affect the spread of cholera?
(ii) What factors determine cholera outbreaks that persists in some countries but not in others?

(iii) Why does cholera typically spread in a certain period of a year (e.g. monsoon or rainy seasons [23,47])?

There have been some studies on the spatial dynamics of cholera transmission. In [17,38,45], some ODE models based on patch/network structures were established to discuss the global dynamics of steady states. Some reaction–diffusion equation models were proposed to discuss the global dynamics of cholera dynamics [6,10,36,48,51,56,57]. In [11], a nonautonomous ODE model was presented to discuss periodic patterns of cholera outbreaks. Posny and Wang [34] studied the threshold dynamics of a nonautonomous ODE cholera model with general incidence rate. Wang et al. [52] studied the threshold dynamics of a reaction–convection–diffusion cholera model with spatial and temporal heterogeneity. However, to the best of our knowledge, very little has been known and undertaken on threshold dynamics for time periodic reaction–diffusion cholera transmission models with latent period. Azman et al. [2] estimated that the median incubation period of cholera is 1.4 days (95% CI 1.3–1.6). Hence, it is reasonable and helpful to incorporate nonlocal time delay into cholera transmission models.

Motivated by the aforementioned, in this study we consider a case where the latent period is taken into account. Thus, the main purpose of this paper is to propose a time periodic reaction–diffusion equation cholera transmission model with latent period and to explore its dynamics. To this end, we organize the structure of the rest of the paper as follows. We formulate a time periodic reaction–diffusion equation model with latent period according to the criteria of structural population and spatial diffusion in Sect. 2. In Sect. 3, we are dedicated ourselves to the threshold dynamics in terms of the basic reproduction number $\mathscr{R}_0$. It is then followed by Sect. 4 where we mainly discuss the global attractivity of the endemic steady state when $\mathscr{R}_0 > 1$. It is done by constructing Lyapunov functionals. We numerically illustrate our theoretical findings. Finally, we conclude our study in Sect. 6.

2. Model formulation

First, we, based on the notations in model (1.1), introduce the time- and location-dependent densities: $S(t, x)$, $V(t, x)$, $I(t, x)$, $R(t, x)$, $B_H(t, x)$ and $B_L(t, x)$, respectively, denote the densities of susceptible individuals, vaccinated individuals, infected individuals, recovered individuals, hyperinfectious vibrios and hypoinfectious vibrios at the time t and location x. Then, we develop the equations they satisfy. Following the standard procedure on developing model involving structured population and spatial diffusion in [20], we can get

$$\begin{aligned}&\frac{\partial I_1(t, h, x)}{\partial t}+\frac{\partial I_1(t, h, x)}{\partial h}\\&= D_3\Delta I_1(t, h, x)-[\mu(t, x)+\Upsilon(t, h, x)+d(t, h, x)]\, I_1(t, h, x),\ \ t>0,\ x\in\Omega,\end{aligned}\tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a general open bounded domain with smooth boundary $\partial\Omega$, $I_1(t, h, x)$ is the density of individuals with infection age h at time t and location x, $\mu(t, x)$ represents the natural human death rate of infected individuals, independent of h, D_3 is the diffusion rate of infected individuals, and Δ is the usual Laplace operator. Assuming that τ is the average incubation period, we have

$$\underbrace{I_m(t, x) = \int_0^{\tau} I_1(t, h, x)\mathrm{d}h,}_{\text{latently infected individuals}}$$

and

$$\underbrace{I(t,\, x) = \int_{\tau}^{\infty} I_1(t,\, h,\, x)\mathrm{d}h}_{\text{infected individuals}}.$$

We make some assumptions for the functions $\Upsilon(t,\, h,\, x)$ and $d(t,\, h,\, x)$ as follows:

$$\Upsilon(t,\, h,\, x) = \begin{cases} \Upsilon_L(t,\, x), & \text{for } t \geq 0,\ h \in [0,\, \tau],\ \text{and } x \in \Omega, \\ \Upsilon_I(t,\, x), & \text{for } t \geq 0,\ h \in [\tau,\, +\infty],\ \text{and } x \in \Omega, \end{cases}$$

and

$$d(t,\, h,\, x) = \begin{cases} d_L(t,\, x), & \text{for } t \geq 0,\ h \in [0,\, \tau],\ \text{and } x \in \Omega, \\ d_I(t,\, x), & \text{for } t \geq 0,\ h \in [\tau,\, +\infty],\ \text{and } x \in \Omega. \end{cases}$$

A straightforward computation yields

$$\frac{\partial I_m(t,\, x)}{\partial t} = D_3 \Delta I_m(t,\, x) - [\mu(t,\, x) + \Upsilon_L(t,\, x) + d_L(t,\, x)]\, I_m(t,\, x) - I_1(t,\, \tau,\, x) + I_1(t,\, 0,\, x),$$

and

$$\frac{\partial I(t,\, x)}{\partial t} = D_3 \Delta I(t,\, x) - [\mu(t,\, x) + \Upsilon_I(t,\, x) + d_I(t,\, x)]\, I(t,\, x) + I_1(t,\, \tau,\, x) - I_1(t,\, \infty,\, x).$$

In biology, we assume that $I(t,\, \infty,\, x) = 0$ as suggested by [27]. The density of newly infected individuals $(I_1(t,\, 0,\, x))$ is adopted by:

$$I_1(t, 0, x) = \left[\beta_h(t, x)\, I(t, x) + \frac{\beta_H(t, x)\, B_H(t, x)}{k_H + B_H(t, x)} + \frac{\beta_L(t, x)\, B_L(t, x)}{k_L + B_L(t, x)}\right] [S(t, x) + \sigma V(t, x)].$$

Based on model (1.1) and the above discussions, we propose a time periodic reaction–diffusion cholera transmission model. To make things not too complicated (as spatial and temporal heterogeneity, and nonlocal time delay have already made the problem challenging), we compromise a little bit by assuming $\eta = 0$, yielding

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_1 \Delta S(t,x) + A(t,x) - \left[\beta_h(t,x)\, I(t,x) + \dfrac{\beta_H(t,x)\, B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L(t,x)\, B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x) \\ \qquad - [\varpi(t,x) + \mu(t,x)]\, S(t,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial V(t,x)}{\partial t} = D_2 \Delta V(t,x) + \varpi(t,x)\, S(t,x) - \sigma \left[\beta_h(t,x)\, I(t,x) + \dfrac{\beta_H(t,x)\, B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L(t,x)\, B_L(t,x)}{k_L + B_L(t,x)}\right] V(t,x) \\ \qquad - \mu(t,x)\, V(t,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial I_m(t,x)}{\partial t} = D_3 \Delta I_m(t,x) - [\mu(t,x) + \Upsilon_L(t,x) + d_L(t,x)]\, I_m(t,x) - I_1(t,\tau,x) + I_1(t,0,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial I(t,x)}{\partial t} = D_3 \Delta I(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)]\, I(t,x) + I_1(t,\tau,x) - I_1(t,\infty,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial R(t,x)}{\partial t} = D_4 \Delta R(t,x) + \Upsilon_I(t,x)\, I(t,x) - \mu(t,x)\, R(t,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial B_H(t,x)}{\partial t} = D_5 \Delta B_H(t,x) + \xi(t,x)\, I(t,x) - \chi(t,x)\, B_H(t,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial B_L(t,x)}{\partial t} = D_6 \Delta B_L(t,x) + \chi(t,x)\, B_H(t,x) - \delta_L(t,x)\, B_L(t,x),\ t > 0,\ x \in \Omega, \\ \dfrac{\partial S(t,\, x)}{\partial \nu} = \dfrac{\partial V(t,\, x)}{\partial \nu} = \dfrac{\partial I_m(t,\, x)}{\partial \nu} = \dfrac{\partial I(t,\, x)}{\partial \nu} = \dfrac{\partial R(t,\, x)}{\partial \nu} = \dfrac{\partial B_H(t,\, x)}{\partial \nu} = \dfrac{\partial B_L(t,\, x)}{\partial \nu} = 0,\ t > 0,\ x \in \partial\Omega, \end{cases} \tag{2.2}$$

where D_i $(i = 1, 2, 3, 4, 5, 6)$ are the diffusion rates of susceptible individuals, vaccinated individuals, infected individuals, recovered individuals, hyperinfectious vibrios and hypoinfectious vibrios, respectively. The corresponding flowchart of cholera transmission in model (2.2) is depicted in Fig. 1.

We further make the following basic hypotheses:

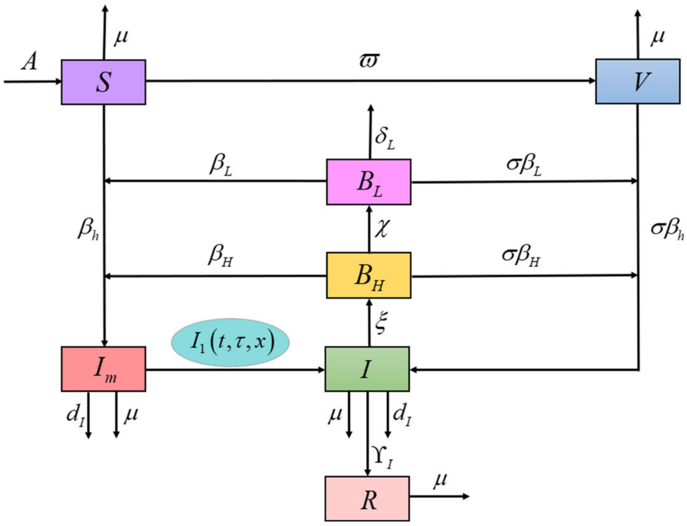

FIG. 1. Schematic diagram for cholera transmission in model (2.2), adapted from Fig. 1 in [5]

(**H**): All functions $A(t,x)$, $\beta_h(t,x)$, $\beta_H(t,x)$, $\beta_L(t,x)$, $\varpi(t,x)$, $\mu(t,x)$, $\Upsilon_L(t,x)$, $d_L(t,x)$, $\Upsilon_I(t,x)$, $d_I(t,x)$, $\xi(t,x)$, $\chi(t,x)$ and $\delta_L(t,x)$ are Hölder continuous and positive on $\mathbb{R}\times\overline{\Omega}$, and ω-periodic in time ($\omega>0$).

Next, we determine $I_1(t,\tau,x)$ by using the characteristics method. For $h\in(0,\tau]$ and $\bar{r}\geq 0$, let $\overline{s}(\bar{r},h,x)=I_1(h+\bar{r},h,x)$. Then, one gets from (2.1) that

$$\begin{cases} \dfrac{\partial\overline{s}(\bar{r},h,x)}{\partial h} = \left[\dfrac{\partial I_1(\bar{r},h,x)}{\partial t}+\dfrac{\partial I_1(\bar{r},h,x)}{\partial h}\right]_{t=h+\bar{r}} \\ \qquad = D_3\Delta\overline{s}(\bar{r},h,x)-\left[\mu(h+\bar{r},x)+\Upsilon_L(h+\bar{r},h,x)+d_L(h+\bar{r},h,x)\right]\bar{s}(\bar{r},h,x), \\ \bar{s}(\bar{r},0,x)=\left[\beta_h(\bar{r},x)I(\bar{r},x)+\dfrac{\beta_H(\bar{r},x)B_H(\bar{r},x)}{k_H+B_H(\bar{r},x)}+\dfrac{\beta_L(\bar{r},x)B_L(\bar{r},x)}{k_L+B_L(\bar{r},x)}\right]\times\left[S(\bar{r},x)+\sigma V(\bar{r},x)\right], \end{cases} \tag{2.3}$$

which has a solution

$$\bar{s}(\bar{r},h,x)=\int_\Omega \Gamma(\bar{r}+h,\bar{r},x,y)\,M\mathrm{d}y,$$

where

$$M=\left[\beta_h(\bar{r},y)I(\bar{r},y)+\frac{\beta_H(\bar{r},y)B_H(\bar{r},y)}{k_H+B_H(\bar{r},y)}+\frac{\beta_L(\bar{r},y)B_L(\bar{r},y)}{k_L+B_L(\bar{r},y)}\right]\times\left[S(\bar{r},y)+\sigma V(\bar{r},y)\right],$$

and $\Gamma(t,s,x,y)$ with $t>s\geq 0$ and $x,y\in\Omega$ is the fundamental solution for the partial differential operator $\partial_t-D_3\Delta+\mu(t,\cdot)+\Upsilon_L(t,\cdot)+d_L(t,\cdot)$ [18, Chapter 1]. We then can verify that $\Gamma(t,s,x,y)=\Gamma(t+\omega,s+\omega,x,y)$ for $t>s\geq 0$, and $x,y\in\Omega$ since $\mu(t+\omega,\cdot)=\mu(t,\cdot)$, $\Upsilon_L(t+\omega,\cdot)=\Upsilon_L(t,\cdot)$ and $d_L(t+\omega,\cdot)=d_L(t,\cdot)$ for $t\geq 0$. Notice $I_1(t,h,x)=\overline{s}(t-h,h,x)$. Then, for $h=\tau$ and $\forall\ t\geq\tau$, we have

$$\begin{aligned} &I_1(t,\tau,x) \\ &=\int_\Omega \Gamma(t,t-\tau,x,y)\times\left[\beta_h(t-\tau,y)+\frac{\beta_H(t-\tau,y)B_H(t-\tau,y)}{k_H+B_H(t-\tau,y)}+\frac{\beta_L(t-\tau,y)B_L(t-\tau,y)}{k_L+B_L(t-\tau,y)}\right] \\ &\times\left[S(t-\tau,y)+\sigma V(t-\tau,y)\right]\mathrm{d}y. \end{aligned}$$

Since I_m in model (2.2) can be decoupled from the other equations, we now reach our main model governed by the following system (2.4) of time periodic reaction–diffusion equations to describe the cholera transmission with latent period. It is then followed by exploring its threshold dynamics and stability in Sects. 3 and 4, respectively.

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_1 \Delta S(t,x) + A(t,x) - [\varpi(t,x) + \mu(t,x)] S(t,x) \\ \qquad - \left[\beta_h(t,x) I(t,x) + \dfrac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x), \ t>0, \ x \in \Omega, \\ \dfrac{\partial V(t,x)}{\partial t} = D_2 \Delta V(t,x) + \varpi(t,x) S(t,x) \\ \qquad -\sigma \left[\beta_h(t,x) I(t,x) + \dfrac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] V(t,x) - \mu(t,x) V(t,x), \ t>0, \ x \in \Omega, \\ \dfrac{\partial I(t,x)}{\partial t} = D_3 \Delta I(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)] I(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y) \left[\beta_h(t-\tau,y) I(t-\tau,y) + \dfrac{\beta_H(t-\tau,y) B_H(t-\tau,y)}{k_H + B_H(t-\tau,y)} + \dfrac{\beta_L(t-\tau,y) B_L(t-\tau,y)}{k_L + B_L(t-\tau,y)}\right] \\ \qquad \times [S(t-\tau,y) + \sigma V(t-\tau,y)] \, \mathrm{d}y, \ t>0, \ x \in \Omega, \\ \dfrac{\partial R(t,x)}{\partial t} = D_4 \Delta R(t,x) + \Upsilon_I(t,x) I(t,x) - \mu(t,x) R(t,x), \ t>0, \ x \in \Omega, \\ \dfrac{\partial B_H(t,x)}{\partial t} = D_5 \Delta B_H(t,x) + \xi(t,x) I(t,x) - \chi(t,x) B_H(t,x), \ t>0, \ x \in \Omega, \\ \dfrac{\partial B_L(t,x)}{\partial t} = D_6 \Delta B_L(t,x) + \chi(t,x) B_H(t,x) - \delta_L(t,x) B_L(t,x), \ t>0, \ x \in \Omega, \\ \dfrac{\partial S(t,x)}{\partial \nu} = \dfrac{\partial V(t,x)}{\partial \nu} = \dfrac{\partial I(t,x)}{\partial \nu} = \dfrac{\partial R(t,x)}{\partial \nu} = \dfrac{\partial B_H(t,x)}{\partial \nu} = \dfrac{\partial B_L(t,x)}{\partial \nu} = 0, \ t>0, \ x \in \partial\Omega. \end{cases} \tag{2.4}$$

3. Threshold dynamics

For the convenience of discussion in the rest study, we introduce some commonly used notations in the literature: Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R}^6)$ denote the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. For $\tau \geq 0$, we define $C_\tau := C([-\tau, 0], \mathbb{X})$ with the norm $\|\cdot\| := \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}$, $\phi \in C_\tau$. Define $\mathbb{X}^+ := C(\overline{\Omega}, \mathbb{R}^6_+)$ and $C_\tau^+ = C([-\tau, 0], \mathbb{X}^+)$. Then $(\mathbb{X}, \mathbb{X}^+)$ and (C_τ, C_τ^+) are strongly ordered spaces. For $\sigma > 0$ and a function $\gamma : [-\tau, \sigma) \mapsto \mathbb{X}$, we define $\gamma_t \in C_\tau$ by

$$\gamma_t(\theta) = \gamma(t+\theta) = \Big(\gamma_1(t+\theta), \gamma_2(t+\theta), \gamma_3(t+\theta), \gamma_4(t+\theta), \gamma_5(t+\theta), \gamma_6(t+\theta)\Big), \\ \theta \in [-\tau, 0], \ t \in [0, \sigma).$$

