Z. Angew. Math. Phys. (2022) 73:100

0044-2275/22/030001-20
published online April 21, 2022
https://doi.org/10.1007/s00033-022-01740-w

Zeitschrift für angewandte Mathematik und Physik ZAMP

Stability and time decay rates of the 2D magneto-micropolar equations with partial dissipation

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Abstract. This paper studies the stability and decay estimates of solutions to the two-dimensional (2D) magneto-micropolar fluid equations with partial dissipation. We first establish the L^2-decay estimates for global solutions and their derivative with initial data in $L^1(\mathbb{R}^2)$. Furthermore, we show the global stability of these solutions in $H^s(\mathbb{R}^2)$, and the decay rates of these global solutions and their higher derivatives when the initial data belongs to the negative Sobolev space $\dot{H}^{-l}(\mathbb{R}^2)$ (for each $0 \leq l < 1$).

Mathematics Subject Classification. 35Q35, 35B40, 76D03.

Keywords. 2D magneto-micropolar equations, Partial dissipation, Stability, Decay estimate.

1. Introduction

The 3D incompressible magneto-micropolar fluid equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla \pi + (\mu + \chi)\Delta u + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega - \gamma \nabla \nabla \cdot \omega + 4\chi \omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x,y,z,0) = u_0(x,y,z), \omega(x,y,z,0) = \omega_0(x,y,z), b(x,y,z,0) = b_0(x,y,z), \end{cases} \tag{1.1}$$

where $(x, y, z) \in \mathbb{R}^3$ and $t \geq 0$, $u(x, y, z, t), \omega(x, y, z, t), b(x, y, z, t)$ and $\pi(x, y, z, t)$ denote the velocity of the fluid, microrotational velocity, the magnetic field and the hydrostatic pressure, respectively. μ, χ and $\frac{1}{\nu}$ are, respectively, kinematic viscosity, vortex viscosity and magnetic Reynolds number. γ and κ are angular viscosities. The 3D magneto-micropolar equations reduce to the 2D micropolar equations when

$$\begin{aligned} u &= (u_1(x,y,t), u_2(x,y,t), 0), \qquad \pi = \pi(x,y,t), \\ b &= (b_1(x,y,t), b_2(x,y,t), 0), \qquad \omega = (0, 0, \omega_3(x,y,t)). \end{aligned}$$

More explicitly, the 2D incompressible magneto-micropolar fluid equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi \omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \omega(x,y,0) = \omega_0(x,y), b(x,y,0) = b_0(x,y), \end{cases} \tag{1.2}$$

where we have written $u = (u_1, u_2)$, $b = (b_1, b_2)$ and ω for ω_3 for notational brevity. It is worth noting that, in the 2D case,

$$\Omega \equiv \nabla \times u = \partial_x u_2 - \partial_y u_1$$

is a scalar function representing the vorticity, and $\nabla \times \omega = (\partial_y \omega, -\partial_x \omega)$.

The magneto-micropolar equations were introduced in [1] to describe the motion of incompressible, electrically conducting micropolar fluids in the presence of an arbitrary magnetic field. It has attracted considerable attention from the community of mathematical fluids (see, e.g., [2–12]). The magneto-micropolar equations share similarities with the Navier–Stokes equations, but they contain much richer structures than Navier–Stokes. It is well-known that the L^2 decay problem of weak solutions to the 3D Navier–Stokes equations, i.e., (1.1) with $\omega = 0$, $b = 0$ and $\chi = 0$, was proposed by the celebrated work of Leray [13]. By introducing the elegant method of Fourier splitting, the algebraic decay rate for weak solutions was first obtained by Schonbek [14]. Later, the result in [14] is sharpened and extended in [15], see also [16]. Recently, in [17] Zhao obtained the decay rates of solutions for the three-dimensional incompressible Navier–Stokes equations with damping term $|u|^{\beta-1}u$ $(\beta \geq 3)$.

When (1.2) has full dissipation (namely, $\mu, \chi, \kappa, \nu > 0$), the global existence and uniqueness of solutions can be obtained easily (see, e.g., [8,18]). For more results related to the well-posedness of solutions, one refers to [19–25] and the reference therein. However, for the inviscid case (namely, (1.2) with $\mu > 0$, $\chi > 0$, $\kappa = \nu = 0$ and Δu replaced by u), the global regularity problem is still a challenging open problem. Therefore, it is natural to study the intermediate cases, namely (1.2) with partial dissipation.

In certain physical regimes and under suitable scaling, the full Laplacian dissipation is reduced to a partial dissipation. One notable example is the Prandtl boundary layer equation in which only the vertical dissipation is included in the horizontal component (see, e.g., [26]). This paper focuses on a system of the 2D incompressible magneto-micropolar equations that is closely related to (1.2),

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega + 4\chi \omega = \kappa \Delta \omega + 2\chi \nabla \times u, \\ \partial_t b_1 + u \cdot \nabla b_1 = \nu \partial_{yy} b_1 + b \cdot \nabla u_1, \\ \partial_t b_2 + u \cdot \nabla b_2 = \nu \partial_{xx} b_2 + b \cdot \nabla u_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \omega(x,y,0) = \omega_0(x,y), b(x,y,0) = b_0(x,y). \end{cases} \tag{1.3}$$

When the magnetic fluid $b_1 = b_2 = 0$, the system (1.3) reduces to the 2D incompressible micropolar fluid equations, which describe some physical phenomena such as the motion of animal blood, liquid crystals and dilute aqueous polymer solutions. Micropolar fluid model was first proposed by Eringen [27], while the existences of weak and strong solutions were studied by Galdi and Rionero [28] and Yamaguchi [29]. The global well-posedness of the micropolar equations with full viscosity was obtained by Łukaszewicz [18]. Dong and Chen [19] via using Fourier splitting method proved the L^2-decay rates for global solutions of the 2D micropolar equations. Guterres, Melo, Nunes and Perusato [12] improved the decay rates of ω in $L^2(\mathbb{R}^2)$ and the decay estimates of the higher-order derivatives of ω.

When $\omega = 0$ and $\chi = 0$, the system (1.3) reduces to the 2D incompressible Magnetohydrodynamic (MHD) equations with partial magnetic diffusion. The global existence and regularity have been obtained in [30] and proved the large time decay rates for smooth solutions of the 2D MHD equations with fractional dissipation partial and magnetic diffusion, that is, Δu in (1.3) is replaced by $(-\Delta)^\alpha u$, and α is required to belong to $(1, \frac{1}{2})$.

Due to the complex structure of (1.2), when there is only partial dissipation, the global regularity problem can be quite difficult. However, many important progress has recently been made on this direction (see, e.g., [7,25,30–35]). In [25,32,34], the global regularity of the 2D magneto-micropolar equations with various partial dissipation cases was obtained. When the magneto-micropolar equations only have velocity dissipation and magnetic diffusion, namely, (1.1) with $\kappa = 0$, Niu and Shang [36] proved the optimal L^2-decay estimates of weak solutions, and also obtained the decay rates of global solutions in $\dot{H}^s (s > \frac{3}{2})$ and $\dot{B}^m_{2,\infty}$ spaces with $0 \leq m \leq s$ and in $\dot{B}^m_{2,1}$ with $0 \leq m \leq \frac{1}{2}$. When (1.2) only has microrotational dissipation and magnetic diffusion(namely, Δu replaced by u), the decay estimates of solutions have been obtained by Shang and Gu [37,38]. Shang and Gu [37] also proved the global existence of classical solutions for (1.3).

In this paper, we study the global well-posedness and the decay estimates of solutions to (1.3). Motivated by [36], we first establish the L^2-decay estimates for global solutions and their derivative with initial data in $L^1(\mathbb{R}^2)$. Secondly, we establish the global existence result to system (1.3) in $H^s(\mathbb{R}^2)$. Furthermore, we show the decay rates of these global solutions and their higher derivatives.

We now list the main results of the paper. The first result is the decay estimates of global solutions in L^2 space.

Theorem 1.1. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume that $(u_0, \omega_0, b_0) \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, the global solutions (u, ω, b) of (1.3) satisfies, for all $t > 0$,*

$$\|u(t)\|_{L^2} + \|\omega(t)\|_{L^2} + \|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}, \tag{1.4}$$

$$\|\nabla u(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}. \tag{1.5}$$

We remark that the magnetic field b only has partial dissipation in (1.3), the classic Fourier splitting method which relies on decomposing the whole space into two time-dependent sub-dependent does not apply. To do so, we used the method of [39], to obtain the L^2-decay estimates for global solutions of (1.3).

The next theorem is devoted to the global existence and the time decay estimates of solutions to system (1.3) with small initial data in Sobolev space. The process of proving the existence of global solutions is similar to [36]. For the sake of simplicity, we only pay attention to the global a priori bounds of (u, ω, b), and we can obtain the global existence in $H^s(\mathbb{R}^2)$. Furthermore, by using the interpolation inequality, energy estimates and the technique of Fourier analysis, we obtain the corresponding time decay rates of these solutions.

More precisely, the following theorem establishes a unique global solution when the initial data (u_0, ω_0, b_0) is sufficiently small in $H^s(\mathbb{R}^2)$, and obtain the decay rates of these global solutions and their higher derivatives, as stated blow.

