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A viscoelastic wave equation with delay and variable exponents: existence and nonexistence

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Abstract. This article deals with the existence and nonexistence of solutions for a viscoelastic wave equation with time delay and variable exponents on the damping and on source term. Firstly, we get the existence of weak solutions by combining the Banach contraction mapping principle and the Faedo–Galerkin method under suitable assumptions on the variable exponents $m(\cdot)$ and $p(\cdot)$. For nonincreasing positive function g, we obtain the nonexistence of solutions with negative initial energy in appropriate conditions.

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1. Introduction

We consider a viscoelastic wave equation with time delay and variable exponents on the damping and on source term given by

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s + \mu_1 u_t(x,t)\,|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)\,|u_t|^{m(x)-2}(x,t-\tau) \\ = bu\,|u|^{p(x)-2} \text{ in } \Omega\times(0,\infty), \end{aligned} \tag{1.1}$$

with boundary conditions

$$u(x,t) = 0 \text{ on } \partial\Omega\times(0,\infty), \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \tag{1.3}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau) \text{ in } \Omega\times(0,\tau), \tag{1.4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n\geq 1$. $\tau>0$ represents the time delay, μ_1 is a positive constant, μ_2 is a real number, and $b\geq 0$ is a constant. u_0, u_1, and f_0 are the initial data functions to be specified later.

The exponents $m(\cdot)$ and $p(\cdot)$ are given as continuous functions on $\overline{\Omega}$ that satisfies

$$\begin{cases} 2\leq m^-\leq m(x)\leq m^+\leq m^*, \\ 2\leq p^-\leq p(x)\leq p^+\leq p^*, \end{cases} \tag{1.5}$$

where

$$m^{-} = ess\inf_{x\in\Omega} m(x), \quad m^{+} = ess\sup_{x\in\Omega} m(x),$$
$$p^{-} = ess\inf_{x\in\Omega} p(x), \quad p^{+} = ess\sup_{x\in\Omega} p(x),$$

and

$$\begin{cases} 2 \le m^{*}, p^{*} < \infty, & \text{if } n < 3, \\ 2 \le m^{*}, p^{*} \le \frac{2n}{n-2} & \text{if } n \ge 3. \end{cases}$$

Generally, the problems with variable exponents arise in many branches in sciences such as electrorheological fluids, nonlinear elasticity theory, and image processing [7,8,36]. Time delay often appears in many practical problems such as economic phenomena, thermal, biological, physical, and chemical [15].

When $m(x)$ and $p(x)$ are constant, the problem (1.1) becomes

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 |u_t|^{m-2} u_t + \mu_2 u_t(x,t-\tau)|u_t|^{m-2}(x,t-\tau) = bu|u|^{p-2}. \quad (1.6)$$

Kang [19] concerned with this problem and established the blow-up of solutions for positive initial energy $E(0) > 0$.

Without the delay term $(\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau))$ the problem (1.1) reduces to the following form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + a|u_t|^{m(x)-2} u_t = bu|u|^{p(x)-2}. \quad (1.7)$$

Related to (1.7), Park and Kang [31] established existence of solutions by using Galerkin method and proved blow-up result for positive initial energy $E(0) > 0$. Pişkin [35] obtained the blow-up of solutions for negative initial energy $E(0) < 0$ with $m(x) < p(x)$. In the presence of dissipative term $(-\Delta u_{tt})$, Gao et al. [11] studied the existence of weak solutions by using the embedding theory and the Faedo–Galerkin method. Messaoudi et al. [28] proved a global existence result using the well-depth method and established explicit and general decay results under a general assumption on the relaxation function.

When $m(x)$ is constant and without the delay term $(\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau))$ and source term $(bu|u|^{p(x)-2})$, the problem (1.1) takes the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{m-2} u_t = 0.$$

Belhannache et al. [6] established an existence result and some general decay results for both cases $m > 2$ and $1 < m < 2$. They improved the work of Messaoudi [29].

In the absence of the viscoelastic term $(g = 0)$, the problem (5) reduces to

$$u_{tt} - \Delta u + \mu_1 |u_t|^{m(x)-2} u_t + \mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2}. \quad (1.8)$$

Kafini and Messaoudi [16] proved the global nonexistence and the decay estimates for (1.8).

In the absence of the viscoelastic term $(g = 0)$ and delay term $(\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau))$, the problem (1.1) becomes in the form with variable exponents as follows:

$$u_{tt} - \Delta u + a|u_t|^{m(x)-2} u_t = bu|u|^{p(x)-2}. \quad (1.9)$$

For equation (1.9), Messaoudi et al. [27] considered the existence of a unique weak local solution by using the Faedo–Galerkin method. The authors also established the blow-up of solutions with negative initial energy $E(0) < 0$ for this equation. Ghegal et al. [13] proved a global result and obtained the stability result by applying an integral inequality due to Komornik.

In the absence of the viscoelastic term ($g = 0$), delay term $(\mu_2 u_t(x, t-\tau) |u_t|^{m(x)-2}(x, t-\tau))$, and when $m(x)$ and $p(x)$ are constant exponents, problem (1.1) leads to

$$u_{tt} - \Delta u + |u_t|^{m-2} u_t = u|u|^{p-2}. \tag{1.10}$$

In the absence of the damping term $|u_t|^{m-2} u_t$, Ball [4] showed that the source term $u|u|^{p-2}$ causes blow-up of solutions for $E(0) < 0$. Haraux and Zuazua [14] established that the damping term assures global existence for arbitrary initial data, in the absence of the source term. When $m = 2$ in the linear damping, Levine [22] proved a finite-time blow-up result with $E(0) < 0$. Georgiev and Todorova [12] improved Levine's result to the nonlinear damping case $m > 2$. They obtained that, if $m \geq p$, the global solution exists for arbitrary initial data. Also, they showed that, if $p > m$, solutions with sufficiently $E(0) < 0$ blow up in finite time. Messaoudi [24] extended the result of Georgiev and Todorova. He obtained blow-up of solutions in a finite time with $E(0) < 0$.

Nicaise and Pignotti [30] considered the following wave equation:

$$u_{tt}(x, t) - \Delta u(x, t) + a_0 u_t(x, t) + a u_t(x, t-\tau) = 0, \tag{1.11}$$

where a_0 and a are positive real parameters. The authors proved that the system is exponentially stable under condition $0 \leq a \leq a_0$. In the case $a \geq a_0$, they obtained a sequence of delays that shows the solution is unstable.

Concerning the hyperbolic-type equations with variable exponent, we refer to the work of Antontsev [1], who studied the wave equation as follows:

$$u_{tt} - div\left(a(x, t) |\nabla u|^{p(x,t)-2} \nabla u\right) - \alpha \Delta u_t = b(x, t) u |u|^{\sigma(x,t)-2}. \tag{1.12}$$

The author established several blow-up results for nonpositive initial energy, under specific conditions on a, b, p, σ and for certain solutions. Moreover, Antontsev [2] obtained the existence of local and global weak solutions of equation (1.12) by using Galerkin's approximations in spaces of Orlic–Sobolev type. Also, he proved the blow-up of weak solutions for nonpositive initial energy functional.

In recent years, some other authors investigate hyperbolic-type equations with delay or variable exponents (see [3,17,18,26,31,33–35,37]). Motivated by the above studies, we deal with the existence of weak solutions and nonexistence of solutions for the viscoelastic wave equation with delay term, source term, and variable exponents. There is no research, to our best knowledge, related to the viscoelastic $(\int_0^t g(t-s) \Delta u(s) \, ds)$ wave equation (1.1) with delay $(\mu_2 u_t(x, t-\tau) |u_t|^{m(x)-2}(x, t-\tau))$ and source terms with variable exponents; hence, our paper is generalization of the previous ones. Our main goal is to get the existence of weak solutions and establish the nonexistence of solutions for negative initial energy $E(0) < 0$ under sufficient conditions on $m(\cdot)$ and $p(\cdot)$ for the problem (1.1).

In addition to the Introduction, this work consists of four sections. Firstly, in Sect. 2, we present the definitions and some properties of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$. The equivalent system to (1.1)–(1.4) with its respective energy functional is presented. In Sect. 3, we introduce some technical lemmas. In Sect. 4, we establish the existence of weak solutions. Finally, in Sect. 5, we prove the nonexistence of solutions for negative initial energy $E(0) < 0$.

2. Preliminaries

Firstly, we state the results related to Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see [2,8,9,21,32]). At the end of the section, we present the equivalent system to (1.1)–(1.4) with its respective energy functional.

Let $p:\Omega\rightarrow[1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega)=\left\{u:\Omega\rightarrow R;\ \text{measurable in } \Omega: \int_{\Omega}|u|^{p(\cdot)}\,\mathrm{d}x<\infty\right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_{\Omega}\left|\frac{u}{\lambda}\right|^{p(x)}\mathrm{d}x\leq 1\right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space, see [8].

We define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows:

$$W^{1,p(\cdot)}(\Omega)=\left\{u\in L^{p(\cdot)}(\Omega):\ \nabla u \text{ exists and } |\nabla u|\in L^{p(\cdot)}(\Omega)\right\}.$$

Variable exponent Sobolev space with the norm

$$\|u\|_{1,p(\cdot)}=\|u\|_{p(\cdot)}+\|\nabla u\|_{p(\cdot)}$$

is a Banach space.

By $W_0^{1,p(\cdot)}(\Omega)$, we denote the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u\in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)}=\|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, similar to Sobolev spaces, where

$$\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1.$$

We also assume the log-Hölder condition,

$$|p(x)-p(y)|\leq-\frac{A}{\log|x-y|}\ \text{and}\ |m(x)-m(y)|\leq-\frac{B}{\log|x-y|}\ \text{for all } x,y\in\Omega, \tag{2.1}$$

$A,B>0$ and $0<\delta<1$ with $|x-y|<\delta$.