3.1. The Well-posedness

Let $\mathbb{Y} := C(\overline{\Omega}, \mathbb{R})$ and $\mathbb{Y}^+ := C(\overline{\Omega}, \mathbb{R}_+)$. We assume that $T_i(t, s)$ $(i = 1, \ldots, 6) : \mathbb{Y} \to \mathbb{Y}$, are, respectively, the evolution operators associated with

$$\begin{aligned} \frac{\partial S(t,x)}{\partial t} &= D_1 \Delta S(t,x) - [\varpi(t,x) + \mu(t,x)] S(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial V(t,x)}{\partial t} &= D_2 \Delta V(t,x) - \mu(t,x) V(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial I(t,x)}{\partial t} &= D_3 \Delta I(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)] I(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial R(t,x)}{\partial t} &= D_4 \Delta R(t,x) - \mu(t,x) R(t,x), \ t>0, \ x \in \Omega, \end{aligned}$$

$$\frac{\partial B_H(t,x)}{\partial t} = D_5\Delta B_H(t,x) - \chi(t,x) B_H(t,x), \ \ t>0, \ x\in\Omega,$$
$$\frac{\partial B_L(t,x)}{\partial t} = D_6\Delta B_L(t,x) - \delta_L(t,x) B_L(t,x), \ \ t>0, \ x\in\Omega,$$

subject to the zero-flux boundary conditions. By the assumption (**H**), and using [14, Lemma 6.1] for $t \geq s$, we then have $T_i(t+\omega, s+\omega) = T_i(t, s)$ for $(t, s) \in \mathbb{R}^2$ with $i = 1, \ldots, 6$. According to [22, Chapter II], for $(t, s) \in \mathbb{R}^2$ with $t > s$, $T_i(t, s)$ $(i = 1, \ldots, 6)$ are compact and strongly positive. Further, denote

$$T(t, s) = \operatorname{diag}(T_1(t, s), T_2(t, s), T_3(t, s), T_4(t, s), T_5(t, s), T_6(t, s)).$$

Then, $T(t, s) : \mathbb{X} \to \mathbb{X}$ is an evolution operator for $(t, s) \in \mathbb{R}^2$ with $t \geq s$.

Define $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6)$: $[0, +\infty) \times C_\tau^+ \to \mathbb{X}$ by

$$\begin{aligned}
\Sigma_1(t,\phi) &= A(t,\cdot) - \left[\beta_h(t,\cdot)\phi_3(0,\cdot) + \frac{\beta_H(t,\cdot)\phi_5(0,\cdot)}{k_H+\phi_5(0,\cdot)} + \frac{\beta_L(t,\cdot)\phi_6(0,\cdot)}{k_L+\phi_6(t,\cdot)}\right]\phi_1(0,\cdot),\\
\Sigma_2(t,\phi) &= \varpi(t,\cdot)\phi_1(0,\cdot) - \sigma\left[\beta_h(t,\cdot)\phi_3(0,\cdot) + \frac{\beta_H(t,\cdot)\phi_5(0,\cdot)}{k_H+\phi_5(0,\cdot)} + \frac{\beta_L(t,\cdot)\phi_6(0,\cdot)}{k_L+\phi_6(0,\cdot)}\right]\phi_2(0,\cdot),\\
\Sigma_3(t,\phi) &= \int_\Omega \Gamma(t,t-\tau,\cdot,y)\left[\beta_h(t-\tau,y)\phi_3(-\tau,y) + \frac{\beta_H(t-\tau,y)\phi_5(-\tau,y)}{k_H+\phi_5(-\tau,y)} + \frac{\beta_L(t-\tau,y)\phi_6(-\tau,y)}{k_L+\phi_6(-\tau,y)}\right]\\
&\quad\times[\phi_1(-\tau,y)+\sigma\phi_2(-\tau,y)]\,\mathrm{d}y,\\
\Sigma_4(t,\phi) &= \Upsilon_I(t,\cdot)\phi_3(0,\cdot),\\
\Sigma_5(t,\phi) &= \xi(t,\cdot)\phi_3(0,\cdot),\\
\Sigma_6(t,\phi) &= \chi(t,\cdot)\phi_5(0,\cdot),
\end{aligned}$$

for $t \geq 0$, $x \in \overline{\Omega}$, and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in C_\tau^+$. Then, model (2.4) can be rewritten as

$$\begin{aligned}
\frac{\partial \Pi(t,x)}{\partial t} &= \mathbb{K}(t)\Pi(t,x) + \Sigma(t,\Pi_t), \ \ t>0, \ x\in\Omega,\\
\Pi(\theta,x) &= \phi(\theta,x), \ \ \theta\in[-\tau,0], \ x\in\Omega,
\end{aligned} \tag{3.1}$$

where $\Pi(t, x) = (\Pi_1(t, x), \Pi_2(t, x), \Pi_3(t, x), \Pi_4(t, x), \Pi_5(t, x), \Pi_6(t, x))$, and $\mathbb{K}(t) = \operatorname{diag}(\mathbb{K}_1(t), \mathbb{K}_2(t), \mathbb{K}_3(t), \mathbb{K}_4(t), \mathbb{K}_5(t), \mathbb{K}_6(t))$. Here $\mathbb{K}_i(t)$ $(i = 1, 2, 3, 4, 5, 6)$ are given by:

$$\begin{aligned}
D(\mathbb{K}_i(t)) &= \left\{\varphi \in C^2(\overline{\Omega}) : \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\right\},\\
\mathbb{K}_1(t)\varphi &= D_1\Delta\varphi - [\varpi(t,x)+\mu(t,x)]\varphi, \ \ \varphi\in D(\mathbb{K}_1(t)),\\
\mathbb{K}_2(t)\varphi &= D_2\Delta\varphi - \mu(t,x)\varphi, \ \ \varphi\in D(\mathbb{K}_2(t)),\\
\mathbb{K}_3(t)\varphi &= D_3\Delta\varphi - [\mu(t,x)+\Upsilon_I(t,x)+d_I(t,x)]\varphi, \ \ \varphi\in D(\mathbb{K}_3(t)),\\
\mathbb{K}_4(t)\varphi &= D_4\Delta\varphi - \mu(t,x)\varphi, \ \ \varphi\in D(\mathbb{K}_4(t)),\\
\mathbb{K}_5(t)\varphi &= D_5\Delta\varphi - \chi(t,x)\varphi, \ \ \varphi\in D(\mathbb{K}_5(t)),\\
\mathbb{K}_6(t)\varphi &= D_6\Delta\varphi - \delta_L(t,x)\varphi, \ \ \varphi\in D(\mathbb{K}_6(t)).
\end{aligned}$$

Theorem 3.1. *For any $\phi \in C_\tau^+$, model (2.4) admits a unique mild solution $z(t, \cdot; \phi)$ on its maximal existence interval $\left[0, \widetilde{t}_\phi\right)$ with $z_0 = \phi$, where $\widetilde{t}_\phi \leq +\infty$. Moreover, $z(t, \cdot; \phi) \in \mathbb{X}^+$, and $z(t, \cdot; \phi)$ is a classical solution of model (2.4) for $t > \tau$.*

Proof. By definition of Σ, we can easily verify that it is locally Lipschitz continuous. The conditions (H_1)–(H_2) in [30, Corollary 4] are obviously satisfied due to $D = \mathbb{X}^+$. Indeed, by [30, Remark 2.2], the condition (H_3) in [30, Corollary 4] is automatically satisfied since $\mathbb{X}^+$ is convex. By the definition of the continuity, the condition (H_4) is satisfied. Then, by [30, Corollary 4], it suffices to prove

$$\lim_{\theta_1\to 0^+}\frac{1}{\theta_1}\text{dist}\left(\phi(0,\,\cdot)+\theta_1\Sigma(t,\,\phi),\,\mathbb{X}^+\right)=0,\ \ (t,\,\phi)\in[0,\,+\infty)\times C_\tau^+.$$

For $(t,\,\phi)\in[0,\,+\infty)\times C_\tau^+$ and $\theta_1\geq 0$, one gets

$$\phi(0,\,x)+\theta_1\Sigma(t,\,\phi)(x)\geq\begin{pmatrix}\left[1-\theta_1\left(\beta_h\left(t,x\right)\phi_3\left(0,x\right)+\dfrac{\beta_H\left(t,x\right)\phi_5\left(0,x\right)}{k_H+\phi_5\left(0,x\right)}+\dfrac{\beta_L\left(t,x\right)\phi_6\left(0,x\right)}{k_L+\phi_6\left(0,x\right)}\right)\right]\phi_1\left(0,x\right)\\ \left[1-\theta_1\left(\dfrac{\sigma\beta_H\left(t,x\right)\phi_5\left(0,x\right)}{k_H+\phi_5\left(0,x\right)}+\dfrac{\sigma\beta_L\left(t,x\right)\phi_6\left(0,x\right)}{k_L+\phi_6\left(0,x\right)}\right)\right]\phi_2\left(0,x\right)\\ \phi_3(0,\,x)\\ \phi_4(0,\,x)\\ \phi_5(0,\,x)\\ \phi_6(0,\,x)\end{pmatrix}.$$

If θ_1 is small enough, then $\phi(0)+\theta_1\Sigma(t,\,\phi)\in\mathbb{X}^+$. It follows from [30, Corollary 4] that there admits a unique mild solution $z(t,\,\cdot;\,\phi)$ with $z_0=\phi$ on $t\in\left[0,\,\widetilde{t}_\phi\right)$ for model (2.4), where $\widetilde{t}_\phi\leq+\infty$, and $z(t,\,\cdot;\,\phi)\in\mathbb{X}^+$. In addition, by the analytic of $T(t,\,s)$, $t,\,s\in\mathbb{R}$ and $t>s$, $z(t,\,\cdot;\,\phi)$ is a classical solution for $t>\tau$. □

Let

$$C_\tau^{++}:=C\left(\left[-\tau,\,0\right],\mathbb{Y}^+\right)\times C\left(\left[-\tau,\,0\right],\mathbb{Y}^+\right)\times C\left(\left[-\tau,\,0\right],\mathbb{Y}^+\right)\times\mathbb{Y}^+\times C\left(\left[-\tau,\,0\right],\mathbb{Y}^+\right)\times C\left(\left[-\tau,\,0\right],\mathbb{Y}^+\right).$$

For any given $\varphi\in C_\tau^{++}$, we define $\widetilde{\varphi}=(\varphi_1,\,\varphi_2,\,\varphi_3,\,\widetilde{\varphi}_4,\,\varphi_5,\,\varphi_6)$, where $\widetilde{\varphi}_4(\theta,\,\cdot)=\varphi_4(\cdot)\in\mathbb{Y}^+$ and $\theta\in[-\tau,\,0]$. Then $\widetilde{\varphi}\in C_\tau^+$. By the uniqueness of solutions, $\Pi(t,\,\cdot;\,\varphi)=z(t,\,\cdot;\,\widetilde{\varphi})$ for $t\geq 0$. It follows from Theorem 3.1 that there exists a unique solution $\Pi(t,\,\cdot;\,\varphi)$ of model (2.4) and $\Pi_0=\varphi$ on $\left[0,\,\widetilde{t}_\varphi\right)$, where

$$\begin{aligned}&\Pi_t(\varphi)(\theta,\,x)\\&=(\Pi_1(t+\theta,\,x;\,\varphi),\,\Pi_2(t+\theta,\,x;\,\varphi),\,\Pi_3(t+\theta,\,x;\,\varphi),\,\Pi_4(t,\,x;\,\varphi),\,\Pi_5(t+\theta,\,x;\,\varphi),\,\varUpsilon_6(t+\theta,\,x;\,\varphi)),\end{aligned}$$

for $t\geq 0$ and $(\theta,\,x)\in[-\tau,\,0]\times\overline{\Omega}$.

Denote

$$\begin{aligned}A_0&=\max_{t\in[0,\omega]}\int_\Omega A\left(t,x\right)\mathrm{d}x,\ \underline{\mu}=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}\mu\left(t,\,x\right),\ \underline{\Upsilon}_I=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}\Upsilon_I\left(t,\,x\right),\ \underline{d}_I=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}d_I\left(t,\,x\right),\\ \underline{\varpi}&=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}\varpi\left(t,\,x\right),\ \overline{\varpi}=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\varpi\left(t,\,x\right),\ \overline{\Upsilon}_I=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\Upsilon_I\left(t,\,x\right),\ \overline{\xi}=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\xi\left(t,\,x\right),\\ \overline{\chi}&=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\chi\left(t,\,x\right),\ \underline{\chi}=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}\chi\left(t,\,x\right),\ \underline{\delta}_L=\min_{t\in[0,\omega],\,x\in\overline{\Omega}}\delta_L\left(t,\,x\right),\ \overline{\beta}_h=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\beta_h\left(t,\,x\right),\\ \overline{\beta}_H&=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\beta_H\left(t,\,x\right),\ \overline{\beta}_L=\max_{t\in[0,\omega],\,x\in\overline{\Omega}}\beta_L\left(t,\,x\right).\end{aligned}$$

Theorem 3.2. *For any $\varphi\in C_\tau^{++}$, model (2.4) admits a unique solution $\Pi(t,\,\cdot;\,\varphi)$ on $[0,\,+\infty)$ with $\Pi_0=\varphi$. In addition, model (2.4) generates an ω-periodic semiflow $W_t:=\Pi_t(\cdot):\,C_\tau^{++}\to C_\tau^{++}$, i.e. $W_t(\varphi)(s,\,x)=\Pi(t+s,\,x;\,\varphi)$ for $\varphi\in C_\tau^{++},\ t\geq 0,\ s\in[-\tau,\,0]$ and $x\in\Omega$, and $W_\omega:\,C_\tau^{++}\to C_\tau^{++}$ has a global attractor in C_τ^{++}.*

Proof. From the first equation of model (2.4), by the comparison theorem, we have that $S\left(t,x;\varphi\right)$ is bounded by $K_1>0$ on $\left[0,\widetilde{t}_\varphi\right)$. Similarly, $V\left(t,x;\varphi\right)$ is bounded by $K_2>0$ on $\left[0,\widetilde{t}_\varphi\right)$. Hence, the $I\left(t,x\right)$,

$R(t,x)$, $B_H(t,x)$ and $B_L(t,x)$ equations of model (2.4) are dominated by the following linear reaction–diffusion system:

$$\begin{cases} \dfrac{\partial I(t,x)}{\partial t} = D_3\Delta I - \left[\underline{\mu} + \underline{\Upsilon}_I + \underline{d}_I\right] I(t,x) + \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\cdot\left[\overline{\beta}_h I(t-\tau,y) + \overline{\beta}_H + \overline{\beta}_L\right]\cdot \\ \qquad\qquad (K_1+\sigma K_2)\,\mathrm{d}y,\ t>0,\ x\in\Omega, \\ \dfrac{\partial R(t,x)}{\partial t} = D_4\Delta R + \overline{\Upsilon}_I I(t,x) - \underline{\mu} R(t,x),\ t>0,\ x\in\Omega, \\ \dfrac{\partial B_H(t,x)}{\partial t} = D_5\Delta B_H + \overline{\xi} I(t,x) - \underline{\chi} B_H(t,x),\ t>0,\ x\in\Omega, \\ \dfrac{\partial B_L(t,x)}{\partial t} = D_6\Delta B_L + \overline{\chi} B_H(t,x) - \underline{\delta}_L B_L(t,x),\ t>0,\ x\in\Omega, \\ \dfrac{\partial I(t,x)}{\partial \nu} = \dfrac{\partial R(t,x)}{\partial \nu} = \dfrac{\partial B_H(t,x)}{\partial \nu} = \dfrac{\partial B_L(t,x)}{\partial \nu} = 0,\ t>0,\ x\in\partial\Omega. \end{cases} \tag{3.2}$$

By the global existence of solutions of the linear system (3.2). (see, e.g. [54, Theorem 2.1.1]), we get $\widetilde{t}_\varphi = +\infty$ for each $\varphi \in C_\tau^{++}$.

