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Existence and stability of traveling curved fronts for nonlocal dispersal equations with bistable nonlinearity

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Abstract. This paper is concerned with existence and stability of V-shaped traveling fronts for a class of nonlocal dispersal equations with unbalanced bistable nonlinearity. The main tool is sub- and supersolution technique combined with a comparison principle.

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1. Introduction

We consider the following equation

$$u_t = J * u - u + f(u), \quad (\boldsymbol{x}, t) \in \mathbb{R}^2 \times (0, +\infty) \tag{1.1}$$

with the nonlocal dispersal operator $(J * u - u)(\boldsymbol{x}, t) = \int_{\mathbb{R}^2} J(\boldsymbol{x} - \boldsymbol{y}) \left(u(\boldsymbol{y}, t) - u(\boldsymbol{x}, t)\right) \mathrm{d}\boldsymbol{y}$. The kernel $J \in C^1(\mathbb{R}^2)$ satisfies

(**J1**) $J \geq 0$ is radial symmetric and has unit integral;

(**J2**) $\int_0^\infty J(r) e^{\lambda r} \mathrm{d}r < \infty$ for some $\lambda > 0$.

The nonlinearity $f \in C^2(\mathbb{R})$ has only three zeros 0, a and 1, and satisfies

(**F1**) $f'(0) < 0, f'(a) > 0, f'(1) < 0$ and $\int_0^1 f(u) \mathrm{d}u \neq 0$;

(**F2**) $\sup_{s \in [0,1]} f'(s) < 1$.

Obviously, if $J(\boldsymbol{x}) = \frac{1}{4\pi\lambda} e^{-\frac{|x|^2}{4\lambda}}$ for any given $\lambda > 0$ or $J(\boldsymbol{x})$ is compactly supported with symmetric property, then it satisfies (J1)–(J2). Condition (F2) guarantees that the solution of the corresponding Cauchy problem of (1.1) has the same regularity with its initial function [13].

Traveling waves of (1.1) are widely used to model nonlocal diffusion phenomena in fields such as physics, chemistry, ecology and epidemiology. In one-dimensional space, traveling wave solutions have the form $u(x, t) = U(\xi)$, $\xi = x + ct$, where U is the wave profile and c is the wave speed. It is referred to [1,6,8,9,19,29] for the mathematical study on traveling waves of (1.1).

Let

$$J_1(x) = \int_{\mathbb{R}} J(x, x_2) \mathrm{d}x_2. \tag{1.2}$$

Then under the condition (J1), J_1 is nonnegative and even, with unit integral on $x \in \mathbb{R}$. It is also not difficult to prove that $J_1'(x) \in L^1(\mathbb{R})$ with the aid of condition (J2). Then under the condition (F1), the following equation

$$-(J_1 * U - U)(\xi) + cU'(\xi) - f(U(\xi)) = 0, \ U'(\xi) > 0, \ \xi \in \mathbb{R} \tag{1.3}$$

admits a unique (up to translation) solution U connecting 0 and 1. Moreover, U is of class C^{k+1} if f is of class C^k for some $k \geq 1$, and its speed c is given by

$$c = \frac{\int_0^1 f(u)\mathrm{d}u}{\int_{-\infty}^{\infty} (U'(\xi))^2 \mathrm{d}\xi},$$

which can be positive or negative [1]. We assume that $c > 0$ in the present paper, and the case $c < 0$ can be dealt with by a same way. Furthermore, the wave profile U and its derivative U' have exponential behaviors near $\pm\infty$:

$$\begin{aligned} B_1 e^{-\delta_1 \xi} \leq 1 - U(\xi) \leq A_1 e^{-\lambda_1 \xi}, & \quad \text{when } \xi \to +\infty, \\ B_2 e^{\delta_2 \xi} \leq U(\xi) \leq A_2 e^{\lambda_2 \xi}, & \quad \text{when } \xi \to -\infty, \\ U'(\xi) \leq A_3 e^{-\lambda_3 |\xi|}, & \quad \text{when } |\xi| \to +\infty, \end{aligned} \tag{1.4}$$

where $A_i, B_i, \lambda_i, \delta_i (i \in \{1, 2, 3\})$ are positive constants, see [9,10]. Following the technique as in [9, Section 1.5], we can also get that U'' has exponential behavior near $\pm\infty$.

Studies on the existence and stability of nonplanar traveling waves for the classical reaction diffusion equations or systems are already quite a lot, see [2–5,11,12,16,17,20–23,26] and references therein for scalar equations, and see [15,18,24,25,27,28] and references therein for reaction–diffusion systems. While there are still few studies on nonplanar traveling waves of nonlocal dispersal equations. Chan and Wei proved the existence of pyramidal traveling wave solutions for the fractional bistable equation [7], and Li et al. proved the existence of pyramidal traveling wave solutions for the bistable nonlocal equation [14]. However, to the best of our knowledge, there is still no result about the stability of nonplanar traveling wave solutions for the nonlocal dispersal equations. In the current paper, we aim to prove the existence and stability of V-shaped traveling fronts for (1.1).

Since the curvature accelerates propagation of waves, it is naturally to assume that the speed s of nonplanar traveling waves satisfies $s > c$. Without loss of generality, we also assume that the solutions travel towards the x_2-direction; thus, they have the form $u(\boldsymbol{x}, t) = \widehat{u}(x_1, x_2 + st, t)$, and $\widehat{u}$ satisfies

$$\begin{cases} \widehat{u}_t - (J * \widehat{u} - \widehat{u}) + s\widehat{u}_{x_2} - f(\widehat{u}) = 0, & (\boldsymbol{x}, t) \in \mathbb{R}^2 \times (0, +\infty), \\ \widehat{u}(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{R}^2. \end{cases} \tag{1.5}$$

Throughout this paper, we denote the solution of (1.5) by $\widehat{u}(\boldsymbol{x}, t; u_0)$. In this paper, we first find a nontrivial steady-state function $V(\boldsymbol{x})$ of (1.5), i.e., $V(\boldsymbol{x})$ satisfies

$$\mathscr{L}[V] := -(J * V - V) + sV_{x_2} - f(V) = 0 \ \text{ in } \ \mathbb{R}^2, \tag{1.6}$$

and then prove its stability.

Let $m_* = \sqrt{s^2 - c^2}/c$ and

$$v^-(\boldsymbol{x}) = U\left(\frac{c}{s}(x_2 + m_*|x_1|)\right) = \max_{1 \leq j \leq n} \left\{ U\left(\frac{c}{s}(x_2 + m_* x_1)\right), U\left(\frac{c}{s}(x_2 - m_* x_1)\right) \right\}. \tag{1.7}$$

Then, $v^-(\boldsymbol{x})$ is a subsolution to (1.6). Actually, denote $v_j^1(\boldsymbol{x}) := U(A_j \cdot \boldsymbol{x})$ with $A_j = \frac{c}{s}(m_*, 1)$. If we let $\boldsymbol{\xi} = \boldsymbol{A}\boldsymbol{y}$ with $\boldsymbol{A}$ a 2×2 orthonormal matrix whose first row is A_j, then we have

$$\int_{\mathbb{R}^2} J(\boldsymbol{x}-\boldsymbol{y})U(A_j \cdot \boldsymbol{y})\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2} J(\boldsymbol{y})U(A_j \cdot (\boldsymbol{x}-\boldsymbol{y}))\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2} J(\boldsymbol{A}^{-1}\boldsymbol{\xi})U(A_j \cdot \boldsymbol{x} - \xi_1)\mathrm{d}\boldsymbol{\xi}$$
$$= \int_{\mathbb{R}^2} J(\boldsymbol{\xi})U(A_j \cdot \boldsymbol{x} - \xi_1)d\boldsymbol{\xi} = \int_{\mathbb{R}} J_1(\xi_1)U(A_j \cdot \boldsymbol{x} - \xi_1)\mathrm{d}\xi_1.$$

See (1.2) for the definition of J_1. It follows that

$$\begin{aligned}\mathscr{L}[v_j^1(\boldsymbol{x})] &= -[(J * U(A_j \cdot))(\boldsymbol{x}) - U(A_j \cdot \boldsymbol{x})] + sU_{x_2}(A_j \cdot \boldsymbol{x}) - f(U(A_j \cdot \boldsymbol{x}))\\ &= -(J_1 * U - U)(A_j \cdot \boldsymbol{x}) - cU'(A_j \cdot \boldsymbol{x}) - f(U(A_j \cdot \boldsymbol{x})) = 0,\end{aligned}$$

which means that $v_j^1(\boldsymbol{x})$ is a planar wave to (1.6). Similarly, denote $v_j^2(\boldsymbol{x}) := U(A_j \cdot \boldsymbol{x})$ with $A_j = \frac{c}{s}(-m_*, 1)$, and then, we can get $\mathscr{L}[v_j^2(\boldsymbol{x})] = 0$. Thus, $v^-(\boldsymbol{x})$ is a subsolution.

Now we give the main results.