In addition, $m(\cdot)$ satisfies

$$m^{*}(x)=\begin{cases}\frac{nm(x)}{n-m(x)}, & \text{if } m(x)<n,\\ \text{any number in } [1,\infty), & \text{if } m(x)\geq n.\end{cases}$$

As usual, the notation $\|\cdot\|_p$ denotes L^p norm, and $(\cdot,\cdot)$ is the L^2 inner product. In particular, we write $\|\cdot\|$ instead of $\|\cdot\|_2$.

We make some assumptions on g:

(A1) Let $g:[0,\infty)\rightarrow(0,\infty)$ be a nonincreasing and differentiable function, satisfying

$$1-\int_0^{\infty}g(s)\,\mathrm{d}s=l>0.$$

(A2) $g(s)\geq 0$, $g'(s)\leq 0$ and

$$\int_0^{\infty}g(s)\,\mathrm{d}s<\frac{(1-a)p^{-}-2-\frac{2C}{m^{-}k^{1-m^{-}}}}{(1-a)p^{-}-2+\frac{1}{2\lambda}},\ 0<a<1.$$

Using the direct calculations, we get

$$\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))\,\mathrm{d}s = -\frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}(g' o \nabla u)(t)$$
$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{(go\nabla u)(t) - \left(\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|^2\right\}, \tag{2.2}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2\,\mathrm{d}s.$$

For coefficients μ_1 and μ_2, we suppose

$$|\mu_2| < \frac{m^-}{m^+}\mu_1. \tag{2.3}$$

Now, as in [30] we introduce the auxiliary unknown

$$z(x,\rho,t) = u_t(x, t-\tau\rho),\ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

It is straightforward to check that z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ \ x\in\Omega,\ \ \rho\in(0,1),\ \ t>0,$$

and consequently, problem (1.1)–(1.4) is equivalent to

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} + \mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2}$$
$$= bu|u|^{p(x)-2}\ \text{ in } \Omega\times(0,\infty), \tag{2.4}$$
$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \tag{2.5}$$

with boundary conditions

$$u(x,t) = 0 \text{ on } \partial\Omega\times(0,\infty), \tag{2.6}$$

and initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega, \tag{2.7}$$
$$z(x,\rho,0) = f_0(x,-\tau\rho)\ \text{ in } \Omega\times(0,1), \tag{2.8}$$
$$z(x,0,t) = u_t(x,t)\ \text{ in } \Omega\times(0,\infty). \tag{2.9}$$

For $t\geq 0$, the energy functional of the system (2.4)–(2.9) is defined by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{2}(g\circ\nabla u)(t)$$
$$+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\mathrm{d}x\mathrm{d}\rho - b\int_\Omega \frac{|u|^{p(x)}}{p(x)}\mathrm{d}x, \tag{2.10}$$

where ξ is a continuous function satisfying (2.11)

$$\tau|\mu_2|(m(x)-1) < \xi(x) < \tau(\mu_1 m(x) - |\mu_2|),\ x\in\overline{\Omega}. \tag{2.11}$$

3. Technical lemmas

Let us start this section by proving that the energy functional $E(t)$ defined by (2.10) is non-increasing.

Lemma 3.1. *Let (u,z) be a solution of (2.4)–(2.9). The energy functional $E(t)$ is nonincreasing, that is,*

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)}\right) \mathrm{d}x < 0, \text{ forsome } C_0 > 0.$$

Proof. Multiplying (2.4) by u_t, integrating over Ω, multiplying (2.5) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$, integrating over $\Omega \times (0,1)$, and summing up, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{2}(g \circ \nabla u)(t)\right.\\
&\qquad\left. + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} \mathrm{d}x\mathrm{d}\rho - b\int_\Omega \frac{|u|^{p(x)}}{p(x)} \mathrm{d}x\right]\\
&= -\mu_1 \int_\Omega |u_t|^{m(x)}\,\mathrm{d}x - \frac{1}{2} g(s)\|\nabla u\|^2 + \frac{1}{2}(g' \circ \nabla u)(t)\\
&\qquad - \frac{1}{\tau}\int_\Omega \int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} z z_\rho(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x\\
&\qquad - \mu_2 \int_\Omega u_t z(x,1,t)|z(x,1,t)|^{m(x)-2}\,\mathrm{d}x
\end{aligned} \tag{3.1}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \int_\Omega [\nabla u(s) - \nabla u(t)]^2\,\mathrm{d}x\mathrm{d}s.$$

Now, we estimate the last two terms of the right-hand side of (3.1) as follows:

$$\begin{aligned}
&-\frac{1}{\tau}\int_\Omega \int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} z z_\rho(x,\rho,t)\,\mathrm{d}\rho\mathrm{d}x\\
&= -\frac{1}{\tau}\int_\Omega \int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right)\mathrm{d}\rho\mathrm{d}x\\
&= \frac{1}{\tau}\int_\Omega \frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)} - |z(x,1,t)|^{m(x)}\right)\mathrm{d}x\\
&= \int_\Omega \frac{\xi(x)}{\tau m(x)}|u_t|^{m(x)}\,\mathrm{d}x - \int_\Omega \frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}\,\mathrm{d}x.
\end{aligned}$$

Using the Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term to obtain

$$|u_t||z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)}|u_t|^{m(x)} + \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

Consequently, we deduce that

$$-\mu_2\int_\Omega u_t z\left|z(x,1,t)\right|^{m(x)-2}\mathrm{d}x$$
$$\le|\mu_2|\left(\int_\Omega\frac{1}{m(x)}\left|u_t(t)\right|^{m(x)}\mathrm{d}x+\int_\Omega\frac{m(x)-1}{m(x)}\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x\right).$$

So,

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t}\le&-\int_\Omega\left[\mu_1-\left(\frac{\xi(x)}{\tau m(x)}+\frac{|\mu_2|}{m(x)}\right)\right]\left|u_t(t)\right|^{m(x)}\mathrm{d}x\\&-\int_\Omega\left(\frac{\xi(x)}{\tau m(x)}-\frac{|\mu_2|(m(x)-1)}{m(x)}\right)\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x.\\&-\frac{1}{2}g(s)\left\|\nabla u\right\|^2+\frac{1}{2}\left(g'\circ\nabla u\right)(t).\end{aligned}$$

For all $x\in\overline{\Omega}$ the relation (2.11) leads to

$$f_1(x)\overset{def}{:=}\mu_1-\left(\frac{\xi(x)}{\tau m(x)}+\frac{|\mu_2|}{m(x)}\right)>0,$$
$$f_2(x)\overset{def}{:=}\frac{\xi(x)}{\tau m(x)}-\frac{|\mu_2|(m(x)-1)}{m(x)}>0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. So, if we define

$$C_0(x)=\min\left\{f_1(x),\ f_2(x)\right\}>0 \text{ for any } x\in\overline{\Omega},$$

and take $C_0(x)=\inf_{\overline{\Omega}}C_0(x)$, so $C_0(x)\ge C_0>0$. Moreover, by using assumptions (A1)–(A2) we have,

$$E'(t)\le-C_0\left[\int_\Omega\left|u_t(t)\right|^{m(x)}\mathrm{d}x+\int_\Omega\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x\right]<0. \tag{3.2}$$

□

Taking into account that signal of $E(t)$ is not defined, (3.2) is an important property that leads $E(t)\le E(0)$. Next, we introduce some technical lemmas.

Lemma 3.2. [2] (Poincare inequality) *Assume that* $p(\cdot)$ *satisfies* (2.1)*. Then,*

$$\|u\|_{p(\cdot)}\le c\left\|\nabla u\right\|_{p(\cdot)} \text{ for all } u\in W_0^{1,p(\cdot)}(\Omega),$$

where $c=c\left(p^-,p^+,|\Omega|\right)>0$, Ω *is a bounded domain of* R^n.

Lemma 3.3. [2] *If* $p:\overline{\Omega}\to[1,\infty)$ *is continuous,*

$$2\le p^-\le p(x)\le p^+\le\frac{2n}{n-2},\ n\ge3, \tag{3.3}$$

holds, then the embedding $H_0^1(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous.*

Lemma 3.4. [1] *If* $p^+<\infty$ *and* $p:\Omega\to[1,\infty)$ *is a measurable function, then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Lemma 3.5. [1] (Hölder' inequality) *Let* $p, q, s \geq 1$ *be measurable functions defined on* Ω *and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}\ , \text{ for a.e. } y \in \Omega,$$

holds. If $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$, *then* $fg \in L^{s(\cdot)}(\Omega)$ *and*

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Lemma 3.6. [1] (Unit ball property) *Let* $p \geq 1$ *be a measurable function on* Ω. *Then,*

$$\|f\|_{p(\cdot)} \leq 1 \text{ if and only if } \varrho_{p(\cdot)}(f) \leq 1,$$

where

$$\varrho_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)}\,\mathrm{d}x.$$

Lemma 3.7. [2] *If* $p \geq 1$ *is a measurable function on* Ω, *then*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \varrho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$ *and for a.e.* $x \in \Omega$.