Since $\frac{\partial S}{\partial t} \le D_1\Delta S + A(t,x) - [\phi(t,x) + \mu(t,x)]S$, there exists a constant $B_1 > 0$ such that for any $\varphi \in C_\tau^{++}$, there is an integer $L_1 = L_1(\varphi) > 0$ satisfying $S(t,x;\varphi) \le B_1$, for $t \ge L_1\omega$ and $x \in \bar{\Omega}$. Similarly, there exists a constant $B_2 > 0$ such that for any $\varphi \in C_\tau^{++}$, there is an integer $L_2 = L_2(\varphi) > 0$ satisfying $V(t,x;\varphi) \le B_2$, for $t \ge L_2\omega$ and $x \in \bar{\Omega}$. In what follows, using similar arguments to [42, Theorem 2.1], we show that the solution of model (2.4) is ultimately bounded. Let $\bar{S}(t) = \int_\Omega S(t,x)\,\mathrm{d}x$, $\bar{V}(t) = \int_\Omega V(t,x)\,\mathrm{d}x$ and $\bar{I}(t) = \int_\Omega I(t,x)\,\mathrm{d}x$. By integrating the first equation of model (2.4), it follows from Green's formula that

$$\begin{aligned} \frac{\mathrm{d}\bar{S}(t)}{\mathrm{d}t} &= \int_\Omega A(t,x)\,\mathrm{d}x - \int_\Omega [\varpi(t,x) + \mu(t,x)]\,S(t,x)\,\mathrm{d}x \\ &\quad - \int_\Omega \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x)\,\mathrm{d}x \\ &\le A_0 - \left(\underline{\varpi} + \underline{\mu}\right)\overline{S}(t) - \int_\Omega \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x)\,\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x)\,\mathrm{d}x \\ &\qquad \le A_0 - \left(\underline{\varpi} + \underline{\mu}\right)\overline{S}(t) - \frac{\mathrm{d}\overline{S}(t)}{\mathrm{d}t}, \quad t > 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}\bar{V}(t)}{\mathrm{d}t} &= \int_\Omega \varpi(t,x)\,S(t,x)\,\mathrm{d}x - \int_\Omega \mu(t,x)\,V(t,x)\,\mathrm{d}x \\ &\quad - \int_\Omega \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] \sigma V(t,x)\,\mathrm{d}x \\ &\le B_1\overline{\varpi}\,|\Omega| - \underline{\mu}\overline{V} - \int_\Omega \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] \sigma V(t,x)\,\mathrm{d}x. \end{aligned}$$

Thus, we have

$$\frac{\mathrm{d}\bar{S}(t)}{\mathrm{d}t}+\frac{\mathrm{d}\bar{V}(t)}{\mathrm{d}t}\le A_0+B_1\overline{\varpi}|\Omega|-\left(\underline{\varpi}+\underline{\mu}\right)\overline{S}(t)-\underline{\mu}\overline{V}(t)$$
$$-\int_{\Omega}\left[\beta_h(t,x)I(t,x)+\frac{\beta_H(t,x)B_H(t,x)}{k_H+B_H(t,x)}+\frac{\beta_L(t,x)B_L(t,x)}{k_L+B_L(t,x)}\right](S+\sigma V(t,x))\,\mathrm{d}x.$$

Then,

$$\int_{\Omega}\left[\beta_h(t,x)I(t,x)+\frac{\beta_H(t,x)B_H(t,x)}{k_H+B_H(t,x)}+\frac{\beta_L(t,x)B_L(t,x)}{k_L+B_L(t,x)}\right](S+\sigma V(t,x))\,\mathrm{d}x$$
$$\le A_0+B_1\overline{\varpi}|\Omega|-\left(\underline{\varpi}+\underline{\mu}\right)\overline{S}(t)-\underline{\mu}\overline{V}(t)-\frac{\mathrm{d}\overline{S}(t)}{\mathrm{d}t}-\frac{\mathrm{d}\overline{V}(t)}{\mathrm{d}t}.$$

By the property of the fundamental solutions [18], integrating the third equation of model (2.4) yields

$$\frac{\mathrm{d}\bar{I}(t)}{\mathrm{d}t}\le-\left(\underline{\mu}+\underline{\Upsilon}_I+\underline{d}_I\right)\overline{I}(t)-k_1\left[\overline{S}(t-\tau)+\overline{V}(t-\tau)\right]-k_2\left[\frac{\mathrm{d}\overline{S}(t-\tau)}{\mathrm{d}t}+\frac{\mathrm{d}\overline{V}(t-\tau)}{\mathrm{d}t}\right]$$
$$+k_3,\ \forall t\ge(L_1+L_2)\omega+\tau,$$

where k_1, k_2 and k_3 are positive constants independent of φ. Choose $k_1\le\left(\underline{\mu}+\underline{\Upsilon}_I+\underline{d}_I\right)k_2$, one gets

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{I}(t)+k_2\left(\overline{S}(t-\tau)+\overline{V}(t-\tau)\right)\right]\\&\le-\left(\underline{\mu}+\underline{\Upsilon}_I+\underline{d}_I\right)\overline{I}(t)-k_1\left[\overline{S}(t-\tau)+\overline{V}(t-\tau)\right]+k_3\\&\le-\frac{k_1}{k_2}\overline{I}(t)-k_1\left[\overline{S}(t-\tau)+\overline{V}(t-\tau)\right]+k_3\\&\le-\frac{k_1}{k_2}\left[\overline{I}(t)+k_2\left(\overline{S}(t-\tau)+\overline{V}(t-\tau)\right)\right]+k_3,\end{aligned}$$

which yields $\overline{I}(t)+k_2\left[\overline{S}(t-\tau)+\overline{V}(t-\tau)\right]\le\frac{k_2k_3}{k_1}+1$, for $t\ge L_1'\omega+\tau$, $L_1'>L_1+L_2$. Since $\Gamma(t,t-\tau,x,y)$, S and V are bounded, then

$$\frac{\partial I}{\partial t}\le D_3\Delta I-\left[\underline{\mu}+\underline{\Upsilon}_I+\underline{d}_I\right]I(t,x)+c\overline{I}(t)+M,$$

where $c>0$ and $M>0$ are constants. According to the standard parabolic maximum principle, there is a constant $B_3>0$ independent of φ, and an integer $L_3=L_3(\varphi)>(L_1+L_2)(\varphi)$, such that $I(t,x,\varphi)\le B_3$, for $t\geqslant L_3\omega+\tau$, $x\in\overline{\Omega}$.

From the last three equations of model (2.4), we obtain

$$\begin{cases}\dfrac{\partial R(t,x)}{\partial t}\le D_4\Delta R(t,x)+\overline{\Upsilon}_I I(t,x)-\underline{\mu}R(t,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial B_H(t,x)}{\partial t}\le D_5\Delta B_H(t,x)+\overline{\xi}I(t,x)-\underline{\chi}B_H(t,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial B_L(t,x)}{\partial t}\le D_6\Delta B_L(t,x)+\overline{\chi}B_H(t,x)-\underline{\delta}_L B_L(t,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial R(t,x)}{\partial\nu}=\dfrac{\partial B_H(t,x)}{\partial\nu}=\dfrac{\partial B_L(t,x)}{\partial\nu}=0, & t>0,\ x\in\partial\Omega.\end{cases}$$

There exists a constant $B_4>0$ such that for any $\varphi\in C_\tau^{++}$, there is an integer $L_4=L_4(\varphi)>0$ satisfying $R(t,x;\varphi)\le B_4$, for $t\ge L_4\omega$ and $x\in\bar{\Omega}$. There is a constant $B_5>0$ such that for any $\varphi\in C_\tau^{++}$, there is an integer $L_5=L_5(\varphi)>0$ satisfying $B_H(t,x;\varphi)\le B_5$, for $t\ge L_5\omega$ and $x\in\bar{\Omega}$. There is a constant $B_6>0$ such that for any $\varphi\in C_\tau^{++}$, there is an integer $L_6=L_6(\varphi)>0$ satisfying $B_L(t,x;\varphi)\le B_6$, for $t\ge L_6\omega$ and $x\in\bar{\Omega}$.

We now define a family of operators $\{W_t\}_{t\geq 0}$ on C_τ^{++} by $W_t(\varphi)(s, x) = \Pi_t(s, x; \varphi) = \Pi(t+s, x; \varphi)$ for $t \geq 0$, $s \in [-\tau, 0]$, $x \in \overline{\Omega}$, and $\varphi \in C_\tau^{++}$. From the above proofs, $W_t : C_\tau^{++} \to C_\tau^{++}$ is point dissipative. Similar to the proof of [61, Lemma 2.1], we can show that $W_{t\geq 0}$ is an ω-periodic semiflow on C_τ^{++}. Furthermore, from [54, Theorem 2.1.8], $W_t : C_\tau^{++} \to C_\tau^{++}$ is compact for each $t > \tau$. In view of [29, Theorem 2.9], we deduce that $W_\omega : C_\tau^{++} \to C_\tau^{++}$ admits a global attractor in C_τ^{++}. □

3.2. Basic reproduction number

Let $\mathbb{M} = C(\overline{\Omega}, \mathbb{R}^3)$ and $\mathbb{M}^+ = C(\overline{\Omega}, \mathbb{R}_+^3)$. Let $C_\omega(\mathbb{R}, \mathbb{M})$ be the Banach space for all ω-periodic functions with $\|\psi\|_{C_\omega(\mathbb{R}, \mathbb{M})} := \max_{\theta\in[0,\omega]} \|\psi(\theta)\|_{\mathbb{M}}$ from $\mathbb{R}$ to $\mathbb{M}$ for $\psi \in C_\omega(\mathbb{R}, \mathbb{M})$.

Consider the following periodic reaction–diffusion equation:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_1\Delta S(t,x) + A(t,x) - [\varpi(t,x) + \mu(t,x)]\, S(t,x), \quad t>0,\ x\in\Omega, \\ \dfrac{\partial S(t,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{3.3}$$

It follows from [61, Lemma 2.1] that model (3.3) has a unique positive ω-periodic solution $S^*(t, \cdot)$, which is globally attractive in $\mathbb{Y}^+$. Then, the second equation of model (2.4) is asymptotic to

$$\begin{cases} \dfrac{\partial V(t,x)}{\partial t} = D_2\Delta V(t,x) + \varpi(t,x)\, S^*(t,x) - \mu(t,x)\, V(t,x), \quad t>0,\ x\in\Omega, \\ \dfrac{\partial V(t,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{3.4}$$

Similarly, model (3.4) has a unique positive ω-periodic solution $V^*(t, \cdot)$, which is globally attractive in $\mathbb{Y}^+$. Setting $I = R = B_H = B_L = 0$ in model (2.4), we have

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_1\Delta S(t,x) + A(t,x) - [\varpi(t,x) + \mu(t,x)]\, S(t,x), \ t>0,\ x\in\Omega, \\ \dfrac{\partial V(t,x)}{\partial t} = D_2\Delta V(t,x) + \varpi(t,x)\, S(t,x) - \mu(t,x)\, V(t,x), \ t>0,\ x\in\Omega, \\ \dfrac{\partial S(t,x)}{\partial \nu} = \dfrac{\partial V(t,x)}{\partial \nu} = 0, \ t>0,\ x\in\partial\Omega. \end{cases} \tag{3.5}$$

Hence, by [61, Lemma 2.1] and the theory of chain transitive sets [59], there is a globally attractive positive ω-periodic solution $(S^*(t, \cdot), V^*(t, \cdot))$ of model (3.5) in $C(\overline{\Omega}, \mathbb{R}_+^2)$. Consequently, model (2.4) has a unique infection-free periodic solution $E_0 = (S^*(t,\cdot), V^*(t,\cdot), 0, 0, 0, 0)$.

Linearizing model (2.4) at E_0, we get the infective compartments:

$$\begin{cases} \dfrac{\partial q_3(t,x)}{\partial t} = D_3\Delta q_3(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)]\, q_3(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y)\, [S^*(t-\tau, y) + \sigma V^*(t-\tau, y)] \\ \qquad \times \left[\beta_h(t-\tau, y)\, q_3(t-\tau, y) + \dfrac{\beta_H(t-\tau, y)}{k_H} q_5(t-\tau, y) + \dfrac{\beta_L(t-\tau, y)}{k_L} q_6(t-\tau, y)\right] \mathrm{d}y, \\ \qquad t>0,\ x\in\Omega, \\ \dfrac{\partial q_5(t,x)}{\partial t} = D_5\Delta q_5(t,x) + \xi(t,x)\, q_3(t,x) - \chi(t,x)\, q_5(t,x), \ t>0,\ x\in\Omega, \\ \dfrac{\partial q_6(t,x)}{\partial t} = D_6\Delta q_6(t,x) + \chi(t,x)\, q_5(t,x) - \delta_L(t,x)\, q_6(t,x), \ t>0,\ x\in\Omega, \\ \dfrac{\partial q_3(t,x)}{\partial \nu} = \dfrac{\partial q_5(t,x)}{\partial \nu} = \dfrac{\partial q_6(t,x)}{\partial \nu} = 0, \ t>0,\ x\in\partial\Omega. \end{cases} \tag{3.6}$$

To introduce the basic reproduction number, we define

$$\widehat{\Sigma}(t)(\phi_3, \phi_5, \phi_6)^T$$

$$= \begin{pmatrix} \int\limits_{\Omega} \Gamma\left(t,\, t-\tau,\, \cdot,\, y\right)\left[S^*\left(t-\tau,\, y\right)+\sigma V^*\left(t-\tau,\, y\right)\right] \\ \times\left[\beta_h\left(t-\tau,\, y\right)\phi_3\left(-\tau,\, y\right)+\frac{\beta_H(t-\tau,\, y)}{k_H}\phi_5\left(-\tau,\, y\right)+\frac{\beta_L(t-\tau,\, y)}{k_L}\phi_6\left(-\tau,\, y\right)\right]\mathrm{d}y \\ \xi(t,\, \cdot)\phi_3(0,\, \cdot) \\ \chi(t,\, \cdot)\phi_5(0,\, \cdot) \end{pmatrix},$$

for $t \in \mathbb{R}$, $(\phi_3,\, \phi_5,\, \phi_6) \in C([-\tau,\, 0],\, \mathbb{M})$, and

$$-\mathbb{V}(t)q = D\Delta q - \Theta(t)q,$$

where $D = \mathrm{diag}(D_3,\, D_5,\, D_6)$, and for $x \in \overline{\Omega}$,

$$[\Theta(t)]\,(x) = \begin{pmatrix} \mu\left(t,\, x\right)+\Upsilon_I\left(t,\, x\right)+d_I\left(t,\, x\right) & 0 & 0 \\ 0 & \chi\left(t,\, x\right) & 0 \\ 0 & 0 & \delta_L\left(t,\, x\right) \end{pmatrix}.$$

Assume that $\Lambda(t,\, s) = \mathrm{diag}(T_3(t,\, s),\, T_5(t,\, s),\, T_6(t,\, s))$ $(t \geq s)$ is the evolution operators associated with

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\mathbb{V}(t)q,$$

subject to zero-flux boundary conditions. According to [14, Theorem 6.6], there are $c_0 \in \mathbb{R}$ and $\mathbb{U} \geq 1$ such that

$$\|\Lambda(t,\, s)\| \leq \mathbb{U}e^{c_0(t-s)}, \quad t \geq s,\ t, s \in \mathbb{R},$$

and $\overline{\omega}(\Lambda) \leq c_0$, where $\overline{\omega}(\Lambda)$ is the exponential growth bound of $\Lambda(t,\, s)$, and

$$\overline{\omega}(\Lambda) = \inf\left\{\widetilde{\omega} : \exists\ \mathbb{U} \geq 1 \text{ such that } \parallel \Lambda(t+s, s) \parallel \leq \mathbb{U}e^{\widetilde{\omega}t},\ \forall s \in \mathbb{R},\ t \geq 0\right\}.$$

By [22, Lemma 14.2 and Krein–Rutman Theorem], it is easy to see that

$$0 < r(\Lambda(\omega,\, 0)) = \max\left\{r\left(T_3(\omega,\, 0\right),\, r\left(T_5(\omega,\, 0)\right),\, r\left(T_6(\omega,\, 0)\right)\right\} < 1.$$

In view of [40, Proposition 5.6], $\overline{\omega}(\Lambda) < 0$. Thus, each $\widehat{\Sigma}(t)$ maps $C([-\tau,\, 0],\, \mathbb{M}^+)$ into $\mathbb{M}^+$, and each matrix-$\Theta(t)$ is cooperative with $\overline{\omega}(\Lambda) < 0$.