Theorem 1.2. *Let $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$. Assume that $(u_0, \omega_0, b_0) \in H^s(\mathbb{R}^2)$ with $s > 0$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, the following two statements hold:*

(I) Let $s > 1$, then there exists a positive constant ϵ_0, such that for all $0 < \epsilon < \epsilon_0$, if

$$\|u_0\|^2_{H^s(\mathbb{R}^2)} + \|\omega_0\|^2_{H^s(\mathbb{R}^2)} + \|b_0\|^2_{H^s(\mathbb{R}^2)} < \epsilon, \tag{1.6}$$

then system (1.3) has a unique global solution (u, ω, b) satisfying, for any $t > 0$,

$$\begin{aligned} &\|u(t)\|^2_{H^s(\mathbb{R}^2)} + \|\omega(t)\|^2_{H^s(\mathbb{R}^2)} + \|b(t)\|^2_{H^s(\mathbb{R}^2)} \\ &\quad + \int_0^t (\|\nabla u(\tau)\|^2_{H^s(\mathbb{R}^2)} + \|\nabla \omega(\tau)\|^2_{H^s(\mathbb{R}^2)} + \|\nabla b(\tau)\|^2_{H^s(\mathbb{R}^2)}) \mathrm{d}\tau \le C\epsilon, \end{aligned} \tag{1.7}$$

where $C > 0$ is a constant independent of t.

(II) Let $s > 1$, suppose that $(u_0, \omega_0, b_0) \in \dot{H}^{-l}(\mathbb{R}^2)$ with $0 \le l < 1$. Then, the global solution (u, ω, b) satisfies the following decay estimates:

i) For all real number m with $0 \le m \le s$, we have

$$\|D^m u(t)\|_{L^2(\mathbb{R}^2)} + \|D^m \omega(t)\|_{L^2(\mathbb{R}^2)} + \|D^m b(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{m}{2}-\frac{l}{2}}. \tag{1.8}$$

ii) For $0 \le m \le s-1$, we have

$$\|D^m \omega(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{(m+1)}{2}-\frac{l}{2}}. \tag{1.9}$$

This decay estimate of $\omega(t)$ is improved from (1.8) to (1.9).

Remark 1.3. Since $L^p(\mathbb{R}^2) \hookrightarrow \dot{H}^{-l}(\mathbb{R}^2)$ when $0 \le l < 1$ and $p \in (1,2]$, thus Theorem 1.2 also holds for $(u_0, \omega_0, b_0) \in L^p(\mathbb{R}^2)$ with $p \in (1,2]$.

The proof of the global existence part of Theorem 1.2 relies on the global a priori bound for $\|(u,\omega,b)\|_{H^s(\mathbb{R}^2)}$. For the proof of (II) in Theorem 1.2, compared with [36], here we used the negative homogeneous Sobolev space $\dot{H}^{-l}(\mathbb{R}^2)$ to study the decay estimates of (1.3). Since the space dimension $n = 2$, Sobolev's inequality in L^2 is invalid, then we used Sobolev–Nirenberg–Gagliardo inequality to overcome this difficulty.

The rest of this paper is divided into four sections. In Sect. 2, we provide some lemmas to be used later. Sections 3 and 4 state the proofs of Theorems 1.1 and 1.2, respectively. To simplify the notation, we will write $\int f$ for $\int_{\mathbb{R}^2} f \mathrm{d}x$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^2)}$, $\|f\|_{\dot{H}^s}$ and $\|f\|_{H^s}$ for $\|f\|_{\dot{H}^s(\mathbb{R}^2)}$ and $\|f\|_{H^s(\mathbb{R}^2)}$, respectively, $\|f\|_{\dot{B}^s_{p,r}}$ for $\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)}$.

2. Preliminaries

As preparation, we recall the following lemmas.

Lemma 2.1. *(commutator estimates, see, e.g., [40,41]) Let $s > 0$, $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1,\infty)$ and $p_1, q_2 \in [1,\infty]$. Then,*

$$\|[\Lambda^s, f \cdot \nabla]g\|_{L^r} \le C(\|\nabla f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|\nabla g\|_{L^{q_2}}), \tag{2.1}$$

and

$$\|\Lambda^s(f\, g)\|_{L^r} \le C\, \left(\|g\|_{L^{p_1}}\, \|\Lambda^s f\|_{L^{q_1}} + \|\Lambda^s g\|_{L^{p_2}}\, \|f\|_{L^{q_2}}\right), \tag{2.2}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator, and $[\Lambda^s, f \cdot \nabla]g = \Lambda^s(f \cdot \nabla g) - f \cdot \nabla \Lambda^s g$ and C's are positive constants depending on the indices s, r, p_1, q_1, p_2 and q_2.

Lemma 2.2. *(see [42]) If p belongs to $(1,2]$, then $L^p(\mathbb{R}^d)$ embeds continuously in $\dot{H}^s(\mathbb{R}^d)$ with $s = \frac{d}{2} - \frac{d}{p}$.*

Next, we state the Sobolev–Nirenberg–Gagliardo inequality. For the sake of simplicity, the proof process will not be described here, and the detailed proof can be found in [43].

Lemma 2.3. *Let $u \in L^q$ in $\mathbb{R}^n$ and its derivatives of order m, $D^m u \in L^r$, $1 \le q, r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequality hold*

$$\|D^j u\|_{L^p} \le C\|D^m u\|^a_{L^r} \|u\|^{1-a}_{L^q}, \tag{2.3}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

for all a in the interval

$$\frac{j}{m} \le a \le 1 \tag{2.4}$$

(the constant depending only on n, m, j, q, r, a), with the following exceptional cases

1, If $j = 0$, $rm < n$, $q = \infty$ then we make the additional assumption that either u tends to zero at infinity or $u \in L^{\tilde{q}}$ for some finite $\tilde{q} > 0$.

2, If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer then (2.3) holds only for a satisfying $j/m \le a < 1$.

Finally, since $\dot{B}^s_{2,2} \sim \dot{H}^s$, we recall the following Besov space interpolation estimate.

Lemma 2.4. *(see [44]) Fixed $m > l > k$, and $1 \le p \le q \le r \le \infty$, we have*

$$\|f\|_{\dot{B}^l_{q,q'}(\mathbb{R}^2)} \le \|f\|^{\theta}_{\dot{B}^k_{r,r'}(\mathbb{R}^2)} \|f\|^{1-\theta}_{\dot{B}^m_{p,p'}(\mathbb{R}^2)}. \tag{2.5}$$

These parameters satisfy the following restrictions

$$l = k\theta + m(1-\theta), \quad \frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}, \quad \frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{p'}.$$

Also $1 \le p' \le q' \le r' \le \infty$ and solving we have $\theta = \frac{m-l}{m-k} \in (0,1]$.

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. As preparations, we verify the following four propositions. The first proposition is as follows.

Proposition 3.1. *Assume $\mu > 0$, $\chi > 0$, $\nu > 0$ and $\kappa > 0$ and $(u_0, \omega_0, b_0) \in H^1(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, the corresponding solution (u, ω, b) of (1.3) obeys the following uniform bounds, for any $t > 0$,*

$$\begin{aligned} \|u(t)\|^2_{L^2} + &\|\omega(t)\|^2_{L^2} + \|b(t)\|^2_{L^2} + \mu \int_0^t \|\nabla u(\tau)\|^2_{L^2} \mathrm{d}\tau + \frac{8\mu\chi}{\mu + 2\chi} \int_0^t \|\omega(\tau)\|^2_{L^2} \mathrm{d}\tau \\ &+ 2\kappa \int_0^t \|\nabla \omega(\tau)\|^2_{L^2} \mathrm{d}\tau + 2\nu \int_0^t \left(\|\partial_y b_1(\tau)\|^2_{L^2} + \|\partial_x b_2(\tau)\|^2_{L^2} \right) \mathrm{d}\tau \le C, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|\Omega(t)\|^2_{L^2} + \|j(t)\|^2_{L^2} + \|\nabla \omega(t)\|^2_{L^2} + \mu \int_0^t \|\nabla \Omega(\tau)\|^2_{L^2} \mathrm{d}\tau + \kappa \int_0^t \|\Delta \omega(\tau)\|^2_{L^2} \mathrm{d}\tau \\ + \int_0^t H(b, \tau) \mathrm{d}\tau \le C, \end{aligned} \tag{3.2}$$

where

$$H(b,t) = \nu \int_{\mathbb{R}^2} \left((\partial_{xx} b_1)^2 + (\partial_{xx} b_2)^2 + (\partial_{yy} b_1)^2 + (\partial_{yy} b_2)^2 \right) \mathrm{d}x.$$

and C's are positive constants depending on μ, ν, χ, κ and $\|(u_0, \omega_0, b_0)\|_{H^1}$ only.