Theorem 1.1. *(Existence) Assume (J1)–(J2) and (F1)–(F2) hold. For each $s > c$, (1.6) admits a solution $V_*(\boldsymbol{x})$ with $\partial_{x_2}V_*(\boldsymbol{x}) > 0$ in $\mathbb{R}^2$ and*

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\geq R} \left|V_*(\boldsymbol{x}) - v^-(\boldsymbol{x})\right| = 0, \tag{1.8}$$

$$v^-(\boldsymbol{x}) < V_*(\boldsymbol{x}) < 1, \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{1.9}$$

Then $u(\boldsymbol{x}, t) = V_*(x_1, x_2 + st)$ is a traveling front of (1.1), whose global average speed tends to c along with time, i.e.,

$$\lim_{|t_1-t_2|\to\infty} \frac{\mathrm{dist}(L_{t_1}, L_{t_2})}{|t_1 - t_2|} = c, \tag{1.10}$$

where L_t represents the level set of $u(\boldsymbol{x}, t)$ at time t.

Theorem 1.2. *(Stability) Let $v_0 \in C(\mathbb{R}^2, [0, 1])$ satisfy $v_0 - v^- \in L^1(\mathbb{R}^2)$ and*

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\geq R} |v_0(\boldsymbol{x}) - v^-(\boldsymbol{x})| = 0,$$

$$v^-(\boldsymbol{x}) \leq v_0(\boldsymbol{x}), \ \forall \ \boldsymbol{x} \in \mathbb{R}^2.$$

Then, we have

$$\lim_{t\to+\infty} ||v(\cdot, t; v_0) - V_*(\cdot)||_{L^\infty(\mathbb{R}^2)} = 0.$$

Remark 1.3. (1.8) implies that $V_*(\boldsymbol{x})$ has V-shaped level sets and behaves like planar traveling waves far away in the space. The speed of $V_*(\boldsymbol{x})$ is a semi-continuum, that is, $s \in (c, +\infty)$, which is quite different to classical bistable case, while (1.10) tells that the average speed of $V_*(\boldsymbol{x})$ is unique and always equals the planar wave's speed c.

In the next section, we establish the existence result. The proof looks simpler than that of [14] but is a little different. In Sect. 3, we obtain the stability result.

2. Existence of V-shaped traveling fronts

For any $s > c$, the equation

$$s = \frac{\varphi_{xx}}{1 + \varphi_x^2} + c\sqrt{1 + \varphi_x^2}$$

admits a unique solution $\varphi(x)$ whose asymptotic line is $y = m_*|x|$ satisfying $m_*|x| \le \varphi(x)$, see [16]. Furthermore, there exist positive constants K_0, K_1, K_2, K_3 such that for all $x \in \mathbb{R}$,

$$\max|\varphi'(x)| \ \le \ K_0, \tag{2.1}$$

$$\max\{|\varphi''(x)|, |\varphi'''(x)|\} \ \le \ K_1 \mathrm{sech}(\gamma x), \tag{2.2}$$

$$K_2 \mathrm{sech}(\gamma x) \ \le \ \frac{s}{\sqrt{1+\varphi'(x)^2}} - c \le K_3 \mathrm{sech}(\gamma x), \tag{2.3}$$

$$\mu_- \ \le \mu(x) = \frac{s(\varphi(x) - m_*|x|)}{s - c\sqrt{1+\varphi'(x)^2}} \le \mu_+, \tag{2.4}$$

where $\mu_\pm > 0$ are constants and $\gamma = sm_* = \frac{s\sqrt{s^2-c^2}}{c} > 0$.

2.1. Construction of a supersolution

By the assumption (F1), there exists a positive constant $\delta_0 \in (0, \frac{1}{4})$ such that

$$-f'(u) \ge \kappa_1 \text{ if } |u| \le 2\delta_0 \text{ or } |1-u| \le 2\delta_0, \tag{2.5}$$

where $\kappa_1 := \frac{1}{2}\min\{-f'(0), -f'(1)\} > 0$.

Lemma 2.1. *There exist a positive constant ε_0^+ and a positive function $\alpha_0^+(\varepsilon)$ such that for any $\varepsilon \in (0, \varepsilon_0^+)$ and $\alpha \in (0, \alpha_0^+(\varepsilon))$, the function*

$$v^+(\boldsymbol{x}; \varepsilon, \alpha) = U\left(\frac{x_2 + \frac{1}{\alpha}\varphi(\alpha x_1)}{\sqrt{1+(\varphi'(\alpha x_1))^2}}\right) + \varepsilon \mathrm{sech}(\gamma\alpha x_1), \tag{2.6}$$

is a supersolution to (1.6) and

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R} |v^+(\boldsymbol{x}; \varepsilon, \alpha) - v^-(\boldsymbol{x})| \le 2\varepsilon, \tag{2.7}$$

$$v^-(\boldsymbol{x}) < v^+(\boldsymbol{x}; \varepsilon, \alpha) \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{2.8}$$

Proof. Assume $\alpha \in (0,1)$ and write $v^+(\boldsymbol{x})$ instead of $v^+(\boldsymbol{x}; \varepsilon, \alpha)$ throughout the proof. Denote

$$\zeta(\boldsymbol{x}) = \frac{x_2 + \frac{1}{\alpha}\varphi(\alpha x_1)}{\sqrt{1+(\varphi'(\alpha x_1))^2}} \text{ and } \sigma(x_1) = \mathrm{sech}(\gamma\alpha x_1),$$

then $v^+(\boldsymbol{x})$ can be rewritten as $v^+(\boldsymbol{x}) = U(\zeta(\boldsymbol{x})) + \varepsilon\sigma(x_1)$. By the equation (1.3) and the definition of $\mathscr{L}$, we have

$$\begin{aligned}\mathscr{L}[v^+(\boldsymbol{x})] &= -(J * v^+ - v^+)(\boldsymbol{x}) + s\partial_{x_2} v^+(\boldsymbol{x}) - f(v^+(\boldsymbol{x})) \\ &= -(J * U(\zeta(\cdot)))(\boldsymbol{x}) - \varepsilon(J*\sigma)(\boldsymbol{x}) + (J_1 * U)(\zeta(\boldsymbol{x})) + \varepsilon\sigma(x_1) \\ &\quad + \frac{s}{\sqrt{1+(\varphi'(\alpha x_1))^2}} U'(\zeta(\boldsymbol{x})) - cU'(\zeta(\boldsymbol{x})) + f(U(\zeta(\boldsymbol{x}))) - f(v^+(\boldsymbol{x})).\end{aligned}$$

Denote

$$\begin{aligned}&I := -(J * U(\zeta(\cdot)))(\boldsymbol{x}) + (J_1 * U)(\zeta(\boldsymbol{x})),\ II := -\varepsilon(J*\sigma)(\boldsymbol{x}) + \varepsilon\sigma(x_1), \\ &III := \left(\frac{s}{\sqrt{1+(\varphi'(\alpha x_1))^2}} - c\right) U'(\zeta(\boldsymbol{x})),\ IV := f(U(\zeta(\boldsymbol{x}))) - f(v^+(\boldsymbol{x})).\end{aligned}$$

Now we estimate the four terms.

(1) Estimate of term I. Since J is radially symmetric, it is easy to see that

$$I = -\int_{\mathbb{R}^2} J(\boldsymbol{y})U(\zeta(\boldsymbol{x}-\boldsymbol{y}))\mathrm{d}\boldsymbol{y} + \int_{\mathbb{R}} J_1(\mu)U(\zeta(\boldsymbol{x})-\mu)\mathrm{d}\mu.$$

Let

$$A = \begin{pmatrix} \frac{\varphi'(\alpha x_1)}{\sqrt{1+(\varphi'(\alpha x_1))^2}} & \frac{1}{\sqrt{1+(\varphi'(\alpha x_1))^2}} \\ \frac{1}{\sqrt{1+(\varphi'(\alpha x_1))^2}} & -\frac{\varphi'(\alpha x_1)}{\sqrt{1+(\varphi'(\alpha x_1))^2}} \end{pmatrix},$$

and $\boldsymbol{\xi} = A\boldsymbol{y}$, where $\boldsymbol{\xi} = (\mu,\nu)^T$. Such an orthogonal transformation gives that $\mu = \frac{\varphi'(\alpha x_1)}{\sqrt{1+(\varphi'(\alpha x_1))^2}} y_1 + \frac{1}{\sqrt{1+(\varphi'(\alpha x_1))^2}} y_2$. Then

$$\begin{aligned}\int_{\mathbb{R}} J_1(\mu)U(\zeta(\boldsymbol{x})-\mu)\mathrm{d}\mu &= \int_{\mathbb{R}^2} J(\boldsymbol{\xi})U(\zeta(\boldsymbol{x})-\mu)\mathrm{d}\mu\mathrm{d}\nu \\ &= \int_{\mathbb{R}^2} J(A\boldsymbol{y}^T)U\left(\zeta(\boldsymbol{x}) - \frac{\varphi'(\alpha x_1)y_1+y_2}{\sqrt{1+(\varphi'(\alpha x_1))^2}}\right)\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{\mathbb{R}^2} J(\boldsymbol{y})U\left(\frac{x_2-y_2+\frac{1}{\alpha}\varphi(\alpha x_1)-\varphi'(\alpha x_1)y_1}{\sqrt{1+(\varphi'(\alpha x_1))^2}}\right)\mathrm{d}y_1\mathrm{d}y_2.\end{aligned}$$