Lemma 3.8. [8] *Let* $m(\cdot) \in C\left(\overline{\Omega}\right)$ *and* $p : \Omega \to [1, \infty)$ *be a measurable function that satisfy*

$$essinf_{x\in\overline{\Omega}}\left(m^*(x) - p(x)\right) > 0.$$

Then, the Sobolev embedding $W_0^{1,m(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous and compact, where*

$$m^*(x) = \begin{cases} \frac{nm^-}{n-m^-} \text{ if } m^- < n, \\ \text{any number in } [1,\infty) \text{ if } m^- \geq n. \end{cases}$$

Lemma 3.9. [17] *Suppose that* $p(\cdot)$ *satisfies*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \infty,$$

for a.e. $x \in \Omega$, *then the function* $h(s) = b|s|^{p(x)-2}s$ *is differentiable function and*

$$|h'(s)| = b|p(x) - 1|\,|s|^{p(x)-2}.$$

Lemma 3.10. [16] *Suppose that condition*

$$2 \leq m^- \leq m(x) \leq m^+ \leq p^- \leq p(x) \leq p^+ \leq \frac{2(n-1)}{n-2},\ n \geq 3$$

holds. Then, depending on Ω *only, there exists a positive* $C > 1$, *such that*

$$\varrho^{s/p^-}(u) \leq C\left(\|\nabla u\|^2 + \varrho(u)\right). \tag{3.4}$$

Moreover, we have the following inequalities:

$$\|u\|_{p^-}^s \leq C\left(\|\nabla u\|^2 + \|u(t)\|_{p^-}^{p^-}\right), \tag{3.5}$$

$$\varrho^{s/p^-}(u) \leq C\left(|H(t)| + \|u_t\|^2 + \varrho(u) + \int_0^1\int_{\Omega} \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\mathrm{d}x\mathrm{d}\rho\right), \tag{3.6}$$

$$\|u\|_{p^-}^s \leq C\left(|H(t)| + \|u_t\|^2 + \|u\|_{p^-}^{p^-} + \int_0^1\int_{\Omega} \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\mathrm{d}x\mathrm{d}\rho\right), \tag{3.7}$$

for any $u \in H_0^1(\Omega)$ *and* $2 \le s \le p^-$.

$$\varrho(u) \ge C \|u\|_{p^-}^{p^-}, \tag{3.8}$$

$$\int_\Omega |u|^{m(x)} \mathrm{d}x \le C\left(\varrho^{m^-/p^-}(u) + \varrho^{m^+/p^-}(u)\right). \tag{3.9}$$

4. Existence of solutions

In this section, combining the contraction mapping theorem and Faedo–Galerkin method similar to [27,31], we obtain the local existence of solution for problem (2.4)–(2.9).

Theorem 4.1. *Assume that assumptions (A1)–(A2) hold. If* $g \in L^2(\Omega \times (0,T))$, μ_1 *and* μ_2 *are under the condition* (2.3), $m(x)$ *satisfies* (1.5) *and* (2.1). *Then, for every initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $f_0 \in L^{m(\cdot)}(\Omega \times (0,1))$ *and* $T > 0$, *the problem* (2.4)–(2.9) *has a unique weak solution* (u, z) *where*

$$u \in L^\infty\left((0,T); H_0^1(\Omega)\right),\ u_t \in L^\infty\left((0,T); L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)),\ z \in L^{m(\cdot)}(\Omega \times (0,1)).$$

Proof. (**Existence**): Firstly, we take a basis $\{\upsilon_j\}_{j=1}^\infty$ to $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$ and define the finite-dimensional subspace $V_k = span\{\nu_1, \ldots, \nu_k\}$. By normalization, we have $\|\upsilon_j\| = 1$.

Similar to [10,20,38], we define the sequence $\varphi_j(x,\rho)$, for $1 \le j \le k$, as follows:

$$\varphi_j(x,0) = \upsilon_j(x).$$

We extend over $L^2(\Omega \times [0,1])$, $\varphi_j(x,0)$ by $\varphi_j(x,\rho)$ and denote $U_k = span\{\varphi_1, \ldots, \varphi_k\}$.

Define

$$u^k(x,t) = \sum_{j=1}^k d_j(t)\upsilon_j(x) \text{ and } z^k(x,t) = \sum_{j=1}^k e_j(t)\varphi_j(x,\rho)$$

where $(u^k(x,t), z^k(x,t))$ are solutions of the following approximate problems as:

$$\begin{aligned}
&\int_\Omega u_{tt}^k(x,t)\upsilon_j(x)\mathrm{d}x + \int_\Omega \nabla u^k(x,t)\nabla\upsilon_j(x)\mathrm{d}x - \int_\Omega\int_0^t g(t-s)\nabla u^k(x,s)\nabla\upsilon_j(x)\mathrm{d}s\mathrm{d}x \\
&\quad + \mu_1 \int_\Omega \left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t)\upsilon_j(x)\mathrm{d}x + \mu_2 \int_\Omega \left|z^k(x,1,t)\right|^{m(x)-2} z^k(x,1,t)\upsilon_j(x)\mathrm{d}x \\
&= \int_\Omega g(x,t)\upsilon_j(x)\mathrm{d}x,
\end{aligned} \tag{4.1}$$

with initial data

$$u^k(x,0) = u_0^k,\ u_t^k(x,0) = u_1^k,\ z^k(x,0,t) = u_t^k(x,t),\ \forall i = 1, \ldots, k, \tag{4.2}$$

and

$$\int_\Omega \tau z_t^k(x,\rho,t)\varphi_j\mathrm{d}x + \int_\Omega z_\rho^k(x,\rho,t)\varphi_j\mathrm{d}x = 0, \quad z^k(0) = z_0^k,\ \forall j = 1, \ldots, k, \tag{4.3}$$

where

$$u_0^k = \sum_{j=1}^k (u_0, \upsilon_j)\upsilon_j \to u_0 \text{ in } H_0^1(\Omega) \text{ as } k \to \infty. \tag{4.4}$$

$$u_1^k = \sum_{j=1}^{k} (u_i, \upsilon_j)\, \upsilon_j \to u_1 \ \text{ in } \ L^2(\Omega) \ \text{ as } k \to \infty. \tag{4.5}$$

$$z_0^k = \sum_{j=1}^{k} (f_0, \varphi_j)\, \varphi_j \to f_0 \ \text{ in } \ L^{m(\cdot)}(\Omega \times (0,1)) \ \text{ as } k \to \infty. \tag{4.6}$$

Considering the standard theory of ordinary differential equations, the finite-dimensional problem (4.1)–(4.3) has solution defined on $[0, t_k)$, $0 < t_k < T$ for $T > 0$.

Now, we will prove that $t_k = T$, $\forall k \geq 1$. We multiply Eq. (4.1) by $d_j'(t)$ and sum up the product result in j; then, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \left|u_t^k(t)\right|^2 \mathrm{d}x + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \left(g \circ \nabla u^k\right)(t)\right\} \\
&\quad + \mu_1 \int_\Omega \left|u_t^k(t)\right|^{m(x)} \mathrm{d}x + \mu_2 \int_\Omega \left|z^k(x,1,t)\right|^{m(x)-2} z^k(x,1,t)\, u_t^k(t)\,\mathrm{d}x \\
&= -\frac{1}{2} g(t) \int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \frac{1}{2}\left(g' \circ \nabla u^k\right)(t) + \int_\Omega g(x,t)\, u_t^k(t)\,\mathrm{d}x.
\end{aligned}$$

Using the hypothesis on g and integrating over $(0, t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\{\int_\Omega \left|u_t^k(t)\right|^2 \mathrm{d}x + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \left(g \circ \nabla u^k\right)(t)\right\} \\
&\quad + \mu_1 \int_0^t \int_\Omega \left|u_t^k(s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s + \mu_2 \int_0^t \int_\Omega \left|z^k(x,1,s)\right|^{m(x)-2} z^k(x,1,s)\, u_t^k(s)\,\mathrm{d}x\mathrm{d}s \\
&= \frac{1}{2}\int_\Omega \left|u_1^k\right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left|\nabla u_0^k\right|^2 \mathrm{d}x + \int_0^t \int_\Omega g(x,s)\, u_t^k(s)\,\mathrm{d}x\mathrm{d}s \\
&\leq \frac{1}{2}\int_\Omega \left|u_1^k\right|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega \left|\nabla u_0^k\right|^2 \mathrm{d}x + \frac{1}{4}\int_0^t \int_\Omega \left|u_t^k(s)\right|^2 \mathrm{d}x\mathrm{d}s + \int_0^T \int_\Omega |g(x,s)|^2 \,\mathrm{d}x\mathrm{d}s \\
&\leq C + \frac{1}{4}\sup_{(0,t_k)} \int_\Omega \left|u_t^k(x,t)\right|^2 \mathrm{d}x, \ \forall t \in [0, t_k).
\end{aligned} \tag{4.7}$$

By (4.3), we have

$$\begin{aligned}
&\int_0^t \int_0^1 \int_\Omega \zeta z_t^k(x,\rho,s)\, z^k(x,\rho,s)^{m(x)-1}\,\mathrm{d}x\mathrm{d}\rho\mathrm{d}s \\
&\quad + \int_0^t \int_0^1 \int_\Omega \frac{\zeta}{\tau} z_\rho^k(x,\rho,s)\, z^k(x,\rho,s)^{m(x)-1}\,\mathrm{d}x\mathrm{d}\rho\mathrm{d}s = 0,
\end{aligned}$$

which provides

$$\int_0^1\int_\Omega \frac{\zeta}{m(x)}\left|z^k(x,\rho,t)\right|^{m(x)}\mathrm{d}x\mathrm{d}\rho-\int_0^1\int_\Omega \frac{\zeta}{m(x)}\left|z^k(x,\rho,0)\right|^{m(x)}\mathrm{d}x\mathrm{d}\rho$$
$$+\int_0^t\int_\Omega \frac{\zeta}{m(x)\tau}\left|z^k(x,1,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega \frac{\zeta}{m(x)\tau}\left|z^k(x,0,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s=0. \tag{4.8}$$