Proof of Proposition 3.1. Taking the L^2-inner product to (1.3) with (u, ω, b_1, b_2) and integrating in time yield

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u(t)\|^2_{L^2} + \|\omega(t)\|^2_{L^2} + \|b(t)\|^2_{L^2} \right) + (\mu + \kappa) \|\nabla u\|^2_{L^2} + \kappa \|\nabla \omega\|^2_{L^2} + 4\chi \|\omega\|^2_{L^2} \\ + \nu (\|\partial_y b_1\|^2_{L^2} + \|\partial_x b_2\|^2_{L^2}) = 4\chi \int \nabla \times u \cdot \omega \mathrm{d}x, \end{aligned} \tag{3.3}$$

where we used the facts that,

$$\int (b \cdot \nabla u_1 \cdot b_1 + b \cdot \nabla u_2 \cdot b_2) \mathrm{d}x = \int b \cdot \nabla u \cdot b \mathrm{d}x = -\int b \cdot \nabla b \cdot u \mathrm{d}x,$$

$$\int \nabla \times u \cdot \omega \mathrm{d}x = \int \nabla \times \omega \cdot u \mathrm{d}x.$$

By Hölder's inequality and the Young inequality, we have

$$
\begin{aligned}
&4\chi\int \nabla\times u\cdot\omega \mathrm{d}x\\
&\le 4\chi\|\nabla u\|_{L^2}\|\omega\|_{L^2}\\
&\le \left(\frac{\mu}{2}+\chi\right)\|\nabla u\|_{L^2}^2+\frac{4\chi^2}{\frac{\mu}{2}+\chi}\|\omega\|_{L^2}^2. \qquad (3.4)
\end{aligned}
$$

Inserting (3.4) into (3.3), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\right)+\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\kappa\|\nabla\omega\|_{L^2}^2+4\chi\left(1-\frac{2\chi}{2\chi+\mu}\right)\|\omega\|_{L^2}^2\\
&\quad+\nu(\|\partial_y b_1\|_{L^2}^2+\|\partial_x b_2\|_{L^2}^2)\le 0. \qquad (3.5)
\end{aligned}
$$

Integrating (3.5) in $[0,t]$, we can get (3.1) immediately.

Now we turn to prove (3.2). The vorticity $\Omega=\nabla\times u$, $\nabla\omega$ and $j=\nabla\times b$ satisfy

$$
\begin{cases}
\partial_t\Omega+u\cdot\nabla\Omega-(\mu+\chi)\Delta\Omega=b\cdot\nabla j-2\chi\Delta\omega,\\
\partial_t\nabla\omega+\nabla(u\cdot\nabla\omega)+4\chi\nabla\omega=\kappa\nabla\Delta\omega+2\chi\nabla\Omega,\\
\partial_t j+u\cdot\nabla j-\nu\partial_{xxx}b_2+\nu\partial_{yyy}b_1=b\cdot\nabla\Omega+T(\nabla u,\nabla b),
\end{cases} \qquad (3.6)
$$

where

$$
T(\nabla u,\nabla b)=2\partial_x b_1(\partial_x u_2+\partial_y u_1)-2\partial_x u_1(\partial_x b_2+\partial_y b_1).
$$

Taking the L^2-inner products of (3.6) with $(\Omega,\nabla\omega,j)$, respectively, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|_{L^2}^2+(\mu+\chi)\|\nabla\Omega\|_{L^2}^2=\int b\cdot\nabla j\ \Omega\mathrm{d}x-2\chi\int\Delta\omega\ \Omega\mathrm{d}x, \qquad (3.7)
$$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+4\chi\|\nabla\omega\|_{L^2}^2+\kappa\|\Delta\omega\|_{L^2}^2=2\chi\int\nabla\omega\cdot\nabla\Omega\mathrm{d}x-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\mathrm{d}x, \qquad (3.8)
$$

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|j\|_{L^2}^2+I=\int(b\cdot\nabla\Omega j+Tj)\mathrm{d}x, \qquad (3.9)
$$

where

$$
\begin{aligned}
I&=\nu\int(-\partial_{xxx}b_2+\partial_{yyy}b_1)j\mathrm{d}x\\
&=\nu\int(-\partial_{xxx}b_2+\partial_{yyy}b_1)(\partial_x b_2-\partial_y b_1)\mathrm{d}x\\
&=\nu\int[(\partial_{xx}b_2)^2+(\partial_{yy}b_2)^2+(\partial_{xx}b_1)^2+(\partial_{yy}b_1)^2]\mathrm{d}x\equiv H(b,t),
\end{aligned}
$$

due to the divergence free condition $\nabla\cdot b=\partial_x b_1+\partial_y b_2=0$. By Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
&\int Tj\mathrm{d}x\le C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}\|j\|_{L^4}\\
&\qquad\le C\|\Omega\|_{L^2}\|j\|_{L^2}\|\nabla j\|_{L^2}\le C(\nu)\|\Omega\|_{L^2}^2\|j\|_{L^2}^2+\frac{\nu}{8}\|\nabla j\|_{L^2}^2,
\end{aligned}
$$

where we have used the fact that the Calderon–Zygmund operators are bounded on L^p $(1<p<\infty)$. It is easy to verify that

$$
\frac{\nu}{4}\|\nabla j\|_{L^2}^2\le H(b,t).
$$

Indeed,

$$
\nu\|\nabla j\|_{L^2}^2=\nu\|(\partial_x j,\partial_y j)\|_{L^2}^2=\nu\|((\partial_{xx}b_2-\partial_{xy}b_1),(\partial_{xy}b_2-\partial_{yy}b_1))\|_{L^2}^2
$$

$$= \nu\|((\partial_{xx}b_2 + \partial_{yy}b_2), -(\partial_{xx}b_1 + \partial_{yy}b_1))\|_{L^2}^2 \le 4H(b,t).$$

Then, it follows from the above bounds and (3.9) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|j\|_{L^2}^2 + \frac{1}{2}H(b,t) \le \int b\cdot\nabla\Omega j\mathrm{d}x + C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2. \tag{3.10}$$

Combining (3.7), (3.8) and (3.10), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + 2(\mu+\chi)\|\nabla\Omega\|_{L^2}^2 + 8\chi\|\nabla\omega\|_{L^2}^2 + 2\kappa\|\Delta\omega\|_{L^2}^2 + H(b,t)\\
&\le 8\chi\int\nabla\omega\cdot\nabla\Omega\mathrm{d}x - 2\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\mathrm{d}x + C\|\Omega\|_{L^2}^2\|j\|_{L^2}^2\\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{3.11}$$

where we have used the fact that,

$$\int b\cdot\nabla\Omega j\mathrm{d}x = \int b\cdot\nabla j\Omega\mathrm{d}x.$$

Next, we consider I_1 and I_2, respectively. Applying the Young inequality,

$$\begin{aligned}
I_1 &= 8\chi\int\nabla\omega\cdot\nabla\Omega\mathrm{d}x\\
&\le (\mu+2\chi)\|\nabla\Omega\|_{L^2}^2 + \frac{8\chi^2}{\frac{\mu}{2}+\chi}\|\nabla\omega\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= -2\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\mathrm{d}x\\
&\le C\|\nabla u\|_{L^2}\|\nabla\omega\|_{L^4}^2\\
&\le \kappa\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\omega\|_{L^2}^2.
\end{aligned}$$

Inserting I_1 and I_2 into (3.11), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \mu\|\nabla\Omega\|_{L^2}^2 + \frac{8\chi\mu}{\mu+2\chi}\|\nabla\omega\|_{L^2}^2 + \kappa\|\Delta\omega\|_{L^2}^2 + H(b,t)\\
&\le C\left(\|j\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right)\left(\|\Omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right).
\end{aligned}$$

Gronwall's inequality, together with the fact that

$$\|j\|_{L^2}^2 \le 2\|\partial_x b_2\|_{L^2}^2 + 2\|\partial_y b_1\|_{L^2}^2$$

allows us to conclude that

$$\begin{aligned}
&\|\Omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \mu\int_0^t\|\nabla\Omega(\tau)\|_{L^2}^2\mathrm{d}\tau + \frac{8\chi\mu}{\mu+2\chi}\int_0^t\|\nabla\omega(\tau)\|_{L^2}^2\mathrm{d}\tau\\
&\quad + \kappa\int_0^t\|\Delta\omega(\tau)\|_{L^2}^2\mathrm{d}\tau + \int_0^t H(b,\tau)\mathrm{d}\tau\\
&\le \left(\|\Omega_0\|_{L^2}^2 + \|\nabla\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2\right)\exp\left\{C\int_0^t\left(\|j(\tau)\|_{L^2}^2 + \|\nabla\omega(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\right\}\\
&\le \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla\omega_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2\right)\exp\left\{C\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right)\right\},
\end{aligned}$$

which immediately yields (3.2). Thus, the proof of Proposition 3.1 is completed. □

Remark 3.2. Because of $\nabla \cdot b = 0$, it is easy to verify that

$$\|\nabla b\|_{L^2(\mathbb{R}^2)} = \|\nabla \times b\|_{L^2(\mathbb{R}^2)} \leq 2\left(\|\partial_y b_1(\tau)\|_{L^2}^2 + \|\partial_x b_2(\tau)\|_{L^2}^2\right).$$

Therefore, we can rewrite (3.1) as

$$\|u(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \mathrm{d}\tau + 8\chi\left(1 - \frac{2\chi}{\mu + 2\chi}\right)\int_0^t \|\omega(\tau)\|_{L^2}^2 \mathrm{d}\tau$$
$$+2\kappa \int_0^t \|\nabla \omega(\tau)\|_{L^2}^2 \mathrm{d}\tau + \nu \int_0^t \|\nabla b(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C. \tag{3.12}$$

The following proposition provides the decay rates for $\|(\nabla u, \nabla \omega, \nabla b)\|_{L^2}$.

