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Normalized solutions to a kind of fractional Schrödinger equation with a critical nonlinearity

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Abstract. In this paper, we study normalized solutions of the fractional Schrödinger equation with a critical nonlinearity

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{2_s^*-2}u \\ \int\limits_{\mathbb{R}^N} u^2 = a^2 \end{cases}$$

where $N \geq 2$, $s \in (0,1)$, $a > 0$, $2 < p < 2_s^* = \frac{2N}{2N-2s}$ and $(-\Delta)^s$ is the fractional Laplace operator. We prove the existence of the normalized solutions under different conditions on a, p, s and N.

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1. Introduction and main results

In this paper, we study normalized solutions for the fractional Schrödinger equation with a critical nonlinearity $|u|^{2_s^*-2}u$

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{2_s^*-2}u \\ \int\limits_{\mathbb{R}^N} u^2 = a^2, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $a > 0$, $N \geq 2$, $2 < p < 2_s^* \doteq 2N/(N-2s)$ and the fractional Laplacian $(-\Delta)^s$ is defined by

$$\begin{aligned} (-\Delta)^s u(x) &= -\frac{C(N,s)}{2} \text{P.V.} \int\limits_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} dy \\ &= C(N,s) \text{P.V.} \int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \end{aligned}$$

with a positive constant $C(N,s)$. For convenience, we normalize the factor $\frac{C(N,s)}{2} = 1$.

The study of (1.1) originates from investigating the standing wave solutions of the following fractional Schrödinger equation with nonlinearities

$$\begin{cases} i\frac{\partial \Phi}{\partial t} = (-\Delta)^s \Phi - f(|\Phi|)\Phi, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ \int\limits_{\mathbb{R}^N} |\Phi(x,t)|^2 dx = a^2, \end{cases} \tag{1.2}$$

where $0 < s < 1$, $N \geq 2$, $2 < p < 2_s^* \doteq 2N/N-2s$, $f(t) = t^{p-2} + t^{2_s^*-2}$ and i denotes the imaginary unit. To search for standing wave solutions $\Phi(x,t) = e^{-i\lambda t}u(x)$ for (1.2), one leads to problem (1.1). In quantum mechanics, for the wave function $\Phi = \Phi(x,t) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{C}$, $|\Phi(x,t)|^2$ represents the probability density of the particles appearing in space x at time t. For single particle system, physicists are interested in normalized solutions, that is, solutions satisfying $\int_{\mathbb{R}^N} |\Phi(x,t)|^2 dx = 1$. For n body system of Bose–Einstein condensate (see [3]), the wave function for the whole condensate is $\phi(x,t) = \sqrt{n}\Phi(x,t)$ and the wave function is normalized according to the total number of the particles, i.e., $\int_{\mathbb{R}^N} |\phi(x,t)|^2 dx = n$.

The operator $(-\Delta)^s$ arises in physics, biology, chemistry and finance and can be seen as the infinitesimal generators of Lévy stable diffusion process (see [1,2]). And, $(-\Delta+m^2)^{\frac{1}{2}}$ appears naturally in quantum mechanics, where m is the mass of the particle under consideration; see [17]. The study of nonlinear equations involving a fractional Laplacian has attracted much attention from many mathematicians working in different fields. Caffarelli et al investigated a fractional Laplacian with free boundary conditions; see [7,8]. Chang and González [9] investigated this operator which also appears in conformal geometry. Silvestre [23] studies the regularity problems for the obstacle problem of the fractional Laplacian. Felmer, Quaas and Tan [14] studied the existence, regularity and symmetry of positive solutions to the fractional Schrödinger equations in the whole space. In [11], Coti Zelati and Nolasco obtained the existence of a ground state of some fractional Schrödinger equation involving the operator $(-\Delta + m^2)^{\frac{1}{2}}$ appearing in quantum mechanics. We refer to [6,13] for more details on the fractional operator and applications.

Normalized solutions to Schrödinger equations, that is, equations similar to (1.1) as $s = 1$, where the energy is unbounded from below on the L^2-constraint, were first studied in the paper [16]. For quite a long time, it is the only one in this aspect. More recently, however, problems of this type received much attention, e.g., see [22] and the references therein. For problem (1.1), $p = 2 + \frac{4s}{N}$ is the L^2-critical exponent. But the appearance of the critical term $|u|^{2_s^*-2}u$ in (1.1) implies that the energy functional is always unbounded below whether p is smaller or larger than the L^2-critical exponent $p = 2 + \frac{4s}{N}$. In the recent paper [18], the authors deal with the existence of normalized ground states for the fractional Schrödinger equations with combined nonlinearities as follows:

$$(-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} u^2 = a^2. \tag{1.3}$$

But they only consider the Sobolev sub-critical case $p, q < 2_s^*$. They found that the exponents p and q affect the geometry of the corresponding energy functional and the solvability of the above problem. Indeed, they obtained the following results.

- In the purely L^2-subcritical case $2 < q < p < 2 + \frac{4s}{N}$, problem (1.3) admits a ground state for any $a > 0$;
- In the purely L^2-subcritical case $2 + \frac{4s}{N} < q < p < 2_s^*$, problem (1.3) admits a radial solution for any $a > 0$;
- If $2 < q < p = 2 + \frac{4s}{N}$, then problem (1.3) admits a positive radial minimizer for any $a \in (0, (\int_{\mathbb{R}^N} |Q_{N,p}|^2)^{\frac{1}{2}})$, where the function $Q_{N,p}$ appears in (1.4);
- If $2 + \frac{4s}{N} = q < p < 2_s^*$, then problem (1.3) admits a radial solution for any $a \in (0, a(s,N,p,q))$, where $a(s,N,p,q)$ is some constant;
- If $2 < q < 2 + \frac{4s}{N} < p < 2_s^*$, then problem (1.3) admits two radial solutions for any $a \in (0, a(s,N,p,q))$, where $a(s,N,p,q)$ is some constant.

Motivated by the above paper, the paper [21] where the normalized ground states for NLS equations in the Sobolev critical case were considered and the pioneering work of Brezis and Nirenberg [5] on investigating Sobolev critical component problem with a lower term perturbation, we address the problem

(1.1):

$$(-\Delta)^s u = \lambda u + |u|^{p-2}u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^N, \int\limits_{\mathbb{R}^N} u^2 = a^2,$$

for some $\lambda \in \mathbb{R}$, where $2 < p < 2_s^*$.

In order to state our main results, we need to know some of the constants from the following fractional Gagliardo–Nirenberg–Sobolev (GNS) inequality. That is, there exists a best constant $C(s, N, p)$ depending on s, N and p such that for any $u \in H^s(\mathbb{R}^N)$,

$$|u|_p^p \leq C(s, N, p)|u|_2^{(1-\gamma_{p,s})p}|(-\Delta)^{\frac{s}{2}}u|_2^{\gamma_{p,s}p} \tag{1.4}$$

where $\gamma_{p,s} \doteq \frac{N(p-2)}{2ps}$. The constant $C(s, N, p)$ can be achieved by $Q_{N,p}$; see [15]. For the problem obtained from (1.1) by removing the critical exponent term, we obtain one of its normalized solutions by rescaling $Q_{N,p}$. From $\gamma_{p,s}p = 2$, we get $p = 2 + \frac{4s}{N}$ which is called the L^2-critical exponent for the problem (1.1).

The main results of this paper are as follows.

Theorem 1.1. *Let $N \geq 2$, $s \in (0, 1)$, $2 < p < 2 + \frac{4s}{N}$ and assume that $0 < a < \min\{\alpha_1, \alpha_2\}$ where*

$$\alpha_1 \doteq \left\{ \frac{p(2_s^* - 2)}{2C(s, N, p)(2_s^* - p\gamma_{p,s})} \left(\frac{2_s^* S_s^{\frac{2_s^*}{2}} (2 - p\gamma_{p,s})}{2(2_s^* - p\gamma_{p,s})} \right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}} \right\}^{\frac{1}{p(1-\gamma_{p,s})}} \tag{1.5}$$

and

$$\alpha_2 \doteq \left\{ \frac{22_s^* s}{N\gamma_{p,s}C(s, N, p)(2_s^* - p\gamma_{p,s})} \left(\frac{\gamma_{p,s} S_s^{\frac{N}{2s}}}{2 - p\gamma_{p,s}} \right)^{\frac{2-p\gamma_{p,s}}{2}} \right\}^{\frac{1}{p(1-\gamma_{p,s})}}, \tag{1.6}$$

where S_s is defined in (2.1). Then problem (1.1) has a couple of solutions $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$. Moreover,

$$E(u_a) = \inf_{u \in V(a)} E(u) = \inf_{u \in V(a)^+} E(u) = \inf_{u \in A_k} E(u), \tag{1.7}$$

for some suitable small constant $k > 0$, where $V(a)$ is the Pohozaev manifold defined in Lemma 2.1, the set $V(a)^+$ is defined in (3.2) and

$$A_k = \left\{ u \in S(a) : \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)} < k \right\}.$$

Theorem 1.2. *Let $N \geq 2$, $s \in (0, 1)$, $N^2 > 8s^2$, $p = 2 + \frac{4s}{N}$ and assume that $0 < a < \alpha_3$ where*

$$\alpha_3 \doteq \left(\frac{p}{2C(s, N, p)} \right)^{\frac{1}{p-2}}. \tag{1.8}$$

Then problem (1.1) has a couple of solutions $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$ such that

$$E(u_a) = \inf_{u \in V(a)} E(u) = \inf_{u \in V(a)^-} E(u)$$

where $V(a)^-$ is defined in (3.2).

Theorem 1.3. *Let $N \geq 2$, $s \in (0, 1)$, $N^2 > 8s^2$, $p > 2 + \frac{4s}{N}$ and assume that $0 < a < \alpha_4$ where*

$$\alpha_4 \doteq (\gamma_{p,s})^{-\frac{1}{p(1-\gamma_{p,s})}} S_s^{\frac{N}{4s}}. \tag{1.9}$$

Then problem (1.1) has a couple of solutions $(u_a, \lambda_a) \in S(a) \times \mathbb{R}$ such that

$$E(u_a) = \inf_{u \in V(a)} E(u) = \inf_{u \in V(a)^-} E(u).$$

Remark 1.4. The threshold α_4 in Theorem 1.3 is finite. Actually we can relax to $\alpha_4 = +\infty$ as $2 \leq N \leq 4s$; we refer to Remark 5.3 for more details.

Dimension $N = 1$ also admits a Sobolev critical exponent for the problem (1.3) if $s < \frac{1}{2}$. But the compact embedding $H^s_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ we need in our proofs only holds for $N \geq 2$ and $p \in (2, 2^*_s)$. For convenience, in this paper we only deal with the case $N \geq 2$, although the case $N = 1$ can also be dealt with by combining some other techniques.

For the proofs of the theorems here, we borrow the main strategy in [21], including the decomposition of corresponding Pohozave manifold into three disjoint submanifolds. But extra difficulties still occur and more complicated calculations are needed due to the nonlocal nature of the problem. For example, in the proofs Theorem 1.2 and Theorem 1.3, in order to control the Gagliardo seminorm of u_ε, pointwise estimates on u_ε are not enough. We need additional skills different from [21] for the estimates of $\|u_\varepsilon\|_p$. Moreover, the condition $N^2 > 8s^2$ is crucial to obtain that the mountain pass level is strictly less than the threshold and the proof is different from the one for the corresponding Laplacian equation in [21].