According to [58], from a biological perspective, we assume that $q(s,\, x) = q(s)(x) \in C_\omega(\mathbb{R},\, \mathbb{M})$ is the initial distribution of infected individuals, hyperinfectious vibrios and hypoinfectious vibrios at time $s \in \mathbb{R}$ and location $x \in \overline{\Omega}$. For a given $s \geq 0$, $\widehat{\Sigma}(t-s)q(t-s+\cdot,\, x)$ denotes the distribution of newly infected individuals, hyperinfectious vibrios and hypoinfectious vibrios at time $t-s$ $(t > s)$ and location x, which is generated by infected individuals, hyperinfectious vibrios and hypoinfectious vibrios who were introduced over $[t-s-\tau,\, t-s]$. Hence, $\Lambda(t,\, t-s)\widehat{\Sigma}(t-s)q(t-s+\cdot,\, x)$ stands for the distribution of those infected individuals, hyperinfectious vibrios and hypoinfectious vibrios who were newly infected at time $t-s$ and still survive at time t for $t \geq s$. Thus,

$$\int\limits_0^\infty \Lambda(t,\, t-s)\widehat{\Sigma}(t-s)q(t-s+\cdot,\, x)\mathrm{d}s$$

is the total distribution of infected individuals, hyperinfectious vibrios and hypoinfectious vibrios at time t and location x, which are produced by all those infected individuals, hyperinfectious vibrios and hypoinfectious vibrios introduced at all previous time to t.

We define two linear next generation operators on $C_\omega(\mathbb{R},\, \mathbb{M})$ are

$$[Lq](t) := \int\limits_0^\infty \Lambda(t,\, t-s)\widehat{\Sigma}(t-s)q(t-s+\cdot)\mathrm{d}s,\ \ t \in \mathbb{R},$$

and

$$\left[\widehat{L}q\right](t) := \widehat{\Sigma}(t)\left(\int_0^\infty \Lambda(t+\cdot,\, t-s+\cdot)q(t-s+\cdot)\mathrm{d}s\right),\ \ t\in\mathbb{R}.$$

Let $\mathbb{A}$ and $\mathbb{B}$ be two bounded linear operators on $C_\omega(\mathbb{R},\,\mathbb{M})$, denoted by

$$[\mathbb{A}q](t) = \int_0^\infty \Lambda(t,\, t-s)q(t-s)\mathrm{d}s,$$

and

$$[\mathbb{B}q](t) = \widehat{\Sigma}(t)q_t, \quad \forall t\in\mathbb{R},\ q\in C_\omega(\mathbb{R},\,\mathbb{M}).$$

We see that $L=\mathbb{A}\circ\mathbb{B}$ and $\widehat{L}=\mathbb{B}\circ\mathbb{A}$, and L and $\widehat{L}$ have the same spectral radius. By [3,16,40,46,58], the basic reproduction number for model (2.4) is

$$\mathscr{R}_0 := r(L),$$

where $r(L)$ is the spectral radius of L, and $\mathscr{R}_0 := r(L) = r\left(\widehat{L}\right)$. It is known that $\mathscr{R}_0$ plays a crucial role in the prevention, intervention and control on the spread of the cholera transmission. In epidemiology, $\mathscr{R}_0$ is regarded as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual [16,26].

For $t\geq 0$, we assume that $\widehat{\mathbb{P}}(t)$ is the solution map of model (3.6) on $C\left([-\tau,\,0],\,\mathbb{M}^+\right)$. That is, $\widehat{\mathbb{P}}(t)\phi = q_t(\phi)$, where

$$q_t(\phi)(\theta,\,x) = q(t+\theta,\,x;\,\phi) = (q_3(t+\theta,\,x;\,\phi),\,q_5(t+\theta,\,x;\,\phi),\,q_6(t+\theta,\,x;\,\phi)),\ \ (\theta,\,x)\in[-\tau,\,0]\times\overline{\Omega},$$

and $q(t,\,x;\,\phi)$ is the unique solution of model (3.6) with $q(\theta,\,x)=\phi(\theta,\,x)$ for $(\theta,\,x)\in[-\tau,\,0]\times\overline{\Omega}$. So we get that $\widehat{\mathbb{P}} := \widehat{\mathbb{P}}(\omega)$ is the Poincaré map for model (3.6). By using the similar arguments to that of [27], we have $q(t,\,x;\,\phi)\gg 0$ for all $t>\tau$, $x\in\overline{\Omega}$ and $\phi\in C\left([-\tau,\,0],\,\mathbb{M}^+\right)$ with $\phi\not\equiv 0$. Moreover, from [54, Theorem 2.1.8], q_t is compact on $C\left([-\tau,\,0],\,\mathbb{M}^+\right)$ for $t>\tau$. It follows that $\widehat{\mathbb{P}}^n$ is strongly positive and compact for $n\omega > 2\tau$. In terms of [25, Lemma 3.1], $r\left(\widehat{\mathbb{P}}\right)$ is a simple eigenvalue of $\widehat{\mathbb{P}}$ which admits a strongly positive eigenvector $\overline{\phi}$. Set $r\left(\widehat{\mathbb{P}}\right)$ as the spectral radius of $\widehat{\mathbb{P}}$, the following results can be obtained from [58]:

Lemma 3.3. *$\mathscr{R}_0-1$ has the same sign as $r\left(\widehat{\mathbb{P}}\right)-1$.*

3.3. Threshold dynamics

Recall that the Kuratowski measure of noncompactness in C_τ^+ [15] can be defined as

$$\kappa(B) := \inf\{d:\ B \text{ has a finite cover of diameter} <\ d\}$$

for any bounded subset B of C_τ^+.

Using an analogous argument to [4, Lemma 8] (see, also [50, Lemma 3.7]) with minor modifications, the following result is obtained.

Lemma 3.4. *For $r>0$, there exists an equivalent norm $\|\cdot\|_r^*$ on C_τ^+ such that the solution map $\widehat{W}(t) := z_t$ of model (2.4) satisfies $\kappa\left(\widehat{W}(t)B\right)\leq e^{-rt}\kappa(B)$, $t>0$, where κ is the Kuratowski measure of noncompactness in $(C_\tau^+,\,\|\cdot\|_r^*)$.*

Lemma 3.5. *Let $\Pi(t,\,x;\,\varphi)$ be the solution of model (2.4) with $\Pi_0=\varphi\in C_\tau^{++}$. If there is some $t_0\geq 0$ satisfying $I(t_0,\,\cdot;\,\varphi)\not\equiv 0$, $R(t_0,\,\cdot;\,\varphi)\not\equiv 0$, $B_H(t_0,\,\cdot;\,\varphi)\not\equiv 0$ and $B_L(t_0,\,\cdot;\,\varphi)\not\equiv 0$, then $I(t,\,x;\,\varphi)>0$, $R(t,\,x;\,\varphi)>0$, $B_H(t,\,x;\,\varphi)>0$ and $B_L(t,\,x;\,\varphi)>0$ for $t>t_0$ and $x\in\overline{\Omega}$. There holds $S(t,\,x;\,\varphi)>0$ and $V(t,\,x;\,\varphi)>0$ for $t>0$ and $x\in\overline{\Omega}$, $\lim_{t\to+\infty}\inf S(t,\,x;\,\varphi)\geq\vartheta$ and $\lim_{t\to+\infty}\inf V(t,\,x;\,\varphi)\geq\vartheta$ uniformly for $x\in\overline{\Omega}$, where $\vartheta>0$ is a φ-independent constant.*

Proof. Let

$$\begin{aligned}&\overline{\delta}_L=\max_{t\in[0,\,\omega],\,x\in\overline{\Omega}}\delta_L(t,\,x),\ \overline{\mu}=\max_{t\in[0,\,\omega],\,x\in\overline{\Omega}}\mu(t,\,x),\ \overline{\Upsilon}_L=\max_{t\in[0,\,\omega],\,x\in\overline{\Omega}}\Upsilon_L(t,\,x),\\&\overline{d}_I=\max_{t\in[0,\,\omega],\,x\in\overline{\Omega}}d_I(t,\,x),\ \underline{A}=\min_{t\in[0,\,\omega],\,x\in\overline{\Omega}}A(t,\,x).\end{aligned}$$

From model (2.4), one gets

$$\begin{cases}\dfrac{\partial I(t,\,x)}{\partial t}\geq D_3\Delta I(t,\,x)-\left(\overline{\mu}+\overline{\Upsilon}_I+\overline{d}_I\right)I(t,\,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial R(t,\,x)}{\partial t}\geq D_4\Delta R(t,\,x)-\overline{\mu}R(t,\,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial B_H(t,\,x)}{\partial t}\geq D_5\Delta B_H(t,\,x)-\overline{\chi}B_H(t,\,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial B_L(t,\,x)}{\partial t}\geq D_6\Delta B_L(t,\,x)-\overline{\delta}_LB_L(t,\,x), & t>0,\ x\in\Omega,\\ \dfrac{\partial I(t,\,x)}{\partial\nu}=\dfrac{\partial R(t,\,x)}{\partial\nu}=\dfrac{\partial B_H(t,\,x)}{\partial\nu}=\dfrac{\partial B_L(t,\,x)}{\partial\nu}=0, & t>0,\ x\in\partial\Omega.\end{cases}$$

Obviously, if there is some $t_0\geq 0$ satisfying $I(t_0,\,\cdot;\,\varphi)\not\equiv 0$, $R(t_0,\,\cdot;\,\varphi)\not\equiv 0$, $B_H(t_0,\,\cdot;\,\varphi)\not\equiv 0$ and $B_L(t_0,\,\cdot;\,\varphi)\not\equiv 0$, then $I(t,\,x;\,\varphi)>0$, $R(t,\,x;\,\varphi)>0$, $B_H(t,\,x;\,\varphi)>0$ and $B_L(t,\,x;\,\varphi)>0$ for $t>t_0$ and $x\in\overline{\Omega}$.

From Theorem 3.2, there is a constant $B>0$ such that $I(t,\,x;\,\varphi)\leq B$, $\forall t>0$, $x\in\overline{\Omega}$. Assume that $v(t,\,x;\,\varphi)$ is the solution of

$$\begin{cases}\dfrac{\partial v(t,\,x)}{\partial t}=D_1\Delta v(t,\,x)+A(t,\,x)-\left(\beta_h(t,\,x)B+\beta_H(t,\,x)+\beta_L(t,\,x)+\varpi(t,\,x)+\mu(t,\,x)\right)v(t,\,x),\\ \qquad\qquad t>0,\,x\in\Omega,\\ \dfrac{\partial v(t,\,x)}{\partial\nu}=0,\quad t>0,\ x\in\partial\Omega,\\ v(0,\,x)=\varphi_1(0,\,x),\quad x\in\Omega.\end{cases}\tag{3.7}$$

It follows that

$$S(t,\,x;\,\varphi)\geq v(t,\,x;\,\varphi_1)>0,\ \forall t>0,\ x\in\overline{\Omega}.$$

Hence, we obtain

$$\liminf_{t\to+\infty}S(t,\,x;\,\varphi)\geq\vartheta_1:=\inf_{t\in[0,\omega],\,x\in\overline{\Omega}}v^*(t,\,x)$$

uniformly for $x\in\overline{\Omega}$, where $v^*(t,\,x)$ is the unique positive ω-periodic solution of model (3.7).

From the first two equations of model (2.4), we have

$$\begin{cases}\dfrac{\partial S\,(t,x)}{\partial t}\geq D_1\Delta S\,(t,x)+A\,(t,x)-[\beta_h(t,\,x)B+\beta_H(t,\,x)+\beta_L(t,\,x)+\varpi(t,\,x)+\mu(t,\,x)]\\ \qquad\qquad\times S\,(t,x)\,,\ t>0,\ x\in\Omega,\\ \dfrac{\partial V\,(t,x)}{\partial t}\geq D_2\Delta V\,(t,x)+\varpi\,(t,x)\,S\,(t,x)-\{\sigma\,[\beta_h\,(t,x)\,B+\beta_H\,(t,x)+\beta_L\,(t,x)]+\mu\,(t,x)\}\\ \qquad\qquad\times V\,(t,x)\,,\ t>0,\ x\in\Omega,\\ \dfrac{\partial S(t,\,x)}{\partial\nu}=\dfrac{\partial V(t,\,x)}{\partial\nu}=0,\ \ t>0,\ x\in\partial\Omega.\end{cases}$$

Consider

$$\begin{cases} \dfrac{\partial \overline{S}\,(t,x)}{\partial t} = D_1 \Delta \overline{S}\,(t,x) + A\,(t,x) - [\beta_h(t,\,x)B + \beta_H(t,\,x) + \beta_L(t,\,x) + \varpi(t,\,x) + \mu(t,\,x)] \\ \qquad \times \overline{S}\,(t,x)\,,\ t>0,\ x\in\Omega, \\ \dfrac{\partial \overline{V}\,(t,x)}{\partial t} = D_2 \Delta \overline{V}\,(t,x) + \varpi\,(t,x)\,\overline{S}\,(t,x) - \{\sigma\,[\beta_h\,(t,x)\,B + \beta_H\,(t,x) + \beta_L\,(t,x)] + \mu\,(t,x)\} \\ \qquad \times \overline{V}\,(t,x)\,,\ t>0,\ x\in\Omega, \\ \dfrac{\partial \overline{S}(t,\,x)}{\partial \nu} = \dfrac{\partial \overline{V}(t,\,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{3.8}$$

Denote $e = \overline{\beta}_h B + \overline{\beta}_H + \overline{\beta}_L$. We easily find that there is a vector with positive elements

$$\kappa = (\kappa_1,\,\kappa_2) = \left(\frac{\underline{A}}{e+\overline{\varpi}+\overline{\mu}},\ \frac{\underline{A\varpi}}{(e+\overline{\varpi}+\overline{\mu})\,(\sigma e+\overline{\mu})}\right)$$

such that

$$A\,(t,x) - [\beta_h(t,\,x)B + \beta_H(t,\,x) + \beta_L(t,\,x) + \varpi(t,\,x) + \mu(t,\,x)]\,\kappa_1 \geq 0,$$

and

$$\varpi\,(t,x)\,\kappa_1 - \{\sigma\,[\beta_h\,(t,x)\,B + \beta_H\,(t,x) + \beta_L\,(t,x)] + \mu\,(t,x)\}\,\kappa_2 \geq 0.$$

This indicates that for any $0 < m \leq 1$, $m\kappa$ is a lower solution of model (3.8). The comparison principle implies that solutions of model (3.8) are uniformly bounded. Thus, there is a constant $\underline{B} > 0$ such that $S(t,\,x;\,\varphi) > \underline{B}$ and $V(t,\,x;\,\varphi) > \underline{B}$, $\forall t>0$, $x\in\overline{\Omega}$. Assume that $u(t,\,x;\,\varphi)$ is the solution of

$$\begin{cases} \dfrac{\partial u(t,\,x)}{\partial t} = D_2\Delta u(t,\,x) + \varpi(t,\,x)\underline{B} - \{\sigma\,[\beta_h\,(t,x)\,B + \beta_H\,(t,x) + \beta_L\,(t,x)] + \mu\,(t,x)\} \\ \qquad \times u(t,\,x),\ t>0,\ x\in\Omega, \\ \dfrac{\partial u(t,\,x)}{\partial \nu} = 0,\ t>0,\ x\in\partial\Omega, \\ u(0,\,x) = \varphi_2(0,\,x),\ x\in\Omega. \end{cases} \tag{3.9}$$

Hence, we obtain $\liminf_{t\to+\infty} V(t,\,x;\,\varphi) \geq \vartheta_2 := \inf_{t\in[0,\omega],\,x\in\overline{\Omega}} u^*(t,\,x)$ uniformly for $x\in\overline{\Omega}$, where $u^*(t,\,x)$ is the unique positive ω-periodic solution of model (3.9). Taking $\vartheta = \min\{\vartheta_1,\,\vartheta_2\}$, we then complete the proof. □

Let

$$\mathbb{C}_0 = \left\{\varphi = (\varphi_1,\,\varphi_2,\,\varphi_3,\,\varphi_4,\,\varphi_5,\,\varphi_6) \in C_\tau^{++}:\ \varphi_3(0,\cdot)\not\equiv 0\,, \varphi_5(0,\cdot)\not\equiv 0 \text{ and } \varphi_6(0,\cdot)\not\equiv 0\right\}.$$

Then

$$\partial\mathbb{C}_0 := C_\tau^{++}\backslash\mathbb{C}_0 = \{\varphi\in C_\tau^{++}:\ \varphi_3(0,\,\cdot)\equiv 0 \text{ or } \varphi_5(0,\,\cdot)\equiv 0 \text{ or } \varphi_6(0,\,\cdot)\equiv 0\}.$$

For $\varphi\in\mathbb{C}_0$, it follows from Lemma 3.5 that $I(t,\,x;\,\varphi)>0$, $B_H(t,\,x;\,\varphi)>0$ and $B_L(t,\,x;\,\varphi)>0$ for $t>0$ and $x\in\overline{\Omega}$. Then $W^n(\mathbb{C}_0)\subset\mathbb{C}_0$ for $n\in\mathbb{N}$ and $W^n = W(n\omega)$. In view of Theorem 3.2, $W_\omega : C_\tau^{++}\to C_\tau^{++}$ admits a global attractor in C_τ^{++}.