Let

$$\mu^*(t) = \frac{x_2-y_2+(1-t)\left(\frac{1}{\alpha}\varphi(\alpha x_1)-\varphi'(\alpha x_1)y_1\right)+t\cdot\frac{1}{\alpha}\varphi(\alpha(x_1-y_1))}{\sqrt{1+(\varphi'(\alpha(x_1-ty_1)))^2}},$$

and define $F(t) = -U(\mu^*(t)), t\in(0,1)$, and then, (2.9) can be written as

$$I = \int_{\mathbb{R}^2} J(\boldsymbol{y})\left(-U(\mu^*(1))+U(\mu^*(0))\right)\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^2} J(\boldsymbol{y})\left(F(1)-F(0)\right)\mathrm{d}\boldsymbol{y}. \tag{2.9}$$

Denote $y_1(t) = \alpha(x_1-ty_1)$. Then, $\mu_t^*(t) = A(t)+B(t)\mu^*(t)$, where $B(t) = \frac{\alpha\varphi'(y_1(t))\varphi''(y_1(t))y_1}{1+(\varphi'(y_1(t)))^2}$ and

$$A(t) = \frac{-\left(\frac{1}{\alpha}\varphi(\alpha x_1)-\varphi'(\alpha x_1)y_1\right)+\frac{1}{\alpha}\varphi(\alpha(x_1-y_1))}{\sqrt{1+(\varphi'(y_1(t)))^2}} = \frac{\frac{\alpha}{2}\varphi''(y_1(\tau))y_1^2}{\sqrt{1+(\varphi'(y_1(t)))^2}},\ \tau\in(0,1).$$

Furthermore, $A_t(t) = A(t)B(t)$, and thus, $\mu_{tt}^*(t) = 2A(t)B(t)+(B_t(t)+B^2(t))\mu^*(t)$, where

$$B_t(t) = \frac{-\alpha^2y_1^2\left[(\varphi''(y_1(t)))^2\left(1-(\varphi'(y_1(t)))^2\right)+\varphi'(y_1(t))\varphi'''(y_1(t))\left(1+(\varphi'(y_1(t)))^2\right)\right]}{\left[1+(\varphi'(y_1(t)))^2\right]^2}.$$

Following from (2.1)–(2.2), we have

$$\begin{aligned}|A(t)| &\le \alpha K_1y_1^2\mathrm{sech}(\gamma\alpha(x_1-\tau y_1)),\\ |B(t)| &\le \alpha K_0K_1|y_1|\mathrm{sech}(\gamma\alpha(x_1-ty_1)),\\ |B_t(t)| &\le \alpha^2K_1(1+K_0)(1+K_0^2)y_1^2\mathrm{sech}(\gamma\alpha(x_1-ty_1)).\end{aligned}$$

Since $F(1)-F(0) = F'(0)+\int_0^1(1-t)F''(t)\mathrm{d}t$ and

$$F'(t) = -U'(\mu^*(t))\mu_t^*(t),\ F''(t) = -U''(\mu^*(t))(\mu_t^*(t))^2-U'(\mu^*(t))\mu_{tt}^*(t),$$

we have

$$\begin{aligned}
&|-U(\mu^*(1))+U(\mu^*(0))|\\
&\le |-U'(\mu^*(0))\left(A(0)+B(0)\mu^*(0)\right)|\\
&\quad+\Big|\int_0^1(1-t)\Big[-U''(\mu^*(t))\left(A^2(t)+2A(t)B(t)\mu^*(t)+B^2(t)(\mu^*(t))^2\right)\\
&\quad-U'(\mu^*(t))\left(2A(t)B(t)+(B_t(t)+B^2(t))\mu^*(t)\right)\Big]\mathrm{d}t\Big|\\
&\le C_U\sigma(x_1)\left(\alpha K_1y_1^2e^{\gamma|y_1|}+\alpha K_0K_1|y_1|\right)+C_U\sigma(x_1)\int_0^1(1-t)\Big(\alpha^2K_1^2y_1^4\\
&\quad+4\alpha^2K_0K_1^2|y_1|^3+2\alpha^2K_0^2K_1^2y_1^2+\alpha^2K_1(1+K_0)(1+K_0^2)\Big)e^{\gamma|y_1|}\mathrm{d}t\\
&\le C_UK^*\alpha\sigma(x_1)\left(|y_1|+4y_1^2+4|y_1|^3+y_1^4\right)e^{\gamma|y_1|},
\end{aligned}\tag{2.10}$$

where $K^*=\max\{K_1,K_1^2,K_0K_1,K_0K_1^2,K_0^2K_1^2,K_1(1+K_0)(1+K_0^2)\}$ and

$$C_U=\max\{||U'(\mu)||_\infty,||U''(\mu)||_\infty,||U'(\mu)\mu||_\infty,||U''(\mu)\mu||_\infty,||U''(\mu)\mu^2||_\infty\}.$$

Here, $||\cdot||_\infty$ is the supremum norm about $\mu\in\mathbb{R}$. And in the above estimate we use the inequality $\mathrm{sech}(x_1+y_1)\le\mathrm{sech}x_1\cdot e^{|y_1|}$. Combining (2.9)–(2.10) and the above estimates, we have

$$I|\le C_UK^*\alpha\sigma(x_1)\int_{\mathbb{R}^2}J(\boldsymbol{y})\left(|y_1|+4y_1^2+4|y_1|^3+y_1^4\right)e^{\gamma|y_1|}\mathrm{d}\boldsymbol{y}.$$

Under the condition (J2), the integral $\int_{\mathbb{R}^2}J(\boldsymbol{y})\left(|y_1|+4y_1^2+4|y_1|^3+y_1^4\right)e^{\gamma|y_1|}\mathrm{d}\boldsymbol{y}$ is bounded for some $\lambda>0$, and thus, there exists a constant $C_1>0$ such that

$$|I|\le C_1\alpha\sigma(x_1).$$

(2) Estimate of term II.

$$\begin{aligned}
II&=-\varepsilon\int_{\mathbb{R}^2}J(\boldsymbol{y})\left(\sigma(x_1-y_1)-\sigma(x_1)\right)\mathrm{d}\boldsymbol{y}\\
&=\varepsilon\gamma\alpha\int_{\mathbb{R}^2}J(\boldsymbol{y})\sigma'(x_1-\theta y_1)y_1\mathrm{d}\boldsymbol{y},
\end{aligned}$$

where $\theta\in(0,1)$. Since $\mathrm{sech}'x=\mathrm{sech}x\cdot\frac{e^{-x}-e^{x}}{e^{x}+e^{-x}}$, we have

$$|II|\le\varepsilon\gamma\alpha\sigma(x_1)\int_{\mathbb{R}^2}J(\boldsymbol{y})|y_1|e^{\gamma|y_1|}\mathrm{d}\boldsymbol{y}.$$

Again the assumption (J2) guarantees that $\int_{\mathbb{R}^2}J(\boldsymbol{y})|y_1|e^{\gamma|y_1|}\mathrm{d}\boldsymbol{y}$ is bounded for some $\lambda>0$, and thus, there exists a constant $C_2>0$ such that

$$|II|\le C_2\varepsilon\gamma\alpha\sigma(x_1).$$

(3) Estimate of terms III and IV. By (2.3), we have

$$0<K_2\sigma(x_1)U'(\zeta(\boldsymbol{x}))\le\left(\frac{s}{\sqrt{1+(\varphi'(\alpha x_1))^2}}-c\right)U'(\zeta(\boldsymbol{x})).$$

About the fourth term, we have

$$IV = -f'\left(U(\zeta(\boldsymbol{x})) + \tilde{\theta}\varepsilon\sigma(x_1)\right) \cdot \varepsilon\sigma(x_1),$$

where $\tilde{\theta} \in (0, 1)$.

In order to prove $\mathscr{L}[v^+(\boldsymbol{x})] \geq 0$, we consider two cases.

Case 1. $U(\zeta(\boldsymbol{x})) \geq 1 - \delta_0$ or $U(\zeta(\boldsymbol{x})) \leq \delta_0$.

Then, $-f'\left(U(\zeta(\boldsymbol{x})) + \tilde{\theta}\varepsilon\sigma(x_1)\right) \geq \kappa_1$ by (2.5), provided that $0 < \varepsilon < \delta_0$. And thus

$$\mathscr{L}[v^+(\boldsymbol{x})] \geq -\left(C_1 + C_2\gamma\varepsilon\right)\alpha\sigma(x_1) + \kappa_1\varepsilon\sigma(x_1) \geq 0$$

for any α satisfying $\alpha \leq \frac{\kappa_1\varepsilon}{C_1 + C_2\gamma}$.

Case 2. $\delta_0 \leq U(\zeta(\boldsymbol{x})) \leq 1 - \delta_0$.