2. Preliminaries

To prove our theorems, we need some notations and useful preliminary results.

Throughout this paper, we denote B_r the open ball of radius r with center at 0 in $\mathbb{R}^N$, and $|\cdot|_p$ the usual norm of the space $L^p(\mathbb{R}^N)$ for $p \geq 1$. Let $H = H^s(\mathbb{R}^N)$ and $\|\cdot\|$ be its norm. A generic positive constant is denoted by C, C_1, or C_2..., which may change from line to line. Let $\mathbb{H} = H \times \mathbb{R}$ with the usual scalar product

$$\langle \cdot, \cdot \rangle_{\mathbb{H}} = \langle \cdot, \cdot \rangle_H + \langle \cdot, \cdot \rangle_{\mathbb{R}}$$

and the corresponding norm

$$\|(\cdot, \cdot)\|^2_{\mathbb{H}} = \|\cdot\|^2 + |\cdot|^2_{\mathbb{R}}.$$

We denote the best constant of the imbedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ by

$$S_s = \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{|(-\Delta)^{\frac{s}{2}} u|_2^2}{|u|^2_{2^*_s}}, \tag{2.1}$$

where $D^{s,2}(\mathbb{R}^N)$ denotes the completion of the space $C^\infty_c(\mathbb{R}^N)$ with the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = |(-\Delta)^{\frac{s}{2}} u|_2.$$

Ii is well-known that S_s is achieved by

$$U_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}}, \tag{2.2}$$

where $\varepsilon > 0$ is a parameter [12].

Solutions of (1.1) can be obtained as the critical points of associated energy functional

$$E(u) = \frac{1}{2} \int\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy - \frac{1}{p} \int_{\mathbb{R}^N} |u(x)|^p dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |u(x)|^{2^*_s} dx$$

defined on the constraint manifold

$$S(a) \doteq \left\{ u \in H^s(\mathbb{R}^N) : \Psi(u) = \frac{1}{2} a^2 \right\},$$

where $\Psi(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} u^2$ and

$$H^s(\mathbb{R}^N) \doteq \left\{ u \in L^2(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 = \int\int\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy < \infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^s}^2 = \int\limits_{\mathbb{R}^N} |u(x)|^2 + \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2.$$

In this paper, we only need to find the critical points of the functional E on $H_r^s(\mathbb{R}^N) \bigcap S(a)$, since $H_r^s(\mathbb{R}^N) \doteq \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}$ is a natural constraint (e.g., see [24]). It is also well-known that $H_r^s(\mathbb{R}^N)$ is compactly embedded into $L^p(\mathbb{R}^N)$ for any $p \in (2, 2_s^*)$.

In this paper, the useful fiber map preserving the L^2-norm

$$\tau \star u \doteq e^{\frac{N\tau}{2}} u(e^\tau x), \qquad \text{for a.e. } x \in \mathbb{R}^N,$$

which was firstly introduced in [16], is also used. By direct calculation, we have

$$|\tau \star u|_2^2 = |u|_2^2, \quad |\tau \star u|_p^p = e^{ps\gamma_{p,s}\tau} |u|_p^p$$

and

$$|(-\Delta)^{\frac{s}{2}} (\tau \star u)|_2^2 = e^{2s\tau} |(-\Delta)^{\frac{s}{2}} u|_2^2.$$

Define a auxiliary functional $I : \mathbb{H} \to \mathbb{R}$ by

$$I(u,\tau) = E(\tau \star u) = \frac{1}{2} e^{2s\tau} |(-\Delta)^{\frac{s}{2}} u|_2^2 - \frac{e^{ps\gamma_{p,s}\tau}}{p} |u|_p^p - \frac{e^{2_s^* s\tau}}{2_s^*} |u|_{2_s^*}^{2_s^*}.$$

The Pohozaev identity plays an important role in our discussion. We give it in the following lemma; for more details, see [10,16].

Lemma 2.1. *Let $(u, \lambda) \in S(a) \times \mathbb{R}$ be a weak solution of problem (1.1). Then u belongs to the set*

$$V(a) \triangleq \{u \in H^s(\mathbb{R}^N) : P(u) = 0\}$$

where

$$P(u) = |(-\Delta)^{\frac{s}{2}} u|_2^2 - \gamma_{p,s} |u|_p^p - |u|_{2_s^*}^{2_s^*}.$$

We define $\varphi_u(\tau) \doteq I(u, \tau)$ for any $u \in S(a)$ and $\tau \in \mathbb{R}$. Then

$$\begin{aligned}(\varphi_u)'(\tau) &= s \left(e^{2s\tau} |(-\Delta)^{\frac{s}{2}} u|_2^2 - \gamma_{p,s} e^{ps\gamma_{p,s}\tau} |u|_p^p - e^{2_s^* s\tau} |u|_{2_s^*}^{2_s^*} \right) \\ &= s \left(|(-\Delta)^{\frac{s}{2}} (\tau \star u)|^2 - \gamma_{p,s} |\tau \star u|_p^p - |\tau \star u|_{2_s^*}^{2_s^*} \right).\end{aligned}$$

Therefore, we have

Lemma 2.2. *For any $u \in S(a)$, $\tau \in \mathbb{R}$ is a critical point of $\varphi_u(\tau)$ if and only if $\tau \star u \in V(a)$. Particularly, $u \in V(a)$ if and only if 0 is a critical point for $\varphi_u(\tau)$.*

Remark 2.3.

$$\text{The map } (u,\tau) \in \mathbb{H} \mapsto \tau \star u \in H \quad \text{is continuous;} \tag{2.3}$$

see [4, Lemma3.5].

3. L^2-subcritical perturbation case

In this section, we deal with the L^2-subcritical perturbation case $2 < p < \frac{4s}{N} + 2$ and give the proof of Theorem 1.1. As in [18,21,22], we firstly consider a decomposition of $V(a)$. In view of Lemma 2.2, we define the following sets:

$$\begin{aligned} V(a)^+ &\doteq \left\{u \in V(a) : 2|(-\Delta)^{\frac{s}{2}}u|_2^2 > p\gamma_{p,s}^2|u|_p^p + 2_s^*|u|_{2_s^*}^{2_s^*}\right\} \\ &= \left\{u \in V(a) : (\varphi_u)''(0) > 0\right\} \end{aligned} \tag{3.1}$$

$$\begin{aligned} V(a)^0 &\doteq \left\{u \in V(a) : 2|(-\Delta)^{\frac{s}{2}}u|_2^2 = p\gamma_{p,s}^2|u|_p^p + 2_s^*|u|_{2_s^*}^{2_s^*}\right\} \\ &= \left\{u \in V(a) : (\varphi_u)''(0) = 0\right\}, \end{aligned}$$

$$\begin{aligned} V(a)^- &\doteq \left\{u \in V(a) : 2|(-\Delta)^{\frac{s}{2}}u|_2^2 < p\gamma_{p,s}^2|u|_p^p + 2_s^*|u|_{2_s^*}^{2_s^*}\right\} \\ &= \left\{u \in V(a) : (\varphi_u)''(0) < 0\right\}. \end{aligned} \tag{3.2}$$

It is clearly that

$$V(a) = V(a)^+ \cup V(a)^0 \cup V(a)^-.$$

Lemma 3.1. *Let $2 < p < \frac{4s}{N} + 2$ and $a < \alpha_1$, where*

$$\alpha_1 \doteq \left\{ \frac{p(2_s^* - 2)}{2C(s,N,p)(2_s^* - p\gamma_{p,s})} \left(\frac{2_s^* S_s^{\frac{2_s^*}{2}} (2 - p\gamma_{p,s})}{2(2_s^* - p\gamma_{p,s})} \right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}} \right\}^{\frac{1}{p(1-\gamma_{p,s})}}.$$

Then $V(a)^0 = \emptyset$ and $V(a)$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^N)$.

Proof. Suppose on the contrary that $V(a)^0 \neq \emptyset$. Taking $u \in V(a)^0$, we have

$$\begin{cases} 2|(-\Delta)^{\frac{s}{2}}u|_2^2 = p\gamma_{p,s}^2|u|_p^p + 2_s^*|u|_{2_s^*}^{2_s^*} \\ |(-\Delta)^{\frac{s}{2}}u|_2^2 = \gamma_{p,s}|u|_p^p + |u|_{2_s^*}^{2_s^*}. \end{cases}$$

Combining the above equalities with (1.4) and (2.1), we deduce that

$$\begin{cases} |(-\Delta)^{\frac{s}{2}}u|_2^2 = \frac{2_s^*-p\gamma_{p,s}}{2-p\gamma_{p,s}}|u|_{2_s^*}^{2_s^*} \le \frac{2_s^*-p\gamma_{p,s}}{(2-p\gamma_{p,s})S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*} \\ |(-\Delta)^{\frac{s}{2}}u|_2^2 = \gamma_{p,s}\frac{2_s^*-p\gamma_{p,s}}{2_s^*-2}|u|_p^p \le C(s,N,p)\gamma_{p,s}\frac{2_s^*-p\gamma_{p,s}}{2_s^*-2}a^{(1-\gamma_{p,s})p}|(-\Delta)^{\frac{s}{2}}u|_2^{\gamma_{p,s}p}. \end{cases}$$

By the fact $p\gamma_{p,s} < 2$ and the above inequality, we obtain that

$$\begin{aligned} a^{(1-\gamma_{p,s})p} &\ge \frac{(2_s^* - 2)}{\gamma_{p,s}C(s,N,p)(2_s^* - p\gamma_{p,s})} \left(\frac{2_s^* S_s^{\frac{2_s^*}{2}} (2 - p\gamma_{p,s})}{2(2_s^* - p\gamma_{p,s})} \right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}} \\ &\ge \frac{p(2_s^* - 2)}{2C(s,N,p)(2_s^* - p\gamma_{p,s})} \left(\frac{2_s^* S_s^{\frac{2_s^*}{2}} (2 - p\gamma_{p,s})}{2(2_s^* - p\gamma_{p,s})} \right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}} = \alpha_1^{(1-\gamma_{p,s})p} \end{aligned}$$

which contradicts to $a < \alpha_1$.