Lemma 3.6. *If $\mathscr{R}_0 > 1$, there is a $\delta^*>0$ such that for $\varphi\in\mathbb{C}_0$, the solution semiflow of model (2.4) satisfies*

$$\limsup_{n\to+\infty} \|W^n(\phi) - \mathbb{Q}\| \geq \delta^*, \quad \varphi\in\mathbb{C}_0,\ \ n\in\mathbb{N},$$

where $\mathbb{Q} = \left(S_0^*, V_0^*, \widehat{0}, 0, \widehat{0}, \widehat{0}\right)$, $S_0^*(\theta,\,\cdot) = S^*(\theta,\,\cdot)$, $V_0^*(\theta,\,\cdot) = V^*(\theta,\,\cdot)$ and $\widehat{0}(\theta,\,\cdot) = 0$ for $\theta\in[-\tau,\,0]$.

Proof. Consider the following system with a parameter $\delta > 0$:

$$\begin{cases} \dfrac{\partial q_3^\delta(t,x)}{\partial t} = D_3\Delta q_3^\delta(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)]\, q_3^\delta(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\left[(S^*(t-\tau,y)-\delta) + \sigma(V^*(t-\tau,y)-\delta)\right] \\ \qquad \times \left[\beta_h(t-\tau,y)\, q_3^\delta(t-\tau,y) + \dfrac{\beta_H(t-\tau,y)}{k_H+\delta} q_5^\delta(t-\tau,y) + \dfrac{\beta_L(t-\tau,y)}{k_L+\delta} q_6^\delta(t-\tau,y)\right] dy, \\ \qquad t > 0,\ x \in \Omega, \\ \dfrac{\partial q_5^\delta(t,x)}{\partial t} = D_5\Delta q_5^\delta(t,x) + \xi(t,x)\, q_3^\delta(t,x) - \chi(t,x)\, q_5^\delta(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial q_6^\delta(t,x)}{\partial t} = D_6\Delta q_6^\delta(t,x) + \chi(t,x)\, q_5^\delta(t,x) - \delta_L(t,x)\, q_6^\delta(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial q_3^\varepsilon(t,x)}{\partial \nu} = \dfrac{\partial q_5^\varepsilon(t,x)}{\partial \nu} = \dfrac{\partial q_6^\varepsilon(t,x)}{\partial \nu} = 0,\ \ t > 0,\ x \in \partial\Omega. \end{cases} \tag{3.10}$$

For $\varphi \in C([-\tau, 0], \mathbb{M}^+)$, let $q^\delta(t, x; \varphi) = \left(q_3^\delta(t, x; \varphi), q_5^\delta(t, x; \varphi), q_6^\delta(t, x; \varphi)\right)$ be the unique solution of model (3.10) with $q_0^\delta(\varphi)(\theta, x) = \varphi(\theta, x)$ for $\theta \in [-\tau, 0]$ and $x \in \overline{\Omega}$, where

$$q_t^\delta(\theta, x) = q^\delta(t+\theta, x; \varphi) = \left(q_3^\delta(t+\theta, x; \varphi), q_5^\delta(t+\theta, x; \varphi), q_6^\delta(t+\theta, x; \varphi)\right),\ t \geq 0.$$

Assume that $\widehat{\mathbb{P}}_\delta : C([-\tau, 0], \mathbb{M}^+) \to C([-\tau, 0], \mathbb{M}^+)$ is the Poincaré map of model (3.10), that is, $\widehat{\mathbb{P}}_\delta(\varphi) = q_\omega^\delta(\varphi)$ for $\varphi \in C([-\tau, 0], \mathbb{M}^+)$. Due to $\lim_{\delta\to 0} r(\widehat{\mathbb{P}}_\delta) = r(\widehat{\mathbb{P}}) > 1$, it follows that $r(\widehat{\mathbb{P}}_\delta) > 1$ by fixing a small enough number $\delta > 0$. By fixing $\delta > 0$, there is a $\delta^* > 0$ satisfying $\|W(t)\varphi - W(t)\mathbb{Q}\| < \delta$ when $\|\varphi - \mathbb{Q}\| < \delta^*$ for $t \in [0, \omega]$.

By proof of contradiction, we assume that $\limsup_{n\to+\infty} \|W^n(\varphi_0) - \mathbb{Q}\| < \delta^*$ for some $\varphi_0 \in \mathbb{C}_0$. Thus, there exists $n_1 \geq 1$ such that $\|W^n(\varphi_0) - \mathbb{Q}\| < \delta^*$ for $n \geq n_1$. For $t \geq n_1\omega$, letting $t = n\omega + t^1$ with $n = [t/\omega]$, and $t^1 \in [0, \omega)$, one gets

$$\|W(t)\varphi_0 - W(t)\mathbb{Q}\| = \|W(t^1)(W^n(\varphi_0)) - W(t^1)\mathbb{Q}\| < \delta,$$

and $S(t, x; \varphi_0) > S^*(t, x) - \delta$, $V(t, x; \varphi_0) > V^*(t, x) - \delta$ and $R(t, x; \varphi_0) < \delta$, for $t \geq n_1\omega$ and $x \in \overline{\Omega}$. Hence, we get

$$\begin{cases} \dfrac{\partial I(t,x)}{\partial t} \geq D_3\Delta I(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)]\, I(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\left[(S^*(t-\tau,y)-\delta) + \sigma(V^*(t-\tau,y)-\delta)\right] \\ \qquad \times \left[\beta_h(t-\tau,y)\, I(t-\tau,y) + \dfrac{\beta_H(t-\tau,y)}{k_H+\delta} B_H(t-\tau,y) + \dfrac{\beta_L(t-\tau,y)}{k_L+\delta} B_L(t-\tau,y)\right] dy, \\ \qquad t > 0,\ x \in \Omega, \\ \dfrac{\partial B_H(t,x)}{\partial t} \geq D_5\Delta B_H(t,x) + \xi(t,x)\, I(t,x) - \chi(t,x)\, B_H(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial B_L(t,x)}{\partial t} \geq D_6\Delta B_L(t,x) + \chi(t,x)\, B_H(t,x) - \delta_L(t,x)\, B_L(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial I(t,x)}{\partial \nu} = \dfrac{\partial B_H(t,x)}{\partial \nu} = \dfrac{\partial B_L(t,x)}{\partial \nu} = 0,\ \ t \geq n_1\omega + \tau,\ x \in \partial\Omega. \end{cases} \tag{3.11}$$

Since $\Pi(t, x; \varphi_0) > 0$ for $t > 0$ and $x \in \overline{\Omega}$, there is an $\alpha_1 > 0$ satisfying

$$(I(t, x; \varphi_0), B_H(t, x; \varphi_0), B_L(t, x; \varphi_0)) \geq \alpha_1 e^{\varsigma_\delta t} q_\delta^*(t, x),\ \ t \in [n_1\omega, n_1\omega + \tau],\ \ x \in \overline{\Omega},$$

where $q_\delta^*(t, x)$ is a positive ω-periodic function such that $e^{\varsigma_\delta t} q_\delta^*(t, x)$ is a solution of model (3.11), and $\varsigma_\delta = \frac{\ln r(\widehat{\mathbb{P}}_\delta)}{\omega}$. The comparison theorem yields $(I(t, x; \varphi_0), B_H(t, x; \varphi_0), B_L(t, x; \varphi_0)) \geq \alpha_1 e^{\varsigma_\delta t} q_\delta^*(t, \cdot)$ for

$t \geq n_1\omega + \tau$ and $x \in \overline{\Omega}$. Since $\varsigma_\delta > 0$, then $I(t, x; \varphi_0) \to +\infty$, $B_H(t, x; \varphi_0) \to +\infty$ and $B_L(t, x; \varphi_0) \to +\infty$ as $t \to \infty$, which leads to a contradiction. □

Theorem 3.7. *Let $\Pi(t, x; \varphi)$ be the solution of model (2.4) with $\Pi_0 = \varphi \in C_\tau^{++}$. Then, the following hold.*

- (i) *If $\mathscr{R}_0 < 1$, the infection-free ω-periodic solution E_0 of model (2.4) is globally attractive.*
- (ii) *If $\mathscr{R}_0 > 1$, there exists at least one endemic ω-periodic solution $(S^*(t, x), V^*(t, x), I^*(t, x), R^*(t, x), B_H^*(t, x), B_L^*(t, x))$ of model (2.4), and there is a $\check{\varrho} > 0$ such that for $\varphi \in C_\tau^{++}$ with $\varphi_3(\cdot, 0) \not\equiv 0$, $\varphi_5(\cdot, 0) \not\equiv 0$ and $\varphi_6(\cdot, 0) \not\equiv 0$, we have $\liminf_{t\to+\infty} \Big(S(t, x; \varphi), V(t, x; \varphi), I(t, x; \varphi), R(t, x; \varphi), B_H(t, x; \varphi), B_L(t, x; \varphi)\Big) \geq (\check{\varrho}, \check{\varrho}, \check{\varrho}, \check{\varrho}, \check{\varrho}, \check{\varrho})$, uniformly for $x \in \overline{\Omega}$.*

Proof. (i) If $\mathscr{R}_0 < 1$, then $r\left(\widehat{\mathbb{P}}\right) < 1$, and $\varsigma = \frac{\ln r(\widehat{\mathbb{P}})}{\omega} < 0$. Consider the following system with a parameter $\varepsilon > 0$:

$$\begin{cases} \dfrac{\partial q_3^\varepsilon(t,x)}{\partial t} = D_3\Delta q_3^\varepsilon(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)]\, q_3^\varepsilon(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y)\left[(S^*(t-\tau, y)+\varepsilon) + \sigma(V^*(t-\tau, y)+\varepsilon)\right] \\ \qquad \times \left[\beta_h(t-\tau, y)\, q_3^\varepsilon(t-\tau, y) + \dfrac{\beta_H(t-\tau, y)}{k_H} q_5^\varepsilon(t-\tau, y) + \dfrac{\beta_L(t-\tau, y)}{k_L} q_6^\varepsilon(t-\tau, y)\right] \mathrm{d}y, \\ \qquad t > 0,\ x \in \Omega, \\ \dfrac{\partial q_5^\varepsilon(t,x)}{\partial t} = D_5\Delta q_5^\varepsilon(t,x) + \xi(t,x)\, q_3^\varepsilon(t,x) - \chi(t,x)\, q_5^\varepsilon(t,x), \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial q_6^\varepsilon(t,x)}{\partial t} = D_6\varepsilon q_6^\varepsilon(t,x) + \chi(t,x)\, q_5^\varepsilon(t,x) - \varepsilon_L(t,x)\, q_6^\varepsilon(t,x), \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial q_3^\varepsilon(t, x)}{\partial \nu} = \dfrac{\partial q_5^\varepsilon(t, x)}{\partial \nu} = \dfrac{\partial q_6^\varepsilon(t, x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{3.12}$$

For $\psi \in C([-\tau, 0], \mathbb{M}^+)$, let $q^\varepsilon(t, x; \psi) = (q_3^\varepsilon(t, x; \psi), q_5^\varepsilon(t, x; \psi), q_6^\varepsilon(t, x; \psi))$ be the unique solution of model (3.12) with $q_0^\varepsilon(\psi) = \psi(\theta, x)$ for $(\theta, x) \in [-\tau, 0] \times \overline{\Omega}$, where

$$q_t^\varepsilon(\theta, x) = q^\varepsilon(t+\theta, x; \psi) = (q_3^\varepsilon(t+\theta, x; \psi), q_5^\varepsilon(t+\theta, x; \psi), q_6^\varepsilon(t+\theta, x; \psi)), \quad t \geq 0.$$

Assume that $\widehat{\mathbb{P}}_\varepsilon : C([-\tau, 0], \mathbb{M}^+) \to C([-\tau, 0], \mathbb{M}^+)$ is the Poincaré map of model (3.12), that is, $\widehat{\mathbb{P}}_\varepsilon(\psi) = q_\omega^\varepsilon(\psi)$ for $\psi \in C([-\tau, 0], \mathbb{M}^+)$. Since $\lim_{\varepsilon\to 0} r(\widehat{\mathbb{P}}_\varepsilon) = r(\widehat{\mathbb{P}}) < 1$, there is a sufficiently small number $\varepsilon > 0$ satisfying $r(\widehat{\mathbb{P}}_\varepsilon) < 1$. Similar to the arguments of those in [4, Lemma 5] (see also, [50, Lemma 3.5]), we can show that there is a positive ω-periodic function $q_\varepsilon^*(t, x)$ such that $q^\varepsilon(t, x) = e^{\varsigma_\varepsilon t} q_\varepsilon^*(t, x)$, which is a solution of model (3.12), where $\varsigma_\varepsilon = \frac{\ln r(\widehat{\mathbb{P}}_\varepsilon)}{\omega} < 0$. For the fixed $\varepsilon > 0$, there is a sufficiently large integer $n_2 > 0$ satisfying $n_2\omega \geq \tau$, $S(t, x) \leq S^*(t, x) + \varepsilon$ and $V(t, x) \leq V^*(t, x) + \varepsilon$ for $t \geq n_2\omega - \tau$ and $x \in \overline{\Omega}$. Consider

$$\begin{cases} \dfrac{\partial I(t,\,x)}{\partial t} = D_3\Delta I\left(t,x\right) - \left[\mu\left(t,x\right) + \Upsilon_I\left(t,x\right) + d_I\left(t,x\right)\right] I\left(t,x\right) \\ \qquad + \displaystyle\int_\Omega \Gamma\left(t,t-\tau,x,y\right)\left[\left(S^*\left(t-\tau,y\right)+\varepsilon\right) + \sigma\left(V^*\left(t-\tau,y\right)+\varepsilon\right)\right] \\ \qquad \times \left[\beta_h\left(t-\tau,y\right) I\left(t-\tau,y\right) + \dfrac{\beta_H\left(t-\tau,y\right)}{k_H} B_H\left(t-\tau,y\right) + \dfrac{\beta_L\left(t-\tau,y\right)}{k_L} B_L\left(t-\tau,y\right)\right] \mathrm{d}y, \\ \qquad t>0,\ x\in\Omega, \\ \dfrac{\partial B_H(t,\,x)}{\partial t} = D_5\Delta B_H\left(t,x\right) + \xi\left(t,x\right) I\left(t,x\right) - \chi\left(t,x\right) B_H\left(t,x\right),\ \ t>0,\ x\in\Omega, \\ \dfrac{\partial B_L(t,\,x)}{\partial t} = D_6\Delta B_L\left(t,x\right) + \chi\left(t,x\right) B_H\left(t,x\right) - \delta_L\left(t,x\right) B_L\left(t,x\right),\ \ t>0,\ x\in\Omega, \\ \dfrac{\partial I(t,\,x)}{\partial \nu} = \dfrac{\partial B_H(t,\,x)}{\partial \nu} = \dfrac{\partial B_L(t,\,x)}{\partial \nu} = 0,\ \ t>0,\ x\in\partial\Omega. \end{cases} \tag{3.13}$$

Then, for any given $\varphi \in C_\tau^{++}$, there is some $\alpha_2 > 0$ such that the solution of (3.13) satisfies

$$(I(t,\,x;\,\varphi),\,B_H(t,\,x;\,\varphi),\,B_L(t,\,x;\,\varphi)) \le \alpha_2 q^\varepsilon(t,\,x),\ t\in[n_2\omega-\tau,\,n_2\omega],\ x\in\overline{\Omega}.$$

Then, the comparison principle yields $(I(t,\,x;\,\varphi),\,B_H(t,\,x;\,\varphi),\,B_L(t,\,x;\,\varphi)) \le \alpha_2 e^{\varsigma_\varepsilon t} q_\varepsilon^*(t,\,\cdot),\ t\ge n_2\omega,\ x\in\overline{\Omega}$. Hence,

$$\lim_{t\to+\infty}(I(t,\,x;\,\varphi),\,R(t,\,x;\,\varphi),\,B_H(t,\,x;\,\varphi),\,B_L(t,\,x;\,\varphi)) = (0,\,0,\,0,\,0)$$

uniformly for $x\in\overline{\Omega}$. Using an analogous argument to [4, Theorem (i)] (see, also [50, Theorem 3.10 (i)]), we can show that $\lim_{t\to+\infty} S(t,\,x;\,\varphi) = S^*(t,\,x)$ and $\lim_{t\to+\infty} V(t,\,x;\,\varphi) = V^*(t,\,x)$ uniformly for $x\in\overline{\Omega}$. Hence,

$$\begin{aligned}&\lim_{t\to+\infty}\|\,(S(t,\,\cdot;\,\varphi),\,V(t,\,\cdot;\,\varphi),\,I(t,\,\cdot;\,\varphi),\,R(t,\,\cdot;\,\varphi),\,B_H(t,\,\cdot;\,\varphi),\,B_L(t,\,\cdot;\,\varphi))\\ &-(S^*(t,\,\cdot),\,V^*(t,\,\cdot),\,0,\,0,\,0,\,0)\|_{\mathbb{X}} = 0.\end{aligned}$$

This completes the proof of the first conclusion. We next prove the second conclusion.