Denote $U_* = \min_{U(x)\in[\delta_0, 1-\delta_0]} U'(x)$ and $f^* = \max_{x\in[-1,2]} |f'(x)|$. We have

$$\mathscr{L}[v^+(\boldsymbol{x})] \geq -\left(C_1 + C_2\gamma\varepsilon\right)\alpha\sigma(x_1) + K_2\sigma(x_1)U_* - \varepsilon f^*\sigma(x_1) \geq 0$$

provided that $\left(C_1 + C_2\gamma\right)\alpha + \varepsilon f^* \leq K_2 U_*$. Let

$$\varepsilon_0^+ := \min\left\{1, \delta_0, \frac{K_2 U_*}{2f^*}\right\}, \quad \alpha_0^+(\varepsilon) := \min\left\{1, \frac{\kappa_1\varepsilon}{C_1 + C_2\gamma}, \frac{K_2 U_*}{2(C_1 + C_2\gamma)}\right\},$$

then $v^+(\boldsymbol{x})$ is a supersolution if $0 < \varepsilon < \varepsilon_0^+$ and $0 < \alpha < \alpha_0^+(\varepsilon)$. A similar argument to that of Taniguchi [20, Lemma 7] yields (2.7)-(2.8). The proof is complete. □

2.2. Proof of the existence result

First, we establish the comparison principle. Define

$$BUC(\mathbb{R}^2) := \{u : \mathbb{R}^2 \to \mathbb{R}, u \text{ is bounded and uniformly continuous in } \mathbb{R}^2\}.$$

Theorem 2.2. *Assume that (J1) and (F1)–(F2) hold. Let $u_0(\boldsymbol{x})$ and $\partial_{x_2}u_0(\boldsymbol{x})$ belong to $BUC(\mathbb{R}^2)$, and then, the following Cauchy problem*

$$\begin{cases} \widehat{u}_t = J * \widehat{u} - \widehat{u} + b\widehat{u}_{x_2} + f(\widehat{u}), & (\boldsymbol{x}, t) \in \mathbb{R}^2 \times [0, \infty), \\ \widehat{u}(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

has a unique solution $\widehat{u}(\boldsymbol{x}, t; u_0) \in C(\mathbb{R}^2 \times [0, \infty), [0, 2])$, which is also differentiable with respect to x_2. Here, $b \in \mathbb{R}$ is a nonzero constant. Moreover, if $u_0(\boldsymbol{x})$ is globally Lipschitz continuous, then $\widehat{u}(\boldsymbol{x}, t; u_0)$ is also a globally Lipschitz solution which is uniform in time.

Lemma 2.3. *(Maximum principle) Assume that (J1) hold and that $\widehat{u} \in C(\mathbb{R}^2 \times [0, \infty))$ is bounded and differentiable with respect to x_2. If $\widehat{u}$ satisfies*

$$\begin{cases} \widehat{u}_t \geq J * \widehat{u} - \widehat{u} + b\widehat{u}_{x_2} + K(\boldsymbol{x}, t)\widehat{u}, & (\boldsymbol{x}, t) \in \mathbb{R}^2 \times [0, \infty), \\ \widehat{u}(\boldsymbol{x}, 0) \geq 0, \ \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

where $K(\boldsymbol{x}, t) : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ is continuous and uniformly bounded, and b is a nonzero constant, then $\widehat{u}(\boldsymbol{x}, t) \geq 0$ for $(\boldsymbol{x}, t) \in \mathbb{R}^2 \times [0, \infty)$. Furthermore, if $\widehat{u}(\boldsymbol{x}, 0) \not\equiv 0$ for $\boldsymbol{x} \in \mathbb{R}^2$, then $\widehat{u}(\boldsymbol{x}, t) > 0$ for $(\boldsymbol{x}, t) \in \mathbb{R}^2 \times (0, \infty)$.

Lemma 2.4. *(Comparison principle) Assume that (J1) holds and* $u_1, u_2 \in C(\mathbb{R}^2 \times [0,\infty))$ *are both bounded and differentiable with respect to* x_2*. Denote* $\mathscr{L}_t[u] = u_t - (J*u-u) + bu_{x_2} - f(u)$*, where* f *is continuously differentiable with respect to* u*,* $f'(u)$ *is uniformly bounded and* b *is a nonzero constant. If* u_1, u_2 *satisfy*

$$\begin{cases} \mathscr{L}_t[u_1] \geq \mathscr{L}_t[u_2], & (\boldsymbol{x}, t) \in \mathbb{R}^2 \times [0,\infty), \\ u_1(\boldsymbol{x}, 0) \geq u_2(\boldsymbol{x}, 0), & \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

then $u_1 \geq u_2$ *on* $(\boldsymbol{x}, t) \in \mathbb{R}^2 \times [0,\infty)$*. Furthermore, if* $u_1(\boldsymbol{x}, 0) \not\equiv u_2(\boldsymbol{x}, 0)$ *for* $\boldsymbol{x} \in \mathbb{R}^2$*, then* $u_1 > u_2$ *for* $(\boldsymbol{x}, t) \in \mathbb{R}^2 \times (0,\infty)$*.*

The proof of the above three results can be referred to [14]. Now we prove Theorem 1.1.

Proof of Theorem 1.1. After making a slight modification of the proof of [14, Theorem 1.1], we can prove the existence of V_* and (1.8)–(1.9). Now we focus on the proof of (1.10).

By the comparison principle, we have

$$v^-(x_1, x_2 + st) < u(\boldsymbol{x}, t) := V_*(x_1, x_2 + st) < v^+(x_1, x_2 + st), \ \forall \boldsymbol{x} \in \mathbb{R}^2, t \in \mathbb{R}, \tag{2.11}$$

where v^- and v^+ are defined by (1.7) and (2.6), respectively. Fix a constant $\delta \in (0,1)$ and denote the level sets of $v^{\pm}(x_1, x_2 + st) = \delta$ at time t by $L_t^{\pm}$. Due to (2.11), the level sets $L_t^{\pm}$ and L_t do not intersect each other at any time t. We know

$$L_t^- = \left\{ (x_1, g^-(x_1, t)) \in \mathbb{R}^2 : g^-(x_1, t) = \frac{s}{c} U^{-1}(\delta) - m_*|x_1| - st \right\}$$

and

$$L_t^+ = \left\{ (x_1, g^+(x_1, t)) \in \mathbb{R}^2 : g^+(x_1, t) = \sqrt{1 + \varphi'(\alpha x_1)^2}\, U^{-1}(\delta - \varepsilon\sigma(x_1)) - \frac{\varphi(\alpha x_1)}{\alpha} - st \right\}.$$

For convenience, we also denote

$$L_t = \left\{ (x_1, g(x_1, t)) \in \mathbb{R}^2 : V_*(x_1, g(x_1, t) + st) = \delta \right\}.$$

Since

$$\delta = v^-(x_1, g^-(x_1, t) + st) = v^+(x_1, g^+(x_1, t) + st) > v^-(x_1, g^+(x_1, t) + st),$$

we know $g^-(x_1, t) > g^+(x_1, t)$ for all $x_1 \in \mathbb{R}$ and $t \in \mathbb{R}$ by the monotonicity of $v^-(\boldsymbol{x})$ on x_2. Similarly, there hold $g^-(x_1, t) > g(x_1, t)$ and $g(x_1, t) > g^+(x_1, t)$ for all $x_1 \in \mathbb{R}$ and $t \in \mathbb{R}$. In summary, there is

$$g^-(x_1, t) > g(x_1, t) > g^+(x_1, t) \text{ for } x_1 \in \mathbb{R}, t \in \mathbb{R}. \tag{2.12}$$

Moreover, we know that

$$\begin{aligned} \operatorname{dist}(L_t^-, L_t^+) &= \inf_{\boldsymbol{x} \in L_t^-, \boldsymbol{y} \in L_t^+} |\boldsymbol{x} - \boldsymbol{y}| \leq \inf_{x_1 \in \mathbb{R}} \left| g^+(x_1, t) - g^-(x_1, t) \right| \\ &\leq \inf_{x_1 \in \mathbb{R}} \left(\left| \frac{s}{c} U^{-1}(\delta) - \sqrt{1 + |\varphi'(\alpha x_1)|^2}\, U^{-1}(\delta - \varepsilon\sigma(x_1)) \right| + \frac{\varphi(\alpha x_1)}{\alpha} - m_*|x_1| \right) = 0 \end{aligned}$$

holds for each fixed $\alpha > 0$. Consequently, $\operatorname{dist}(L_t^{\pm}, L_t) = 0$. Define

$$M^* := \sup_{x_1 \in \mathbb{R}, t \in \mathbb{R}} \left| g^+(x_1, t) - g^-(x_1, t) \right| > 0.$$

Obviously, $M^* < +\infty$ for each fixed α. Now we take two moments t_1, $t_2 \in \mathbb{R}$ and assume $s(t_2 - t_1) > M_*$ without loss of generality. Under the assumption $s(t_2 - t_1) > M_*$, there holds $g^-(x_1, t_2) < g^+(x_1, t_1)$, and thus, $L_{t_1}^+$ does not intersect $L_{t_2}^-$.