Next we verify that $V(a)$ is a smooth manifold of codimension 2 in H. Notice that

$$V(a) = \{u \in H : F(u) = (P(u), |u|_2^2 - a^2) = (0,0)\},$$

where P and $G(u) \doteq |u|_2^2 - a^2$ are of class C^1 in H. It suffices to prove that for any $u \in V(a)$ the range of $F'(u)$ is $\mathbb{R}^2$. Since

$$F'(u)v = \left(\int_{\mathbb{R}^N} 2(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v - p\gamma_{p,s}|u|^{p-2}uv - 2_s^*|u|^{2_s^*-2}uv, \int_{\mathbb{R}^N} 2uv \right)$$

for any $v \in H$ and $F'(u)u = (b, 2a^2)$ for $b = (2-p)\gamma_{p,s}|u|_p^p - (2_s^* - p)|u|_{2_s^*}^{2_s^*} < 0$ (by $u \in V(a)$), we have that $F'(u)$ is not surjective if and only if

$$\int_{\mathbb{R}^N} 2(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v - p\gamma_{p,s}|u|^{p-2}uv - 2_s^*|u|^{2_s^*-2}uv = \frac{b}{2a^2}\int_{\mathbb{R}^N} 2uv, \quad \forall v \in H.$$

Hence, u is a solution of the following equation

$$2(-\Delta)^s u = \frac{b}{a^2}u + p\gamma_{p,s}|u|^{p-2}u + 2_s^*|u|^{2_s^*-2}u \qquad \text{in } \mathbb{R}^N.$$

We can easily conclude that the following Pohozaev type identity

$$2|(-\Delta)^{\frac{s}{2}}u|_2^2 = p{\gamma_{p,s}}^2|u|_p^p + 2_s^*|u|_{2_s^*}^{2_s^*},$$

which is a contradiction to the fact that $u \in V(a)$. □

By the fractional GNS inequality (1.4), we deduce that for every $u \in H^s(\mathbb{R}^N) \bigcap S(a)$

$$E(u) \geq \frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2 - \frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}|(-\Delta)^{\frac{s}{2}}u|_2^{p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*}. \tag{3.3}$$

In order to explore the geometry of the functional $E(u)$, we introduce a function $h : \mathbb{R}^+ \to \mathbb{R}$ related to the right-hand side of (3.3)

$$h(t) \doteq \frac{1}{2}t^2 - \frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}t^{p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}t^{2_s^*}.$$

Since $p\gamma_{p,s} < 2 < 2_s^*$, it is easy to see that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$.

Lemma 3.2. *Assume that $a < \alpha_1$, where α_1 is defined in (1.5). Then the function h has a local strict minimum at negative level and a global strict maximum at positive level, and there exist two positive constants $R_1 > R_0$, both depending on a, such that $h(R_0) = h(R_1) = 0$ and $h(t) > 0$ if and only if $t \in (R_0, R_1)$.*

Proof. Notice that

$$h(t) = t^{p\gamma_{p,s}}\left(\frac{1}{2}t^{2-p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}t^{2_s^*-p\gamma_{p,s}} - \frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}\right).$$

So, we have $h(t) > 0$ if and only if $g(t) > 0$ for $t > 0$, where

$$g(t) \doteq \frac{1}{2}t^{2-p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}t^{2_s^*-p\gamma_{p,s}} - \frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}.$$

Since

$$g'(t) = \frac{2-p\gamma_{p,s}}{2}t^{1-p\gamma_{p,s}} - \frac{2_s^*-p\gamma_{p,s}}{2_s^* S_s^{\frac{2_s^*}{2}}}t^{2_s^*-p\gamma_{p,s}-1},$$

we have that g is strictly increasing on $(0,t_1)$ and decreasing $(t_1,+\infty)$ where

$$t_1=\left(\frac{2_s^* S_s^{\frac{2_s^*}{2}}(2-p\gamma_{p,s})}{2(2_s^*-p\gamma_{p,s})}\right)^{\frac{1}{2_s^*-2}}.$$

The maximum value of g on $(0,+\infty)$ is

$$\begin{aligned} g(t_1)&=\frac{(2_s^*-2)}{2(2_s^*-p\gamma_{p,s})}\left(\frac{2_s^* S_s^{\frac{2_s^*}{2}}(2-p\gamma_{p,s})}{2(2_s^*-p\gamma_{p,s})}\right)^{\frac{2-p\gamma_{p,s}}{2_s^*-2}}-\frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}\\ &=\frac{C(s,N,p)}{p}\alpha_1^{p-p\gamma_{p,s}}-\frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}. \end{aligned}$$

Since $a<\alpha_1$, there exist two constants R_0 and R_1 such that

$$h(t)\begin{cases} <0, & t\in(0,R_0)\text{ or }(R_1,+\infty),\\ =0, & t=R_0\text{ or }R_1,\\ >0, & t\in(R_0,R_1). \end{cases}$$

It follows that $h(t)$ has a global maximum at positive level in (R_0,R_1) and, by the fact that $h(0^+)=0^-$, a local minimum at negative level in $(0,R_0)$. By a direct calculation

$$h'(t)=t^{p\gamma_{p,s}-1}\left(t^{2-p\gamma_{p,s}}-\frac{1}{S_s^{\frac{2_s^*}{2}}}t^{2_s^*-p\gamma_{p,s}}-\gamma_{p,s}C(s,N,p)a^{p-p\gamma_{p,s}}\right)^{p\gamma_{p,s}-1}f(t)$$

and

$$f'(t)=(2-p\gamma_{p,s})t^{1-p\gamma_{p,s}}-\frac{2_s^*-p\gamma_{p,s}}{S_s^{\frac{2_s^*}{2}}}t^{2_s^*-p\gamma_{p,s}-1},$$

it is easy to see that $h'(t)=0$ if and only if $f=0$ for $t>0$, and f is strictly increasing on $(0,t_2)$, decreasing on $(t_2,+\infty)$ where

$$t_2=\left(\frac{S_s^{\frac{2_s^*}{2}}(2-p\gamma_{p,s})}{2_s^*-p\gamma_{p,s}}\right)^{\frac{1}{2_s^*-2}}.$$

Thus, f has at most two zeros on $(0,+\infty)$, which are necessarily the previously found local minimum and the global maximum of h. □

By the properties of the function h, we give the following lemma.

Lemma 3.3. *Let $2<p<\frac{4s}{N}+2$ and $a<\alpha_1$. Then for every $u\in S(a)$, $\varphi_u(\tau)$ has two critical points $s_u<t_u\in\mathbb{R}$ and two zeros $c_u<d_u$ with $s_u<c_u<t_u<d_u$. Moreover,*

(1) *$s_u\star u\in V(a)^+$ and $t_u\star u\in V(a)^-$, and if $\tau\star u\in V(a)$, then either $\tau=s_u$ or $\tau=t_u$;*
(2) *$|(-\Delta)^{\frac{s}{2}}(\tau\star u)|_2^2\le R_0$ for every $\tau<c_u$ and*

$$E(s_u\star u)=\min\left\{E(\tau\star u):\tau\in\mathbb{R}\text{ and }|(-\Delta)^{\frac{s}{2}}(\tau\star u)|_2^2\le R_0\right\}<0;\tag{3.4}$$

(3) *we have*

$$E(t_u\star u)=\max\left\{E(\tau\star u):\tau\in\mathbb{R}\right\}>0,\tag{3.5}$$

and $\varphi_u(\tau)$ is strictly decreasing and concave on $(t_u,+\infty)$;
(4) *the maps $u\in V(a)\mapsto s_u\in\mathbb{R}$ and $u\in V(a)\mapsto t_u\in\mathbb{R}$ are of class C^1.*

Proof. First, we show that $\varphi_u(\tau)$ has at least two critical points. From (3.3), we have

$$\varphi_u(\tau) = E(\tau \star u) \geq h(|(-\Delta)^{\frac{s}{2}}(\tau \star u)|_2) = h(e^{2s\tau}|(-\Delta)^{\frac{s}{2}}u|_2).$$

By Lemma 3.2, we deduce that $\varphi_u(\tau) > 0$ on $(\xi(R_0), \xi(R_1))$ where

$$\xi(R) = \frac{\log R - 2\log|(-\Delta)^{\frac{s}{2}}u|_2}{2s}.$$

Combining $\varphi_u(-\infty) = 0^-$, $\varphi_u(+\infty) = -\infty$ and noticing that $\varphi_u(\tau)$ is a C^2 function, we deduce that $\varphi_u(\tau)$ has at least two critical points $s_u < t_u$, with s_u a local minimum point on $(-\infty, \xi(R_0))$ at negative level and t_u a global maximum point at positive level. By the arguments similar to those in Lemma 3.2, we can get that $\varphi_u(\tau)$ have no other critical points. Then (3.5) holds and (3.4) follows from

$$|(-\Delta)^{\frac{s}{2}}(s_u \star u)|_2^2 = e^{2s_u s}|(-\Delta)^{\frac{s}{2}}u|_2^2 \leq e^{2\xi(R_0)s}|(-\Delta)^{\frac{s}{2}}u|_2^2 < R_0. \tag{3.6}$$

By the minimality of s_u, $(\varphi_u)''(s_u) \geq 0$. Recalling that $V(a)^0 = \emptyset$, we obtain that $s_u \star u \in V(a)^+$. Similarly, we have $t_u \star u \in V(a)^-$.

Moreover, noticing that $\varphi_u(s_u) < 0$, $\varphi_u(t_u) > 0$ and $\varphi_u(+\infty) = -\infty$, we can deduce that $\varphi_u(\tau)$ has two zeros $c_u < d_u$ with $s_u < c_u < t_u < d_u$. It is easy to verify that $\varphi_u(\tau)$ have no other zeros; otherwise, $\varphi_u(\tau)$ will has a third critical point.

Since

$$(\varphi_u)''(\tau) = s^2\left(2e^{2s\tau}|(-\Delta)^{\frac{s}{2}}u|_2^2 - p\gamma_{p,s}{}^2 e^{ps\gamma_{p,s}\tau}|u|_p^p - 2_s^* e^{2_s^* s\tau}|u|_{2_s^*}^{2_s^*}\right),$$

we have that $(\varphi_u)''(-\infty) = 0^-$. Combining $(\varphi_u)''(s_u) > 0$ and $(\varphi_u)''(t_u) < 0$, we deduce that $(\varphi_u)''(\tau)$ has two zeroes, which implies that $\varphi_u(\tau)$ has two inflection points. Arguing as before, $(\varphi_u)''(\tau)$ has exactly two inflection points. So $\varphi_u(\tau)$ is strictly decreasing and concave on $(t_u, +\infty)$.

It remains to prove that the maps $u \in V(a) \mapsto s_u \in \mathbb{R}$ and $u \in V(a) \mapsto t_u \in \mathbb{R}$ are of class C^1. Here, we apply the implicit function theorem to the C^1 function $\Phi(\tau, u) \doteq (\varphi_u)'(\tau)$ and use the facts that $\Phi(s_u, u) = 0$ and $\partial_\tau(\Phi(s_u, u) > 0$. Hence, $u \in V(a) \mapsto s_u \in \mathbb{R}$ is of class C^1. The proof for $u \in V(a) \mapsto t_u \in \mathbb{R}$ is similarly given. □

For $k > 0$, we define

$$A_k \doteq \left\{u \in S(a) : |(-\Delta)^{\frac{s}{2}}u|_2 < k\right\} \quad \text{and } m(a) \doteq \inf_{A_{R_0}} E.$$

As an immediate corollary, we have

Corollary 3.4. $\sup_{V(a)^+} E \leq 0 \leq \inf_{V(a)^-} E$ *and* $V(a)^+ \subset A_{R_0}$.

Furthermore, we have the following lemma.