Proposition 3.3. *Suppose $(u,\ \omega,\ b)$ is a solution of (1.3) with the corresponding initial data $(u_0, \omega_0, b_0) \in H^1(\mathbb{R}^2)$. Then,*

$$\|\nabla u(t)\|_{L^2} + \|\nabla \omega(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}. \tag{3.13}$$

Proof of Proposition 3.3. As in proof of Proposition 3.1, we have, for all $0 \leq s < t \leq \infty$,

$$\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \mu \int_s^t \|\nabla u(\tau)\|_{L^2}^2 \mathrm{d}\tau + 2\kappa \int_s^t \|\nabla \omega(\tau)\|_{L^2}^2 \mathrm{d}\tau$$
$$+ 2\nu \int_s^t \left(\|\partial_y b_1(\tau)\|_{L^2}^2 + \|\partial_x b_2(\tau)\|_{L^2}^2\right) \mathrm{d}\tau \leq \|u(s)\|_{L^2}^2 + \|\omega(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2, \tag{3.14}$$

and

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \mu \int_s^t \|\nabla \Omega(\tau)\|_{L^2}^2 \mathrm{d}\tau$$
$$+ \kappa \int_s^t \|\Delta \omega(\tau)\|_{L^2}^2 \mathrm{d}\tau + \int_s^t H(b,\tau) \mathrm{d}\tau$$
$$\leq \left(\|\Omega(s)\|_{L^2}^2 + \|\nabla \omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2\right) \exp\left\{C\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right)\right\}. \tag{3.15}$$

Therefore,

$$\int_0^\infty \|\nabla b(\tau)\|_{L^2}^2 \mathrm{d}\tau = \int_0^\infty \|j(\tau)\|_{L^2}^2 \mathrm{d}\tau$$
$$\leq 2 \int_0^\infty \left(\|\partial_x b_2(\tau)\|_{L^2}^2 + \|\partial_y b_1(\tau)\|_{L^2}^2\right) \mathrm{d}\tau \leq C\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right),$$

and

$$\int_0^\infty \|\nabla \omega(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right),$$
$$\int_0^\infty \|\nabla u(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C\left(\|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right).$$

A special consequence is that

$$\int_{\frac{t}{2}}^{t}\left(\|\nabla u(\tau)\|_{L^2}^2+\|\nabla \omega(\tau)\|_{L^2}^2+\|\nabla b(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \longrightarrow 0, \quad as \quad t\to\infty.$$

Integrating (3.15) with respect to s in $(\frac{t}{2},t)$, we have

$$\begin{aligned}
&t\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla \omega(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\right)\\
&\le 2\exp\left\{C\left(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\|b_0\|_{L^2}^2\right)\right\}\int_{\frac{t}{2}}^{t}\left(\|\nabla u(\tau)\|_{L^2}^2+\|\nabla \omega(\tau)\|_{L^2}^2+\|\nabla b(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le C.
\end{aligned}$$

Therefore, for any $t\ge 1$, we get

$$\|\nabla u(t)\|_{L^2}^2+\|\nabla \omega(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\le Ct^{-1}\le C(1+t)^{-1}. \tag{3.16}$$

Moreover, for any $0<t<1$, it follows from (3.15),

$$\|\nabla u(t)\|_{L^2}^2+\|\nabla \omega(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\le C\le C(1+t)^{-1}. \tag{3.17}$$

Then, (3.16) and (3.17) yield (3.13).

□

The following proposition which will play an important role to drive the decay estimates of Theorem 1.1.

Proposition 3.4. *Let (u,ω,b) be a global solution of system (1.3), with $\nabla\cdot u_0=\nabla\cdot b_0=0$ and $(u_0,\omega_0,b_0)\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Then, there exists a constant $C>0$, such that*

$$\begin{aligned}
&|\hat{u}(\xi,t)|+|\hat{\omega}(\xi,t)|+|\hat{b}_1(\xi,t)|+|\hat{b}_2(\xi,t)|\\
&\le C+C|\xi|\int_0^t\left(\|u(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2\right)\mathrm{d}\tau.
\end{aligned} \tag{3.18}$$

Proof of Proposition 3.3. Applying the Fourier transform to system (1.3), we obtain:

$$\begin{cases}
\partial_t\hat{u}+(\mu+\chi)|\xi|^2\hat{u}=-\mathcal{F}(\nabla\pi)+\mathcal{F}(b\cdot\nabla b)+2\chi i\xi\times\hat{\omega}-\mathcal{F}(u\cdot\nabla u),\\
\partial_t\hat{\omega}+\kappa|\xi|^2\hat{\omega}+4\chi\hat{\omega}=2\chi i\xi\times\hat{u}-\mathcal{F}(u\cdot\nabla\omega),\\
\partial_t\hat{b}_1+\nu|\xi_2|^2\hat{b}_1=\mathcal{F}[b\cdot\nabla u_1-u\cdot\nabla b_1],\\
\partial_t\hat{b}_2+\nu|\xi_1|^2\hat{b}_2=\mathcal{F}[b\cdot\nabla u_2-u\cdot\nabla b_2].
\end{cases} \tag{3.19}$$

Multiplying the $(3.19)_1$, $(3.19)_2$, $(3.19_3$ and $(3.19)_4$ by $\bar{\hat{u}})$, $\bar{\hat{\omega}}$, $\bar{\hat{b}}_1$ and $\bar{\hat{b}}_2$, respectively, and summing up, we have, noting that $|\hat{u}|^2=\hat{u}\bar{\hat{u}}$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\hat{u}|^2+|\hat{\omega}|^2+|\hat{b}_1|^2+|\hat{b}_2|^2)+(\mu+\chi)|\xi|^2|\hat{u}|^2+\kappa|\xi|^2|\hat{\omega}|^2\\
&\quad+\nu(|\xi_2|^2|\hat{b}_1|^2+|\xi_1|^2|\hat{b}_2|^2)+4\chi|\hat{\omega}|^2\\
&=-\mathcal{F}(\nabla\pi)\bar{\hat{u}}+\mathcal{F}(b\cdot\nabla b)\bar{\hat{u}}-\mathcal{F}(u\cdot\nabla u)\bar{\hat{u}}-\mathcal{F}(u\cdot\nabla\omega)\bar{\hat{\omega}}+\mathcal{F}(b\cdot\nabla u_1)\bar{\hat{b}}_1\\
&\quad-\mathcal{F}(u\cdot\nabla b_1)\bar{\hat{b}}_1+\mathcal{F}(b\cdot\nabla u_2)\bar{\hat{b}}_2-\mathcal{F}(u\cdot\nabla b_2)\bar{\hat{b}}_2+2\chi i\xi\times\hat{\omega}\bar{\hat{u}}+2\chi i\xi\times\hat{u}\bar{\hat{\omega}}\\
&=K_1+K_2+\cdots+K_{10}.
\end{aligned} \tag{3.20}$$

For K_1, taking divergence to the first equation of (1.3), one yields

$$\pi=(-\Delta)^{-1}(\nabla\otimes\nabla)(b\otimes b-u\otimes u).$$

And taking Fourier transformation obeys, nothing that $|\hat{u}| = |\bar{\hat{u}}|$

$$\begin{aligned} K_1 &\le |\xi||\hat{\pi}||\bar{\hat{u}}| \\ &\le |\xi|(\|b\otimes b\|_{L^1} + \|u\otimes u\|_{L^1})|\bar{\hat{u}}| \\ &\le |\xi|(\|b\|_{L^2}^2 + \|u\|_{L^2}^2)|\hat{u}|. \end{aligned}$$

For K_2,

$$K_2 \le |\xi||\widehat{b\otimes b}||\bar{\hat{u}}| \le |\xi|\|b\otimes b\|_{L^1}|\bar{\hat{u}}| \le |\xi|\|b\|_{L^2}^2|\hat{u}|.$$

Similarly, we obtain

$$\begin{aligned} |K_3+K_4| &\le 2|\xi|(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2)(|\hat{u}| + |\hat{\omega}|), \\ |K_5+K_6| &\le |\xi|(\|b\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \|b_1\|_{L^2}^2 + \|u\|_{L^2}^2)|\hat{b}_1| \\ &\le 2|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2)|\hat{b}_1|, \\ |K_7+K_8| &\le 2|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2)|\hat{b}_2|, \\ |K_9+K_{10}| &\le 4\chi|\xi||\hat{\omega}||\hat{u}| \\ &\le (\frac{\mu}{2}+\chi)|\xi|^2|\hat{u}|^2 + \frac{8\chi^2}{\mu+2\chi}|\hat{\omega}|^2. \end{aligned}$$

Inserting K_1–K_{10} into (3.20), we derive that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(|\hat{u}|^2 + |\hat{\omega}|^2 + |\hat{b}_1|^2 + |\hat{b}_2|^2) + \mu|\xi|^2|\hat{u}|^2 + 2\kappa|\xi|^2|\hat{\omega}|^2 \\ &\quad + 2\nu(|\xi_2|^2|\hat{b}_1|^2 + |\xi_1|^2|\hat{b}_2|^2) + \frac{16\chi\mu}{\mu+2\chi}|\hat{\omega}|^2 \\ &\le C|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\omega\|_{L^2}^2)(|\hat{u}| + |\hat{\omega}| + |\hat{b}_1| + |\hat{b}_2|), \end{aligned}$$

which immediately yields

$$\partial_t\sqrt{|\hat{u}|^2 + |\hat{\omega}|^2 + |\hat{b}_1|^2 + |\hat{b}_2|^2} \le C|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\omega\|_{L^2}^2). \tag{3.21}$$

Integrating (3.21) in $[0,t]$, we obtain

$$\begin{aligned} &\sqrt{|\hat{u}(t)|^2 + |\hat{\omega}(t)|^2 + |\hat{b}_1(t)|^2 + |\hat{b}_2(t)|^2} \\ &\le \sqrt{|\hat{u}(0)|^2 + |\hat{\omega}(0)|^2 + |\hat{b}_1(0)|^2 + |\hat{b}_2(0)|^2} + C|\xi|\int_0^t(\|u(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2)\mathrm{d}\tau \\ &\le C + C|\xi|\int_0^t(\|u(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2)\mathrm{d}\tau. \end{aligned}$$

Thus, the proof of Proposition 3.3 is completed. □

The following proposition provides an auxiliary logarithmic decay estimate.