The inequality (2.12) means that the level set L_t of $u(\boldsymbol{x}, t)$ is between those of $L_t^{\pm}$ for all $x_1 \in \mathbb{R}$ at any time $t \in \mathbb{R}$. Thus by the definition of M^* and the choice of t_i $(i = 1, 2)$, it is straightforward that

$$\operatorname{dist}(L_{t_1}, L_{t_2}) \leq \operatorname{dist}(L_{t_1}^-, L_{t_2}^+). \tag{2.13}$$

To obtain $\mathrm{dist}(L^-_{t_1}, L^+_{t_2})$, it is sufficient to consider all the perpendicular segments from $L^+_{t_2}$ to $L^-_{t_1}$ except those intersecting $L^+_{t_2}$ once more. Now take an arbitrary point $\boldsymbol{x}^+_2 \in L^+_{t_2}$ and find the corresponding point $\boldsymbol{x}^-_1 \in L^-_{t_1}$ such that the segment generated by $\boldsymbol{x}^+_2$ and $\boldsymbol{x}^-_1$ is perpendicular to $L^-_{t_1}$. We denote this perpendicular segment by $l_{\boldsymbol{x}^-_1\boldsymbol{x}^+_2}$. Obviously, $l_{\boldsymbol{x}^-_1\boldsymbol{x}^+_2}$ is also perpendicular to $L^-_{t_2}$, and we denote their intersection point by $\boldsymbol{x}^-_2$. It follows immediately that $\left|\boldsymbol{x}^-_2 - \boldsymbol{x}^-_1\right| = c|t_2 - t_1|$. Then for any $\boldsymbol{x}^+_2 \in L^+_{t_2}$ and $\boldsymbol{x}^-_i$ $(i = 1, 2)$ chosen by this way, we have

$$\begin{aligned}\mathrm{dist}(L^-_{t_1}, L^+_{t_2}) &= \inf_{\boldsymbol{x}^+_2 \in L^+_{t_2}} \left|\boldsymbol{x}^+_2 - \boldsymbol{x}^-_1\right| \\ &= \inf_{\boldsymbol{x}^+_2 \in L^+_{t_2}} \left(\left|\boldsymbol{x}^+_2 - \boldsymbol{x}^-_2\right| + \left|\boldsymbol{x}^-_2 - \boldsymbol{x}^-_1\right|\right) \\ &= \inf_{\boldsymbol{x}^+_2 \in L^+_{t_2}} \left(\left|\boldsymbol{x}^+_2 - \boldsymbol{x}^-_2\right| + c|t_2 - t_1|\right) \\ &= \mathrm{dist}(L^-_{t_2}, L^+_{t_2}) + c|t_2 - t_1| = c|t_2 - t_1|.\end{aligned}$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$ $(N \geq 1)$. This fact and (2.13) yield that

$$\lim_{|t_1 - t_2| \to \infty} \frac{\mathrm{dist}(L_{t_1}, L_{t_2})}{|t_1 - t_2|} \leq c.$$

Now we prove that the average speed is not less than c. For any given point $\boldsymbol{x}_i \in L_{t_i}$ $(i = 1, 2)$, draw a line passing through $\boldsymbol{x}_i$ and parallel to the x_2 axis. Necessarily, this line intersects $L^-_{t_i}$ at a point, which is still denoted by $\boldsymbol{x}^-_i$. Since $\boldsymbol{x}_i \in L_{t_i}$ is arbitrary, $\boldsymbol{x}^-_i$ is also arbitrary, and vice versa. Obviously,

$$\begin{aligned}|\boldsymbol{x}_2 - \boldsymbol{x}_1| &\geq |\boldsymbol{x}_2 - \boldsymbol{x}^-_1| - |\boldsymbol{x}^-_1 - \boldsymbol{x}_1| \\ &\geq |\boldsymbol{x}^-_1 - \boldsymbol{x}^-_2| - |\boldsymbol{x}^-_2 - \boldsymbol{x}_2| - |\boldsymbol{x}^-_1 - \boldsymbol{x}_1| \\ &\geq |\boldsymbol{x}^-_1 - \boldsymbol{x}^-_2| - 2M_* \\ &\geq c|t_2 - t_1| - 2M_*.\end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\begin{aligned}\lim_{|t_1 - t_2| \to \infty} \frac{\mathrm{dist}(L_{t_1}, L_{t_2})}{|t_1 - t_2|} &= \lim_{|t_1 - t_2| \to \infty} \frac{\inf_{\boldsymbol{x}_1 \in L_{t_1}, \boldsymbol{x}_2 \in L_{t_2}} |\boldsymbol{x}_2 - \boldsymbol{x}_1|}{|t_1 - t_2|} \\ &\geq \lim_{|t_1 - t_2| \to \infty} \frac{\inf_{\boldsymbol{x}^-_1 \in L^-_{t_2}, \boldsymbol{x}^-_2 \in L^-_{t_2}} \left|\boldsymbol{x}^-_2 - \boldsymbol{x}^-_1\right| - 2M^*}{|t_1 - t_2|} \\ &\geq \lim_{|t_1 - t_2| \to \infty} \frac{c|t_2 - t_1| - 2M^*}{|t_1 - t_2|} = c.\end{aligned}$$

This implies the average speed is larger than or equal to c. This completes the proof. □

3. Stability of V-shaped traveling fronts in $\mathbb{R}^2$

This section establishes the stability result.

Lemma 3.1. *For any $M > 0$, there exists a constant $C > 0$ such that*

$$\partial_{x_2} V_*(\boldsymbol{x}) \geq C \text{ and } \partial_{x_2} v^+(\boldsymbol{x}) \geq C \text{ if } |x_2 + m_*|x_1|| \leq M,$$

where V_* and v^+ are given by Theorem 1.1 and (2.6), respectively. Moreover, we have

$$\lim_{R \to \infty} \sup_{|x_2 + m_*|x_1|| \geq R} \partial_{x_2} v^+(\boldsymbol{x}) = \lim_{R \to \infty} \sup_{|x_2 + m_*|x_1|| \geq R} \partial_{x_2} V_*(\boldsymbol{x}) = 0.$$

Proof. The assertions about v^+ are very straightforward. Now we prove the assertions about V_*. Since $\partial_{x_2} V_* > 0$ in $\mathbb{R}^2$, it suffices to prove that for any sequence $\{(x_n, z_n)\}_{n\geq 1} \subseteq \mathbb{R}^2$ satisfying $|(x_n, z_n)| \to \infty$ and $|z_n + m_*|x_n|| \leq M$, there is

$$\lim_{n\to\infty} \partial_{x_2} V_*(x_n, z_n) > 0.$$

Now we prove this result by contradiction. Assume $\lim_{n\to\infty} \partial_{x_2} V_*(x_n^*, z_n^*) = 0$ for a certain sequence $\{(x_n^*, z_n^*)\}_{n\geq 1}$. Since $v^- < V < v^+$ in $\mathbb{R}^2$ and notice that v^- is a subsolution, we have

$$\begin{aligned} s\partial_{x_2} V_*|_{(x_n^*,z_n^*)} &= J * V_* - V_* + f(V_*)|_{(x_n^*,z_n^*)} \\ &> J * v^- - v^+ + f(v^-) + f(V_*) - f(v^-)|_{(x_n^*,z_n^*)} \\ &= J * v^- - v^- + f(v^-) + f'(V_\tau)(V_* - v^-) + v^- - v^+|_{(x_n^*,z_n^*)} \\ &\geq s\partial_{x_2} v^- + f'(V_\tau)(V_* - v^-) + v^- - v^+|_{(x_n^*,z_n^*)}, \end{aligned} \tag{3.1}$$

where V_τ is between V_* and v^-. Under the condition $|z_n^* + m_*|x_n^*|| \leq M$ and $|(x_n^*, z_n^*)| \to \infty$, there must be $|x_n^*| \to \infty$, and thus,

$$\lim_{n\to\infty} \operatorname{sech}(\gamma\alpha x_n^*) = 0 \text{ and } \lim_{n\to\infty} \left[\frac{z_n^* + \varphi(\alpha x_n^*)/\alpha}{\sqrt{1 + (\varphi'(\alpha x_n^*))^2}} - \frac{c}{s}(z_n^* + m_*|x_n^*|) \right] = 0.$$

Then, it follows that

$$\lim_{n\to\infty} (v^-(x_n^*, z_n^*) - v^+(x_n^*, z_n^*)) = 0,$$

which further implies that

$$\lim_{n\to\infty} (V_*(x_n^*, z_n^*) - v^-(x_n^*, z_n^*)) = 0.$$

Let $n \to \infty$ in (3.1), and then, we have

$$0 \geq \liminf_{n\to\infty} cU'\left(\frac{c}{s}(z_n^* + m_*|x_n^*|\right) > 0,$$

which is a contradiction. Similarly,

$$\begin{aligned} s\partial_{x_2} V_* &= J * V_* - V_* + f(V_*) \\ &< J * v^+ - v^- + f(v^+) + f(V_*) - f(v^+) \\ &= J * v^+ - v^+ + f(v^+) + f'(V_\tau)(V_* - v^+) + v^+ - v^- \\ &\leq s\partial_{x_2} v^+ + f'(V_\tau)(V_* - v^+) + v^- - v^+, \end{aligned}$$

and thus $\partial_{x_2} V_* \to 0$ as $\left|\frac{c}{s}(x_2 + m_*|x_1|)\right| \to \infty$. This completes the proof. □

Lemma 3.2. *There exist positive constants $\rho > 0, \kappa > 0$ and $\delta \in (0, \delta_0)$ such that*

$$u^+(\boldsymbol{x}, t) = V_*(x_1, x_2 + \xi + \rho\delta(1 - e^{-\kappa t})) + \delta e^{-\kappa t}$$

is a supersolution, where $\xi \in \mathbb{R}$ is a constant and δ_0 is defined in (2.5).

Proof. Let $\tilde{\mathscr{L}}[u] = u_t - (J * u - u) + s\partial_{x_2} u - f(u)$. Then, we have

$$\begin{aligned} \tilde{\mathscr{L}}[u^+] &= \rho\delta\kappa e^{-\kappa t}\partial_{x_2} V_* - \delta\kappa e^{-\kappa t} - (J * V_* - V_*) + s\partial_{x_2} V_* - f(u^+) \\ &= \rho\delta\kappa e^{-\kappa t}\partial_{x_2} V_* - \delta\kappa e^{-\kappa t} + f(V_*) - f(u^+) \\ &= \delta e^{-\kappa t}\left(\rho\kappa\partial_{x_2} V_* - \kappa - f'(V_* + \tau\delta e^{-\kappa t})\right), \end{aligned}$$

where $\tau \in (0, 1)$. To prove the lemma, we argue as follows.