Summing up the identities (4.7) and (4.8), we get

$$\frac{1}{2}\left\{\int_\Omega \left|u_t^k(t)\right|^2\mathrm{d}x+\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega\left|\nabla u^k\right|^2\mathrm{d}x+\left(g\circ\nabla u^k\right)(t)\right\}$$
$$+\int_0^1\int_\Omega \frac{\zeta}{m(x)}\left|z^k(x,\rho,t)\right|^{m(x)}\mathrm{d}x\mathrm{d}\rho+\int_0^t\int_\Omega \frac{\zeta}{m(x)\tau}\left|z^k(x,1,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_\Omega\left(\mu_1-\frac{\zeta}{m(x)\tau}\right)\left|u_t^k(s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s$$
$$\leq \frac{1}{2}\int_\Omega\left|u_1^k\right|^2\mathrm{d}x+\frac{1}{2}\int_\Omega\left|\nabla u_0^k\right|^2\mathrm{d}x+\frac{1}{4}\int_0^t\int_\Omega\left|u_t^k(s)\right|^2\mathrm{d}x\mathrm{d}s+\int_0^T\int_\Omega|g(x,s)|^2\mathrm{d}x\mathrm{d}s$$
$$+\int_0^1\int_\Omega\frac{\zeta}{m(x)}\left|z_0^k\right|^{m(x)}\mathrm{d}x\mathrm{d}\rho-\mu_2\int_0^t\int_\Omega\left|z^k(x,1,s)\right|^{m(x)-2}z^k(x,1,s)\,u_t^k(s)\,\mathrm{d}x\mathrm{d}s. \tag{4.9}$$

Utilizing Young's inequality, we get

$$-\mu_2\int_0^t\int_\Omega\left|z^k(x,1,s)\right|^{m(x)-2}z^k(x,1,s)\,u_t^k(s)\,\mathrm{d}x\mathrm{d}s$$
$$\leq\int_0^t\int_\Omega\frac{|\mu_2|(m(x)-1)}{m(x)}\left|z^k(x,1,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s+\int_0^t\int_\Omega\frac{|\mu_2|}{m(x)}\left|u_t^k(s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s. \tag{4.10}$$

By combining (4.9) and (4.10), we obtain

$$\frac{1}{2}\left\{\int_\Omega \left|u_t^k(t)\right|^2\mathrm{d}x+\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega\left|\nabla u^k(t)\right|^2\mathrm{d}x+\left(g\circ\nabla u^k\right)(t)\right\}$$
$$+\frac{\zeta}{m^+}\int_0^1\int_\Omega\left|z^k(x,\rho,t)\right|^{m(x)}\mathrm{d}x\mathrm{d}\rho$$
$$+\int_0^t\int_\Omega\left(\frac{\zeta}{m^+\tau}-\frac{(m^+-1)|\mu_2|}{m^+}\right)\left|z^k(x,1,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_\Omega\left(\mu_1-\frac{\zeta}{m^-\tau}-\frac{|\mu_2|}{m^-}\right)\left|u_t^k(s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s$$

$$\leq \frac{1}{2}\int_{\Omega}\left|u_1^k\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega}\left|\nabla u_0^k\right|^2 \mathrm{d}x + \frac{1}{4}\int_0^t\int_{\Omega}\left|u_t^k(s)\right|^2 \mathrm{d}x\mathrm{d}s$$
$$+\int_0^T\int_{\Omega}|g(x,s)|^2 \mathrm{d}x\mathrm{d}s + \frac{\zeta}{m^-}\int_0^1\int_{\Omega}\left|z_0^k\right|^{m(x)} \mathrm{d}x\mathrm{d}\rho$$
$$\leq C + \frac{1}{4}\sup_{(0,t_k)}\int_{\Omega}\left|u_t^k(t)\right|^2 \mathrm{d}x,\ \forall t\in[0,t_k).$$

From (2.11), we can find c_2 and c_3 positive constants, such that

$$\sup_{(0,t_k)}\int_{\Omega}\left|u_t^k(t)\right|^2 \mathrm{d}x + \sup_{(0,t_k)} l\int_{\Omega}\left|\nabla u^k(t)\right|^2 \mathrm{d}x$$
$$+\sup_{(0,t_k)}\int_0^1\int_{\Omega}\left|z^k(x,\rho,t)\right|^{m(x)} \mathrm{d}x\mathrm{d}\rho$$
$$+c_2\int_0^{t_k}\int_{\Omega}\left|u_t^k(s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s + c_3\int_0^{t_k}\int_{\Omega}\left|z^k(x,1,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s \leq C.$$

Hence, the solution can be extended to $[0,T)$ and we obtain, for all $k\in N$,

$$\begin{aligned}
&(u^k) \text{ is a bounded sequence in } L^\infty\left((0,T);H_0^1(\Omega)\right),\\
&(u_t^k) \text{ is a bounded sequence in } L^\infty\left((0,T);L^2(\Omega)\right)\cap L^{m(\cdot)}(\Omega\times(0,T)),\\
&(z^k) \text{ is a bounded sequence in } L^\infty\Big((0,T);L^{m(\cdot)}(\Omega\times(0,1))\Big),\\
&(z^k(1)) \text{ is a bounded sequence in } L^{m(\cdot)}(\Omega\times(0,T)).
\end{aligned}$$

Then, there exists a subsequence (u^μ) of (u^k) such that

$$\begin{aligned}
&u^\mu \to u \text{ weak star in } L^\infty\left((0,T);H_0^1(\Omega)\right),\\
&u_t^\mu \to u_t \text{ weak star in } L^\infty\left((0,T);L^2(\Omega)\right) \text{ and weakly in } L^{m(\cdot)}(\Omega\times(0,T))
\end{aligned}$$

and subsequence (z^μ) of (z^k) such that

$$\begin{aligned}
&z^\mu \to z \text{ weak star in } L^\infty\Big((0,T);L^{m(\cdot)}(\Omega\times(0,1))\Big)\\
&z^\mu(1) \to z(1) \text{ weak star in } L^{m(\cdot)}(\Omega\times(0,T)).
\end{aligned}$$

On the other hand, utilizing Lions lemma (see [23]), we infer that $u\in C\left([0,T];L^2(\Omega)\right)$. Since u_t^μ and $z^\mu(1)$ are bounded in $L^{m(\cdot)}(\Omega\times(0,T))$, then $|u_t^\mu|^{m(x)-2}u_t^\mu$ and $|z^\mu(1)|^{m(x)-2}z^\mu(1)$ are bounded in $L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T))$.

As in [27], we have

$$\begin{aligned}
&|u_t^\mu|^{m(\cdot)-2}u_t^\mu \to |u_t|^{m(\cdot)-2}u_t \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T)),\\
&|z^\mu(1)|^{m(\cdot)-2}z^\mu(1) \to |z(1)|^{m(\cdot)-2}z(1) \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega\times(0,T)).
\end{aligned}$$

Thus, we obtain, for all $\upsilon \in L^{m(\cdot)}\left((0,T)\times H_0^1(\Omega)\right)$,

$$\int_\Omega \Bigg(u_{tt}\upsilon + \nabla u \nabla \upsilon - \int_0^t g(t-s)\nabla u(s)\nabla \upsilon \mathrm{d}s + \mu_1 |u_t|^{m(x)-2} u_t \upsilon + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t)\upsilon \Bigg) \mathrm{d}x = \int_\Omega g\upsilon \mathrm{d}x,$$

which gives

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t) = g \text{ in } D'(\Omega\times(0,T)).$$

(**Uniqueness**): Assume that (u^1, z^1) and (u^2, z^2) are two-pair solution to the problem (2.4)–(2.9). We define $\widetilde{u} = u^1 - u^2$ and $\widetilde{z} = z^1 - z^2$, then $(\widetilde{u}, \widetilde{z})$ satisfy

$$\widetilde{u}_{tt} - \Delta\widetilde{u} + \int_0^t g(t-s)\Delta\widetilde{u}(s)\,\mathrm{d}s + \mu_1 \left|u_t^1\right|^{m(x)-2} u_t^1 - \mu_1 \left|u_t^2\right|^{m(x)-2} u_t^2 + \mu_2 \left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) - \mu_2 \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t) = 0 \text{ in } \Omega\times(0,T) \tag{4.11}$$

$$\tau\widetilde{z}_t(x,\rho,t) + \widetilde{z}_\rho(x,\rho,t) = 0 \text{ in } \Omega\times(0,1)\times(0,T), \tag{4.12}$$

boundary condition

$$\widetilde{u}(x,t) = 0 \text{ on } \partial\Omega\times(0,T) \tag{4.13}$$

and initial conditions

$$\widetilde{z}(x,0,t) = \widetilde{u}_t(x,t) \quad \text{in } \Omega\times(0,T), \tag{4.14}$$

$$\widetilde{z}(x,\rho,0) = 0 \text{ in } \Omega\times(0,1), \tag{4.15}$$

$$\widetilde{u}(x,0) = 0,\ \widetilde{u}_t(x,0) = 0 \text{ in } \Omega. \tag{4.16}$$

Multiplying (4.11) by $\widetilde{u}_t$ and integrating over Ω, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \left|\widetilde{u}_t(t)\right|^2 \mathrm{d}x + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega \left|\nabla\widetilde{u}(t)\right|^2 \mathrm{d}x + \left(g\circ\nabla\widetilde{u}\right)(t)\right\} + \frac{1}{2}g(t)\int_\Omega \left|\nabla\widetilde{u}(t)\right|^2 \mathrm{d}x - \frac{1}{2}\left(g'\circ\nabla\widetilde{u}\right)(t)$$
$$+ \mu_1 \int_\Omega \left(\left|u_t^1(t)\right|^{m(x)-2} u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2} u_t^2(x,t)\right)\widetilde{u}_t(t)\,\mathrm{d}x$$
$$+ \mu_2 \int_\Omega \left(\left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t)\right)\widetilde{u}_t(t)\,\mathrm{d}x = 0. \tag{4.17}$$

Multiplying (4.12) by $\widetilde{z}$ and integrating over $\Omega \times (0,1)$, we obtain

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_\Omega \left|\widetilde{z}(x,\rho,t)\right|^2 \mathrm{d}x\mathrm{d}\rho + \frac{1}{2}\left(\left\|\widetilde{z}(x,1,t)\right\|^2 - \left\|\widetilde{u}_t(t)\right\|^2\right) = 0. \tag{4.18}$$