Proposition 3.5. *Assume that $(u_0,\omega_0,b_0) \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, then the global solution (u,ω,b) of (1.3), satisfies*

$$\|u(t)\|_{L^2(\mathbb{R}^2)}^2 + \|b(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\omega(t)\|_{L^2(\mathbb{R}^2)}^2 \le C\ln^{-2}(e+t). \tag{3.22}$$

Proof of Proposition 3.4. Choosing a positive smooth function $f(t) \in C^\infty[0,\infty)$ satisfying $f(0) = 1$, $f'(t) > 0$ and

$$\frac{f'(t)}{2f(t)} < 1, \quad t > t_0 > 0$$

multiplying both sides of inequality (3.5) with $f(t)$, one shows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right)\right] + c_1 f(t)\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2\right)\\ &\quad\le f'(t)\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right),\end{aligned}$$

where $c_1 = min\{\mu, 2\kappa, \nu\}$. From this and together with Plancherel Theorem,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\left(\|\hat{u}(t)\|_{L^2}^2 + \|\hat{b}(t)\|_{L^2}^2 + \|\hat{\omega}(t)\|_{L^2}^2\right)\right] + c_1 f(t)\int_{\mathbb{R}^2}|\xi|^2\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi\\ &\quad\le f'(t)\int_{\mathbb{R}^2}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi.\end{aligned} \tag{3.23}$$

Letting $B(t) = \left\{\xi \in \mathbb{R}^2 \mid |\xi|^2 \le \frac{f'(t)}{c_1 f(t)}\right\}$, then we obtain

$$\begin{aligned}&c_1 f(t)\int_{\mathbb{R}^2}|\xi|^2\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi\\ &\ge c_1 f(t)\int_{B^c(t)}|\xi|^2\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi\\ &\ge c_1 f(t)\left(\frac{f'(t)}{c_1 f(t)}\right)\int_{B^c(t)}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi\\ &= f'(t)\int_{\mathbb{R}^2}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi - f'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi.\end{aligned}$$

Substituting this result into (3.23), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[f(t)\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right)\right] \le f'(t)\int_{B(t)}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi. \tag{3.24}$$

Employing (3.18), we have

$$\begin{aligned}&\int_{B(t)}\left(|\hat{u}(\xi)|^2 + |\hat{\omega}(\xi)|^2 + |\hat{b}(\xi)|^2\right)\mathrm{d}\xi\\ &= C\int_{B(t)}\left\{|\xi|^2\left(\int_0^t(\|u(\tau)\|_{L^2}^2 + \|\omega(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^2}^2)\mathrm{d}\tau\right)^2 + 1\right\}\mathrm{d}\xi\\ &\le \frac{Cf'(t)}{f(t)} + \frac{C(f'(t))^2 t}{f^2(t)}\int_0^t\left(\|u(\tau)\|_{L^2}^4 + \|\omega(\tau)\|_{L^2}^4 + \|b(\tau)\|_{L^2}^4\right)\mathrm{d}\tau.\end{aligned}$$

Inserting the above inequality into (3.24) and integrating in time from 0 to t ($t > t_0$), one shows the following integral inequality which is crucial for the large time decay for global solutions

$$\begin{aligned} & f(t)\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right) \\ & \leq f(0)\left(\|u_0\|_{L^2}^2+\|b_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2\right)+C\int_0^t \frac{(f'(\tau))^2}{f(\tau)}\mathrm{d}\tau \\ & \quad + C\int_0^t \frac{(f'(\tau))^3\tau}{f^2(\tau)}\int_0^\tau \left(\|u(s)\|_{L^2}^4+\|\omega(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4\right)\mathrm{d}s\mathrm{d}\tau. \end{aligned} \tag{3.25}$$

Letting $f(t) = (\ln(e+t))^3$ in (3.25) and applying energy inequality (3.1), one yields

$$\begin{aligned} & \ln^3(e+t)\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right) \\ & \leq C + C\ln(e+t) + \int_0^t \frac{\tau^2}{(e+\tau)^3}\mathrm{d}\tau\left(\|u_0\|_{L^2}^4+\|\omega_0\|_{L^2}^4+\|b_0\|_{L^2}^4\right) \\ & \leq C + C\ln(e+t). \end{aligned}$$

Thus, we have,

$$\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2 \leq C\ln^{-2}(e+t).$$

This completes the proof of Proposition 3.4. □

With the above propositions at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.3, we completed the proof of (1.5) of Theorem 1.1, next we start to prove (1.4) of Theorem 1.1.

Letting $f(t) = (1+t)^2$ and inserting it into (3.25), we obtain

$$\begin{aligned} & (1+t)^2\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right) \\ & \leq C + C(1+t) + C\int_0^t \frac{\tau}{1+\tau}\int_0^\tau \left(\|u(s)\|_{L^2}^4+\|\omega(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4\right)\mathrm{d}s\mathrm{d}\tau \\ & \leq C(1+t) + C\int_0^t\int_0^\tau \left(\|u(s)\|_{L^2}^4+\|\omega(s)\|_{L^2}^4+\|b(s)\|_{L^2}^4\right)\mathrm{d}s\mathrm{d}\tau \\ & \leq C(1+t) + C(1+t)\int_0^t \left(\|u(s)\|_{L^2}^2+\|b(s)\|_{L^2}^2+\|\omega(s)\|_{L^2}^2\right)\ln^{-2}(e+s)\mathrm{d}s. \end{aligned}$$

Denoting

$$\mathcal{N}(t) = \sup_{0\leq\tau\leq t}\left\{(1+\tau)(\|u(\tau)\|_{L^2}^2+\|b(\tau)\|_{L^2}^2+\|\omega(\tau)\|_{L^2}^2)\right\}.$$

Then, we have

$$\mathcal{N}(t) \leq C + C\int_0^t \mathcal{N}(s)(1+s)^{-1}\ln^{-2}(e+s)\mathrm{d}s.$$

Applying Gronwall's inequality, we obtain

$$\mathcal{N}(t) \le C \exp\left\{ \int_0^\infty (1+s)^{-1} \ln^{-2}(e+s) \mathrm{d}s \right\} < C,$$

which implies the desired decay

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} + \|\omega(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}},$$

That is (1.4), and completed the proof of Theorem 1.1. □

4. The proof of Theorem 1.2

This section is devoted to proving Theorem 1.2.

We first prove the global stability part (I) of Theorem 1.2. As we know, it suffices to establish the global *a priori* H^s estimates.

Proof of (I) of Theorem 1.2. Applying Λ^s to (1.3), we have

$$\partial_t \Lambda^s u + u \cdot \nabla \Lambda^s u - (\mu+\chi)\Delta \Lambda^s u = -\Lambda^s \nabla \pi - [\Lambda^s, u \cdot \nabla] u + \Lambda^s (b \cdot \nabla b) + 2\chi \Lambda^s \nabla \times \omega, \tag{4.1}$$

$$\partial_t \Lambda^s \omega + u \cdot \nabla \Lambda^s \omega - \kappa \Delta \Lambda^s \omega + 4\chi \Lambda^s \omega = -[\Lambda^s, u \cdot \nabla]\omega + 2\chi \Lambda^s \nabla \times u, \tag{4.2}$$

$$\partial_t \Lambda^s b_1 + u \cdot \nabla \Lambda^s b_1 - \nu \partial_{yy} \Lambda^s b_1 = -[\Lambda^s, u \cdot \nabla] b_1 + \Lambda^s (b \cdot \nabla u_1), \tag{4.3}$$

$$\partial_t \Lambda^s b_2 + u \cdot \nabla \Lambda^s b_2 - \nu \partial_{xx} \Lambda^s b_2 = -[\Lambda^s, u \cdot \nabla] b_2 + \Lambda^s (b \cdot \nabla u_2), \tag{4.4}$$

where $[\Lambda^s, f \cdot \nabla] g = \Lambda^s (f \cdot \nabla g) - f \cdot \nabla \Lambda^s g$ is commutator.