Case 1. $|x_2 + m_*|x_1|| > R_0$ for some $R_0 > 0$ large enough.

Without loss of generality, assume that $R_0 > 0$ is large enough such that $V_* < \delta_0$ or $V_* > 1 - \delta_0$. Then, we have

$$0 < V_* + \tau\delta e^{-\kappa t} < 2\delta_0 \text{ or } 1 - \delta_0 < V_* + \tau\delta e^{-\kappa t} < 1 + \delta_0.$$

It then follows by (2.5) that

$$-f'(V_* + \tau\delta e^{-\kappa t}) > \kappa_1.$$

Thus,

$$\tilde{\mathscr{L}}[u^+] > \delta e^{-\kappa t}\left(-\kappa - f'(V_* + \tau\delta e^{-\kappa t})\right) > \delta e^{-\kappa t}\left(-\kappa + \kappa_1\right) > 0$$

provided that $\kappa < \kappa_1$.

Case 2. $|x_2 + m_*|x_1|| \le R_0$.

In this case, it follows from Lemma 3.1 that $\partial_{x_2} V_* \ge C := C(R_0)$. Thus,

$$\begin{aligned}\tilde{\mathscr{L}}[u^+] &> \left(\rho C - \kappa - \max_{u\in[0,1+\delta_0]} |f'(u)|\right)\\ &> \delta\kappa e^{-\kappa t}\left(\rho C - 1 - \frac{\max_{u\in[0,1+\delta_0]} |f'(u)|}{\kappa}\right)\\ &> 0\end{aligned}$$

provided that $\rho > 1 + \frac{\max_{u\in[0,1+\delta_0]} |f'(u)|}{\kappa}$. Taking $0 < \kappa < \kappa_1$, $\rho > 1 + \frac{\max_{u\in[0,1+\delta_0]} |f'(u)|}{\kappa}$ and combining the above two cases, we know $\tilde{\mathscr{L}}[u^+] > 0$ in $\mathbb{R}^2$. This completes the proof. □

The proof of the following lemma is very similar to that of Lemma 3.2, and we omit it here.

Lemma 3.3. *There exist positive constants $\rho > 0, \kappa > 0$ and $\delta \in (0, \delta_0)$ such that*

$$w^+(\boldsymbol{x}, t) = v^+(x_1, x_2 + \xi + \rho\delta(1 - e^{-\kappa t})) + \delta e^{-\kappa t}$$

is a supersolution, where $\xi \in \mathbb{R}$ is a constant δ_0 is defined in (2.5).

Lemma 3.4. *If $u(\boldsymbol{x}, t; u_0)$ is the solution of the Cauchy problem*

$$\begin{cases} u_t = J * u - u - s u_{x_2} + f(u), \ \boldsymbol{x} \in \mathbb{R}^2, \ t > 0\\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

where the initial function $u_0 \in C(\mathbb{R}^2)$ satisfies $u_0 - v^- \in L^1(\mathbb{R}^2)$ and

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\ge R} |u_0(\boldsymbol{x}) - v^-(\boldsymbol{x})| = 0,$$

then for any fixed $T > 0$, we have

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\ge R} |u(\boldsymbol{x}, T; u_0) - v^-(\boldsymbol{x})| = 0.$$

Proof. Let $w(\boldsymbol{x}) = U\left(\frac{c}{s}(x_2 + \varphi(x_1))\right)$. First we show that $w - v^- \in L^1(\mathbb{R}^2)$. In fact,

$$\begin{aligned}\int_{\mathbb{R}^2} |w(\boldsymbol{x}) - v^-(\boldsymbol{x})|\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}^2} \left|U\left(\frac{c}{s}(x_2 + \varphi(x_1))\right) - U\left(\frac{c}{s}(x_2 + m_*|x_1|)\right)\right| \mathrm{d}\boldsymbol{x}\\ &= \int_{\mathbb{R}^2} |U'(\tau\varsigma(\boldsymbol{x}) + (1-\tau)\eta(\boldsymbol{x}))| \left(\varsigma(\boldsymbol{x}) - \eta(\boldsymbol{x})\right)\mathrm{d}\boldsymbol{x}\\ &\le \int_{\mathbb{R}^2} A_3 e^{-\lambda_3|\tau\varsigma(\boldsymbol{x}) + (1-\tau)\eta(\boldsymbol{x})|}(\varsigma(\boldsymbol{x}) - \eta(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\end{aligned}$$

$$\leq \frac{c}{s}\left(1-\frac{c}{s}\right)\mu_+ A_3 \int\limits_{\mathbb{R}^2} e^{-\lambda_3|\tau\varsigma(\boldsymbol{x})+(1-\tau)\eta(\boldsymbol{x})|} d\boldsymbol{x} < +\infty,$$

where $\tau \in (0,1)$, $\varsigma(\boldsymbol{x}) = \frac{c}{s}(x_2 + \varphi(x_1))$ and $\eta(\boldsymbol{x}) = \frac{c}{s}(x_2 + m_*|x_1|)$. See (1.4) for A_3, λ_3 and see (2.4) for μ_+. It follows that $w - u_0 \in L^1(\mathbb{R}^2)$. It is also not difficult to prove that

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\geq R} |w(\boldsymbol{x}) - v^-(\boldsymbol{x})| = 0.$$

Thus, it suffices to prove that

$$\lim_{R\to+\infty} \sup_{|\boldsymbol{x}|\geq R} |u(\boldsymbol{x}, T; u_0) - w(\boldsymbol{x})| = 0$$

for any given $T > 0$. Let $\Phi(\boldsymbol{x}, t) := u(\boldsymbol{x}, t; u_0) - w(\boldsymbol{x})$. Then, it satisfies

$$\begin{cases} \Phi_t = J * \Phi - \Phi - s\partial_{x_2}\Phi + f'(\Phi_\tau)\Phi, \ \boldsymbol{x} \in \mathbb{R}^2, \ t > 0, \\ \Phi(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) - w(\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

where $\Phi_\tau = \tau u + (1-\tau)w$ with $\tau \in (0,1)$. Let $\hat{\Phi}$ be the solution of the following Cauchy problem

$$\begin{cases} \hat{\Phi}_t = J * \hat{\Phi} - \hat{\Phi} - s\partial_{x_2}\hat{\Phi} + M\hat{\Phi}, \ \boldsymbol{x} \in \mathbb{R}^2, \ t > 0, \\ \hat{\Phi}(\boldsymbol{x}, 0) = |u_0(\boldsymbol{x}) - w(\boldsymbol{x})|, \ \boldsymbol{x} \in \mathbb{R}^2, \end{cases}$$

where $M := \max_{u\in[0,1]} |f'(u)|$. By the maximum principle, $\hat{\Phi} \geq 0$ in $\mathbb{R}^2$. By the comparison principle, it is easy to verify that

$$|\Phi(\boldsymbol{x})| \leq \hat{\Phi}(\boldsymbol{x}), \ \forall \boldsymbol{x} \in \mathbb{R}^2. \tag{3.2}$$

In the following, we estimate $\hat{\Phi}(\boldsymbol{x}, t)$. Let $\Psi(\boldsymbol{x}, t) = \hat{\Phi}(x_1, x_2 + st, t)$, then Ψ satisfies

$$\begin{cases} \Psi_t = J * \Psi - \Psi + M\Psi, \ \boldsymbol{x} \in \mathbb{R}^2, \ t > 0, \\ \Psi(\boldsymbol{x}, 0) = |u_0(\boldsymbol{x}) - w(\boldsymbol{x})|, \ \boldsymbol{x} \in \mathbb{R}^2. \end{cases} \tag{3.3}$$

The solution of (3.3) can be expressed as

$$\begin{aligned} \Psi(\boldsymbol{x}, t) &= e^{Mt} \int\limits_{\mathbb{R}^2} S(x_1 - x, x_2 - z, t)\Psi(x, z, 0)\mathrm{d}x\mathrm{d}z \\ &= e^{Mt} \int\limits_{\mathbb{R}^2} S(x, z, t)\Psi(x_1 - x, x_2 - z, 0)\mathrm{d}x\mathrm{d}z, \end{aligned}$$

where $S(\boldsymbol{x}, t) = e^{-t}\delta_0(\boldsymbol{x}) + K_t(\boldsymbol{x})$ is the fundamental solution of (3.3) with initial data δ_0, the Dirac measure at zero and $K_t(\boldsymbol{x}) = \int\limits_{\mathbb{R}^2} e^{-t}(e^{\hat{J}(\boldsymbol{y})t} - 1)e^{i(\boldsymbol{x})\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y}$ with $\hat{J}$ the Fourier transform of J. It is not difficult to verify that $||S(\boldsymbol{x}, t)||_{L^1(\mathbb{R}^2)} \leq 3$. Then for any given $T > 0$,

$$\Psi(\boldsymbol{x}, T) \leq e^{MT} \left(\int\limits_{|(x,z)|\leq R} + \int\limits_{|(x,z)|>R} \right) |S(x, z, T)|\Psi(x_1 - x, x_2 - z, 0)\mathrm{d}x\mathrm{d}z.$$