(ii) If $\mathscr{R}_0 > 1$, then $r(\widehat{\mathbb{P}}) > 1$, and $\varsigma = \frac{\ln r(\widehat{\mathbb{P}})}{\omega} > 0$. Let $M_\partial := \{\varphi\in\partial\mathbb{C}_0 : W^n(\varphi)\in\partial\mathbb{C}_0,\ n\in\mathbb{N}\}$, and $\omega(\varphi)$ be the omega limit set of the orbit $\Pi^+(\varphi) = \{W^n(\varphi) : \forall n\in\mathbb{N}\}$. For any given $\psi\in M_\partial$, we see that $W^n(\psi)\in\partial\mathbb{C}_0$, and either $I(n\omega,\,\cdot;\,\psi)\equiv 0$ or $B_H(n\omega,\,\cdot;\,\psi)\equiv 0$ or $B_L(n\omega,\,\cdot;\,\psi)\equiv 0$. For $t\ge 0$, either $I(t,\,\cdot;\,\psi)\equiv 0$ or $B_H(t,\,\cdot;\,\psi)\equiv 0$ or $B_L(t,\,\cdot;\,\psi)\equiv 0$. If $B_H(t,\,\cdot;\,\psi)\equiv 0$ for $t\ge 0$, then $I(t,\,\cdot;\,\psi)\equiv 0$. Hence, $\lim_{t\to+\infty} R(t,\,x;\,\psi) = 0$ and $\lim_{t\to+\infty} B_L(t,\,x;\,\psi) = 0$ uniformly for $x\in\overline{\Omega}$. Thus, S and V equations of model (2.4) are asymptotic to (3.5). Then, $\lim_{t\to+\infty} S(t,\,x;\,\psi) = S^*(t,\,x)$ and $\lim_{t\to+\infty} V(t,\,x;\,\psi) = V^*(t,\,x)$ uniformly for $x\in\overline{\Omega}$. If $B_H(t_0,\,\cdot;\,\psi)\not\equiv 0$ for some $t_0\ge 0$, from Lemma 3.5, we have $B_H(t,\,\cdot;\,\psi) > 0$, $t\ge t_0$, and further get that $B_L(t,\,\cdot;\,\psi)\equiv 0$ or $I(t,\,\cdot;\,\psi)\equiv 0$, $t\ge t_0$. For the case $I(t,\,\cdot;\,\psi)\equiv 0$, $t\ge t_0$, then $\lim_{t\to+\infty} R(t,\,x;\,\psi) = 0$, and $\lim_{t\to+\infty} B_H(t,\,x;\,\psi) = 0$, $\lim_{t\to+\infty} B_L(t,\,x;\,\psi) = 0$ uniformly for $x\in\overline{\Omega}$. It follows that $\lim_{t\to+\infty} S(t,\,x;\,\psi) = S^*(t,\,x)$ and $\lim_{t\to+\infty} V(t,\,x;\,\psi) = V^*(t,\,x)$ uniformly for $x\in\overline{\Omega}$. If $I(t_1,\,\cdot;\,\psi)\not\equiv 0$ for some $t_1 > t_0 \ge 0$, from Lemma 3.5, we have $I(t,\,\cdot;\,\psi) > 0$, $t\ge t_1$, and further get that $B_L(t,\,\cdot;\,\psi)\equiv 0$, and thus $B_H(t,\,\cdot;\,\psi)\equiv 0$. From the fifth equation of model (2.4), we have $I(t,\,\cdot;\,\psi)\equiv 0$ for $t\ge t_1$, which leads to a contradiction. Hence, $\omega(\psi) = \mathbb{Q}$ for any $\psi\in M_\partial$, and $\mathbb{Q}$ cannot form a cycle for W in $\partial\mathbb{C}_0$.

Lemma 3.6 reveals that $\mathbb{Q}$ is an isolated invariant set for W in C_τ^{++}, and $W^s(\mathbb{Q})\cap\mathbb{C}_0 = \varnothing$. It follows from [59, Theorem 1.3.1 and Remark 1.3.1] that $W : C_\tau^{++}\to C_\tau^{++}$ is uniformly persistent for $(\mathbb{C}_0,\,\partial\mathbb{C}_0)$. That is, there is $\eta > 0$ such that

$$\liminf_{n\to+\infty} d(W^n(\varphi),\,\partial\mathbb{C}_0) \ge \eta,\ \ \varphi\in\mathbb{C}_0.$$

Since $W^n = W(n\omega)$ is compact for an integer n with $n\omega > \tau$, W is asymptotically smooth on C_τ^{++}. Theorem 3.2 reveals that W admits a global attractor on C_τ^{++}. According to [29, Theorem 3.7], W admits a global attractor D_0 in $\mathbb{C}_0$. Note that $D_0 = W(D_0) = W(\omega)(D_0)$. We have $\varphi_3(0,\cdot) > 0$, $\varphi_5(0,\cdot) > 0$ and $\varphi_6(0,\cdot) > 0$ for $\varphi \in D_0$. Let $F_0 := \bigcup_{t\in[0,\,\omega]} W(t)D_0$. In view of [59, Theorem 3.1.1], we deduce that $F_0 \subset \mathbb{C}_0$ and $\lim_{t\to+\infty} d(W(t)\varphi, F_0) = 0$ for $\varphi \in \mathbb{C}_0$, where $d(W(t)\varphi, F_0) = \sup_{x\in W(t)\varphi} d(x, F_0)$. Define a continuous function $\mathscr{P} : C_\tau^{++} \to \mathbb{R}^+$ by

$$\mathscr{P}(\varphi) = \min\left\{\min_{x\in\overline{\Omega}} \varphi_3(0,x),\ \min_{x\in\overline{\Omega}} \varphi_5(0,x),\ \min_{x\in\overline{\Omega}} \varphi_6(0,x)\right\},\ \ \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6) \in C_\tau^{++}.$$

Notice that F_0 is a compact subset of $\mathbb{C}_0$, we obtain $\inf_{\varphi\in F_0} \mathscr{P}(\varphi) = \min_{\varphi\in F_0} \mathscr{P}(\varphi) > 0$. Thus, there is a $\check{\rho}^* > 0$ such that

$$\liminf_{t\to+\infty} \min\{I(t,\cdot;\varphi),\ B_H(t,\cdot;\varphi),\ B_L(t,\cdot;\varphi)\} = \liminf_{t\to+\infty} \mathscr{P}(W(t)\varphi) \geq \check{\rho}^*,\ \ \varphi \in \mathbb{C}_0.$$

Furthermore, there exists a $\check{\rho} \in (0, \check{\rho}^*)$ such that

$$\liminf_{t\to+\infty}(S(t,\cdot;\varphi), V(t,\cdot;\varphi), I(t,\cdot;\varphi), R(t,\cdot;\varphi), B_H(t,\cdot;\varphi), B_L(t,\cdot;\varphi)) \geq (\check{\rho}, \check{\rho}, \check{\rho}, \check{\rho}, \check{\rho}, \check{\rho}),\ \ \varphi \in \mathbb{C}_0.$$

In what follows, we demonstrate that model (2.4) has at least one endemic ω-periodic solution. For a given real number $r > 0$, from Lemma 3.4, we equip $\mathbb{C}$ with an equivalent norm $\|\cdot\|_r^*$. Define

$$\widetilde{W}_0 = \left\{\varphi \in C_\tau^+ : \varphi_3(0,\cdot) \not\equiv 0,\ \varphi_5(0,\cdot) \not\equiv 0 \text{ and } \varphi_6(0,\cdot) \not\equiv 0\right\},$$

and

$$\partial\widetilde{W}_0 := C_\tau^+ \backslash \widetilde{W}_0 = \left\{\varphi \in C_\tau^+ : \varphi_3(0,\cdot) \equiv 0 \text{ or } \varphi_5(0,\cdot) \equiv 0 \text{ or } \varphi_6(0,\cdot) \equiv 0\right\}.$$

Let $\widehat{W} = \widehat{W}(\omega)$. By the uniqueness of solutions, $\widehat{W}$ is point dissipative and ρ-uniformly persistent with $\rho(\psi) = d\left(\psi, \partial\widetilde{W}_0\right)$, and $\widehat{W}^n = \widehat{W}(n\omega)$ is compact for $n\omega > \tau$. By Lemma 3.4, $\widehat{W}$ is k-condensing. Thus, by [29, Theorem 4.5], model (3.1) has an ω-periodic solution $\left(\Pi_1^*(t,\cdot), \Pi_2^*(t,\cdot), \Pi_3^*(t,\cdot), \Pi_4^*(t,\cdot), \Pi_5^*(t,\cdot),\right.$ $\left.\Pi_6^*(t,\cdot)\right)$ with $(\Pi_{1t}^*, \Pi_{2t}^*, \Pi_{3t}^*, \Pi_{4t}^*, \Pi_{5t}^*, \Pi_{6t}^*) \in \widetilde{W}_0$. Let $S_0^* = \Pi_{10}^*$, $V_0^* = \Pi_{20}^*$, $I_0^* = \Pi_{30}^*$, $R_0^* = \Pi_{40}^*$, $B_{H_0}^* = \Pi_{50}^*$ and $B_{L_0}^* = \Pi_{60}^*$. The uniqueness of solutions yields that $(S^*(t,\cdot), V^*(t,\cdot), I^*(t,\cdot), R^*(t,\cdot),$ $B_H^*(t,\cdot), B_L^*(t,\cdot))$ is a strictly positive periodic solution of model (2.4) . □

4. Global stability of the endemic steady state

In the case where all the parameters of model (2.4) are positive constants, model (2.4) reduces to the following autonomous reaction–diffusion equations in the absence of nonlocal time delay:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_1\Delta S(t,x) + A - \left[\beta_h I(t,x) + \dfrac{\beta_H B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x) \\ \qquad - (\varpi + \mu)\, S(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial V(t,x)}{\partial t} = D_2\Delta V(t,x) + \varpi S(t,x) - \sigma\left[\beta_h I(t,x) + \dfrac{\beta_H B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L B_L(t,x)}{k_L + B_L(t,x)}\right] V(t,x) \\ \qquad - \mu V(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial I(t,x)}{\partial t} = D_3\Delta I(t,x) - (\mu + \Upsilon_I + d_I)\, I(t,x) + \left[\beta_h I(t,x) + \dfrac{\beta_H B_H(t,x)}{k_H + B_H(t,x)} + \dfrac{\beta_L B_L(t,x)}{k_L + B_L(t,x)}\right] \\ \qquad \times (S(t,x) + \sigma V(t,x)),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial R(t,x)}{\partial t} = D_4\Delta R(t,x) + \Upsilon_I I(t,x) - \mu R(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial B_H(t,x)}{\partial t} = D_5\Delta B_H(t,x) + \xi I(t,x) - \chi B_H(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial B_L(t,x)}{\partial t} = D_6\Delta B_L(t,x) + \chi B_H(t,x) - \delta_L B_L(t,x),\ \ t > 0,\ x \in \Omega, \\ \dfrac{\partial S(t,x)}{\partial \nu} = \dfrac{\partial V(t,x)}{\partial \nu} = \dfrac{\partial I(t,x)}{\partial \nu} = \dfrac{\partial R(t,x)}{\partial \nu} = \dfrac{\partial B_H(t,x)}{\partial \nu} = \dfrac{\partial B_L(t,x)}{\partial \nu} = 0,\ \ t > 0,\ x \in \partial\Omega. \end{cases} \tag{4.1}$$

Obviously, model (4.1) always admits a disease-free steady state $E_0\left(S_0,\, V_0,\, 0,\, 0,\, 0,\, 0\right)$, where

$$S_0=\frac{A\mu}{\mu\left(\mu+\varpi\right)},\ V_0=\frac{A\varpi}{\mu\left(\mu+\varpi\right)}.$$

For model (4.1), we get

$$\mathscr{R}_0=\left(\beta_h+\frac{\beta_H\xi}{k_H\chi}+\frac{\beta_L\xi}{k_L\delta_L}\right)\frac{A\left(\mu+\sigma\varpi\right)}{\mu\left(\mu+\varpi\right)\left(\Upsilon_I+\mu+d\right)}.$$

If $\mathscr{R}_0>1$, from Theorem 3.7, there exists at least an endemic steady state $E^*\left(S^*,\, V^*,\, I^*,\, R^*,\, B_H^*,\, B_L^*\right)$. We are concerned with the global stability of the endemic steady state of model (4.1), that will be proven by Lyapunov's stability theorem [28].

Theorem 4.1. *If $\mathscr{R}_0>1$, the endemic steady state E^* of model (4.1) is globally attractive.*

Proof. Denote

$$l_1=\frac{\left(S^*+\sigma V^*\right)}{\chi B_H^*}\left(\frac{\beta_H B_H^*}{k_H+B_H^*}+\frac{\beta_L B_L^*}{k_L+B_L^*}\right),\ l_2=\frac{\beta_L\left(S^*+\sigma V^*\right)}{\delta_L\left(k_L+B_L^*\right)}.$$

Then, we can construct a Lyapunov function as follows:

$$\begin{aligned}V_1=\ &S^*\int_\Omega\left[\tfrac{S(t,x)}{S^*}-1-\ln\tfrac{S(t,x)}{S^*}\right]\mathrm{d}x+V^*\int_\Omega\left[\tfrac{V(t,x)}{V^*}-1-\ln\tfrac{V(t,x)}{V^*}\right]\mathrm{d}x+I^*\int_\Omega\left[\tfrac{I(t,x)}{I^*}-1-\ln\tfrac{I(t,x)}{I^*}\right]\mathrm{d}x\\&+l_1B_H^*\int_\Omega\left[\tfrac{B_H(t,x)}{B_H^*}-1-\ln\tfrac{B_H(t,x)}{B_H^*}\right]\mathrm{d}x+l_2\int_\Omega\left[\tfrac{B_L(t,x)}{B_L^*}-1-\ln\tfrac{B_L(t,x)}{B_L^*}\right]\mathrm{d}x.\end{aligned}$$

Differentiate V_1 with respect to t and evaluate the result along the solution of model (4.1). We obtain

$$\begin{aligned}\frac{\partial V_1}{\partial t}=&-D_1S^*\int_\Omega\frac{\|\nabla S\|^2}{S^2}dx-D_2V^*\int_\Omega\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x-D_3I^*\int_\Omega\frac{\|\nabla I\|^2}{I^2}dx\\&-D_5B_H^*l_1\int_\Omega\frac{\|\nabla B_H^*\|^2}{B_H^*}\mathrm{d}x-D_6l_2\int_\Omega\frac{\|\nabla B_L^*\|^2}{B_L^*}\mathrm{d}x\\&+S^*\left(\mu+\beta_hI^*\right)\int_\Omega\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)\mathrm{d}x+V^*\left(\mu+\beta_h\sigma I^*\right)\int_\Omega\left(3-\frac{S}{S^*}-\frac{V}{V^*}-\frac{S^*}{S}\frac{V^*}{S}\right)\mathrm{d}x\\&+\frac{\beta_HS^*B_H^*}{k_H+B_H^*}\int_\Omega\left(4-\frac{S^*}{S}-\frac{I}{I^*}\frac{B_H^*}{B_H}-\frac{k_H+B_H}{k_H+B_H^*}-\frac{S}{S^*}\frac{I^*}{I}\frac{B_H}{B_H^*}\frac{k_H+B_H^*}{k_H+B_H}\right)\mathrm{d}x\\&+\frac{\beta_LS^*B_L^*}{k_L+B_L^*}\int_\Omega\left(5-\frac{S^*}{S}-\frac{I}{I^*}\frac{B_H^*}{B_H}-\frac{B_L^*}{B_L}\frac{B_H}{B_H^*}-\frac{k_L+B_L}{k_L+B_L^*}-\frac{S}{S^*}\frac{I^*}{I}\frac{B_L}{B_L^*}\frac{k_L+B_L^*}{k_L+B_L}\right)\mathrm{d}x\\&+\frac{\sigma\beta_HV^*B_H^*}{k_H+B_H^*}\int_\Omega\left(5-\frac{S^*}{S}-\frac{S}{S^*}\frac{V^*}{V}-\frac{I}{I^*}\frac{B_H^*}{B_H}-\frac{k_H+B_H}{k_H+B_H^*}-\frac{V}{V^*}\frac{I^*}{I}\frac{B_H}{B_H^*}\frac{k_H+B_H^*}{k_H+B_H}\right)\mathrm{d}x\\&+\frac{\sigma\beta_LV^*B_L^*}{k_L+B_L^*}\int_\Omega\left(6-\frac{S^*}{S}-\frac{S}{S^*}\frac{V^*}{V}-\frac{I}{I^*}\frac{B_H^*}{B_H}-\frac{B_L^*}{B_L}\frac{B_H}{B_H^*}-\frac{k_L+B_L}{k_L+B_L^*}-\frac{V}{V^*}\frac{I^*}{I}\frac{B_L}{B_L^*}\frac{k_L+B_L^*}{k_L+B_L}\right)\mathrm{d}x\\&-\left(S^*+\sigma V^*\right)\int_\Omega\frac{\beta_Hk_H\left(B_H-B_H^*\right)^2}{\left(k_H+B_H^*\right)^2\left(k_H+B_H\right)}\mathrm{d}x-\left(S^*+\sigma V^*\right)\int_\Omega\frac{\beta_Lk_L\left(B_L-B_L^*\right)^2}{\left(k_L+B_L^*\right)^2\left(k_L+B_L\right)}\mathrm{d}x.\end{aligned}$$

Then, when $\mathscr{R}_0 > 1$ we know $V_1^{'} < 0$ for $(S(t,x), V(t,x), I(t,x), B_H(t,x), B_L(t,x)) \neq (S^*, V^*, I^*, B_H^*, B_L^*)$. It follows from Lyapunov's stability theorem in [28] that the endemic steady state E^* of model (4.1) is globally attractive. □

5. Numerical simulations

In this section, we carry out numerical simulations to discuss the influence of spatial mobility, spatial heterogeneity and the seasonality on the transmission of cholera.