Dotting (4.1)–(4.4) by $\Lambda^s u$, $\Lambda^s \omega$, $\Lambda^s b_1$ and $\Lambda^s b_2$, respectively, integrating the resulting equations in $\mathbb{R}^2$, and adding them together, we easily obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s \omega\|_{L^2}^2 + \|\Lambda^s b_1\|_{L^2}^2 + \|\Lambda^s b_2\|_{L^2}^2) + (\mu+\chi)\|\Lambda^s \nabla u\|_{L^2}^2 + \kappa \|\Lambda^s \nabla \omega\|_{L^2}^2 \\
&\quad + 4\chi \|\Lambda^s \omega\|_{L^2}^2 + \nu(\|\Lambda^s \partial_y b_1\|_{L^2}^2 + \|\Lambda^s \partial_x b_2\|_{L^2}^2) \\
&\le -\int [\Lambda^s, u \cdot \nabla] u \cdot \Lambda^s u + \int [\Lambda^s, b \cdot \nabla] b \cdot \Lambda^s u - \int [\Lambda^s, u \cdot \nabla]\omega \cdot \Lambda^s \omega \\
&\quad + 4\chi \int \Lambda^s \nabla \times u \cdot \Lambda^s \omega - \int [\Lambda^s, u \cdot \nabla] b_1 \cdot \Lambda^s b_1 - \int [\Lambda^s, u \cdot \nabla] b_2 \cdot \Lambda^s b_2 \\
&\quad + \int [\Lambda^s, b \cdot \nabla] u_1 \cdot \Lambda^s b_1 + \int [\Lambda^s, b \cdot \nabla] u_2 \cdot \Lambda^s b_2,
\end{aligned}$$

where we used the facts that,

$$\int b \cdot \nabla \Lambda^s u_1 \cdot \Lambda^s b_1 + \int b \cdot \nabla \Lambda^s u_2 \cdot \Lambda^s b_2 = \int b \cdot \nabla \Lambda^s u \cdot \Lambda^s b = -\int b \cdot \nabla \Lambda^s b \cdot \Lambda^s u,$$

and

$$\int \Lambda^s \nabla \times u \cdot \Lambda^s \omega = \int \Lambda^s \nabla \times \omega \cdot \Lambda^s u.$$

Due to the divergence free condition $\nabla \cdot b = 0$, we have

$$\|\Lambda^s \nabla b\|_{L^2}^2 = \|\Lambda^s \nabla \times b\|_{L^2}^2 \le 2\left(\|\Lambda^s \partial_x b_2\|_{L^2}^2 + \|\Lambda^s \partial_y b_1\|_{L^2}^2\right),$$

$$\|\Lambda^s b\|_{L^2}^2 = \|\Lambda^s b_1\|_{L^2}^2 + \|\Lambda^s b_2\|_{L^2}^2.$$

Then, it follows from the above bounds and facts that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\omega\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2\right)+(\mu+\chi)\|\Lambda^s\nabla u\|_{L^2}^2+\kappa\|\Lambda^s\nabla\omega\|_{L^2}^2\\
&\quad+4\chi\|\Lambda^s\omega\|_{L^2}^2+\frac{\nu}{2}\|\Lambda^s\nabla b\|_{L^2}^2\\
&\le-\int[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u+\int[\Lambda^s,b\cdot\nabla]b\cdot\Lambda^s u-\int[\Lambda^s,u\cdot\nabla]\omega\cdot\Lambda^s\omega\\
&\quad+4\chi\int\Lambda^s\nabla\times u\cdot\Lambda^s\omega-\int[\Lambda^s,u\cdot\nabla]b\cdot\Lambda^s b+\int[\Lambda^s,b\cdot\nabla]u\cdot\Lambda^s b\\
&=I_1+\cdots+I_6.
\end{aligned}\tag{4.5}$$

We first bound I_1. Resorting to the commutator estimate (2.1), we have

$$\begin{aligned}
|I_1|&=\left|-\int[\Lambda^s,u\cdot\nabla]u\cdot\Lambda^s u\right|\\
&\le\|[\Lambda^s,u\cdot\nabla]u\|_{L^2}\|\Lambda^{s-1}\nabla u\|_{L^2}\\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}\|\Lambda^{s-1}\nabla u\|_{L^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|I_2|&=\left|\int[\Lambda^s,b\cdot\nabla]b\cdot\Lambda^s u\right|\\
&\le\|[\Lambda^s,b\cdot\nabla]b\|_{L^2}\|\Lambda^{s-1}\nabla u\|_{L^2}\\
&\le C\|\nabla b\|_{L^\infty}\|\Lambda^s b\|_{L^2}\|\Lambda^{s-1}\nabla u\|_{L^2}.
\end{aligned}$$

Again using the commutator estimate (2.1), we get

$$\begin{aligned}
|I_3|&=\left|-\int[\Lambda^s,u\cdot\nabla]\omega\cdot\Lambda^s\omega\right|\\
&\le\|[\Lambda^s,u\cdot\nabla]\omega\|_{L^2}\|\Lambda^{s-1}\nabla\omega\|_{L^2}\\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\omega\|_{L^2}+\|\nabla\omega\|_{L^\infty}\|\Lambda^s u\|_{L^2})\|\Lambda^{s-1}\nabla\omega\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
|I_5|+|I_6|&=\left|-\int[\Lambda^s,u\cdot\nabla]b\cdot\Lambda^s b\right|+\left|\int[\Lambda^s,b\cdot\nabla]u\cdot\Lambda^s b\right|\\
&\le(\|[\Lambda^s,u\cdot\nabla]b\|_{L^2}+\|[\Lambda^s,b\cdot\nabla]u\|_{L^2})\|\Lambda^{s-1}\nabla b\|_{L^2}\\
&\le C(\|\nabla b\|_{L^\infty}\|\Lambda^s u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2})\|\Lambda^{s-1}\nabla b\|_{L^2}.
\end{aligned}$$

For I_4, by the Young inequality

$$\begin{aligned}
|I_4|&\le4\chi\|\Lambda^s\nabla u\|_{L^2}\|\Lambda^s\omega\|_{L^2}\\
&\le\left(\frac{\mu}{2}+\chi\right)\|\Lambda^s\nabla u\|_{L^2}^2+\frac{4\chi^2}{\frac{\mu}{2}+\chi}\|\Lambda^s\omega\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (4.5), and note that $\|f\|_{L^\infty}\le C\|f\|_{H^s}$ with $s>1$, and using the Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s\omega\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2)+\frac{\mu}{2}\|\Lambda^s\nabla u\|_{L^2}^2+\kappa\|\Lambda^s\nabla\omega\|_{L^2}^2\\
&\quad+\frac{8\mu\chi}{\mu+2\chi}\|\Lambda^s\omega\|_{L^2}^2+\frac{\nu}{2}\|\Lambda^s\nabla b\|_{L^2}^2\\
&\le C\|\Lambda^s u\|_{L^2}(\|\nabla u\|_{H^s}^2+\|\nabla\omega\|_{H^s}^2+\|\nabla b\|_{H^s}^2)+C\|\Lambda^s\omega\|_{L^2}\|\nabla u\|_{H^s}\|\nabla\omega\|_{H^s}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\Lambda^s b\|_{L^2}\|\nabla u\|_{H^s}\|\nabla b\|_{H^s}\\
\le&C(\|\nabla u\|_{H^s}^2+\|\nabla\omega\|_{H^s}^2+\|\nabla b\|_{H^s}^2)(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2)^{\frac12}.
\end{aligned}\tag{4.6}
$$

Substituting (4.6) and (3.5) together, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2)+c_1(\|\nabla u\|_{H^s}^2+\|\nabla\omega\|_{H^s}^2+\|\nabla b\|_{H^s}^2)\\
&\quad\le C(\|\nabla u\|_{H^s}^2+\|\nabla\omega\|_{H^s}^2+\|\nabla b\|_{H^s}^2)(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2)^{\frac12}
\end{aligned}\tag{4.7}
$$

where $c_1=\min\{\mu,2\kappa,\nu\}$. This inequality indicates that, if the initial (u_0,ω_0,b_0) satisfy, for $0<\epsilon<\epsilon_0=(\frac{c_1}{C})^2$

$$
\|u_0\|_{H^s}^2+\|\omega_0\|_{H^s}^2+\|b_0\|_{H^s}^2<\epsilon,
$$

then the corresponding solution remains for all time. That is

$$
\|u(t)\|_{H^s}^2+\|\omega(t)\|_{H^s}^2+\|b(t)\|_{H^s}^2<\epsilon.\tag{4.8}
$$

Next, we need to prove that (4.8) is correct. In fact, if suppose (4.8) is not true and T_0 is the first time such that (4.8) is violated, i.e.,

$$
\|u(T_0)\|_{H^s}^2+\|\omega(T_0)\|_{H^s}^2+\|b(T_0)\|_{H^s}^2=\epsilon,
$$

and (4.8) holds for any $0\le t<T_0$. We can deduce from (4.7) that for any $0\le t\le T_0$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2)+(c_1-C\sqrt{\epsilon})(\|\nabla u\|_{H^s}^2+\|\nabla\omega\|_{H^s}^2+\|\nabla b\|_{H^s}^2)\le 0.
$$

Therefore,

$$
\|u\|_{H^s}^2+\|\omega\|_{H^s}^2+\|b\|_{H^s}^2\le\|u_0\|_{H^s}^2+\|\omega_0\|_{H^s}^2+\|b_0\|_{H^s}^2<\epsilon.
$$

This is a contradiction. Thus, we get the uniform bound of (4.8). In addition,

$$
\int_0^t(\|\nabla u(\tau)\|_{H^s}^2+\|\nabla\omega(\tau)\|_{H^s}^2+\|\nabla b(\tau)\|_{H^s}^2)\mathrm{d}\tau\le C\epsilon.
$$

Therefore, the proof of (I) of Theorem 1.2 is completed. □

Next, we start to prove (II) of Theorem 1.2. For the simplicity of the proof, we divide the main proof into two parts. Firstly, we give the proof of (1.8), then prove that (1.9) is true. As a tool, we verify the following proposition in the negative Sobolev space $\dot{H}^{-l}$ with $0\le l<1$.