For any $\epsilon > 0$ small enough, there exist $R_1 > 0$ and $R_2 > 0$ big enough such that

$$\begin{aligned} &\int\limits_{|(x,z)|>R_1} |S(x, z, T)|\Psi(x_1 - x, x_2 - z, 0)\mathrm{d}x\mathrm{d}z \\ &< \sup_{\boldsymbol{x}\in\mathbb{R}^2} \Psi(\boldsymbol{x}, 0) \int\limits_{|(x,z)|>R_1} |S(x, z, T)|\mathrm{d}x\mathrm{d}z < \frac{\epsilon}{2e^{MT}} \end{aligned}$$

and

$$\int\limits_{|(x,z)|\le R_1} |S(x,z,T)|\Psi(x_1-x,x_2-z,0)\mathrm{d}x\mathrm{d}z < \frac{\epsilon}{2e^{MT}},\ \forall x_1^2+x_2^2\ge R_2^2.$$

This implies that $\Psi(\boldsymbol{x},T)<\epsilon$ for $x_1^2+x_2^2\ge R_2^2$ and thus $\lim_{R\to+\infty}\sup_{x_1^2+x_2^2\ge R^2}\Psi(\boldsymbol{x},T)=0$. Recall $\hat{\Phi}(\boldsymbol{x},t)=\Psi(x_1,x_2-st,t)$ and we have $\lim_{R\to+\infty}\sup_{x_1^2+x_2^2\ge R^2}\hat{\Phi}(\boldsymbol{x},T)=0$. Then, the proof completes following (3.2). □

Lemma 3.5. *The solution $u(\boldsymbol{x},t;u_0)$ of the Cauchy problem*

$$\begin{cases} u_t = J*u-u-s\partial_{x_2}u+f(u),\ \boldsymbol{x}\in\mathbb{R}^2,\ t>0,\\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\ \boldsymbol{x}\in\mathbb{R}^2, \end{cases} \tag{3.4}$$

depends continuously on the initial function $u_0(\boldsymbol{x})$. That is, if $u_1(\boldsymbol{x},t;u_{0,1})$ and $u_2(\boldsymbol{x},t;u_{0,2})$ are two solutions of (3.4) with initial values $u_{0,1}$ and $u_{0,2}$, respectively, then we have

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2}|u_1(\boldsymbol{x},t;u_{0,1})-u_1(\boldsymbol{x},t;u_{0,2})|\le A(t)\sup_{\boldsymbol{x}\in\mathbb{R}^2}|u_{0,1}(\boldsymbol{x})-u_{0,2}(\boldsymbol{x})|$$

for some $A(t)$ depending only on t.

Proof. Let $v(\boldsymbol{x},t)=u(x_1,x_2+st)$. Then $v(\boldsymbol{x},t)$ satisfies

$$\begin{cases} v_t(\boldsymbol{x},t)=J*v(\boldsymbol{x},t)-v(\boldsymbol{x},t)+f(v(\boldsymbol{x},t)),\ \boldsymbol{x}\in\mathbb{R}^2,\ t>0,\\ v(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\ \boldsymbol{x}\in\mathbb{R}^2. \end{cases}$$

The above problem is equivalent to the following integral equation

$$v(\boldsymbol{x},t)=e^{-\mu t}u_0(\boldsymbol{x})+\int\limits_0^t e^{-\mu(t-s)}(J*v-v)(\boldsymbol{x},s)+\mu v(\boldsymbol{x},s)+f(v(\boldsymbol{x},s))\mathrm{d}s,$$

where $\mu>0$ is a constant. Let $w(\boldsymbol{x},t):=v_2(\boldsymbol{x},t)-v_1(\boldsymbol{x},t)$, then it satisfies

$$\begin{aligned} w(\boldsymbol{x},t) &= e^{-\mu t}\left(u_{0,1}(\boldsymbol{x})-u_{0,2}(\boldsymbol{x})\right)\\ &\quad+\int\limits_0^t e^{-\mu(t-s)}(J*w-w)(\boldsymbol{x},s)+\mu w(\boldsymbol{x},s)+f'(w_\tau)w(\boldsymbol{x},s)\mathrm{d}s\\ &\le \sup_{\boldsymbol{x}\in\mathbb{R}^2}|u_{0,1}(\boldsymbol{x})-u_{0,2}(\boldsymbol{x})|\\ &\quad+\int\limits_0^t e^{-\mu(t-s)}\left(\mu+||f'||_{L^\infty(\mathbb{R}^2)}\right)||w(\cdot,s)||_{L^\infty(\mathbb{R}^2)}\mathrm{d}s \end{aligned}$$

It then follows that

$$||w(\cdot,t)||_{L^\infty(\mathbb{R}^2)}\le||u_{0,1}-u_{0,2}||_{L^\infty(\mathbb{R}^2)}+\int\limits_0^t e^{-\mu(t-s)}\left(\mu+||f'||_{L^\infty(\mathbb{R}^2)}\right)||w(\cdot,s)||_{L^\infty(\mathbb{R}^2)}\mathrm{d}s$$

By the Gronwall's inequality, we have

$$||w(\cdot,t)||_{L^\infty(\mathbb{R}^2)}\le||u_{0,1}-u_{0,2}||_{L^\infty(\mathbb{R}^2)}\left(1+C_1te^{C_1t}\right),$$

where $C_1=\mu+||f'||_{L^\infty(\mathbb{R}^2)}$. Let $A(t)=1+C_1te^{C_1t}$. Then, the proof is complete. □

Now, define

$$V^*(\boldsymbol{x}) = \lim_{t\to\infty} v(\boldsymbol{x},t;v^+).$$

Then, V^* satisfies

$$-(J * V^* - V^*) + s\partial_{x_2}V^* - f(V^*) = 0 \text{ in } \mathbb{R}^2.$$

And by the comparison principle, there holds

$$v^-(\boldsymbol{x}) < V_*(\boldsymbol{x}) \le V^*(\boldsymbol{x}) < \min\{1, v^+(\boldsymbol{x})\}.$$

Lemma 3.6. *$V_*(\boldsymbol{x}) \equiv V^*(\boldsymbol{x})$ in $\mathbb{R}^2$.*

Proof. Assume on the contrary that $V_*(\boldsymbol{x}) \not\equiv V^*(\boldsymbol{x})$ in $\mathbb{R}^2$. Then, they must be $V_*(\boldsymbol{x}) < V^*(\boldsymbol{x})$. By the aid of (2.7), we can find a $\delta > 0$ small enough and a proper $\xi > 0$ such that

$$V^*(\boldsymbol{x}) \le V_*(x_1, x_2 + \xi) + \delta.$$

Then, the comparison principle yields

$$V^*(\boldsymbol{x}) \le u^+(\boldsymbol{x},t),\ \forall \boldsymbol{x} \in \mathbb{R}^2,\ t > 0.$$

Letting $t \to \infty$ in the above inequality, we have

$$V^*(\boldsymbol{x}) \le V_*(x_1, x_2 + \xi + \rho\delta).$$

Define

$$\Lambda := \inf\{\lambda > 0 : V^*(\boldsymbol{x}) \le V_*(x_1, x_2 + \lambda), \forall \boldsymbol{x} \in \mathbb{R}^2\}.$$

Obviously $\Lambda \ge 0$ and $V^*(\boldsymbol{x}) \le V_*(x_1, x_2 + \Lambda)$. If $\Lambda = 0$, then the proof is done. Thus, we assume $\Lambda > 0$ to derive a contradiction. It follows from $v^-(\boldsymbol{x}) < V_*(\boldsymbol{x}) < V^*(\boldsymbol{x}) < v^+(\boldsymbol{x})$ that

$$\lim_{x_1\to\infty} V^*(x_1, -m_*x_1) = U(0) \text{ and } \lim_{x_1\to\infty} V_*(x_1, -m_*x_1 + \Lambda) = U\left(\frac{c}{s}\Lambda\right) > U(0),$$

which implies that there must be

$$V^*(\boldsymbol{x}) < V_*(x_1, x_2 + \Lambda),\ \forall \boldsymbol{x} \in \mathbb{R}^2.$$

By Lemma 3.1, there exists a constant $R^* > 0$ large enough such that

$$\sup_{|x_2+m_*|x_1||\ge R^*-\Lambda} \partial_{x_2} V_*(\boldsymbol{x}) < \frac{1}{4\rho}. \tag{3.5}$$

Define

$$\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 | |x_2 + m_*|x_1|| \le R^*\}.$$

Since

$$\begin{aligned}
&\lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R, \boldsymbol{x}\in\Omega} (V_*(x_1, x_2 + \Lambda) - V^*(\boldsymbol{x}))\\
&\ge \lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R, \boldsymbol{x}\in\Omega} \left(v^-(x_1, x_2 + \Lambda) - v^+(\boldsymbol{x})\right)\\
&= \lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R, \boldsymbol{x}\in\Omega} \left(v^-(x_1, x_2 + \Lambda) - v^-(\boldsymbol{x}) + v^-(\boldsymbol{x}) - v^+(\boldsymbol{x})\right)\\
&= \lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R, \boldsymbol{x}\in\Omega} \left(v^-(x_1, x_2 + \Lambda) - v^-(\boldsymbol{x})\right)\\
&= \lim_{R\to\infty} \sup_{|\boldsymbol{x}|\ge R, \boldsymbol{x}\in\Omega} U'\left(\frac{c}{s}(x_2 + m_*|x_1| + \tau\Lambda)\right)\frac{c}{s}\Lambda > 0,
\end{aligned}$$

there exists a $\sigma \in (0, \delta_0)$ small enough such that

$$V^*(\boldsymbol{x}) \le V_*(x_1, x_2 + \Lambda - 2\rho\sigma),\ \forall \boldsymbol{x} \in \Omega.$$