According to the report of World Health Organization for cholera [9], the first round of oral cholera vaccination campaign targeting 650,000 people for the age of 1 year and above was started on 22 June and completed on 30 June 2019 in Somalia. Hence, to make better understandings on the transmission dynamics of historical human infection with *Vibrio cholerae* in Somalia after June 30, model (2.4) becomes:

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = D_1 \Delta S(t,x) + A(t,x) - \mu(t,x) S(t,x) \\ \qquad - \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] S(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial V(t,x)}{\partial t} = D_2 \Delta V(t,x) - \mu(t,x) V(t,x) \\ \qquad - \sigma \left[\beta_h(t,x) I(t,x) + \frac{\beta_H(t,x) B_H(t,x)}{k_H + B_H(t,x)} + \frac{\beta_L(t,x) B_L(t,x)}{k_L + B_L(t,x)}\right] V(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial I(t,x)}{\partial t} = D_3 \Delta I(t,x) - [\mu(t,x) + \Upsilon_I(t,x) + d_I(t,x)] I(t,x) \\ \qquad + \int_\Omega \Gamma(t, t-\tau, x, y) \left[\beta_h(t-\tau,y) I(t-\tau,y) + \frac{\beta_H(t-\tau,y) B_H(t-\tau,y)}{k_H + B_H(t-\tau,y)} + \frac{\beta_L(t-\tau,y) B_L(t-\tau,y)}{k_L + B_L(t-\tau,y)}\right] \\ \qquad \times [S(t-\tau,y) + \sigma V(t-\tau,y)] \, \mathrm{d}y, \ t>0, \ x \in \Omega, \\ \frac{\partial R(t,x)}{\partial t} = D_4 \Delta R(t,x) + \Upsilon_I(t,x) I(t,x) - \mu(t,x) R(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial B_H(t,x)}{\partial t} = D_5 \Delta B_H(t,x) + \xi(t,x) I(t,x) - \chi(t,x) B_H(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial B_L(t,x)}{\partial t} = D_6 \Delta B_L(t,x) + \chi(t,x) B_H(t,x) - \delta_L(t,x) B_L(t,x), \ t>0, \ x \in \Omega, \\ \frac{\partial S(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial R(t,x)}{\partial \nu} = \frac{\partial B_H(t,x)}{\partial \nu} = \frac{\partial B_L(t,x)}{\partial \nu} = 0, \ t>0, \ x \in \partial\Omega, \end{cases} \tag{5.1}$$

in which the number of vaccinated individuals at the initial time equals 650,000. The values of initial variables and parameters of model (5.1) are shown in Table 2.

5.1. Numerical computation of $\mathscr{R}_0$

For any $\lambda > 0$, we consider the following linear system:

$$\frac{\partial q}{\partial t} = \frac{1}{\lambda} \widehat{\Sigma}(t) q_t - \mathbb{V}(t) q, \ t \geq 0, \tag{5.2}$$

subject to the zero-flux boundary conditions. Assume that $Q(t,s,\lambda)$ $(t \geq s)$ is the evolution operator on $C([-\tau, 0], \mathbb{M})$ for model (5.2). Applying arguments similar to those in [58, Theorem 2.2] (see, also [26, Theorem 3.8]), one gets the following result.

Lemma 5.1. *If $\mathscr{R}_0 > 0$, then $\mathscr{R}_0 = \lambda$ is the unique solution of $r(Q(\omega, 0, \lambda)) = 1$.*

According to Lemma 5.1, we can use the standard bisection method to obtain the numerical solution λ to $r(Q(\omega, 0, \lambda)) = 1$, and thus, $\mathscr{R}_0 = \lambda$. Note that for each $\lambda > 0$, $r(Q(\omega, 0, \lambda))$ can be computed numerically via the algorithm developed in [26, Lemma 2.5].

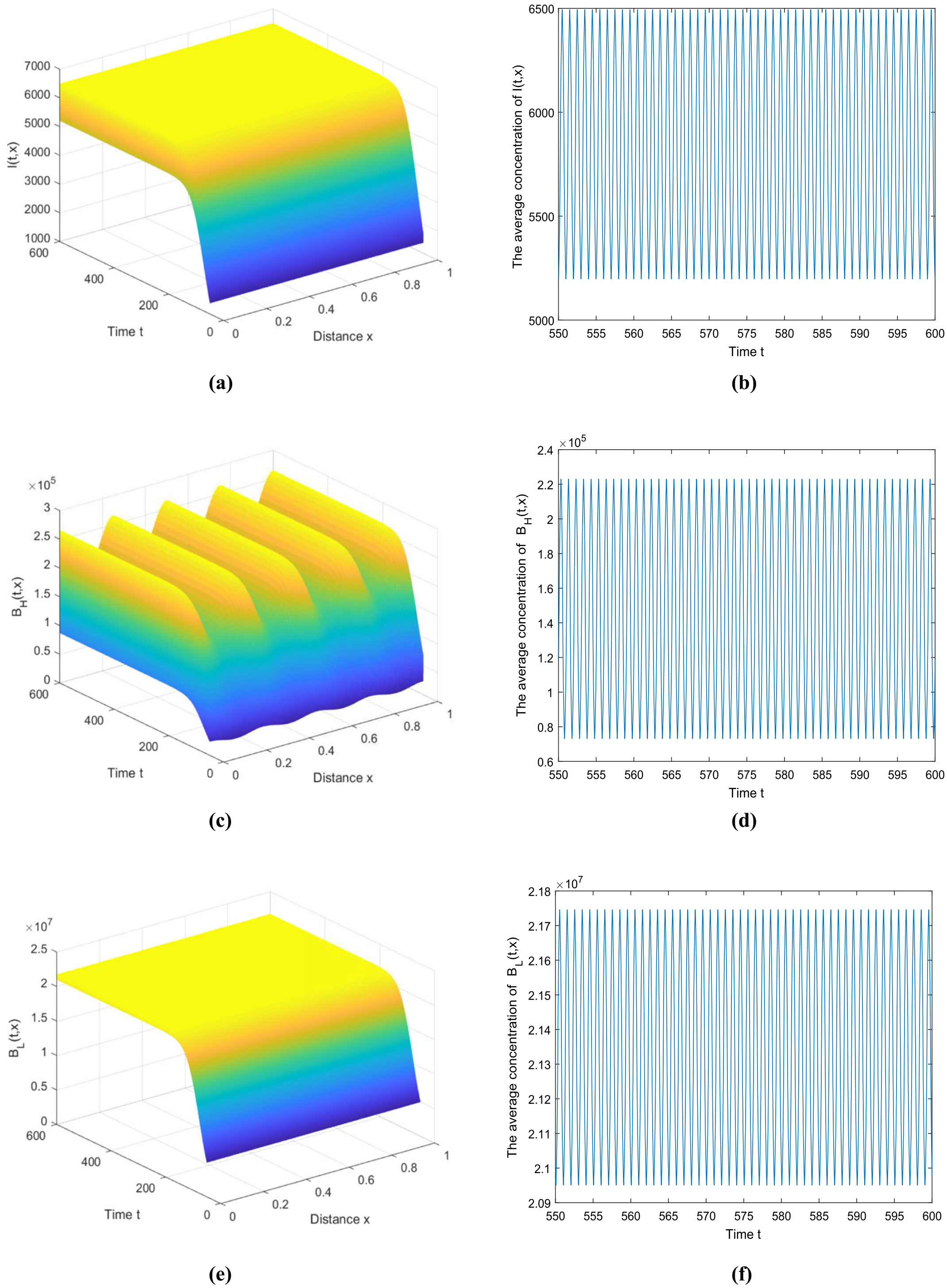

FIG. 2. Time variation of $I(t,x)$, $B_H(t,x)$ and $B_L(t,x)$ with $\xi(t,x) = 12 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t))$, $\beta_h(t,x) = 180 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t))$, $\beta_H(t,x) = 180 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t))$, $\beta_L(t,x) = 180 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t))$

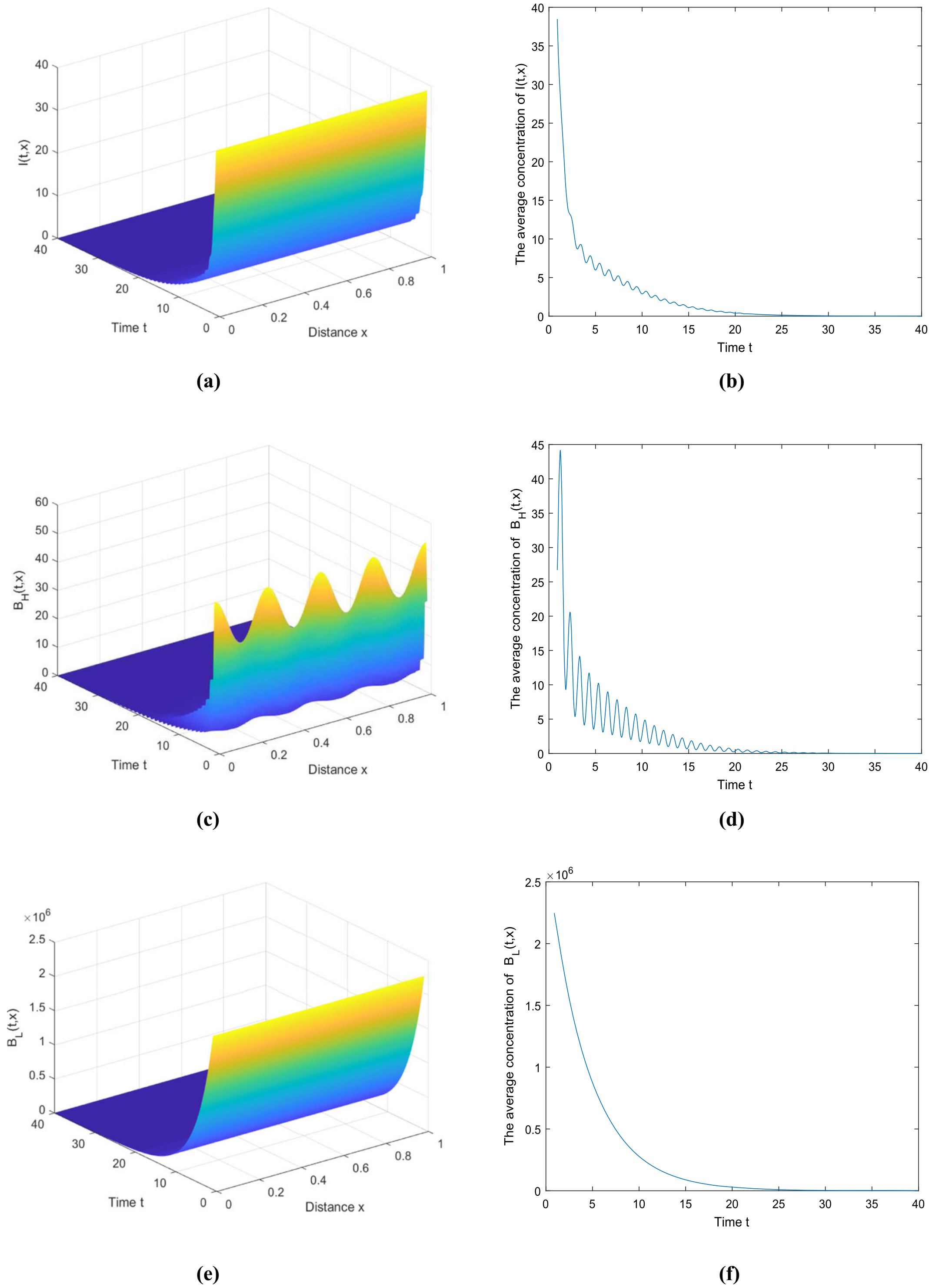

FIG. 3. Time variation of $I(t,x)$, $B_H(t,x)$ and $B_L(t,x)$ with $\xi(t,x) = 0.5 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t))$, $\beta_h(t,x) = 0.05 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t))$, $\beta_H(t,x) = 0.05 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t))$, $\beta_L(t,x) = 0.05 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t))$

TABLE 2. *Descriptions and estimations of initial variables and parameters of model* (5.1)

	Values	Units	Sources
Initial variables			
$S(0,x)$	14,349,974	Persons	[33]
$V(0,x)$	650,000	Persons	[9]
$I(0,x)$	114	Persons	[9]
$R(0,x)$	8138	Persons	[9]
$B_H(0,x)$	$\xi I(0,x)$	Persons	Assumed
$B_L(0,x)$	$\chi B_H(0,x)$	Persons	Assumed
$Q(0,x)$	8252	Persons	[9]
Parameters			
A	8359	Persons week^{-1}	[33]
μ	3.3×10^{-4}	Week^{-1}	[33]
d_I	0	Week^{-1}	[9]
k_H	$10^6/700$	Cells ml^{-1}	[21]
k_L	10^6	Cells ml^{-1}	[21]
σ	0.5	–	[55]
Υ	1.4	Week^{-1}	[21]
ξ	70	Cells ml^{-1} persons^{-1} week^{-1}	[21]
χ	33.6	Week^{-1}	[21]
δ_L	0.23	Week^{-1}	[21]
β_h	9.1161×10^{-9}	Persons week^{-1}	[5]
β_H	1.2667×10^{-6}	Week^{-1}	[5]
β_L	2.4301×10^{-5}	Week^{-1}	[5]
τ	1.4	Day^{-1}	[2]

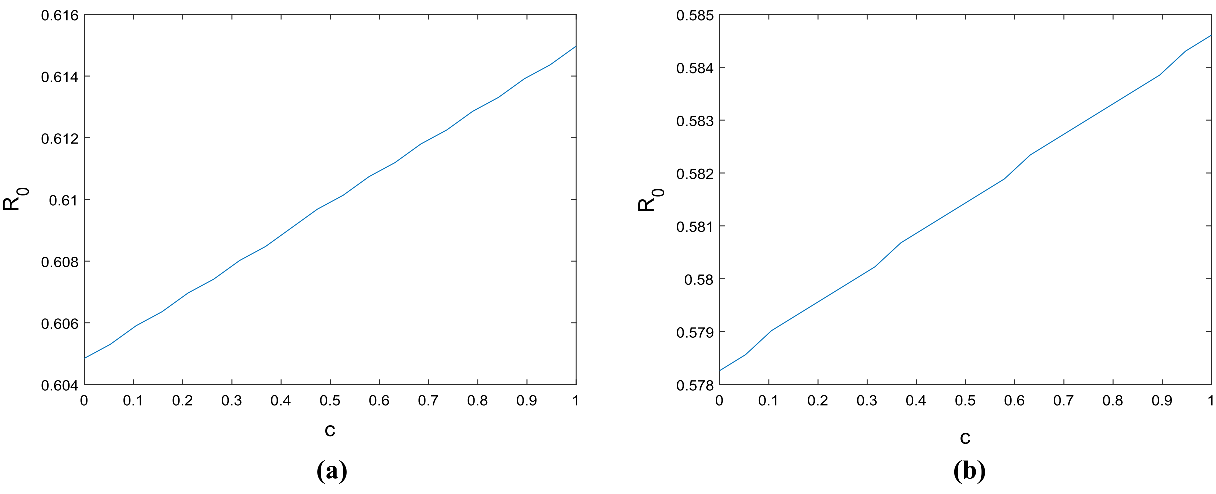

FIG. 4. The effects of spatial heterogeneous and time periodictiy on $\mathscr{R}_0$. **a** $\xi(t,x) = \overline{\xi}\times(1+0.8c\cos(8\pi x))\times(1+0.5\sin(2\pi t))$, $\beta_h(t,x)=\overline{\beta}_h\times(1+0.5\cos(2\pi t))$, $\beta_H(t,x)=\overline{\beta}_H\times(1+0.5\cos(2\pi t))$, $\beta_L(t,x)=\overline{\beta}_L\times(1+0.5\cos(2\pi t))$. **b** $\xi(t,x)=\overline{\xi}\times(1+0.8c\sin(8\pi x))\times(1+0.5\sin(2\pi t))$, $\beta_h(t,x)=\overline{\beta}_h\times(1+0.5\sin(2\pi t))$, $\beta_H(t,x)=\overline{\beta}_H\times(1+0.5\sin(2\pi t))$, $\beta_L(t,x)=\overline{\beta}_L\times(1+0.5\sin(2\pi t))$