Proposition 4.1. *Let $c_1=\min\{\mu,2\kappa,\nu\}$. Then, for $0\le l<1$, we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{\dot H^{-l}}^2+\|b\|_{\dot H^{-l}}^2+\|\omega\|_{\dot H^{-l}}^2)+c_1(\|\nabla u\|_{\dot H^{-l}}^2+\|\nabla b\|_{\dot H^{-l}}^2+\|\nabla\omega\|_{\dot H^{-l}}^2)\\
\le&C(\|\nabla u\|_{L^2}+\|\nabla\omega\|_{L^2}+\|\nabla b\|_{L^2})(\|\nabla u\|_{L^2}^{1-l}+\|\nabla b\|_{L^2}^{1-l})(\|u\|_{L^2}^{l}+\|b\|_{L^2}^{l})\\
&\times(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}}).
\end{aligned}\tag{4.9}
$$

Proof of Proposition 4.1. For $0\le l<1$, similar to the process of (4.5) and the divergence free condition $\nabla\cdot u=\nabla\cdot b=0$, we drive that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{\dot H^{-l}}^2+\|b\|_{\dot H^{-l}}^2+\|\omega\|_{\dot H^{-l}}^2)+2(\mu+\chi)\|\nabla u\|_{\dot H^{-l}}^2+2\kappa\|\nabla\omega\|_{\dot H^{-l}}^2+8\chi\|\omega\|_{\dot H^{-l}}^2+\nu\|\nabla b\|_{\dot H^{-l}}^2\\
\le&2\|u\cdot\nabla u\|_{\dot H^{-l}}\|u\|_{\dot H^{-l}}+2\|b\cdot\nabla b\|_{\dot H^{-l}}\|u\|_{\dot H^{-l}}+2\|u\cdot\nabla\omega\|_{\dot H^{-l}}\|\omega\|_{\dot H^{-l}}+(\mu+2\chi)\|\nabla u\|_{\dot H^{-l}}^2\\
&+\frac{8\chi^2}{\frac{\mu}{2}+\chi}\|\omega\|_{\dot H^{-l}}^2+2(\|u\cdot\nabla b\|_{\dot H^{-l}}+\|b\cdot\nabla u\|_{\dot H^{-l}})\|b\|_{\dot H^{-l}}.
\end{aligned}\tag{4.10}
$$

By Lemma 2.2, Hölder's inequality and Gagliardo–Nirenberg inequality, we have

$$\|u\cdot\nabla u\|_{\dot H^{-l}}\le C\|u\cdot\nabla u\|_{L^{\frac{2}{l+1}}},$$
$$\|u\cdot\nabla u\|_{L^{\frac{2}{l+1}}}\le C\|u\|_{L^{\frac{2}{l}}}\|\nabla u\|_{L^2},$$

and

$$\|u\|_{L^{\frac{2}{l}}}\le C\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^{l}.$$

Then,

$$\|u\cdot\nabla u\|_{\dot H^{-l}}\le C\|\nabla u\|_{L^2}^{2-l}\|u\|_{L^2}^{l}. \tag{4.11}$$

Similarly, we have

$$\|b\cdot\nabla b\|_{\dot H^{-l}}\le C\|\nabla b\|_{L^2}^{2-l}\|b\|_{L^2}^{l}, \tag{4.12}$$
$$\begin{aligned}&\|u\cdot\nabla b\|_{\dot H^{-l}}+\|b\cdot\nabla u\|_{\dot H^{-l}}\\&\quad\le C(\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^{l}\|\nabla b\|_{L^2}+\|\nabla b\|_{L^2}^{1-l}\|b\|_{L^2}^{l}\|\nabla u\|_{L^2}),\end{aligned} \tag{4.13}$$
$$\|u\cdot\nabla \omega\|_{\dot H^{-l}}\le C\|\nabla \omega\|_{L^2}^{1-l}\|\nabla u\|_{L^2}^{1-l}\|u\|_{L^2}^{l}. \tag{4.14}$$

Combining (4.10) and (4.11)–(4.14), we obtain

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}(\|u\|_{\dot H^{-l}}^2+\|b\|_{\dot H^{-l}}^2+\|\omega\|_{\dot H^{-l}}^2)+c_1(\|\nabla u\|_{\dot H^{-l}}^2+\|\nabla b\|_{\dot H^{-l}}^2+\|\nabla \omega\|_{\dot H^{-l}}^2)\\&\le C(\|\nabla u\|_{L^2}+\|\nabla \omega\|_{L^2}+\|\nabla b\|_{L^2})(\|\nabla u\|_{L^2}^{1-l}+\|\nabla b\|_{L^2}^{1-l})(\|u\|_{L^2}^{l}+\|b\|_{L^2}^{l})\\&\quad\times(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}}),\end{aligned}$$

where $c_1=\min\{\mu,2\kappa,\nu\}$. Thus, the proof of Proposition 4.1 is completed. □

Proof of (II) of Theorem 1.2. (1) We first prove (1.8). For $0\le m\le s$, similarly to the process of (4.5), together with Hölder's inequality and the divergence free condition $\nabla\cdot u=\nabla\cdot b=0$, we get

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}(\|\Lambda^m u\|_{L^2}^2+\|\Lambda^m \omega\|_{L^2}^2+\|\Lambda^m b\|_{L^2}^2)+(\mu+\chi)\|\Lambda^m\nabla u\|_{L^2}^2+\kappa\|\Lambda^m\nabla\omega\|_{L^2}^2\\&\quad+4\chi\|\Lambda^m\omega\|_{L^2}^2+\frac{\nu}{2}\|\Lambda^m\nabla b\|_{L^2}^2\\&\le-\int\Lambda^m(u\cdot\nabla u)\cdot\Lambda^m u+\int\Lambda^m(b\cdot\nabla b)\cdot\Lambda^m u-\int\Lambda^m(u\cdot\nabla b)\cdot\Lambda^m b\\&\quad+\int\Lambda^m(b\cdot\nabla u)\cdot\Lambda^m b-\int\Lambda^m(u\cdot\nabla\omega)\cdot\Lambda^m\omega+4\chi\int\Lambda^m\nabla\times u\cdot\Lambda^m\omega\\&\le\|\Lambda^{m+1}(u\otimes u)\|_{L^2}\|\Lambda^m u\|_{L^2}+\|\Lambda^{m+1}(b\otimes b)\|_{L^2}\|\Lambda^m u\|_{L^2}+\|\Lambda^{m+1}(u\otimes b)\|_{L^2}\|\Lambda^m b\|_{L^2}\\&\quad+\|\Lambda^{m+1}(b\otimes u)\|_{L^2}\|\Lambda^m b\|_{L^2}+\|\Lambda^{m+1}(u\otimes \omega)\|_{L^2}\|\Lambda^m \omega\|_{L^2}+4\chi\|\Lambda^m\nabla u\|_{L^2}^2\|\Lambda^m\nabla\omega\|_{L^2}^2\\&=A_1+A_2+A_3+A_4+A_5+A_6.\end{aligned} \tag{4.15}$$

For A_1, applying the commutator estimate (2.2) and Lemma 2.3 yield

$$A_1\le C(\|u\|_{L^\infty}\|\nabla u\|_{\dot H^m})\|u\|_{\dot H^m}\le C\|u\|_{H^s}\|\nabla u\|_{\dot H^m}^2,$$

where we used the facts that

$$\|u\|_{\dot H^m}\le C\|\nabla u\|_{\dot H^m}^{\frac{m}{m+1}}\|u\|_{L^2}^{\frac{1}{m+1}},\quad \|u\|_{L^\infty}\le C\|u\|_{L^2}^{\frac{m}{m+1}}\|\nabla u\|_{\dot H^m}^{\frac{1}{m+1}},\quad m>0.$$

Using the commutator estimate (2.2), Lemma 2.3 and the Young inequality lead to

$$\begin{aligned}A_2&\le C(\|b\|_{L^\infty}\|\nabla b\|_{\dot H^m})\|u\|_{\dot H^m}\\&\le C(\|b\|_{L^2}^{\frac{m}{m+1}}\|\nabla b\|_{\dot H^m}^{\frac{1}{m+1}}\|\nabla u\|_{\dot H^m}^{\frac{m}{m+1}}\|u\|_{L^2}^{\frac{1}{m+1}})\|\nabla b\|_{\dot H^m}\end{aligned}$$

$$\leq C(\|b\|_{L^2}^{\frac{m}{m+1}}\|u\|_{L^2}^{\frac{1}{m+1}}\|\nabla b\|_{\dot{H}^m})(\|\nabla b\|_{\dot{H}^m}+\|\nabla u\|_{\dot{H}^m})$$
$$\leq (\|u\|_{H^s}+\|b\|_{H^s})(\|\nabla b\|_{\dot{H}^m}^2+\|\nabla u\|_{\dot{H}^m}^2).$$

Similarly,

$$A_3+A_4\leq C(\|u\|_{H^s}+\|b\|_{H^s})(\|\nabla b\|_{\dot{H}^m}^2+\|\nabla u\|_{\dot{H}^m}^2),$$
$$A_5\leq C(\|u\|_{H^s}+\|\omega\|_{H^s})(\|\nabla \omega\|_{\dot{H}^m}^2+\|\nabla u\|_{\dot{H}^m}^2).$$