By combining (4.17) and (4.18), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \left|\widetilde{u}_t(t)\right|^2 \mathrm{d}x + \left(1-\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega \left|\nabla\widetilde{u}(t)\right|^2 \mathrm{d}x\right.\\
&\quad\left. + \left(g\circ\nabla\widetilde{u}\right)(t) + \tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\}\\
&\quad + \frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2 + \frac{1}{2}g(t)\int_\Omega \left|\nabla\widetilde{u}(t)\right|^2 \mathrm{d}x\\
&\quad + \mu_1\int_\Omega \left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2}u_t^2(t)\right)\widetilde{u}_t(t)\,\mathrm{d}x\\
&\quad + \mu_2\int_\Omega \left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,\mathrm{d}x\\
&= \frac{1}{2}\left(g'\circ\nabla\widetilde{u}\right)(t) + \frac{1}{2}\left\|\widetilde{u}_t(t)\right\|^2.
\end{aligned} \tag{4.19}$$

Since the equation $y \to |y|^{m(\cdot)-2}y$ is increasing, we get

$$\int_\Omega \left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2}u_t^2(x,t)\right)\widetilde{u}_t(t)\,\mathrm{d}x \geq 0, \tag{4.20}$$

$$\int_\Omega \left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,\mathrm{d}x \geq 0. \tag{4.21}$$

By using (4.19)–(4.21), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\widetilde{u}_t(t)\right\|^2 + l\left\|\widetilde{\nabla u}(t)\right\|^2 + \tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\} + \frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2\\
&\quad \leq c_4\left(\left\|\widetilde{u}_t(t)\right\|^2 + l\left\|\widetilde{\nabla u}(t)\right\|^2\right),
\end{aligned}$$

which implies that $\widetilde{u} = 0$, $\widetilde{z} = 0$. □

The following theorem shows that the problem (2.4)–(2.9) has a unique local solution under suitable condition.

Theorem 4.2. *Assume that assumptions (A1)–(A2) hold. Let μ_1 and μ_2 satisfy the condition (2.3). If $m(x)$ satisfies (1.5), (2.1); $p(x)$ satisfies (2.1) and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2(n-1)}{n-2}, \text{ if } n \geq 3, \tag{4.22}$$

then, for every $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, $f_0 \in L^{m(\cdot)}(\Omega\times(0,1))$ and $T>0$, the problem (2.4)–(2.9) has a unique local solution such that

$$u \in C\left([0,T];H_0^1(\Omega)\right),\ u_t \in C\left([0,T];L^2(\Omega)\right)\cap L^{m(\cdot)}(\Omega\times(0,T)),\ z \in L^{m(\cdot)}(\Omega\times(0,1)).$$

Proof. (**Existence**) Suppose that $v \in L^{\infty}\left((0,T);H_0^1(\Omega)\right)$. Since $2(p^- - 1) \leq 2(p^+ - 1) \leq \frac{2n}{n-2}$, then

$$\|h(v)\|^2 = |b|^2 \int_\Omega |v|^{2(p(x)-1)}\,\mathrm{d}x \leq |b|^2 \left\{ \int_\Omega |v|^{2(p^- -1)}\,\mathrm{d}x + \int_\Omega |v|^{2(p^+ -1)}\,\mathrm{d}x \right\} < \infty.$$

Therefore, we have

$$h(v) \in L^{\infty}\left((0,T);L^2(\Omega)\right) \subset L^2(\Omega \times (0,T)).$$

Hence, from Theorem 4.1, for each $v \in L^{\infty}\left((0,T);H_0^1(\Omega)\right)$, there exists a unique solution

$$u \in L^{\infty}\left((0,T);H_0^1(\Omega)\right),\ u_t \in L^{\infty}\left((0,T);L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)),\ z \in L^{m(\cdot)}(\Omega \times (0,1))$$

satisfying the problem as follows:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t)$$
$$= h(v) \quad \text{in } \Omega \times (0,T), \tag{4.23}$$
$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } \Omega \times (0,1) \times (0,T), \tag{4.24}$$

boundary condition

$$u(x,t) = 0 \ \text{ on } \ \partial\Omega \times (0,T), \tag{4.25}$$

and initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \text{ in } \ \Omega, \tag{4.26}$$
$$z(x,0,t) = u_t(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{4.27}$$
$$z(x,\rho,0) = f_0(x,-\tau\rho) \quad \text{in} \quad \Omega \times (0,1). \tag{4.28}$$

Similar to [5,25], we prove that the sequence (u^k) is Cauchy in

$$X_T := C\left([0,T];H_0^1(\Omega)\right) \cap C^1\left([0,T];L^2(\Omega)\right),$$

equipped with the norm

$$\|u\|_{X_T}^2 := \max_{0\leq t\leq T} \left\{ \|u_t\|^2 + l\,\|\nabla u\|^2 \right\}.$$

Hence, the problem (4.23) has a unique weak solution. Next, we will verify that the problem (2.4) has a unique weak solution.

We define the nonlinear mapping $K : X_T \to X_T$ by $K(v) = u$, where u is the unique solution of the problem (4.23).

Next, we shall show that there exist $T > 0$, such that

(i) $K : X_T \to X_T$,

(ii) K is a contraction mapping in X_T.

To show (i), we multiply the first equation in (4.23) by u_t and integrate over $\Omega \times (0,t)$, and then we get

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|^2 + \frac{1}{2}(g \circ \nabla u)(t)$$
$$+ \frac{1}{2}\int_0^t g(s)\|\nabla u\|^2\,\mathrm{d}s - \frac{1}{2}\int_0^t (g' \circ \nabla u)(s)\,\mathrm{d}s$$

$$+ \mu_1 \int_0^t \int_\Omega |u_t(s)|^{m(x)} \,\mathrm{d}x\mathrm{d}s + \mu_2 \int_0^t \int_\Omega |z(x,1,s)|^{m(x)-2} z(x,1,s)\, u_t(s) \,\mathrm{d}x\mathrm{d}s$$

$$= \frac{1}{2} \int_\Omega u_1^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla u_0|^2 \,\mathrm{d}x + b \int_0^t \int_\Omega |v|^{p(x)-2} v u_t(s) \,\mathrm{d}x\mathrm{d}s. \tag{4.29}$$

We multiply the second equation in (4.23) by $\frac{\zeta}{\tau} z^{m(x)-1}$ and integrate over $\Omega \times (0,1) \times (0,t)$, and then we get

$$\int_0^1 \int_\Omega \frac{\zeta}{m(x)} \left(|z(x,\rho,t)|^{m(x)} - |z(x,\rho,0)|^{m(x)} \right) \mathrm{d}x\mathrm{d}\rho$$

$$= \int_0^t \int_\Omega \frac{\zeta}{m(x)\tau} \left(|z(x,0,s)|^{m(x)} - |z(x,1,s)|^{m(x)} \right) \mathrm{d}x\mathrm{d}s. \tag{4.30}$$

From assumptions (A1)–(A2), we have

$$\frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{2} \int_0^t (g' \circ \nabla u)(s) \,\mathrm{d}s + \frac{1}{2} \int_0^t g(s) \|\nabla u\|^2 \,\mathrm{d}s \geq 0. \tag{4.31}$$

By combining (4.29), (4.30), and (4.31), we obtain

$$\frac{1}{2} \|u_t\|^2 + \frac{l}{2} \|\nabla u\|^2 + \int_0^1 \int_\Omega \frac{\zeta}{m(x)} |z(x,\rho,t)|^{m(x)} \,\mathrm{d}x\mathrm{d}\rho$$

$$+ \mu_1 \int_0^t \int_\Omega |u_t(s)|^{m(x)} \,\mathrm{d}x\mathrm{d}s + \mu_2 \int_0^t \int_\Omega |z(x,1,s)|^{m(x)-2} z(x,1,s)\, u_t(s) \,\mathrm{d}x\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \frac{\zeta}{m(x)\tau} \left(|z(x,1,s)|^{m(x)} - |u_t(s)|^{m(x)} \right) \mathrm{d}x\mathrm{d}s$$

$$= \frac{1}{2} \int_\Omega u_1^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla u_0|^2 \,\mathrm{d}x + \int_0^1 \int_\Omega \frac{\zeta}{m(x)} |f_0(x,-\tau\rho)|^{m(x)} \,\mathrm{d}x\mathrm{d}\rho$$

$$+ b \int_0^t \int_\Omega |v|^{p(x)-2} v u_t(s) \,\mathrm{d}x\mathrm{d}s. \tag{4.32}$$

Utilizing Young's inequality and (1.5), we get

$$- \mu_2 \int_\Omega |z(x,1,s)|^{m(x)-2} z(x,1,s)\, u_t(s) \,\mathrm{d}x$$

$$\leq \frac{|\mu_2|}{m^-} \int_\Omega |u_t(s)|^{m(x)} \,\mathrm{d}x + \frac{(m^+ - 1)|\mu_2|}{m^+} \int_\Omega |z(x,1,s)|^{m(x)} \,\mathrm{d}x. \tag{4.33}$$