For $\boldsymbol{x} \in \mathbb{R}\backslash\Omega$, by (3.5) we have

$$V_*(x_1, x_2 + \Lambda - 2\rho\sigma) - V_*(x_1, x_2 + \Lambda) = \partial_{x_2} V_*(x_1, x_2 + \Lambda - 2\tau\rho\sigma) \cdot (-2\rho\sigma)$$
$$\geq \frac{1}{4\rho} \cdot (-2\rho\sigma) = -\frac{1}{2}\sigma.$$

To sum up, there is

$$V^*(\boldsymbol{x}) \leq V_*(x_1, x_2 + \Lambda - 2\rho\sigma) + \frac{1}{2}\sigma, \ \forall \boldsymbol{x} \in \mathbb{R}^2.$$

Then by the comparison principle, we have

$$V^*(\boldsymbol{x}) \leq u^+(\boldsymbol{x}, t)$$

with $\xi = \Lambda - 2\rho\sigma$. Let $t \to \infty$ in the above inequality, then we have

$$V^*(\boldsymbol{x}) \leq V_*(x_1, x_2 + \Lambda - \rho\sigma),$$

which contradicts to the definition of Λ. Thus, $\Lambda = 0$ and the proof is complete. □

Now we show that the curved fronts V_* are asymptotically stable under the condition that the initial perturbation is positive.

Proof of Theorem 1.2. Denote $v(\boldsymbol{x}, t; v_0)$ by $v(\boldsymbol{x}, t)$ for simplicity. On the one hand, since $v^-(\boldsymbol{x}) \leq v_0(\boldsymbol{x})$, by the comparison principle, we have

$$v^-(\boldsymbol{x}) \leq v(\boldsymbol{x}, t; v^-) \leq v(\boldsymbol{x}, t) < 1, \ \forall \boldsymbol{x} \in \mathbb{R}^2, t > 0. \tag{3.6}$$

On the other hand, a similar argument as [13, proposition 2.5] can deduce that

$$|v(\boldsymbol{x}, t; v^-) - v(\boldsymbol{y}, t; v^-)| \leq L|\boldsymbol{x} - \boldsymbol{y}|, \ |v_t(\boldsymbol{x}, t; v^-) - v_t(\boldsymbol{y}, t; v^-)| \leq L|\boldsymbol{x} - \boldsymbol{y}|$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and $|v_t| \leq C$, where L and C are constants independent of $\boldsymbol{x}, \boldsymbol{y}$ and t. Thus, we have

$$\lim_{t \to +\infty} ||v(\cdot, t; v^-) - V_*(\cdot)||_{L^\infty(\mathbb{R}^2)} = 0. \tag{3.7}$$

Similarly, we have

$$\lim_{t \to +\infty} ||v(\cdot, t; v^+) - V^*(\cdot)||_{L^\infty(\mathbb{R}^2)} = 0. \tag{3.8}$$

(3.6) and (3.7) imply that it suffices to prove that for any $\epsilon > 0$, there exists $T^* > 0$ such that

$$v(\boldsymbol{x}, t) \leq V_*(\boldsymbol{x}) + \epsilon \quad \text{for } t \geq T^*.$$

Step 1. Let $A^* = \sup_{\boldsymbol{x} \in \mathbb{R}^2} \partial_{x_2} V_*(\boldsymbol{x})$, and take $\rho > 0, \kappa > 0$ as in Lemma 3.2. Then,

$$V_*(x_1, x_2 + \rho\delta) - V_*(\boldsymbol{x}) = \rho\delta \int_0^1 \partial_{x_2} V_*(x_1, x_2 + \rho\delta\tau) d\tau < \rho\delta A^* < \frac{\epsilon}{3}$$

provided that $\delta < \frac{\epsilon}{3\rho A^*}$. In other words,

$$V(x_1, x_2 + \rho\delta) < V_*(\boldsymbol{x}) + \frac{\epsilon}{3}, \ \forall \boldsymbol{x} \in \mathbb{R}^2. \tag{3.9}$$

Step 2. Fix $\delta > 0$ in step 1. By (3.6), for any $T_\delta > 0$, we have

$$v(\boldsymbol{x}, t; v^-) \leq v(\boldsymbol{x}, t) < 1, \ \forall \boldsymbol{x} \in \mathbb{R}^2, t \geq T_\delta > 0.$$

Following from Lemma 3.4, there exists a $R_\delta > 0$ such that

$$v(\boldsymbol{x}, T_\delta) < v^-(\boldsymbol{x}) + \frac{\delta}{2} \text{ for } |\boldsymbol{x}| \geq R_\delta. \tag{3.10}$$

Let $\alpha > 0$ be small enough such that

$$U\left(\frac{x_2 + \frac{1}{\alpha}\varphi(\alpha x_1)}{\sqrt{1+\varphi'(\alpha x_1)^2}}\right) > U\left(-R_\delta + \frac{c\varphi(0)}{s\alpha}\right) \geq 1 - \frac{\delta}{2} \text{ for } |\boldsymbol{x}| \leq R_\delta.$$

In other words, if α is chosen to satisfy

$$0 < \alpha < \min\left\{\alpha^+(\beta,\delta), \frac{c\varphi(0)}{s\left[U^{-1}\left(1-\frac{\delta}{2}\right)+R_\delta\right]}\right\},$$

then

$$v^+(\boldsymbol{x}) \geq 1 - \frac{\delta}{2} \text{ for } |\boldsymbol{x}| \leq R_\delta. \tag{3.11}$$

Combining the inequalities (3.10) and (3.11), we have

$$v(\boldsymbol{x}, T_\delta) < v^+(\boldsymbol{x}) + \delta,\ \forall \boldsymbol{x} \in \mathbb{R}^2.$$

Then, Lemma 3.3 and the comparison principle yield that

$$v(\boldsymbol{x}, t + T_\delta; v^-) \leq v(\boldsymbol{x}, t + T_\delta) < w^+(\boldsymbol{x}, t)$$

for $t \geq 0$. Denote $w_+^t := w^+(\boldsymbol{x}, t)$ and applying the comparison principle again, we have

$$v(\boldsymbol{x}, t' + t + T_\delta; v^-) \leq v(\boldsymbol{x}, t' + t + T_\delta) < v(\boldsymbol{x}, t'; w_+^t). \tag{3.12}$$

Since $v(\boldsymbol{x}, t; v^+)$ converges monotonically to $V^*(\boldsymbol{x})$ as $t \to +\infty$, it follows from (3.8) that there exists a $t_1 > 0$ such that

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2} |v(\boldsymbol{x}, t_1; v^{+,\delta}) - V^*(x_1, x_2 + \rho\delta)| \leq \frac{\epsilon}{3},$$

where $v^{+,\delta} = v^+(x_1, x_2 + \rho\delta)$. On the other hand, Lemma 3.5 yields that

$$\sup_{\boldsymbol{x}\in\mathbb{R}^2} |v(\boldsymbol{x}, t_1; u_0) - v(\boldsymbol{x}, t_1; v^{+,\delta})| \leq A(t_1) \sup_{\boldsymbol{x}\in\mathbb{R}^2} |u_0(\boldsymbol{x}) - v^{+,\delta}|.$$

From the definition of w^+, we know that there exists a $T_1 > 0$ such that

$$A(t_1) \sup_{\boldsymbol{x}\in\mathbb{R}^2} |w_+^t - v^{+,\delta}| \leq \frac{\epsilon}{3} \text{ for } t \geq T_1.$$

Combining the above facts, we obtain that

$$\begin{aligned} &|v(\boldsymbol{x}, t_1; w_+^t) - V^*(x, x_2 + \rho\delta)| \\ &\leq |v(\boldsymbol{x}, t_1; w_+^t) - v(\boldsymbol{x}, t_1; v^{+,\delta})| + |v(\boldsymbol{x}, t_1; v^{+,\delta}) - V^*(x_1, x_2 + \rho\delta)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} \end{aligned}$$

for $t \geq T_1$ and $\boldsymbol{x} \in \mathbb{R}^2$. It follows from (3.12) that

$$v(\boldsymbol{x}, t_1 + t + T_\delta) < v(\boldsymbol{x}, t_1; w_+^t) \leq V^*(x_1, x_2 + \rho\delta) + \frac{2\epsilon}{3}$$

for $t \geq T_1$ and $\boldsymbol{x} \in \mathbb{R}^2$. Take $T^* = t_1 + T_1 + T_\delta$. By Lemma 3.6, we have

$$v(\boldsymbol{x}, t) \leq V^*(x_1, x_2 + \rho\delta) + \frac{2\epsilon}{3} = V_*(x_1, x_2 + \rho\delta) + \frac{2\epsilon}{3}$$

for $t \geq T^*$ and $\boldsymbol{x} \in \mathbb{R}^2$. Combining the above inequality and (3.9), we obtain

$$v(\boldsymbol{x}, t) \leq V_*(\boldsymbol{x}) + \epsilon \quad \text{for } \boldsymbol{x} \in \mathbb{R}^2,\ t \geq T^*.$$

This completes the proof. □

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