Lemma 3.5. $-\infty < m(a) = \inf_{V(a)} E = \inf_{V(a)^+} E < 0$ *and*

$$m(a) < \inf_{\overline{A_{R_0}} \setminus A_{R_0-\rho}} E$$

for $\rho > 0$ *small enough.*

Proof. For $u \in A_{R_0}$, by (3.3), one has

$$E(u) \geq h(|(-\Delta)^{\frac{s}{2}}u|_2) \geq \min_{t\in[0,R_0]} h(t) > -\infty.$$

Moreover, by (3.6), we have $m(a) < 0$. By the above corollary, we know $m(a) \leq \inf_{V(a)^+} E$. On the other hand, if $u \in A_{R_0}$, we have $s_u \star u \in V(a)^+$. Hence, by (3.4), we have $m(a) \geq \inf_{V(a)^+} E$. From the above corollary, we get that $0 \leq \inf_{V(a)^-} E$, which implies that

$$\inf_{V(a)} E = \inf_{V(a)^+} E.$$

Finally by the continuity of h and $h(R_0)=0$, there exists $\rho>0$ such that $h(t)\geq m(a)/2$ on $[R_0-\rho,R_0]$. Therefore,

$$E(u)\geq h(|(-\Delta)^{\frac{s}{2}}u|_2)\geq \frac{m(a)}{2}>m(a),$$

for any $u\in\overline{A_{R_0}}\backslash A_{R_0-\rho}$, which completes the proof. □

Proof of Theorem 1.1. Take a minimizing sequence $\{w_n\}\subset H\bigcap S(a)$ for $E|_{A_{R_0}}$. We can assume that $\{w_n\}\subset H_r$ are radially decreasing for every n. Otherwise, we replace w_n by $|w_n|^*$, which is the Schwarz rearrangement of $|w_n|$. By Lemma 3.3, there exists a sequence $\{s_{w_n}\}$ such that $s_{w_n}\star w_n\in V(a)^+$ and $E(s_{w_n}\star w_n)\leq E(w_n)$ for every n. Besides, by Lemma 3.5, we obtain that $s_{w_n}\star w_n\in\overline{A_{R_0}}\backslash A_{R_0-\rho}$. Thus, $\{\bar{w}_n=s_{w_n}\star w_n\}$ is a new minimizing sequence for $E|_{A_{R_0}}$ with $\bar{w}_n\in H_r\cap V(a)^+$ and $|(-\Delta)^{\frac{s}{2}}\bar{w}_n|_2<R_0-\rho$.

By Ekeland's variational principle, there exists a new minimizing sequence $\{u_n\}$ satisfying

$$\begin{cases}\|u_n-\bar{w}_n\|\to 0, & \text{as } n\to\infty,\\ E(u_n)\to m(a), & \text{as } n\to\infty,\\ P(u_n)\to 0, & \text{as } n\to\infty,\\ E'|_{S(a)}(u_n)\to 0, & \text{as } n\to\infty.\end{cases}\tag{3.7}$$

Now, from its last property in (3.7), we apply the Lagrange multipliers rule to $\{u_n\}$. Then there exists a sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$E'(u_n)-\lambda_n\Psi'(u_n)\to 0 \quad \text{in} \quad H^{-1}.\tag{3.8}$$

Since $\{u_n\}\subset A_{R_0}$, we deduce that there exists $u_a\in H_r(\mathbb{R}^N)$ such that, up to a subsequence, $u_n\rightharpoonup u_a$ in H.

Step 1: We prove that, up to a subsequence, $\lambda_n\to\lambda<0$.

Since $\{u_n\}$ is bounded in H, by (3.8), we have

$$E'(u_n)u_n-\lambda_n\Psi'(u_n)u_n=o_n(1).\tag{3.9}$$

Therefore,

$$\lambda_n|u_n|_2^2=|(-\Delta)^{\frac{s}{2}}u_n|_2^2-|u_n|_p^p-|u_n|_{2_s^*}^{2_s^*}+o_n(1).\tag{3.10}$$

Using the fact $|u_n|_2^2=a^2$ and $\{u_n\}$ is bounded in H, we deduce that $\{\lambda_n\}$ is bounded. Hence, up to subsequence, $\lambda_n\to\lambda\in\mathbb{R}$. By noticing that $P(u_n)\to 0$, (3.10) and the embedding $H_r(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p\in(2,2_s^*)$, we have

$$\begin{aligned}\lambda a^2&=\lim_{n\to\infty}(|(-\Delta)^{\frac{s}{2}}u_n|_2-|u_n|_p^p-|u_n|_{2_s^*}^{2_s^*})\\&=(\gamma_{p,s}-1)|u_a|_p^p\leq 0,\end{aligned}\tag{3.11}$$

with $\lambda=0$ if and only if $u_a\equiv 0$. Therefore, we only need to prove that $u_a\neq 0$. We assume by contradiction that $u_a\equiv 0$. Up to a subsequence, let $|(-\Delta)^{\frac{s}{2}}u_n|_2^2\to l\in\mathbb{R}$. Since $P(u_n)\to 0$ and $u_n\to 0$ in $L^p(\mathbb{R}^N)$, then we have $|u_n|_{2_s^*}^{2_s^*}\to l$. By (2.1), one has

$$(l)^{\frac{2}{2_s^*}}\leq\frac{l}{S_s}.$$

So, we have

$$l=0 \;\text{ or }\; l\geq S_s^{\frac{N}{2s}}.$$

If $l>0$, we have

$$\begin{aligned} m(a)+o_n(1) &= E(u_n)=E(u_n)-\frac{1}{2_s^*}P(u_n)+o_n(1)\\ &=\frac{s}{N}|(-\Delta)^{\frac{s}{2}}u_n|_2^2+o_n(1)\\ &=\frac{s}{N}l+o_n(1), \end{aligned}$$

which contradicts to $m(a)<0$ in Lemma 3.5. If $l=0$, we have

$$|(-\Delta)^{\frac{s}{2}}u_n|_2^2\to 0,\quad |u_n|_{2_s^*}^{2_s^*}\to 0,$$

implying that $E(u_n)\to 0$, which is also a contradiction to $m(a)<0$.
Step 2: Since $\lambda<0$, we can define an equivalent norm of H, that is

$$\|u\|^2=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx-\lambda\int_{\mathbb{R}^N}|u|^2dx.$$

Since $u_n\rightharpoonup u_a$ in H, applying the Lebesgue convergence theorem to (3.8), one has

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_a(-\Delta)^{\frac{s}{2}}v-\lambda\int_{\mathbb{R}^N}u_av-|u|^{p-2}uv-|u_a|^{2_s^*-2}u_av=0,\quad \forall v\in H. \tag{3.12}$$

It follows from the Pohozaev identity that $P(u_a)=0$. Let $v_n=u_n-u_a\rightharpoonup 0$, by Brezis–Lieb lemma we have

$$\begin{cases}|(-\Delta)^{\frac{s}{2}}v_n|_2^2=|(-\Delta)^{\frac{s}{2}}u_n|_2^2-|(-\Delta)^{\frac{s}{2}}u|_2^2+o_n(1);\\ |v_n|_{2_s^*}^{2_s^*}=|u_n|_{2_s^*}^{2_s^*}-|u|_{2_s^*}^{2_s^*}+o_n(1).\end{cases} \tag{3.13}$$

Since $P(u_n)=P(u_n)-P(u_a)\to 0$ and $u_n\to u$ in $L^p(\mathbb{R}^N)$, we obtain $|(-\Delta)^{\frac{s}{2}}v_n|_2^2=|v_n|_{2_s^*}^{2_s^*}+o_n(1)$. Up to subsequence, we assume that

$$|(-\Delta)^{\frac{s}{2}}v_n|_2^2=|v_n|_{2_s^*}^{2_s^*}\to l.$$

So, we have

$$l=0,\quad\text{or}\quad l\geq S_s^{\frac{N}{2s}}.$$

If $l\geq S_s^{\frac{N}{2s}}$, by (3.13), we deduce that

$$\begin{aligned} m(a)&=\lim_{n\to\infty}E(u_n)=\lim_{n\to\infty}\left(E(u_a)+\frac{1}{2}|(-\Delta)^{\frac{s}{2}}v_n|_2^2-\frac{1}{2_s^*}|v_n|_{2_s^*}^{2_s^*}\right)\\ &=E(u_a)+\frac{s}{N}l\geq E(u_a)+\frac{s}{N}S_s^{\frac{N}{2s}}. \end{aligned}$$

Step 3: We prove that $l\geq S_s^{\frac{N}{2s}}$ will lead to a contradiction.

We first give a lower bounded estimate on $E(u_a)$. By (1.4), we have

$$\begin{aligned} E(u_a)&=E(u_a)-\frac{1}{2_s^*}P(u_a)\\ &\geq\frac{s}{N}|(-\Delta)^{\frac{s}{2}}u_a|_2^2-(\frac{1}{p}-\frac{\gamma_{ps}}{2_s^*})C(s,N,p)a^{p-p\gamma_{p,s}}|(-\Delta)^{\frac{s}{2}}u_a|_2^{p\gamma_{p,s}}. \end{aligned}$$

Define

$$m(t)\doteq\frac{s}{N}t^2-(\frac{1}{p}-\frac{\gamma_{ps}}{2_s^*})C(s,N,p)a^{p-p\gamma_{p,s}}t^{p\gamma_{p,s}}.$$

By a direct calculation, we have that $m(t)$ is strictly decreasing on $(0,t_3)$ and increasing on $(t_3,+\infty)$, where

$$t_3^{2-p\gamma_{p,s}} = \frac{N}{2s}\left(\frac{1}{p} - \frac{\gamma_{p,s}}{2_s^*}\right)p\gamma_{p,s}C(s,N,p)a^{p-p\gamma_{p,s}}.$$

Hence, the minimum of $m(t)$ on $(0,+\infty)$ is

$$m(t_3) = -\left(1-\frac{p\gamma_{p,s}}{2}\right)\left(a^{p-p\gamma_{p,s}}C(s,N,p)\left(\frac{1}{p}-\frac{\gamma_{p,s}}{2_s^*}\right)\right)^{\frac{2}{2-p\gamma_{p,s}}}\left(\frac{Np\gamma_{p,s}}{2s}\right)^{\frac{p\gamma_{p,s}}{2-p\gamma_{p,s}}}.$$

Define

$$\alpha_2 \doteq \left\{\frac{22_s^* s}{N\gamma_{p,s}C(s,N,p)(2_s^*-p\gamma_{p,s})}\left(\frac{\gamma_{p,s}S_s^{\frac{N}{2s}}}{2-p\gamma_{p,s}}\right)^{\frac{2-p\gamma_{p,s}}{2}}\right\}^{\frac{1}{p(1-\gamma_{p,s})}}. \tag{3.14}$$

Since $a \le \alpha_2$, we have that $m(t) > -\frac{s}{N}S_s^{\frac{N}{2s}}$ on $(0,+\infty)$. Moreover, by Step 2, we have $m(a) > 0$, which is a contradiction.

Step 3: We complete the proof at this step.