By applying Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ and Young's inequality, we get

$$\left| \int_\Omega |v|^{p(x)-2} v u_t(s)\,\mathrm{d}x \right| \le \frac{\epsilon}{4} \int_\Omega |u_t(s)|^2\,\mathrm{d}x + \frac{1}{\epsilon} \int_\Omega |v|^{2(p(x)-1)}\,\mathrm{d}x$$
$$\le \frac{\epsilon}{4} \int_\Omega |u_t(s)|^2\,\mathrm{d}x + \frac{c_e}{\epsilon} \left\{ \|\nabla v\|^{2\left(p^- -1\right)} + \|\nabla v\|^{2\left(p^+ -1\right)} \right\}, \tag{4.34}$$

where c_e is the embedding constant. We insert (4.33) and (4.34) into (4.32); then, we obtain

$$\frac{1}{2}\|u_t\|^2 + \frac{l}{2}\|\nabla u\|^2 + \int_0^1 \int_\Omega \frac{\zeta}{m(x)} |z(x,\rho,t)|^{m(x)}\,\mathrm{d}x\mathrm{d}\rho$$
$$+ \left(\mu_1 - \frac{|\mu_2|}{m^-} - \frac{\zeta}{m^-\tau} \right) \int_0^t \int_\Omega |u_t(s)|^{m(x)}\,\mathrm{d}x\mathrm{d}s$$
$$+ \left(\frac{\zeta}{m^+\tau} - \frac{(m^+-1)|\mu_2|}{m^+} \right) \int_0^t \int_\Omega |z(x,1,s)|^{m(x)}\,\mathrm{d}x\mathrm{d}s$$
$$\le \frac{1}{2}\int_\Omega u_1^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \int_0^1 \int_\Omega \frac{\zeta}{m(x)} |f_0(x,-\tau\rho)|^{m(x)}\,\mathrm{d}x\mathrm{d}\rho$$
$$+ \frac{\epsilon bT}{4} \sup_{(0,T)} \int_\Omega |u_t|^2\,\mathrm{d}x + \frac{c_e b}{\epsilon} \left\{ \int_0^T \|\nabla v\|^{2\left(p^- -1\right)}\,\mathrm{d}s + \int_0^T \|\nabla v\|^{2\left(p^+ -1\right)}\,\mathrm{d}s \right\}.$$

From (2.11), we get

$$\frac{1}{2}\sup_{(0,T)} \|u_t\|^2 + \frac{l}{2}\sup_{(0,T)} \|\nabla u\|^2 + \frac{\zeta}{m^+} \|z(x,\rho,t)\|_{L^{m(\cdot)}(\Omega\times(0,1))}^{m(\cdot)}$$
$$\le \frac{1}{2}\int_\Omega u_1^2\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \frac{\zeta}{m^-}\int_0^1 \int_\Omega |f_0(x,-\tau\rho)|^{m(x)}\,\mathrm{d}x\mathrm{d}\rho$$
$$+ \frac{\epsilon bT}{4} \sup_{(0,T)} \|u_t\|^2 + \frac{c_e bT}{\epsilon l} \left\{ \|v\|_{X_T}^{2\left(p^- -1\right)} + \|v\|_{X_T}^{2\left(p^+ -1\right)} \right\}.$$

Taking ϵ such that $\epsilon bT = 1$, we have

$$\|u\|_{X_T}^2 \le \frac{c^*}{2}\int_\Omega u_1^2\mathrm{d}x + \frac{c^*}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \frac{c^*\zeta}{m^-}\int_0^1 \int_\Omega |f_0(x,-\tau\rho)|^{m(x)}\,\mathrm{d}x\mathrm{d}\rho$$
$$+ c_* T \left\{ \|v\|_{X_T}^{2\left(p^- -1\right)} + \|v\|_{X_T}^{2\left(p^+ -1\right)} \right\},$$

where $\frac{1}{c^*} = \min\left\{ \frac{1}{4}, \frac{\zeta}{m^+} \right\}$ and $c_* = \frac{c^* c_e b}{\epsilon l}$. For some $M > 0$ large, we assume that $\|v\|_{X_T} \le M$. For M large enough so that

$$c^* \int_\Omega u_1^2\mathrm{d}x + c^* \int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \frac{2c^*\zeta}{m^-}\int_0^1 \int_\Omega |f_0(x,-\tau\rho)|^{m(x)}\,\mathrm{d}x\mathrm{d}\rho \le M^2$$

and T sufficiently small so that

$$T \leq \frac{1}{2c_* \left(M^{2(p^- -2)} + M^{2(p^+ -2)} \right)},$$

we infer that

$$\|u\|_{X_T}^2 \leq M^2.$$

This proves that $K : Z \to Z$, where

$$Z = \left\{ u \in X_T \text{ such that } \|u\|_{X_T} \leq M \right\}.$$

Next, we will show that K is a contraction mapping. For this goal, let $K\left(v^1\right) = u^1$ and $K\left(v^2\right) = u^2$ and set $u = u^1 - u^2$ and $z = z^1 - z^2$ and then u and z satisfy

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(s) \, \mathrm{d}s + \mu_1 \left| u_t^1(x,t) \right|^{m(x)-2} u_t^1(x,t) - \mu_1 \left| u_t^2(x,t) \right|^{m(x)-2} u_t^2(x,t) \\ + \mu_2 \left| z^1(x,1,t) \right|^{m(x)-2} z^1(x,1,t) - \mu_2 \left| z^2(x,1,t) \right|^{m(x)-2} z^2(x,1,t) \\ = b \left| v^1 \right|^{p(x)-2} v^1 - b \left| v^2 \right|^{p(x)-2} v^2 \quad \text{in} \quad \Omega \times (0,T), \end{aligned} \tag{4.35}$$

with boundary condition

$$u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \tag{4.36}$$

and initial data

$$z(x,0,t) = u_t(x,t) \text{ in } \Omega \times (0,T), \tag{4.37}$$

$$z(x,\rho,0) = 0 \text{ in } \Omega \times (0,1), \tag{4.38}$$

$$u(x,0) = 0, \; u_t(x,0) = 0 \text{ in } \Omega. \tag{4.39}$$

Multiplying equation (4.35) by u_t and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left(1 - \int_0^t g(s) \, \mathrm{d}s \right) \|\nabla u\|^2 \\ &+ \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{2} \int_0^t g(s) \|\nabla u\|^2 \, \mathrm{d}s - \frac{1}{2} \int_0^t (g' \circ \nabla u)(s) \, \mathrm{d}s \\ &+ \mu_1 \int_0^t \int_\Omega \left(\left| u_t^1(s) \right|^{m(x)-2} u_t^1(s) - \left| u_t^2(s) \right|^{m(x)-2} u_t^2(s) \right) u_t(s) \, \mathrm{d}x \mathrm{d}s \\ &+ \mu_2 \int_0^t \int_\Omega \left(\left| z^1(x,1,s) \right|^{m(x)-2} z^1(x,1,s) - \left| z^2(x,1,s) \right|^{m(x)-2} z^2(x,1,s) \right) u_t(s) \, \mathrm{d}x \mathrm{d}s \\ &= \int_0^t \int_\Omega \left(h(v_1) - h(v_2) \right) u_t(s) \, \mathrm{d}x \mathrm{d}s, \end{aligned}$$

where $h(v) = b|v|^{p(x)-2} v$.

Since the function $u \to |u|^{m(x)-2} u$ is increasing, we infer that

$$\frac{1}{2}\|u_t\|^2 + \frac{l}{2}\|\nabla u\|^2 \le \int_0^t \int_\Omega (h(v_1) - h(v_2))\, u_t(s)\, \mathrm{d}x\mathrm{d}s. \tag{4.40}$$

Utilizing (4.22), Young's inequality, and Sobolev embedding, we get

$$\begin{aligned}
&\int_\Omega |h(v_1) - h(v_2)|\, |u_t(s)|\, \mathrm{d}x \\
&\quad = \int_\Omega |h'(\varrho)|\, \|v\|\, |u_t(s)|\, \mathrm{d}x \le \frac{\delta_0}{2}\int_\Omega |u_t(s)|^2\, \mathrm{d}x + \frac{1}{2\delta_0}\int_\Omega |h'(\varrho)|^2 |v|^2\, \mathrm{d}x \\
&\quad \le \frac{\delta_0}{2}\int_\Omega |u_t(s)|^2\, \mathrm{d}x \\
&\qquad + \frac{b^2(p^+-1)^2}{2\delta_0}\left[\left(\int_\Omega |\varrho|^{n(p^--2)}\, \mathrm{d}x\right)^{\frac{2}{n}} + \left(\int_\Omega |\varrho|^{n(p^+-2)}\, \mathrm{d}x\right)^{\frac{2}{n}}\right]\left(\int_\Omega |v|^{\frac{2n}{n-2}}\, \mathrm{d}x\right)^{\frac{n-2}{n}} \\
&\quad \le \frac{\delta_0}{2}\|u_t(s)\|^2 + \frac{b^2(p^+-1)^2 c_e}{2\delta_0}\left[\|\nabla\varrho\|^{2(p^--2)} + \|\nabla\varrho\|^{2(p^+-2)}\right]\|\nabla v\|^2 \\
&\quad \le \frac{\delta_0}{2}\|u_t(s)\|^2 + \frac{b^2(p^+-1)^2 c_e}{\delta_0 l^{p^+-2}}\left(M^{2(p^--2)} + M^{2(p^+-2)}\right)\|\nabla v\|^2,
\end{aligned} \tag{4.41}$$

where $v = v_1 - v_2$ and $\varrho = \vartheta v_1 + (1-\vartheta) v_2$, $0 \le \vartheta \le 1$. Inserting (4.41) into (4.40) and choosing δ_0 small enough, we get

$$\|u\|_{X_T}^2 \le d\,\|v\|_{X_T}^2, \tag{4.42}$$

where $d = \frac{4b^2(p^+-1)^2 c_e T}{\delta_0 l^{p^+-2}}\left(M^{2(p^--2)} + M^{2(p^+-2)}\right)$.

We choose T small enough that $0 < d < 1$; therefore, (4.42) proves that K is a contraction. The Banach fixed theorem implies the existence of a unique $u \in Z$ satisfies $K(u) = u$. Thus, it is a solution of (2.4).