For A_6, applying the Young inequality, yields

$$A_6\leq (\frac{\mu}{2}+\chi)\|\Lambda^m\nabla u\|_{L^2}^2+\frac{8\chi^2}{\mu+2\chi}\|\Lambda^m\omega\|_{L^2}^2.$$

Inserting A_1–A_6 into (4.15), we obtain,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{\dot{H}^m}^2+\|\omega\|_{\dot{H}^m}^2+\|b\|_{\dot{H}^m}^2)+c_1(\|\nabla u\|_{\dot{H}^m}^2+\|\nabla \omega\|_{\dot{H}^m}^2+\|\nabla b\|_{\dot{H}^m}^2)\\ &\quad\leq C(\|u\|_{H^s}+\|\omega\|_{H^s}+\|b\|_{H^s})(\|\nabla u\|_{\dot{H}^m}^2+\|\nabla \omega\|_{\dot{H}^m}^2+\|\nabla b\|_{\dot{H}^m}^2),\end{aligned}\tag{4.16}$$

where for $0\leq m\leq s$. Choosing the ϵ in (4.8) sufficiently small (i.e.,$\epsilon<(\frac{c_1}{2C})^2$), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{\dot{H}^m}^2+\|\omega\|_{\dot{H}^m}^2+\|b\|_{\dot{H}^m}^2)+\frac{c_1}{2}(\|\nabla u\|_{\dot{H}^m}^2+\|\nabla \omega\|_{\dot{H}^m}^2+\|\nabla b\|_{\dot{H}^m}^2)\leq 0.\tag{4.17}$$

Because $\dot{B}^s_{2,2}\sim \dot{H}^s$, applying Lemma 2.4, we get

$$\|u\|_{\dot{H}^m}\leq C\|u\|_{\dot{H}^{-l}}^{\frac{1}{m+l+1}}\|\nabla u\|_{\dot{H}^m}^{\frac{m+l}{m+l+1}},\tag{4.18}$$
$$\|b\|_{\dot{H}^m}\leq C\|b\|_{\dot{H}^{-l}}^{\frac{1}{m+l+1}}\|\nabla b\|_{\dot{H}^m}^{\frac{m+l}{m+l+1}},\tag{4.19}$$
$$\|\omega\|_{\dot{H}^m}\leq C\|\omega\|_{\dot{H}^{-l}}^{\frac{1}{m+l+1}}\|\nabla \omega\|_{\dot{H}^m}^{\frac{m+l}{m+l+1}}.\tag{4.20}$$

Therefore, if

$$\|u\|_{\dot{H}^{-l}}+\|b\|_{\dot{H}^{-l}}+\|\omega\|_{\dot{H}^{-l}}\leq C,\tag{4.21}$$

and inserting (4.18)–(4.20) into (4.17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{\dot{H}^m}^2+\|\omega\|_{\dot{H}^m}^2+\|b\|_{\dot{H}^m}^2)+C(\|\nabla u\|_{\dot{H}^m}^2+\|\nabla \omega\|_{\dot{H}^m}^2+\|\nabla b\|_{\dot{H}^m}^2)^{\frac{m+l+1}{m+l}}\leq 0.\tag{4.22}$$

It implies that

$$\|u\|_{\dot{H}^m}^2+\|\omega\|_{\dot{H}^m}^2+\|b\|_{\dot{H}^m}^2\leq C(1+t)^{-l-m}.\tag{4.23}$$

To make the proof more completed, therefore, we need to verify that (4.21) holds for $0\leq l<1$. Next, we prove that (4.21). Where we will applying the bootstrapping argument. Assume that

$$\|u_0\|_{\dot{H}^{-l}}^2+\|\omega_0\|_{\dot{H}^{-l}}^2+\|b_0\|_{\dot{H}^{-l}}^2\leq C_0.\tag{4.24}$$

Suppose that for all $t\in[0,T]$,

$$\|u(t)\|_{\dot{H}^{-l}}^2+\|\omega(t)\|_{\dot{H}^{-l}}^2+\|b(t)\|_{\dot{H}^{-l}}^2\leq 2C_0.\tag{4.25}$$

If we can drive that for all $t\in[0,T]$,

$$\|u(t)\|_{\dot{H}^{-l}}^2+\|\omega(t)\|_{\dot{H}^{-l}}^2+\|b(t)\|_{\dot{H}^{-l}}^2\leq \frac{3C_0}{2},\tag{4.26}$$

then an application of the bootstrapping argument would imply that the solution (u,ω,b) of system (1.3) satisfies (4.26) for all $t\in[0,T]$, which implies (4.21). With (4.24) and (4.25) at our disposal, we shall show that (4.26) holds. Integrating (4.9) in $[0,t]$ with $0<t\le T$, together with (4.23), (4.24) and (4.8), one infers that

$$
\begin{aligned}
&\|u(t)\|_{\dot H^{-l}}^2+\|b(t)\|_{\dot H^{-l}}^2+\|\omega(t)\|_{\dot H^{-l}}^2\\
&\quad\le\|u_0\|_{\dot H^{-l}}^2+\|b_0\|_{\dot H^{-l}}^2+\|\omega_0\|_{\dot H^{-l}}^2\\
&\qquad+C\sup_{0\le\tau\le t}(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}})\int_0^t(\|\nabla u(\tau)\|_{L^2}+\|\nabla b(\tau)\|_{L^2}+\|\nabla\omega(\tau)\|_{L^2})\\
&\qquad\times\left(\|\nabla u(\tau)\|_{L^2}^{1-l}+\|\nabla b(\tau)\|_{L^2}^{1-l}\right)\left(\|u(\tau)\|_{L^2}^l+\|b(\tau)\|_{L^2}^l\right)\mathrm{d}\tau\\
&\quad\le C_0+C\epsilon^l\sup_{0\le\tau\le t}(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}})\int_0^t(1+\tau)^{-\left(\frac{l(1-l)}{2}+1\right)}\mathrm{d}\tau\\
&\quad\le C_0+C\epsilon^l\sup_{0\le\tau\le t}(\|u\|_{\dot H^{-l}}+\|b\|_{\dot H^{-l}}+\|\omega\|_{\dot H^{-l}}).
\end{aligned}
$$

By choosing ϵ in (4.8) sufficiently small, then the above inequality together with the Young inequality yield (4.26) for all $t\in[0,T]$, which close the proof. Then, we complete the proof of (4.21).

(2) Now, we start to prove (1.9). Applying D^m to the second equation of (1.3), dotting $D^m\omega$ and integrating in $\mathbb{R}^2$, we get

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|D^m\omega\|_{L^2}^2+\kappa\|D^{m+1}\omega\|_{L^2}^2+4\chi\|D^m\omega\|_{L^2}^2\\
&\quad\le 2\chi\int D^m\nabla\times u\cdot D^m\omega\mathrm{d}x-\int D^m(u\cdot\nabla\omega)\cdot D^m\omega\mathrm{d}x\\
&\quad\le C\left(\|D^{m+1}u\|_{L^2}+\|D^{m+1}u\|_{L^2}\|\omega\|_{L^\infty}+\|u\|_{L^\infty}\|D^{m+1}\omega\|_{L^2}\right)\|D^m\omega\|_{L^2}\\
&\quad\le C(\|D^{m+1}u\|_{L^2}+\|D^{m+1}\omega\|_{L^2}),
\end{aligned}
$$

where we used the commutator estimate (2.2), Hölder's inequality and (1.7). Multiplying this equation by $e^{4\chi t}$, integrating the resulting inequality in $[0,t]$, together with (1.7) and (1.8), we have

$$
\begin{aligned}
\|D^m\omega\|_{L^2}&\le e^{-4\chi t}\|D^m\omega_0\|_{L^2}+C\int_0^t e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2}+\|D^{m+1}\omega(\tau)\|_{L^2})\mathrm{d}\tau\\
&\le Ce^{-4\chi t}+C\int_0^{\frac t2} e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2}+\|D^{m+1}\omega(\tau)\|_{L^2})\mathrm{d}\tau\\
&\quad+C\int_{\frac t2}^t e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2}+\|D^{m+1}\omega(\tau)\|_{L^2})\mathrm{d}\tau\\
&\le Ce^{-4\chi t}+C\int_{\frac t2}^t e^{-4\chi(t-\tau)}(1+\tau)^{-\frac{m+l}{2}-\frac l2}\mathrm{d}\tau\\
&\quad+Ce^{-2\chi t}\left(\int_0^{\frac t2}\mathrm{d}\tau\right)^{\frac12}\left(\int_0^{\frac t2}e^{-4\chi(t-\tau)}(\|D^{m+1}u(\tau)\|_{L^2}^2+\|D^{m+1}\omega(\tau)\|_{L^2}^2)\mathrm{d}\tau\right)^{\frac12}
\end{aligned}
$$

$$\leq Ce^{-4\chi t} + Ce^{-2\chi t} t^{\frac{1}{2}} + C(1+t)^{-\frac{m+1}{2}-\frac{l}{2}}$$
$$\leq C(1+\tau)^{-\frac{m+1}{2}-\frac{l}{2}}.$$

Thus, we complete (1.9). Therefore, the proof of Theorem 1.2 is completed.

□

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(Received: September 22, 2021; revised: March 22, 2022; accepted: March 23, 2022)