From the discussion of Step 2, only $l = 0$ is possible. Therefore, we obtain that $|u_n|_{2_s^*}^{2_s^*} \to |u_a|_{2_s^*}^{2_s^*}$. From (3.12), we get that

$$E'(u_a)u_a - \lambda\Psi'(u_a)u_a = 0.$$

Subtracting the left hand of (3.9) from that of the above equality and using the fact that $|u_n|_p \to |u_a|_p$, we have that $\|u_n\|^2 \to \|u_a\|^2$. In view of $u_n \rightharpoonup u_a$ in H, we have $u_n \to u_a$ in H. Since $E(u_a) = \inf_{V(a)} E$, we deduce that u_a is a ground state. Besides, by Lemma 3.5, we have

$$E(u_a) = \inf_{u\in V(a)} E(u) = \inf_{u\in A_{R_0}} E(u).$$

□

4. L^2-critical perturbation case

In the section, we consider the case $p = 2 + \frac{4s}{N}$ and prove Theorem 1.2. First, we prove that $I(u,s)$ has the mountain pass geometry on $S_r(a)\times\mathbb{R}$ in the following lemmas, where $S_r(a) = H_r^s(\mathbb{R}^N)\bigcap S(a)$.

Lemma 4.1. *Assume that* $p = 2+\frac{4s}{N}$ *and* $0 < a < \alpha_3$, *where*

$$\alpha_3 \doteq \left(\frac{p}{2C(s,N,p)}\right)^{\frac{1}{p-2}}$$

is defined in (1.8). *Then there exist two constants* $k_2 > k_1 > 0$ *such that*

$$P(u), E(u) > 0 \text{ for all } u \in A_{k_1}, \quad \text{and} \quad 0 < \sup_{u\in A_{k_1}} E(u) < \inf_{u\in B_{k_2}} E(u)$$

where

$$A_k \doteq \left\{u\in S(a) : \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)} < k\right\},$$
$$B_k \doteq \{u\in S_r(a) : |(-\Delta)^{\frac{s}{2}}u|_2^2 = 2k\}.$$

Proof. Let $k>0$ be arbitrary fixed and suppose that $u,v\in S_r(a)$ are such that $|(-\Delta)^{\frac{s}{2}}u|_2^2<k$ and $|(-\Delta)^{\frac{s}{2}}v|_2^2=ck$ where $c>0$ is determined later. Then, in view of (1.4) and $p\gamma_{p,s}=2$, we have

$$\begin{aligned}
P(u)&\geq |(-\Delta)^{\frac{s}{2}}u|_2^2-C(s,N,p)\gamma_{p,s}a^{p-2}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*}\\
&\geq (1-C(s,N,p)\frac{2}{p}a^{p-2})|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*},\\
E(u)&\geq \frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{2_s^*S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*}\\
&\geq \frac{1}{2}(1-C(s,N,p)\frac{2}{p}a^{p-2})|(-\Delta)^{\frac{s}{2}}u|_2^2-\frac{1}{2_s^*S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*},
\end{aligned}$$

and

$$\begin{aligned}
E(v)-E(u)&\geq E(v)-\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2\\
&\geq \frac{1}{2}(1-C(s,N,p)\frac{2}{p}a^{p-2})c^2k^2-\frac{c^{2_s^*}}{2_s^*S_s^{\frac{2_s^*}{2}}}k^{2_s^*}-\frac{1}{2}k^2\\
&\geq \frac{1}{2}\left\{(1-C(s,N,p)\frac{2}{p}a^{p-2})c^2-1\right\}k^2-\frac{c^{2_s^*}}{2_s^*S_s^{\frac{2_s^*}{2}}}k^{2_s^*}.
\end{aligned}$$

In view of the definition of α_3, we can take $c>0$ such that $c^2>\frac{1}{1-C(s,N,p)\frac{2}{p}a^{p-2}}$. Since $a<\alpha_3$, hence, for $k_2>k_1>0$ small enough, we have

$$P(u),E(u)>0 \text{ for all } x\in A_{k_1},\quad \text{and}\quad 0<\sup_{u\in A_{k_1}}E(u)<\inf_{u\in B_{k_2}}E(u).$$

□

Next, we give characterizations of the mountain pass levels for $I(\tau,u)$ and $E(u)$. E^d denotes the closed set $\{u\in S_r(a):E(u)\leq d\}$.

Proposition 4.2. *Under the assumptions* $p=2+\frac{4s}{N}$ *and* $0<a<\alpha_3$, *let*

$$\widetilde{\sigma_r}(a)=\inf_{\widetilde{\eta}\in\widetilde{\Gamma_a}}\max_{t\in[0,1]}I(\widetilde{\eta}(t))$$

with

$$\widetilde{\Gamma_a}=\{\widetilde{\eta}\in C([0,1],S_r(a)\times\mathbb{R}):\widetilde{\eta}(0)\in(A_{k_1},0),\ \widetilde{\eta}(1)\in(E^0,0)\},$$

and

$$\sigma_r(a)=\inf_{\eta\in\Gamma_a}\max_{t\in[0,1]}E(\eta(t))$$

with

$$\Gamma_a=\{\eta\in C([0,1],S_r(a)):\eta(0)\in A_{k_1},\ \eta(1)\in E^0\}.$$

Then, we have

$$\widetilde{\sigma_r}(a)=\sigma_r(a).$$

Proof. Since $\Gamma_a \times \{0\} \subseteq \widetilde{\Gamma_a}$, we have that $\widetilde{\sigma_r}(a) \le \sigma_r(a)$. So, it remains to prove $\widetilde{\sigma_r}(a) \le \sigma_r(a)$. For any $\widetilde{\eta}(t) = (\widetilde{\eta_1}(t), \widetilde{\eta_2}(t)) \in \widetilde{\Gamma}_a$, set $\eta(t) = \widetilde{\eta_2}(t) \star \widetilde{\eta_1}(t)$. Then $\eta(t) \in \Gamma_a$ and

$$\max_{t\in[0,1]} I(\widetilde{\eta}(t)) = \max_{t\in[0,1]} E(\widetilde{\eta_2}(t) \star \widetilde{\eta_1}(t)) = \max_{t\in[0,1]} E(\eta(t)).$$

It shows that $\widetilde{\sigma_r}(a) \ge \sigma_r(a)$. □

In the following proposition, we give the existence of a $(PS)_{\widetilde{\sigma}_r(a)}$ sequence for $I(u,s)$, whose proof can be given by standard arguments by using Ekeland Variational principle and constructing pseudo-gradient flows [19].

Proposition 4.3. [16, Proposition2.2] *Let $\{g_n\} \subset \widetilde{\Gamma}_a$ be such that*

$$\max_{t\in[0,1]} I(g_n(t)) \le \widetilde{\sigma_r}(a) + \frac{1}{n}.$$

Then there exists a sequence $\{(u_n,\tau_n)\} \subset S(a) \times \mathbb{R}$ such that

(1) $I(u_n,\tau_n) \in [\widetilde{\sigma_r}(a) - \frac{1}{n}, \widetilde{\sigma_r}(a) + \frac{1}{n}]$;

(2) $\min_{t\in[0,1]} \|(u_n,\tau_n) - g_n(t)\|_{\mathbb{H}} \le \frac{1}{\sqrt{n}}$;

(3) $\|I'|_{S(a)\times\mathbb{R}}(u_n,\tau_n)\| \le \frac{2}{\sqrt{n}}$ *i.e.*

$$|\langle I'(u_n,\tau_n), z\rangle_{\mathbb{H}^{-1}\times\mathbb{H}}| \le \frac{2}{\sqrt{n}}\|z\|_{\mathbb{H}}$$

for all

$$z \in \widetilde{T}_{(u_n,\tau_n)} \doteq \{(z_1,z_2) \in \mathbb{H} : \langle u_n, z_1\rangle_{L^2} = 0\}.$$

Proposition 4.4. *Under the assumptions $p = 2 + \frac{4s}{N}$ and $0 < a < \alpha_3$, there exists a sequence $\{v_n\} \subset S_r(a)$ such that*

(1) $E(v_n) \to \sigma_r(a)$, *as* $n \to \infty$;

(2) $P(v_n) \to 0$, *as* $n \to \infty$;

(3) $E'|_{S_r(a)}(v_n) \to 0$, *as* $n \to \infty$, *i.e.*,

$$|\langle E'(v_n), h\rangle_{H^{-1}\times H}| \to 0$$

uniformly for all $h \in T_{v_n}$, $\|h\| \le 1$, where $T_{v_n} \doteq \{h \in H : \langle v_n, h\rangle_{L^2} = 0\}$.

Proof. By Proposition 4.2, $\widetilde{\sigma_r}(a) = \sigma_r(a)$. Pick $\{g_n = ((g_n)_1, 0)\} \subset \widetilde{\Gamma}_a$ such that

$$\max_{t\in[0,1]} I(g_n(t)) \le \widetilde{\sigma_r}(a) + \frac{1}{n}.$$

It follows from Proposition 4.3 that there exists a sequence $\{(u_n,\tau_n)\} \subset S_r(a) \times \mathbb{R}$ such that, as $n \to \infty$, one has

$$I(u_n,\tau_n) \to \sigma_r(a), \tag{4.1}$$

$$\tau_n \to 0, \tag{4.2}$$

$$\partial_\tau I(u_n,\tau_n) \to 0. \tag{4.3}$$

Let $v_n = \tau_n \star u_n$. Then $E(v_n) = I(u_n,\tau_n)$ and, by (4.1), (1) holds. For the proof of (2), we notice that

$$\begin{aligned}\partial_\tau I(u_n,\tau_n) &= s\left(e^{2s\tau_n}|(-\Delta)^{\frac{s}{2}}u_n|_2^2 - \gamma_{p,s}e^{ps\gamma_{p,s}\tau_n}|u_n|_p^p - e^{2_s^* s\tau_n}|u_n|_{2_s^*}^{2_s^*}\right)\\ &= s\left(|(-\Delta)^{\frac{s}{2}}(\tau_n\star u_n)|^2 - \gamma_{p,s}|\tau_n\star u_n|_p^p - |\tau_n\star u_n|_{2_s^*}^{2_s^*}\right)\\ &= sP(v_n)\end{aligned}$$

which, by (4.3), implies (2).