(**Uniqueness**): Assume that (u^1, z^1) and (u^2, z^2) are two-pair solution to the problem (4.23)–(4.28). We define $\widetilde{u} = u^1 - u^2$ and $\widetilde{z} = z^1 - z^2$, then $(\widetilde{u}, \widetilde{z})$ satisfy

$$\begin{aligned}
&\widetilde{u}_{tt} - \Delta\widetilde{u} + \int_0^t g(t-s)\,\Delta\widetilde{u}(s)\,\mathrm{d}s \\
&\quad + \mu_1 \left|u_t^1\right|^{m(x)-2} u_t^1 - \mu_1 \left|u_t^2\right|^{m(x)-2} u_t^2 \\
&\quad + \mu_2 \left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) - \mu_2 \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t) = 0 \quad \text{in} \quad \Omega \times (0,T)
\end{aligned} \tag{4.43}$$

$$\tau\widetilde{z}_t(x,\rho,t) + \widetilde{z}_\rho(x,\rho,t) = 0 \ \text{ in } \ \Omega \times (0,1) \times (0,T) \tag{4.44}$$

with boundary condition

$$\widetilde{u}(x,t) = 0 \ \text{ on } \ \partial\Omega \times (0,T), \tag{4.45}$$

and initial data

$$\widetilde{z}(x,0,t) = \widetilde{u}_t(x,t) \quad \text{in} \quad \Omega \times (0,T), \tag{4.46}$$

$$\widetilde{z}(x,\rho,0) = 0 \ \text{ in } \ \Omega \times (0,1), \tag{4.47}$$

$$\widetilde{u}(x,0)=0,\ \widetilde{u}_t(x,0)=0 \text{ in } \Omega. \tag{4.48}$$

Multiplying (4.43) by $\widetilde{u}_t$ and integrating over $\Omega\times(0,t)$, we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega \left|\widetilde{u}_t(t)\right|^2 \mathrm{d}x+\frac{1}{2}\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega\left|\nabla\widetilde{u}(t)\right|^2\mathrm{d}x+\frac{1}{2}\left(g\circ\nabla\widetilde{u}\right)(t)\\&+\frac{1}{2}\int_0^t g(s)\int_\Omega\left|\nabla\widetilde{u}(s)\right|^2\mathrm{d}x\mathrm{d}s-\frac{1}{2}\int_0^t\left(g'\circ\nabla\widetilde{u}\right)(s)\,\mathrm{d}s\\&+\mu_1\int_0^t\int_\Omega\left(\left|u_t^1(s)\right|^{m(x)-2}u_t^1(s)-\left|u_t^2(s)\right|^{m(x)-2}u_t^2(s)\right)\widetilde{u}_t(s)\,\mathrm{d}x\mathrm{d}s\\&+\mu_2\int_0^t\int_\Omega\left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,s)-\left|z^2(x,1,s)\right|^{m(x)-2}z^2(x,1,s)\right)\widetilde{u}_t(s)\,\mathrm{d}x\mathrm{d}s=0.\end{aligned}\tag{4.49}$$

Multiplying (4.44) by $\widetilde{z}$ and integrating over $\Omega\times(0,1)\times(0,t)$, we obtain

$$\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_0^1\int_\Omega\left|\widetilde{z}(x,\rho,t)\right|^2\mathrm{d}x\mathrm{d}\rho\mathrm{d}s+\frac{1}{2}\int_0^t\left(\left\|\widetilde{z}(x,1,t)\right\|^2-\left\|\widetilde{u}_t(t)\right\|^2\right)\mathrm{d}s=0. \tag{4.50}$$

By combining (4.49)–(4.50), and similar to (4.40)–(4.41), we obtain

$$\begin{aligned}&\left\{\left\|\widetilde{u}_t(t)\right\|^2+l\left\|\nabla\widetilde{u}(t)\right\|^2+\tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\}+\left\|\widetilde{z}(x,1,t)\right\|^2\\&\quad\le c_4\int_0^t\left(\left\|\widetilde{u}_t(t)\right\|^2+l\left\|\nabla\widetilde{u}(t)\right\|^2\right)\mathrm{d}s.\end{aligned}$$

Gronwall inequality yields

$$\left\{\left\|\widetilde{u}_t(t)\right\|^2+l\left\|\nabla\widetilde{u}(t)\right\|^2+\tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\}+\left\|\widetilde{z}(x,1,t)\right\|^2=0.$$

Thus, $\widetilde{u}=0$, $\widetilde{z}=0$. Hence, the proof is completed. □

5. Nonexistence of solutions

In this section, for $b>0$, we prove the nonexistence of solutions to the problem (2.4)–(2.9) taking into account the negative initial energy, that is, $E(0)<0$.

We set

$$H(t)=-E(t), \tag{5.1}$$

and hence,

$$\begin{aligned}&H'(t)=-E'(t)\ge 0,\\&0<H(0)\le H(t)\le b\int_\Omega\frac{|u|^{p(x)}}{p(x)}\mathrm{d}x\le\frac{b}{p^-}\varrho(u),\end{aligned}\tag{5.2}$$

where

$$\varrho(u) = \varrho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

The following theorem gives the nonexistence of the solution.

Theorem 5.1. *Let $m(x), p(x)$ satisfies the condition*

$$2 \le m^- \le m(x) \le m^+ \le p^- \le p(x) \le p^+ \le \frac{2(n-1)}{n-2},\ n \ge 3,$$

and $m(x), p(x)$ satisfying the log-Hölder condition (2.1). If $E(0) < 0$, then the solution of (2.4)–(2.9) blows up in finite time T^* and

$$T^* \le \frac{1-\alpha}{\Psi\alpha\,[L(0)]^{\alpha/(1-\alpha)}},$$

where $L(t)$ and α are given in (5.3) and (5.4), respectively.

Proof. Define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t \mathrm{d}x, \tag{5.3}$$

where ε small to be chosen later and

$$0 \le \alpha \le \min\left\{\frac{p^- - 2}{2p^-},\ \frac{p^- - m^-}{p^-(m^+ - 1)},\ \frac{p^- - m^+}{p^-(m^+ - 1)}\right\}. \tag{5.4}$$

Differentiation $L(t)$ with respect to t, and using (2.4), we get

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left\|u_t^2\right\| - \varepsilon \|\nabla u\|^2 \\ &\quad + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) \nabla u(s)\,\mathrm{d}x\mathrm{d}s \\ &\quad + \varepsilon b \int_\Omega |u|^{p(x)}\,\mathrm{d}x - \varepsilon\mu_1 \int_\Omega u u_t(x,t) \left|u_t(x,t)\right|^{m(x)-2} \mathrm{d}x \\ &\quad - \varepsilon\mu_2 \int_\Omega u z(x,1,t) \left|z(x,1,t)\right|^{m(x)-2} \mathrm{d}x. \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left\|u_t^2\right\| - \varepsilon \|\nabla u\|^2 \\ &\quad + \varepsilon \int_0^t g(s)\,\mathrm{d}s\, \|\nabla u\|^2 + \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) \left[\nabla u(s) - \nabla u(t)\right] \mathrm{d}x\mathrm{d}s \\ &\quad + \varepsilon b \int_\Omega |u|^{p(x)}\,\mathrm{d}x - \varepsilon\mu_1 \int_\Omega u u_t(x,t) \left|u_t(x,t)\right|^{m(x)-2} \mathrm{d}x \\ &\quad - \varepsilon\mu_2 \int_\Omega u z(x,1,t) \left|z(x,1,t)\right|^{m(x)-2} \mathrm{d}x. \end{aligned} \tag{5.5}$$

By using Young's and Cauchy–Schwarz inequalities, we have

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \nabla u(t)\left[\nabla u(s)-\nabla u(t)\right]\mathrm{d}x\mathrm{d}s \\
&\quad\le \int_0^t g(t-s)\left\|\nabla u(t)\right\|\left\|\nabla u(s)-\nabla u(t)\right\|\mathrm{d}s \\
&\quad\le \lambda\left(g\circ\nabla u\right)(t)+\frac{1}{4\lambda}\int_0^t g(s)\,\mathrm{d}s\left\|\nabla u\right\|^2,\ \lambda>0.
\end{aligned}\tag{5.6}$$

Substituting (5.6) into (5.5), we have

$$\begin{aligned}
L'(t) \ge{}& (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\left\|u_t^2\right\|-\varepsilon\left\|\nabla u\right\|^2 \\
&+\varepsilon\int_0^t g(s)\,\mathrm{d}s\left\|\nabla u\right\|^2-\varepsilon\lambda\left(g\circ\nabla u\right)(t)-\frac{\varepsilon}{4\lambda}\int_0^t g(s)\,\mathrm{d}s\left\|\nabla u\right\|^2 \\
&+\varepsilon b\int_\Omega |u|^{p(x)}\,\mathrm{d}x-\varepsilon\mu_1\int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}\mathrm{d}x \\
&-\varepsilon\mu_2\int_\Omega u z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}\mathrm{d}x.
\end{aligned}\tag{5.7}$$

By using the definition of the $H(t)$ and for $0<a<1$, such that

$$\begin{aligned}
L'(t) \ge{}& C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x+\int_\Omega \left|z(x,1,t)\right|^{m(x)}\mathrm{d}x\right] \\
&+\varepsilon(1-a)p^-H(t)+\varepsilon\frac{(1-a)p^-}{2}\left\|u_t\right\|^2 \\
&+\varepsilon\frac{(1-a)p^-}{2}\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\left\|\nabla u\right\|^2 \\
&+\varepsilon\frac{(1-a)p^-}{2}\left(g\circ\nabla u\right)(t) \\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}\mathrm{d}x\mathrm{d}\rho \\
&+\varepsilon\left\|u_t^2\right\|-\varepsilon\left\|\nabla u\right\|^2+\varepsilon ab\int_\Omega |u|^{p(x)}\,\mathrm{d}x \\
&+\varepsilon\int_0^t g(s)\,\mathrm{d}s\left\|\nabla u\right\|^2-\varepsilon\lambda\left(g\circ\nabla u\right)(t) \\
&-\frac{\varepsilon}{4\lambda}\int_0^t g(s)\,\mathrm{d}s\left\|\nabla u\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon\mu_1 \int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} \mathrm{d}x \\
&- \varepsilon\mu_2 \int_\Omega u z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2} \mathrm{d}x. \qquad (5.8)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L'(t) \geq\ & C_0(1-\alpha) H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)} \mathrm{d}x + \int_\Omega \left|z(x,1,t)\right|^{m(x)} \mathrm{d}x\right] \\
&+ \varepsilon(1-a)p^- H(t) + \varepsilon\frac{(1-a)p^- + 2}{2}\|u_t\|^2 + \varepsilon\left(\frac{(1-a)p^-}{2} - \lambda\right)(g\circ\nabla u)(t) \\
&+ \varepsilon\left[\frac{(1-a)p^-}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right) - 1 + \left(1 - \frac{1}{4\lambda}\right)\int_0^t g(s)\,\mathrm{d}s\right]\|\nabla u\|^2 \\
&+ \varepsilon(1-a)p^- \int_0^1\int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}\mathrm{d}x\mathrm{d}\rho \\
&+ \varepsilon a b \varrho(u) - \varepsilon\mu_1\int_\Omega u u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}\mathrm{d}x \\
&- \varepsilon\mu_2\int_\Omega u z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}\mathrm{d}x. \qquad (5.9)
\end{aligned}
$$