5.2. Long-term behaviour

We assume that β_h, β_H and β_L represent, respectively, the transmission coefficient of contact with infected individuals, hyperinfectious vibrios and hypoinfectious vibrios. Denote by $\overline{f}$ the mean value of parameter

f, where f is one of A, μ, d, β_h, β_H, β_L, k_H, k_L, ϖ, σ, Υ, ξ, χ, δ_L. For the sake of simplicity, we assume that the region is a cross-sectional area of the $1\ mm \times 1\ mm$ thin wire and $\Omega = (0, 1)$. We assume that the diffusion coefficients of S, V, I, R, B_H, B_L, are, respectively, $D_1 = 0.09648$ mm^2 day^{-1}, $D_2 = 0.05$ mm^2 day^{-1}, $D_3 = 0.08$ mm^2 day^{-1}, $D_4 = 0.08$ mm^2 day^{-1}, $D_5 = 0.17$ mm^2 day^{-1}, $D_6 = 0.11$ mm^2 day^{-1}. Assume that

$$\xi(t,x) = 12 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t)),\ \beta_h(t,x) = 180 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t)),$$
$$\beta_H(t,x) = 180 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t)),\ \beta_L(t,x) = 180 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t)),$$

and other parameters are fixed as the mean values. We find that the solution of model (5.1) converges to a positive periodic steady state (see Fig. 2) and the disease is persistent. We take

$$\xi(t,x) = 0.5 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t)),\ \beta_h(t,x) = 0.05 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t)),$$
$$\beta_H(t,x) = 0.05 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t)),\ \beta_L(t,x) = 0.05 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t)),$$

and other parameters are fixed as the mean values. We observe that the solution of model (5.1) converges to zero (see Fig. 3), and the disease vanishes.

5.3. Effects of parameters on $\mathscr{R}_0$

It is well known that $\mathscr{R}_0$ plays an important role in the prevention, intervention and control on the spread of the cholera transmission, which determines transmission risk whether or not the disease persists. However, it is difficult to theoretically study the influence of environmental heterogeneity on the transmission risk $\mathscr{R}_0$ in the periodic system. We now illustrate the effects of spatial heterogeneity and temporal periodicity on $\mathscr{R}_0$ numerically. Let

$$\xi(t,x) = \overline{\xi} \times (1 + 0.8c\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t)),\ \beta_h(t,x) = \overline{\beta}_h \times (1 + 0.5\cos(2\pi t)),$$
$$\beta_H(t,x) = \overline{\beta}_H \times (1 + 0.5\cos(2\pi t)),\ \beta_L(t,x) = \overline{\beta}_L \times (1 + 0.5\cos(2\pi t)).$$

We find that $\mathscr{R}_0$ increases as heterogeneous coefficient c increases in the periodic system (see Fig. 4a). This implies that spatial heterogeneity may increase transmission risk $\mathscr{R}_0$ in the periodic system, and we may underestimate $\mathscr{R}_0$ if the spatial averaged system is used. Let

$$\xi(t,x) = \overline{\xi} \times (1 + 0.8c\sin(8\pi x)) \times (1 + 0.5\sin(2\pi t)),\ \beta_h(t,x) = \overline{\beta}_h \times (1 + 0.5\sin(2\pi t)),$$
$$\beta_H(t,x) = \overline{\beta}_H \times (1 + 0.5\sin(2\pi t)),\ \beta_L(t,x) = \overline{\beta}_L \times (1 + 0.5\sin(2\pi t)).$$

We also find that $\mathscr{R}_0$ increases as heterogeneous coefficient c increases in the periodic system (see Fig. 4b). This implies that spatial heterogeneity may increase transmission risk $\mathscr{R}_0$ in the periodic system, and we may underestimate $\mathscr{R}_0$ if the spatial averaged system is used. Figure 5 shows that decreasing the transmission coefficient of contact with infected individuals, the transmission coefficient of contact with hyperinfectious vibrios and the transmission coefficient of contact with hypoinfectious vibrios can reduce $\mathscr{R}_0$. Figures 6 and 7 show that increasing the vaccination rate of susceptible individuals ϖ and vaccine protective efficacy $1 - \sigma$ can reduce $\mathscr{R}_0$ and decrease the number of infected individuals, which has a positive impact on cholera control.

6. Conclusion

In this paper, we developed a time periodic reaction–diffusion equation model with latent period and explored the multiple effects of spatial mobility, spatial heterogeneity and the seasonality on the transmission of cholera. We first introduced the basic reproduction number $\mathscr{R}_0$ and then discussed the threshold dynamics in terms of $\mathscr{R}_0$. It has shown that the infection-free ω-periodic solution of model (2.4) is globally attractive if $\mathscr{R}_0 < 1$, while there is at least one endemic ω-periodic solution and the disease is uniformly

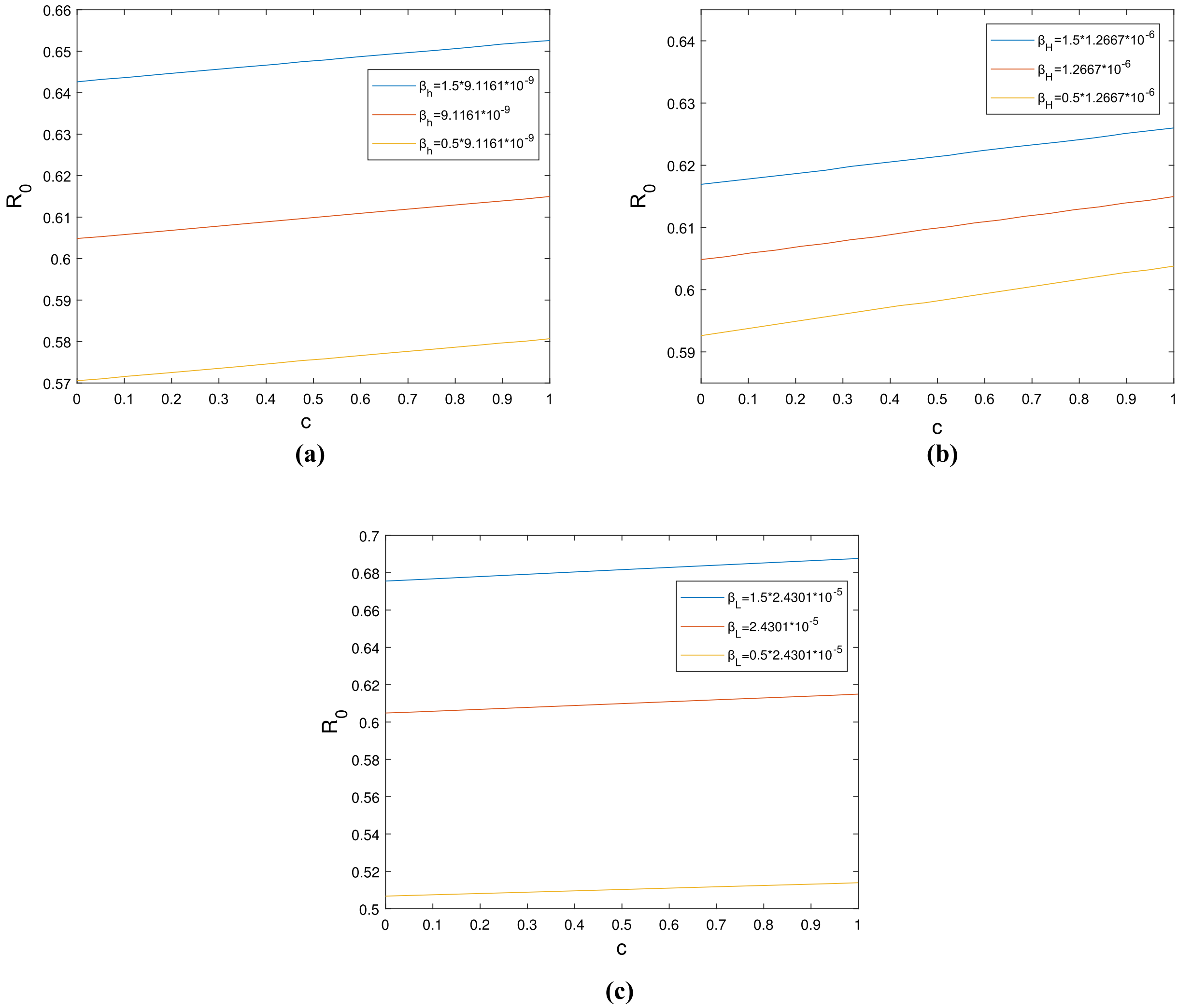

FIG. 5. The effects of parameters on $\mathscr{R}_0$. **a** The effect of the transmission coefficient of contact with infected individuals on $\mathscr{R}_0$. **b** The effect of the transmission coefficient of contact with hyperinfectious vibrios on $\mathscr{R}_0$. **c** The effect of the transmission coefficient of contact with hypoinfectious vibrios on $\mathscr{R}_0$

persistent. In the case where all the parameters are constants, we investigated the global attractivity of the endemic steady state by using Lyapunov functionals when $\mathscr{R}_0 > 1$. Finally, a case study of the cholera outbreak in Somalia was presented numerically. Note that in this paper we only studied the global stability of the endemic steady state in the case where all coefficients are constants. In [12,13,39], Cui et al. discussed asymptotic profiles of endemic steady states for the epidemic models in spatially heterogeneous case. In [53], Wu and Zou discussed profiles of a diffusive host-pathogen model with different diffusion rates. However, to the best of our knowledge, all these methods cannot be directly applied to time periodic reaction–diffusion equation models with latent period. We leave these interesting problems for further investigations.

It is well known that $\mathscr{R}_0$ plays an important role in the prevention, intervention and control on the spread of the cholera transmission, which determines transmission risk whether or not the disease

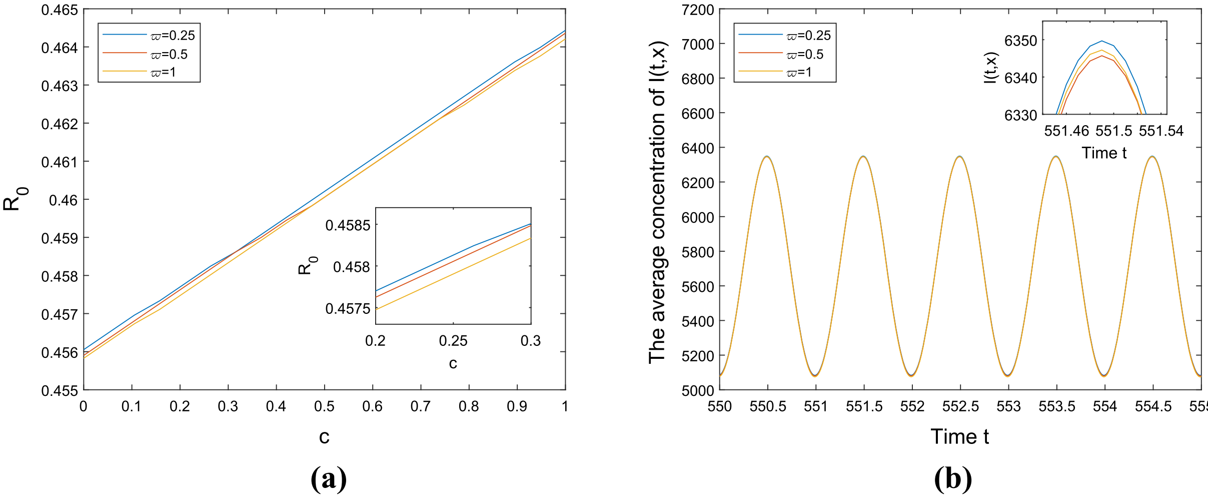

Fig. 6. **a** The effects of the vaccination rate of susceptible individuals on $\mathscr{R}_0$. **b** The effects of the vaccination rate of susceptible individuals on infected individuals with $\xi(t,x) = 12 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t))$, $\beta_h(t,x) = 180 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t))$, $\beta_H(t,x) = 180 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t))$, $\beta_L(t,x) = 180 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t))$

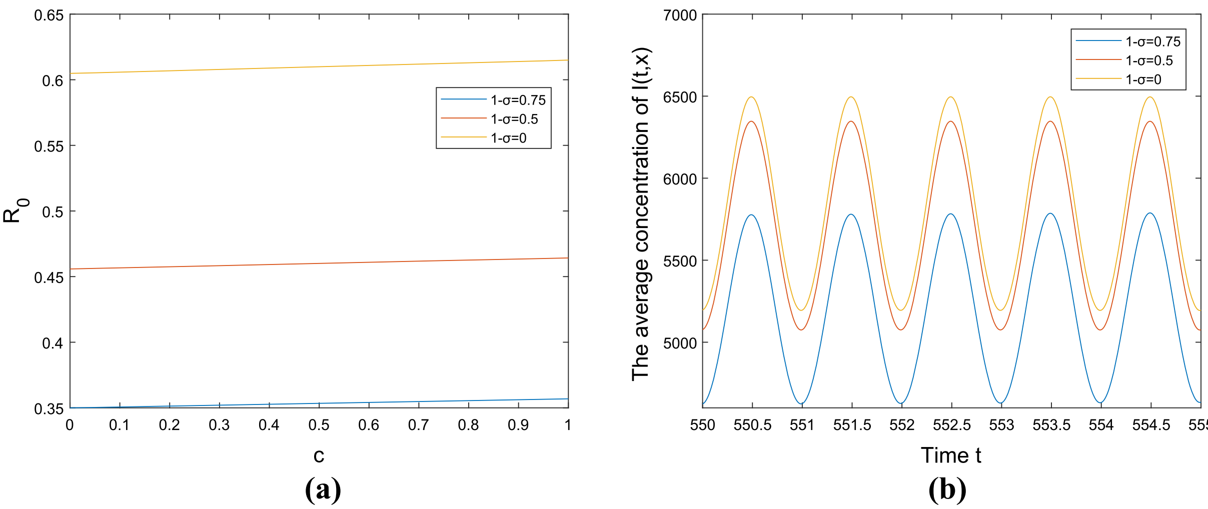

Fig. 7. **a** The effects of the protective efficacy of vaccine on $\mathscr{R}_0$. **b** The effects of the protective efficacy of vaccine on infected individuals with $\xi(t,x) = 12 \times \overline{\xi} \times (1 + 0.8\cos(8\pi x)) \times (1 + 0.5\sin(2\pi t))$, $\beta_h(t,x) = 180 \times \overline{\beta}_h \times (1 + 0.5\sin(2\pi t))$, $\beta_H(t,x) = 180 \times \overline{\beta}_H \times (1 + 0.5\sin(2\pi t))$, $\beta_L(t,x) = 180 \times \overline{\beta}_L \times (1 + 0.5\sin(2\pi t))$

persists. In the current work, we defined the basic reproduction number as the spectral radius of the next-generation infection operator. Unfortunately, we could not derive a clear formula of $\mathscr{R}_0$ if the parameters are spatially and temporally heterogeneous. From the numerical computations, we observed that environmental heterogeneity has an effect on the transmission risk $\mathscr{R}_0$. Our results have suggested that it is possible to underestimate the transmission risk $\mathscr{R}_0$ in the periodic system if the spatial averaged system is used, based on some experimental data.

In addition, we have observed that decreasing the transmission coefficient of contact with infected individuals, the transmission coefficient of contact with hyperinfectious vibrios and the transmission coefficient of contact with hypoinfectious vibrios can reduce the basic reproduction number $\mathscr{R}_0$. We have also found that increasing the vaccination rate of susceptible individuals ϖ and vaccine protective efficacy $1-\sigma$ can reduce the basic reproduction number $\mathscr{R}_0$ and decrease the number of infected individuals, which has a positive impact on cholera control in the population.

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