For the proof of (3), let $h_n \in T_{v_n}$. We have

$$\begin{aligned}\langle E'(v_n), h_n\rangle_{H^{-1}\times H} &= \int\int_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(h_n(x)-h_n(y))}{|x-y|^{N+2s}} \\ &\quad - \int_{\mathbb{R}^N} |v_n(x)|^{p-2}v_n(x)h_n(x) - \int_{\mathbb{R}^N} |v_n(x)|^{2_s^*-2}v_n(x)h_n(x) \\ &= e^{2s\tau_n}\int\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(h_n(e^{-\tau_n}x)-h_n(e^{-\tau_n}y))e^{-\frac{N\tau_n}{2}}}{|x-y|^{N+2s}} \\ &\quad - e^{\frac{N(p-2)\tau_n}{2}}\int_{\mathbb{R}^N} |u_n(x)|^{p-2}u_n(x)h_n(e^{-\tau_n}x)e^{-\frac{N\tau_n}{2}} \\ &\quad - e^{\frac{N(2_s^*-2)\tau_n}{2}}\int_{\mathbb{R}^N} |u_n(x)|^{2_s^*-2}u_n(x)h_n(e^{-\tau_n}x)e^{-\frac{N\tau_n}{2}}.\end{aligned}$$

Setting $\widehat{h}_n(x) = e^{\frac{-N\tau_n}{2}}h_n(e^{-\tau_n}x)$, then

$$\langle I'(u_n,\tau_n), (\widehat{h}_n, 0)\rangle_{\mathbb{H}^{-1}\times\mathbb{H}} = \langle E'(v_n), h_n\rangle_{H^{-1}\times H}.$$

It is easy to see that

$$\begin{aligned}\langle u_n(x), \widehat{h}_n(x)\rangle_{L^2} &= \int_{\mathbb{R}^N} u_n(x)e^{\frac{-N\tau_n}{2}}h_n(e^{-\tau_n}x) \\ &= \int_{\mathbb{R}^N} u_n(e^{\tau_n}x)e^{\frac{3\tau_n}{2}}h_n(x) \\ &= \int_{\mathbb{R}^3} v_n(x)h_n(x) = 0.\end{aligned}$$

So, we have that $(\widehat{h}_n(x), 0) \in \widetilde{T}_{(u_n,\tau_n)}$. On the other hand,

$$\begin{aligned}\|(\widehat{h}_n(x), 0)\|_{\mathbb{H}}^2 &= \|\widehat{h}_n(x)\|^2 \\ &= |h_n(x)|_2^2 + e^{-2\tau_n}|\nabla h_n(x)|_2^2 \\ &\le C\|h_n(x)\|^2,\end{aligned}$$

where the last inequality holds by (4.2). Thus, (3) is proved. □

Let $m_r(a) \doteq \inf_{u\in V_r(a)} E(u)$, where $V_r(a) = V(a)\bigcap S_r(a)$. We have the following relationship between $\sigma_r(a)$ and $m_r(a)$.

Lemma 4.5. *Under the assumptions $p = 2 + \frac{4s}{N}$ and $0 < a < \alpha_3$, we have that*

$$m_r(a) = \inf_{u\in V_r(a)^-} E(u) = \sigma_r(a) > 0,$$

where $V_r(a)^- = V(a)^-\bigcap S_r(a)$.

Proof. **Step 1:** We claim that for every $u \in S_r(a)$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in V_r(a)$, where t_u is the strict maximum point for the function $\phi_u(s) \doteq I(u,s) = E(s\star u)$ on $(0,+\infty)$ on a positive level. Moreover, $V_r(a) = V_r(a)^-$.

The existence of t_u follows from Lemma 2.2. The uniqueness is from the following reasoning. Noticing that

$$(\varphi_u)''(\tau) = s^2 \left(2e^{2s\tau}|(-\Delta)^{\frac{s}{2}}u|_2^2 - p\gamma_{p,s}{}^2 e^{ps\gamma_{p,s}\tau}|u|_p^p - 2_s^* e^{2_s^* s\tau}|u|_{2_s^*}^{2_s^*}\right)$$

combining with $(\varphi_u)'(t_u) = 0$, we have

$$(\varphi_u)''(t_u) = -s^2(2_s^* - 2)|u|_{2_s^*}^{2_s^*} < 0.$$

By noticing that $0 < a < \alpha_3$, it is clear that $\varphi_u(-\infty) = 0^+$ and $\varphi_u(+\infty) = -\infty$. Hence, $\varphi_u(\tau)$ has a global maximum point at a positive level.

Step 2: We claim that $E(u) \leq 0$ implies $P(u) < 0$.

Let $E(u) \leq 0$. Since $(\varphi_u)''(t_u) < 0$, we know that φ_u is strictly decreasing and concave on $(t_u, +\infty)$. Since $\phi_u(0) = E(0 \star u) = E(u) \leq 0$, by the properties of the function $\phi_u(s)$ presented in Step 1, we have that $t_u < 0$. Moreover, since

$$P(t_u \star u) = (\phi_u)'(t_u) = 0 \quad \text{and} \quad P(u) = P(0 \star u) = (\phi_u)'(0),$$

we have that $P(u) < 0$.

Step 3: $\sigma_r(a) = m_r(a)$.

Let $u \in S_r(a)$. We take $\tau^- << 0$ and $\tau^+ >> 0$ such that $\tau^- \star u \in A_{k_1}$ and $E(\tau^+ \star u) < 0$, respectively. Then we define a path

$$\eta_u : t \in [0,1] \mapsto ((1-t)\tau^- + t\tau^+) \star u \in \Gamma_a. \tag{4.4}$$

By the definition of $\sigma_r(a)$, we have

$$\max_{t\in[0,1]} E(\eta_u(t)) \geq \sigma_r(a).$$

So, we have $m_r(a) \geq \sigma_r(a)$ by Step 1. On the other hand, for any $\widetilde{\eta}(t) = (\widetilde{\eta_1}(t), \widetilde{\eta_2}(t)) \in \widetilde{\Gamma}_a$, we consider the function

$$\widetilde{P}(t) = P(\widetilde{\eta_2}(t) \star \widetilde{\eta_1}(t)) \in \mathbb{R}.$$

Since $\widetilde{\eta_2}(0) \star \widetilde{\eta_1}(0)) = \widetilde{\eta_1}(0) \in A_{k_1}$ and $\widetilde{\eta_2}(1) \star \widetilde{\eta_1}(1)) = \widetilde{\eta_1}(1) \in E^0$, hence by Lemma 4.1, we deduce that

$$\widetilde{P}(0) = \widetilde{P}(\widetilde{\eta_1}(0)) > 0,$$

and using the result in Step 2,

$$\widetilde{P}(1) = \widetilde{P}(\widetilde{\eta_1}(1)) < 0.$$

By (2.3), the function $\widetilde{P}(t)$ is continuous and hence we deduce that there exists $\bar{t} \in (0,1)$ such that $\widetilde{P}(\bar{t}) = 0$, which implies that $\widetilde{\eta_2}(\bar{t}) \star \widetilde{\eta_1}(\bar{t})) \in V_r(a)$. Therefore,

$$\max_{t\in[0,1]} I(\widetilde{\eta}(t)) = \max_{t\in[0,1]} E(\widetilde{\eta_2}(t) \star \widetilde{\eta_1}(t)) \geq \inf_{u\in V_r(a)} E(u).$$

So, we infer that $\widetilde{\sigma_r}(a) = \sigma_r(a) \geq m_r(a)$.

Step 4: At this step, we prove that $\sigma_r(a) > 0$.

If $u \in V_r(a)$, then $P(u) = 0$. By GNS inequality (1.4), we deduce that

$$(1 - C(s,N,p)\frac{2}{p}a^{p-2})|(-\Delta)^{\frac{s}{2}}u|_2^2 \leq \frac{1}{S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*}.$$

Noticing $a < \alpha_3$, this implies that there exists $\delta > 0$ such that $\inf_{V_r(a)} |(-\Delta)^{\frac{s}{2}}u|_2^2 \geq \delta$. Then, in view of $p\gamma_{p,s} = 2$, for any $u \in V_r(a)$, we have that

$$E(u) = E(u) - \frac{1}{2_s^*}P(u) = \frac{s}{N}|(-\Delta)^{\frac{s}{2}}u|_2^2 + \frac{2s}{pN}|u|_p^p \geq \frac{s\delta}{N} > 0.$$

Thus, $\sigma_r(a) > 0$. □

In the following lemma, we give an upper bound estimate for the mountain pass level $\sigma_r(a)$.

Lemma 4.6. *Let $N^2 > 8s^2$, $p = 2 + \frac{4s}{N}$ and $0 < a < \alpha_3$. Then $\sigma_r(a) < \frac{s}{N} S_s^{\frac{N}{2s}}$, where S_s is defined in (2.1).*

Proof. Let $\varphi(x) \in C_c^\infty(B_2(0))$ be a radial cutoff function such that $0 \le \varphi(x) \le 1$ and $\varphi(x) \equiv 1$ on $B_1(0)$. We take $u_\varepsilon = \varphi(x) U_\varepsilon$ and

$$v_\varepsilon = a \frac{u_\varepsilon}{|u_\varepsilon|_2} \in S(a) \cap H_r^s,$$

where U_ε is defined in (2.2). In the following, we take $\varepsilon = 1$ and define

$$K_1 \doteq |(-\Delta)^{\frac{s}{2}} U_1|_2^2, \quad K_2 \doteq |U_1|_{2_s^*}^2, \quad K_3 \doteq |U_1|_2^2, \quad K_4 \doteq |U_1|_p^p.$$

It is obvious that $K_1/K_2 = S_s$. As proved in [20], u_ε satisfies the following useful estimates:

$$|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^2 = K_1 + O(\varepsilon^{N-2s}) \quad \text{and} \quad |u_\varepsilon|_{2_s^*}^2 = K_2 + O(\varepsilon^N). \tag{4.5}$$

$$|u_\varepsilon|_2^2 = \begin{cases} K_3 \varepsilon^{2s} + O(\varepsilon^{N-2s}), & N > 4s, \\ C_s \omega \varepsilon^{2s} |\log \varepsilon| + O(\varepsilon^{2s}), & N = 4s, \\ C_s \omega \varepsilon^{N-2s} + O(\varepsilon^{2s}), & N < 4s, \end{cases} \tag{4.6}$$

where ω is the area of the unit sphere in $\mathbb{R}^N$. For $|u_\varepsilon|_p^p$, we have the following estimate:

$$\begin{aligned} |u_\varepsilon|_p^p &= \int_{\mathbb{R}^N} \left(\frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}} \right)^p |\varphi(x)|^p dx \\ &= \varepsilon^{N - \frac{p(N-2s)}{2}} \int_{\mathbb{R}^N} \left(\frac{1}{(1 + |x|^2)^{\frac{N-2s}{2}}} \right)^p |\varphi(\varepsilon x)|^p dx \\ &= K_4 \varepsilon^{N - \frac{p(N-2s)}{2}} + \varepsilon^{N - \frac{p(N-2s)}{2}} \int_{\mathbb{R}^N} \left(\frac{1}{(1 + |x|^2)^{\frac{N-2s}{2}}} \right)^p |\varphi(\varepsilon x) - 1|^p dx. \end{aligned}$$

By $N^2 > 8s^2$ and $p = 2 + \frac{4s}{N}$, it is easy to obtain that $p > \frac{N}{N-2s}$. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\frac{1}{(1 + |x|^2)^{\frac{N-2s}{2}}} \right)^p |\varphi(\varepsilon x) - 1|^p dx \\ &\le \int_{\mathbb{R}^N \setminus B_{1/\varepsilon}} \left(\frac{1}{(1 + |x|^2)^{\frac{N-2s}{2}}} \right)^p dx \\ &\le \omega \int_{1/\varepsilon}^{+\infty} r^{N-1} \left(\frac{1}{(|r|^2)^{\frac{N-2s}{2}}} \right)^p dx \\ &= \omega \int_{1/\varepsilon}^{+\infty} r^{N-1-p(N-2s)} dr = \frac{\omega}{(N-2s)p - N} \varepsilon^{N - p(N-2s)}. \end{aligned}$$

Thus,

$$|u_\varepsilon|_p^p = \varepsilon^{N - \frac{p(N-2s)}{2}} \left(K_4 + O(\varepsilon^{p(N-2s)-N}) \right). \tag{4.7}$$