Utilizing Young's inequality, we get

$$
\begin{aligned}
\int_\Omega |u_t|^{m(x)-1}|u|\,\mathrm{d}x \leq\ & \frac{1}{m^-}\int_\Omega \delta^{m(x)}|u|^{m(x)}\mathrm{d}x \\
&+ \frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\mathrm{d}x, \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega \left|z(x,1,t)\right|^{m(x)-1}|u|\,\mathrm{d}x \leq\ & \frac{1}{m^-}\int_\Omega \delta^{m(x)}|u|^{m(x)}\mathrm{d}x \\
&+ \frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x. \qquad (5.11)
\end{aligned}
$$

As in [27], estimates (5.10) and (5.11) remain valid even if δ is time-dependent. Hence, taking δ such that

$$
\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),
$$

for large $k \geq 1$ to be specified later, we obtain

$$
\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\mathrm{d}x = kH^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}\mathrm{d}x, \qquad (5.12)
$$

$$
\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x = kH^{-\alpha}(t)\left|z(x,1,t)\right|^{m(x)}\mathrm{d}x \qquad (5.13)
$$

and

$$\int_\Omega \delta^{m(x)} |u|^{m(x)} \, \mathrm{d}x = \int_\Omega k^{1-m(x)} H^{\alpha(m(x)-1)}(t) |u|^{m(x)} \, \mathrm{d}x$$
$$\leq \int_\Omega k^{1-m^-} H^{\alpha\left(m^+-1\right)}(t) \int_\Omega |u|^{m(x)} \, \mathrm{d}x.$$

By using (3.8) and (3.9), we obtain

$$H^{\alpha\left(m^+-1\right)}(t) \int_\Omega |u|^{m(x)} \, \mathrm{d}x \leq C \left[(\varrho(u))^{m^-/p^- + \alpha\left(m^+-1\right)} + (\varrho(u))^{m^+/p^- + \alpha\left(m^+-1\right)} \right]. \tag{5.14}$$

From (5.4), we deduce that

$$m^- + \alpha p^- \left(m^+ - 1\right) \leq p^- \quad \text{and} \quad m^+ + \alpha p^- \left(m^+ - 1\right) \leq p^+.$$

Then, by using lemma 3.10, we get

$$H^{\alpha\left(m^+-1\right)}(t) \int_\Omega |u|^{m(x)} \, \mathrm{d}x \leq C \left(\|\nabla u\|^2 + \varrho(u) \right). \tag{5.15}$$

Combining (5.10)–(5.15), we get

$$\begin{aligned}
L'(t) \geq\ & (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck \right] \int_\Omega |u_t|^{m(x)} \, \mathrm{d}x \\
& + (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck \right] \int_\Omega |z(x,1,t)|^{m(x)} \, \mathrm{d}x \\
& + \varepsilon \left[\frac{(1-a) p^-}{2} \left(1 - \int_0^t g(s) \, \mathrm{d}s \right) - 1 \right. \\
& \left. + \left(1 - \frac{1}{4\lambda} \right) \int_0^t g(s) \, \mathrm{d}s - \frac{C}{m^- k^{1-m^-}} \right] \|\nabla u\|^2 \\
& + \varepsilon \left(\frac{(1-a) p^-}{2} - \lambda \right) (g \circ \nabla u)(t) + \varepsilon (1-a) p^- H(t) \\
& + \varepsilon \frac{(1-a) p^- + 2}{2} \|u_t\|^2 + \varepsilon \left(ab - \frac{C}{m^- k^{1-m^-}} \right) \varrho(u) \\
& + \varepsilon (1-a) p^- \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} \mathrm{d}x \mathrm{d}\rho.
\end{aligned} \tag{5.16}$$

Let us choose a small enough such that

$$\frac{(1-a) p^- + 2}{2} > 0$$

and k large enough so that

$$ab - \frac{C}{m^- k^{1-m^-}} > 0,$$

and

$$\frac{(1-a)\,p^{-}}{2}\left(1-\int\limits_{0}^{t}g\left(s\right)\mathrm{d}s\right)-1+\left(1-\frac{1}{4\lambda}\right)\int\limits_{0}^{t}g\left(s\right)\mathrm{d}s-\frac{C}{m^{-}k^{1-m^{-}}}>0.$$

Once k and a are fixed, picking ε small enough such that

$$C_0-\varepsilon\left(\frac{m^{+}-1}{m^{+}}\right)ck>0$$

and

$$L\left(0\right)=H^{1-\alpha}\left(0\right)+\varepsilon\int\limits_{\Omega}u_0u_1\mathrm{d}x>0.$$

Consequently, (5.16) yields

$$L'\left(t\right)\geq\varepsilon\eta\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left(g\circ\nabla u\right)\left(t\right)+\varrho\left(u\right)+\int\limits_{0}^{1}\int\limits_{\Omega}\frac{\xi\left(x\right)\left|z\left(x,\rho,t\right)\right|^{m(x)}}{m\left(x\right)}\mathrm{d}x\mathrm{d}\rho\right] \tag{5.17}$$

for a constant $\eta>0$. Thus, we get

$$L\left(t\right)\geq L\left(0\right)>0,\ \forall t\geq0.$$

Now, for some constants σ, $\Gamma>0$ we denote

$$L'\left(t\right)\geq\Gamma L^{\sigma}\left(t\right).$$

Also, utilizing Hölder inequality, we get

$$\left|\int\limits_{\Omega}uu_t\mathrm{d}x\right|^{1/(1-\alpha)}\leq C\left\|u\right\|_{p^{-}}^{1/(1-\alpha)}\left\|u_t\right\|_{2}^{1/(1-\alpha)},$$

and by using Young's inequality gives

$$\left|\int\limits_{\Omega}uu_t\mathrm{d}x\right|^{1/(1-\alpha)}\leq C\left[\left\|u\right\|_{p^{-}}^{mu/(1-\alpha)}+\left\|u_t\right\|_{2}^{\Theta/(1-\alpha)}\right],$$

where $1/\mu+1/\Theta=1$. From (5.4), the choice of $\Theta=2\left(1-\alpha\right)$ will make $\mu/\left(1-\alpha\right)=2/\left(1-2\alpha\right)\leq p^{-}$. Hence,

$$\left|\int\limits_{\Omega}uu_t\mathrm{d}x\right|^{1/(1-\alpha)}\leq C\left[\left\|u\right\|_{p^{-}}^{s}+\left\|u_t\right\|^2\right],$$

where $s=\mu/\left(1-\alpha\right)$. From (3.7), we have

$$\left|\int\limits_{\Omega}uu_t\mathrm{d}x\right|^{1/(1-\alpha)}\leq C\left[H\left(t\right)+\left\|u_t\right\|^2+\varrho\left(u\right)+\int\limits_{0}^{1}\int\limits_{\Omega}\frac{\xi\left(x\right)\left|z\left(x,\rho,t\right)\right|^{m(x)}}{m\left(x\right)}\mathrm{d}x\mathrm{d}\rho\right]. \tag{5.18}$$

Hence, we get

$$L^{1/(1-\alpha)}\left(t\right)=\left[H^{(1-\alpha)}\left(t\right)+\varepsilon\int\limits_{\Omega}uu_t\mathrm{d}x\right]^{1/(1-\alpha)}$$

$$
\begin{aligned}
&\leq 2^{\alpha/(1-\alpha)}\left[H\left(t\right)+\varepsilon^{1/(1-\alpha)}\left|\int\limits_{\Omega}uu_t\mathrm{d}x\right|^{1/(1-\alpha)}\right]\\
&\leq C\left[H\left(t\right)+\left\|u_t\right\|^2+\varrho\left(u\right)\right]\\
&\leq C\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left(g\circ\nabla u\right)\left(t\right)\right.\\
&\left.+\varrho\left(u\right)+\int\limits_0^1\int\limits_{\Omega}\frac{\xi\left(x\right)\left|z\left(x,\rho,t\right)\right|^{m(x)}}{m\left(x\right)}\mathrm{d}x\mathrm{d}\rho\right].
\end{aligned}
\tag{5.19}
$$

So, for some $\Psi>0$, from (5.17) we arrive

$$
L'\left(t\right)\geq\Psi L^{1/(1-\alpha)}\left(t\right). \tag{5.20}
$$

Integration of (5.20) over $(0,t)$ yields

$$
L^{\alpha/(1-\alpha)}\left(t\right)\geq\frac{1}{L^{-\alpha/(1-\alpha)}\left(0\right)-\Psi\alpha t/\left(1-\alpha\right)},
$$

which implies that $\Psi\left(t\right)$ blows up in a finite time

$$
T^*\leq\frac{1-\alpha}{\Psi\alpha\left[L\left(0\right)\right]^{\alpha/(1-\alpha)}}.
$$

As a result, the proof is completed. □

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