Define the following function on $(-\infty,+\infty)$

$$\psi_{v_\varepsilon}(\tau)\doteq\varphi_{v_\varepsilon}(\tau)+\frac{1}{p}|\tau\star v_\varepsilon|_p^p=\frac{1}{2}e^{2s\tau}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2-\frac{e^{2_s^*s\tau}}{2_s^*}|v_\varepsilon|_{2_s^*}^{2_s^*}.$$

Then

$$\psi'_{v_\varepsilon}(\tau)=se^{2s\tau}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2-se^{2_s^*s\tau}|v_\varepsilon|_{2_s^*}^{2_s^*}.$$

It is easy to see that $\psi_{v_\varepsilon}(\tau)$ has a unique critical point $\tau_{\varepsilon,0}$, which is a strict maximum point such that

$$e^{s\tau_{\varepsilon,0}}=\left(\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{1}{2_s^*-2}}\tag{4.8}$$

and the maximum level of $\psi_{v_\varepsilon}(\tau)$ is

$$\begin{aligned}\psi_{v_\varepsilon}(\tau_{\varepsilon,0})&=\frac{s}{N}\left(\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{2_s^*}{2_s^*-2}}=\frac{s}{N}\left(\frac{|(-\Delta)^{\frac{s}{2}}u_\varepsilon|_2^2}{|u_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{2_s^*}{2_s^*-2}}\\&=\frac{s}{N}\left(\frac{K_1+O(\varepsilon^{N-2s})}{K_2+O(\varepsilon^N)}\right)^{\frac{2_s^*}{2_s^*-2}}=\frac{s}{N}S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})\end{aligned}\tag{4.9}$$

as $\varepsilon\to0$, where the estimates in (4.5) are used. Next, we give an upper bound estimate for the function $\varphi_{v_\varepsilon}(\tau)=I(v_\varepsilon,\tau)$ on $(-\infty,+\infty)$. Note that

$$\varphi'_{v_\varepsilon}(\tau)=s\left(e^{2s\tau}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2-\gamma_{p,s}e^{ps\gamma_{p,s}\tau}|v_\varepsilon|_p^p-e^{2_s^*s\tau}|v_\varepsilon|_{2_s^*}^{2_s^*}\right).$$

Obviously, $\varphi_{v_\varepsilon}(\tau)$ has a unique critical point $\tau_{\varepsilon,1}$ and in view of $p\gamma_{p,s}=2$, we have

$$\begin{aligned}e^{(2_s^*-2)s\tau_{\varepsilon,1}}&=\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}-\frac{2}{p}\frac{|v_\varepsilon|_p^p}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\\&\ge(1-\frac{2}{p}C(s,N,p)a^{p-2})\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}.\end{aligned}$$

Combining (4.9) with the above inequality, we obtain

$$\begin{aligned}\sup_{\mathbb{R}}\varphi_{v_\varepsilon}(\tau)&\le\sup_{\mathbb{R}}\psi_{v_\varepsilon}(\tau)-\frac{1}{p}e^{2s\tau_{\varepsilon,1}}|v_\varepsilon|_p^p\\&\le\frac{s}{N}S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})\\&\quad-\frac{1}{p}\left(1-\frac{2}{p}C(s,N,p)a^{p-2}\right)^{\frac{2}{2_s^*-2}}\left(\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{2}{2_s^*-2}}|v_\varepsilon|_p^p\\&\le\frac{s}{N}S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})\\&\quad-\frac{1}{p}\left(1-\frac{2}{p}C(s,N,p)a^{p-2}\right)^{\frac{2}{2_s^*-2}}a^{p-2}\frac{|(-\Delta)^{\frac{s}{2}}u_\varepsilon|_2^{\frac{4}{2_s^*-4}}|u_\varepsilon|_p^p}{|u_\varepsilon|_{2_s^*}^{\frac{22_s^*}{2_s^*-4}}|u_\varepsilon|_2^{p-2}}.\end{aligned}\tag{4.10}$$

By the facts (4.6), (4.7) and a simple calculation, we have

$$\frac{|u_\varepsilon|_p^p}{|u_\varepsilon|_2^{p-2}}=\frac{|u_\varepsilon|_p^p}{|u_\varepsilon|_2^{\frac{4s}{N}}}\ge\begin{cases}C\varepsilon^{N-(N-2s)(1+\frac{2s}{N})-\frac{4s^2}{N}}=C, & N>4s,\\C\varepsilon^{N-(N-2s)(1+\frac{2s}{N})-s}|\log\varepsilon|^{-\frac{1}{2}}=C|\log\varepsilon|^{-\frac{1}{2}}, & N=4s,\\C\varepsilon^{N-(N-2s)(1+\frac{2s}{N})-\frac{2s}{N}(N-2s)}=C\varepsilon^{\frac{8s^2}{N}-2s} & N<4s.\end{cases}\tag{4.11}$$

Moreover, in view of $a < \alpha_3$, putting (4.5) and (4.11) into (4.10), we obtain that

$$\sup_{\mathbb{R}} \varphi_{v_\varepsilon}(\tau) < \frac{s}{N} S_s^{\frac{N}{2s}}.$$

As in (4.4), we define a path

$$\eta_{v_\varepsilon} : t \in [0,1] \mapsto ((1-t)\tau^- + t\tau^+) \star v_\varepsilon \in \Gamma_a.$$

Therefore,

$$m_r(a) \leq \max_{t\in[0,1]} E(\eta_{v_\varepsilon}(t)) = \sup_{\mathbb{R}} \varphi_{v_\varepsilon}(\tau) < \frac{s}{N} S_s^{\frac{N}{2s}}.$$

□

Proof of Theorem 1.2. Choosing a PS sequence $\{u_n\}$ as in Proposition 4.4 and applying the Lagrange multipliers rule to (3) of Proposition 4.4, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$E'(u_n) - \lambda_n \Psi'(u_n) \to 0 \quad \text{in} \quad H^{-1} \quad \text{and} \quad E(u_n) \to \sigma_r(a) \quad \text{as } n \to \infty. \tag{4.12}$$

As in the proof of Theorem 1.1, we obtain, up to a subsequence,

$$\begin{cases} \lambda_n \to \lambda < 0 \quad \text{as } n \to \infty; \\ u_n \rightharpoonup u_a \in H \quad \text{as } n \to \infty; \\ u_n \to u_a \in L^p(\mathbb{R}) \quad \text{as } n \to \infty \end{cases}$$

and $P(u_a) = 0$. Denote $v_n \doteq u_n - u_a \rightharpoonup 0$. By the similar equalities as in (3.13), we can assume that, up to a subsequence,

$$|(-\Delta)^{\frac{s}{2}} v_n|_2^2 = |v_n|_{2_s^*}^{2_s^*} \to l.$$

So, we have

$$l = 0 \ \text{ or } \ l \geq S_s^{\frac{N}{2s}}.$$

If $l \geq S_s^{\frac{N}{2s}}$, we deduce by the similar equalities as in (3.13), that

$$\begin{aligned} \sigma_r(a) = \lim_{n\to\infty} E(u_n) &= \lim_{n\to\infty} \left(E(u_a) + \frac{1}{2} |(-\Delta)^{\frac{s}{2}} v_n|_2^2 - \frac{1}{2_s^*} |v_n|_{2_s^*}^{2_s^*} \right) \\ &= E(u_a) + \frac{s}{N} l \geq E(u_a) + \frac{s}{N} S_s^{\frac{N}{2s}}. \end{aligned}$$

On the other hand, $E(u_a) = E(u_a) - \frac{1}{2}P(u_a) = \frac{s}{N}|u_a|_{2_s^*}^{2_s^*} > 0$. Moreover, by combining with Lemma 4.6, we get a contradiction. This implies that $u_n \to u_a$ in H by the similar arguments as in the end of the proof of Theorem 1.1. Let $m(a) \doteq \inf_{u\in V(a)} E(u)$. As at the Step 1 of the proof of Lemma 4.5, it is easy to obtain $V(a) = V(a)^-$. Finally, we prove that $m(a) = m_r(a)$.

Since $V_r(a) \subset V(a)$, we obtain $m(a) \leq m_r(a)$. It remains to prove that $m(a) \geq m_r(a)$. Otherwise, we assume that there exists $w \in V(a) \backslash S_r(a)$ such that

$$E(w) < \inf_{V_r(a)} E(u). \tag{4.13}$$

Then we let $v \doteq |w|^*$. By the properties of the Schwarz rearrangement, we know that

$$E(v) \leq E(w) \quad \text{and} \quad P(v) \leq P(w) = 0.$$

If $P(v) = 0$, it contradicts to (4.13). If $P(v) < 0$, noticing that $(\varphi_v)'(0) = P(v) < 0$ and the claim of step 1 of the proof of Lemma 4.5, we get that $t_v < 0$, which still leads to a contradiction. In fact, since $t_v \star v \in V_r(a)$ and (4.13), one has

$$
\begin{aligned}
E(w) &< E(t_v \star v) = E(t_v \star v) - \frac{1}{2}P(t_v \star v) \\
&= \frac{se^{2_s^* st_v}}{N}|v|_{2_s^*}^{2_s^*} = \frac{se^{2_s^* st_v}}{N}|w|_{2_s^*}^{2_s^*} = e^{2_s^* st_v}E(w) < E(w).
\end{aligned}
$$

So,

$$
E(u_a) = \sigma_r(a) = m_r(a) = m(a) = \inf_{u\in V(a)} E(u) = \inf_{u\in V(a)^-} E(u)
$$

and u_a is a ground state. □

5. L^2-supercritical perturbation case

In this section, we deal with the case $2_s^* > p > 2 + \frac{4s}{N}$ and prove Theorem 1.3. For convenience, we still use the notations and definitions in Sect. 4.

Lemma 5.1. *Let $2_s^* > p > 2 + \frac{4s}{N}$. Then we have*
(1) *there exists a sequence $\{u_n\} \subset S_r(a)$ such that*

$$
\begin{cases}
E(u_n) \to \sigma_r(a) \quad \text{as } n \to \infty, \\
P(u_n) \to 0 \quad \text{as } n \to \infty, \\
E'|_{S_r(a)}(u_n) \to 0 \quad \text{as } n \to \infty;
\end{cases}
$$

(2) $\sigma_r(a) = m_r(a) > 0$, *where* $m_r(a) \doteq \inf_{u\in V_r(a)} E(u)$ *and* $\sigma_r(a) \doteq \inf_{\eta\in\Gamma_a} \max_{t\in[0,1]} E(\eta(t))$ *where*

$$
\Gamma_a = \{\eta \in C([0,1], S_r(a)) : \eta(0) \in A_{k_1},\ \eta(1) \in E^0\}.
$$

Proof. Noticing that

$$
\begin{aligned}
E(u) &\geq \frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|_2^2 - \frac{C(s,N,p)}{p}a^{p-p\gamma_{p,s}}|(-\Delta)^{\frac{s}{2}}u|_2^{p\gamma_{p,s}} - \frac{1}{2_s^* S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*}, \\
P(u) &\geq |(-\Delta)^{\frac{s}{2}}u|_2^2 - C(s,N,p)a^{p-p\gamma_{p,s}}\gamma_{p,s}|(-\Delta)^{\frac{s}{2}}u|_2^{p\gamma_{p,s}} - \frac{1}{S_s^{\frac{2_s^*}{2}}}|(-\Delta)^{\frac{s}{2}}u|_2^{2_s^*},
\end{aligned}
$$

and also recalling that $p\gamma_{p,s} > 2$, the results follow from the similar arguments of Lemma 4.1, Propositions 4.2–4.4 and Lemma 4.5. □

In the following lemma, we give an upper bound estimate for the mountain pass level $\sigma_r(a)$.

Lemma 5.2. *Let $N^2 > 8s^2$, $2_s^* > p > 2 + \frac{4s}{N}$ and $0 < a < \alpha_4$, where α_4 is defined in (1.9). Then $\sigma_r(a) < \frac{s}{N}S_s^{\frac{N}{2s}}$, where S_s is defined in (2.1).*

Proof. We define, as in Lemma 4.6, u_ε and v_ε. It suffices to prove that $\sup_{\mathbb{R}} \varphi_{v_\varepsilon}(\tau) < \frac{s}{N}S_s^{\frac{N}{2s}}$. It is easy to see that $\varphi_{v_\varepsilon}(\tau)$ has a unique critical point $\tau_{\varepsilon,1}$ and

$$
e^{2_s^* s\tau_{\varepsilon,1}}|v_\varepsilon|_{2_s^*}^{2_s^*} = e^{2s\tau_{\varepsilon,1}}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2 - \gamma_{p,s}e^{ps\gamma_{p,s}\tau_{\varepsilon,1}}|v_\varepsilon|_p^p \leq e^{2s\tau_{\varepsilon,1}}|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2.
$$

It follows that

$$
e^{s\tau_{\varepsilon,1}} \leq \left(\frac{|(-\Delta)^{\frac{s}{2}}v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{1}{2_s^*-2}}.
$$

Combining the above inequality with $p\gamma_{p,s} > 2$, we deduce that

$$\begin{aligned} e^{(2_s^*-2)s\tau_{\varepsilon,1}} &= \frac{|(-\Delta)^{\frac{s}{2}} v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}} - \gamma_{p,s} e^{(p\gamma_{p,s}-2)s\tau_{\varepsilon,1}} \frac{|v_\varepsilon|_p^p}{|v_\varepsilon|_{2_s^*}^{2_s^*}} \\ &\geq \frac{|(-\Delta)^{\frac{s}{2}} v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}} - \left(\frac{|(-\Delta)^{\frac{s}{2}} v_\varepsilon|_2^2}{|v_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{p\gamma_{p,s}-2}{2_s^*-2}} \frac{|v_\varepsilon|_p^p}{|v_\varepsilon|_{2_s^*}^{2_s^*}} \\ &= \frac{|u_\varepsilon|_2^{2_s^*-2}}{a^{2_s^*-2}} \frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^2}{|u_\varepsilon|_{2_s^*}^{2_s^*}} - \gamma_{p,s} \frac{|u_\varepsilon|_2^{p-2}}{a^{2_s^*-p}} \frac{|u_\varepsilon|_p^p}{|u_\varepsilon|_{2_s^*}^{2_s^*}} \left(\frac{|u_\varepsilon|_2^{2_s^*-2} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^2}{a^{2_s^*-2} |u_\varepsilon|_{2_s^*}^{2_s^*}}\right)^{\frac{p\gamma_{p,s}-2}{2_s^*-2}} \\ &= \frac{|u_\varepsilon|_2^{2_s^*-2}}{a^{2_s^*-2}} \frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^2}{|u_\varepsilon|_{2_s^*}^{2_s^*}} \left(|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^{\frac{2(2_s^*-p\gamma_{p,s})}{2_s^*-2}} - \frac{\gamma_{p,s} a^{(1-\gamma_{p,s})p} |u_\varepsilon|_p^p}{|u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p\gamma_{p,s}-2)}{2_s^*-2}} |u_\varepsilon|_2^{p(1-\gamma_{p,s})}} \right). \end{aligned} \tag{5.1}$$

We claim that there exists $\varepsilon_0 > 0$ small enough and a positive constant $C = C(s, N, p, a)$ such that

$$e^{(2_s^*-2)s\tau_{\varepsilon,1}} \geq C |u_\varepsilon|_2^{2_s^*-2} \quad \text{for any } 0 < \varepsilon < \varepsilon_0. \tag{5.2}$$

Indeed, by (4.5), (4.6) and (4.7), there exist positive constants C_1, C_2 and C_3 depending on N, s and p such that

$$|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2 \geq C_1, \qquad \frac{1}{C_2} \geq |u_\varepsilon|_{2_s^*} \geq C_2, \tag{5.3}$$

and

$$\frac{|u_\varepsilon|_p^p}{|u_\varepsilon|_2^{(1-\gamma_{p,s})p}} \leq \begin{cases} C_3 \varepsilon^{N-\frac{p(N-2s)}{2}-ps(1-\gamma_{p,s})} = C_3, & N > 4s, \\ C_3 \varepsilon^{N-\frac{p(N-2s)}{2}-ps(1-\gamma_{p,s})} |\log\varepsilon|^{-\frac{(1-\gamma_{p,s})p}{2}} = C_3 |\log\varepsilon|^{-\frac{(1-\gamma_{p,s})p}{2}}, & N = 4s, \\ C_3 \varepsilon^{N-\frac{p(N-2s)}{2}-\frac{p(N-2s)}{2}(1-\gamma_{p,s})} = C_3 \varepsilon^{(\frac{N^2}{4s}+2s-\frac{3N}{2})p+2N-\frac{N^2}{2s}} & N < 4s. \end{cases} \tag{5.4}$$

For $N > 4s$, by (5.1) and (5.3), it is sufficient to verify that

$$\gamma_{p,s} a^{(1-\gamma_{p,s})p} < \frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^{\frac{2(2_s^*-p\gamma_{p,s})}{2_s^*-2}} |u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p\gamma_{p,s}-2)}{2_s^*-2}} |u_\varepsilon|_2^{p(1-\gamma_{p,s})}}{|u_\varepsilon|_p^p}. \tag{5.5}$$

Using the interpolation inequalities, we have that

$$\begin{aligned} &\frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^{\frac{2(2_s^*-p\gamma_{p,s})}{2_s^*-2}} |u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p\gamma_{p,s}-2)}{2_s^*-2}} |u_\varepsilon|_2^{p(1-\gamma_{p,s})}}{|u_\varepsilon|_p^p} \\ &\quad \geq \frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^{\frac{2(2_s^*-p\gamma_{p,s})}{2_s^*-2}} |u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p\gamma_{p,s}-2)}{2_s^*-2}} |u_\varepsilon|_2^{p(1-\gamma_{p,s})}}{|u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p-2)}{2_s^*-2}} |u_\varepsilon|_2^{\frac{2(2_s^*-p)}{2_s^*-2}}} \\ &\quad = \frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^{\frac{2(2_s^*-p\gamma_{p,s})}{2_s^*-2}}}{|u_\varepsilon|_{2_s^*}^{\frac{2_s^*(p-p\gamma_{p,s})}{2_s^*-2}}} = \left(\frac{|(-\Delta)^{\frac{s}{2}} u_\varepsilon|_2^2}{|u_\varepsilon|_{2_s^*}^2}\right)^{\frac{2_s^*(p-p\gamma_{p,s})}{2(2_s^*-2)}} \\ &\quad = S_s^{\frac{N}{4s}(1-\gamma_{p,s})p} + O(\varepsilon^{N-2s}). \end{aligned}$$

Therefore, (5.5) holds by $a < \alpha_4$.

For $N = 4s$, it easy to obtain (5.2) by (5.1), (5.3), (5.4) and the fact $0 < \gamma_{p,s} < 1$.

For $N<4s$, it suffices to prove that from $(\frac{N^2}{4s}+2s-\frac{3N}{2})p+2N-\frac{N^2}{2s}>0$. From $N^2>8s^2$, we can deduce that

$$\frac{N^2}{4s}+2s-\frac{3N}{2}=\frac{1}{4s}(N-2s)(N-4s)<0.$$

So,

$$\left(\frac{N^2}{4s}+2s-\frac{3N}{2}\right)p+2N-\frac{N^2}{2s}>\left(\frac{N^2}{4s}+2s-\frac{3N}{2}\right)2_s^*+2N-\frac{N^2}{2s}=0.$$

Finally, we estimate upper bound for $\varphi_{v_\varepsilon}(\tau)$ on $\mathbb{R}$. By (4.9) and (5.5)

$$\begin{aligned}\sup_{\mathbb{R}}\varphi_{v_\varepsilon}(\tau)&\le\sup_{\mathbb{R}}\psi_{v_\varepsilon}(\tau)-\frac{1}{p}e^{ps\gamma_{p,s}\tau_{\varepsilon,1}}|v_\varepsilon|_p^p\\&=\frac{s}{N}S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})-\frac{C}{p}|u_\varepsilon|_2^{p\gamma_{p,s}}\frac{a^p|u_\varepsilon|_p^p}{|u_\varepsilon|_2^p}\\&=\frac{s}{N}S_s^{\frac{N}{2s}}+O(\varepsilon^{N-2s})-C\frac{a^p|u_\varepsilon|_p^p}{|u_\varepsilon|_2^{p(1-\gamma_{p,s})}},\end{aligned}$$

where $C>0$ is independent of ε. Similarly as in (5.4), we have that

$$\frac{|u_\varepsilon|_p^p}{|u_\varepsilon|_2^{(1-\gamma_{p,s})p}}\ge\begin{cases}C_4\varepsilon^{N-\frac{p(N-2s)}{2}-ps(1-\gamma_{p,s})}=C_4, & N>4s,\\ C_4\varepsilon^{N-\frac{p(N-2s)}{2}-ps(1-\gamma_{p,s})}|\log\varepsilon|^{-\frac{(1-\gamma_{p,s})p}{2}}=C_4|\log\varepsilon|^{-\frac{(1-\gamma_{p,s})p}{2}}, & N=4s,\\ C_4\varepsilon^{N-\frac{p(N-2s)}{2}-\frac{p(N-2s)}{2}(1-\gamma_{p,s})}=C_4\varepsilon^{(\frac{N^2}{4s}+2s-\frac{3N}{2})p+2N-\frac{N^2}{2s}}, & N<4s,\end{cases}$$

for a constant $C_4>0$, hence we can infer that $\sup_{\mathbb{R}}\varphi_{v_\varepsilon}(\tau)<\frac{s}{N}S_s^{\frac{N}{2s}}$ for any $\varepsilon>0$ small enough. The result follows. □

Proof of Theorem 1.2. We can proceed exactly as in the proof of Theorem, using Lemma 5.1 and Lemma 5.2.

Remark 5.3. Actually $\sigma_r(a)<\frac{s}{N}S_s^{\frac{N}{2s}}$ for any $0<a<\infty$ when $N\le 4s$ (see the proof of Lemma 5.2). Thus, Theorem 1.3 holds for any $0<a<\infty$ when $N\le 4s$. □

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