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# Analysis of a reaction–diffusion system about West Nile virus with free boundaries in the almost periodic heterogeneous environment

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**Abstract.** We put forward a reaction–diffusion cooperative system for the West Nile virus in a spatial heterogeneous and time almost periodic environment with free boundaries. The existence, uniqueness and regularity estimates of the global solution for this epidemic model are given. Focused on the effects of spatial heterogeneity and time almost periodicity, we introduce the principal Lyapunov exponent $\lambda(t)$ dependent on $t$ and get the initial infected critical size $L^*$ which plays a vital part in analyzing the threshold dynamics and the long-time asymptotic behaviors of the solution. We build the spreading–vanishing dichotomy regimes of this model and obtain several criteria determining the spreading or vanishing. Our analysis result suggests that the solution converges to a positive time almost periodic function $(U^*(x,t), V^*(x,t))$ locally uniformly when the spreading occurs. We discover that the initial disease infected domain and the front expanding rate have momentous impacts on the permanence and extinction of the disease. Moreover, we give the lower and the upper bound estimates about the asymptotic spreading speeds of the double free fronts when the spreading happens.

**Mathematics Subject Classification.** 35B15, 35K57, 35B40, 92D25.

**Keywords.** Almost periodic, Free boundary, Principal Lyapunov exponent, Spreading–vanishing, Asymptotic spreading speed.

## 1. Introduction

West Nile virus (WNv) is one of the most serious mosquito-borne epidemic diseases threatening people's lives by invading people's nervous system, which spreads mainly through mosquitoes as the vectors and biting birds as the hosts. Concentrated on the temporal transmission of the WNv, there have been many studies by ordinary differential equations to explore the existence and stability of the equilibrium, and introduce the basic reproduction number as a threshold value to study the transmission dynamics of WNv, such as [1–5] and references therein.

In reality, the outbreak of the disease is not always caused by a single factor. The casual migration movements of the infected bird populations and mosquitoes populations are usually random, so the spatial diffusion term should be into consideration. Therefore, only using ordinary differential systems to describe the spatial propagation of the West Nile virus is no more suitable. To investigate the spatial dependence of WNv, Lewis et al. [6] investigated the following simplified reaction–diffusion cooperative system for WNv model.

$$\begin{cases} \frac{\partial U}{\partial t} = D_1 \frac{\partial^2 U}{\partial x^2} + \alpha_1 \beta \frac{V}{N_1}(N_1 - U) - \gamma U, \\ \frac{\partial V}{\partial t} = D_2 \frac{\partial^2 V}{\partial x^2} + \alpha_2 \beta \frac{U}{N_1}(N_2 - V) - dV, \end{cases} \tag{1.1}$$

where $U(x,t)$ and $V(x,t)$ are the densities of the infected birds and mosquitoes at location $x$ and time $t$, respectively; the diffusion coefficients for birds and mosquitoes are $D_1$ and $D_2$ which $D_2$ is much less than $D_1$; $\alpha_1, \alpha_2$ are the WNv transmission probability per bite to birds and mosquitoes; $\beta$ is the biting rate of mosquitoes on birds; $d$ is the adult mosquito death rate; $\gamma$ is the bird recovery rate from WNv; $N_1, N_2$ are constants denoting the total population of birds and adult mosquitoes. Under the assumptions that the whole parameters in (1.1) are positive constants, they discussed the long-time dynamics of the solution and gave an estimate for the spreading speed of (1.1) by comparison theorem. Maidana and Yang [7] used the traveling wave solution of the WNv model to study the spatial spreading of the disease across North America and analyzed the dependence of the wave speed on several factors, such as vertical transmission, recovery ratio, disease death rate, diffusion rate and advection rate.

The infected boundaries driven by birds and mosquitoes migrating from one habitat to another usually change with respect to time. Thus, applying the fixed domain to describe the moving fields of the vectors and hosts is not appropriate. Free boundaries conditions have largely attracted lots of concentrations recently and they are frequently used in biological mathematical models, for instance, [8–13]. Given the moving infected boundaries, Lin and Zhu [14] investigated a reaction–diffusion system to explore the spatial spreading of WNv using free boundaries to represent the disease spreading fronts. Tarboush et al.[15] studied a WNv model which incorporates a partial differential equation and an ordinary differential equation with moving boundaries. Cheng and Zheng [16] considered a reaction–advection–diffusion WNv model with double free boundaries and studied the influence of advection terms on the boundary asymptotic spreading speeds.

In our living real world, the habitats for birds and mosquitoes are not usually homogeneous. The environmental diversity is a pretty worthwhile factor to consider in studying epidemic models. In view of the spatial heterogeneity, Allen et al. [17] studied the following SIS reaction–diffusion model in 2008,

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega, \quad t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, \quad t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.2}$$

where domain $\Omega \subset \mathbb{R}^k (k \geq 1)$ is bounded with smooth boundary $\partial\Omega$; $S(x,t)$ and $I(x,t)$ are the population densities of susceptible and infected individuals at position $x$ and time $t$; positive constants $d_S$ and $d_I$ represent diffusion rates for the susceptible and infected populations; $\beta(x)$ is the disease transmission rate at position $x$ and $\gamma(x)$ is the disease recovery rate at position $x$, both of which are positive Hölder continuous functions, respectively. They studied the effects of the heterogeneous media and the individual movement of susceptible and infected populations on the permanence and eradication of the disease and obtained the global dynamics of model (1.2) by the basic reproduction number. Zhou and Xiao [18] explored a diffusive logistic system in a heterogeneous environment with free boundary conditions. Zhao and Wang [19] considered a prey–predator model in higher spatial dimensions and heterogeneous environment. There are also many other studies concentrated on spatial heterogeneity, such as [20–23].

Apart from the spatial heterogeneity, the temporal heterogeneity caused by alternations of seasonality is also a significant factor in influencing the propagation of the disease. Peng and Zhao [24] investigated the model (1.2) in a time-periodic heterogeneous environment in which the transmission rate $\beta(x,t)$ and recovery rate $\gamma(x,t)$ are periodic in $t$. Zhang and Wang [25] studied a diffusive SIR time-periodic system and investigated the spatial dynamics of this epidemic model. Shan et al. [26] investigated a periodic compartmental WNv model with time delay and obtained the effects of seasonal recurrent phenomena on the spreading and recurrence of the epidemic disease.

From a biological view, the effects of the alternation of seasons on the disease transmission rate, disease recovery rate and the disease death rate are not same. Thus, the periods of these parameters for the epidemic model are usually different. Therefore, we had to look for a more reasonable mathematical model. Considering the differences of the periodic coefficients, it is significant to study the time almost periodic system. Shen and Yi [27] studied the convergence of the positive solution for almost periodic models of Fisher and Kolmogorov type. Huang and Shen [28] investigated the spreading dynamics of KPP models in time almost periodic and space periodic environment and gave the estimates of the spreading speed. Wang and Zhao [29] discussed the basic reproduction ratio $R_0$ and obtained its computing formula for almost periodic compartmental ordinary differential epidemic models. Wang et al. [30] investigated a reaction–diffusion SIS model in a time almost periodic environment and discussed the influences of the basic reproduction number $R_0$ on the persistence or extinction of the solution for the epidemic model. Recently, Qiang et al. [31] studied a nonlocal reaction–diffusion model with time delay in almost periodic media and discussed the threshold dynamics.

However, there are few studies on mosquito-borne diseases using the almost periodic systems. For the sake of better exploring the mechanisms of the disease outbreak and more reasonably describing the transmission rules of WNv, almost periodic mathematical biological models incorporate spatial heterogeneity with time almost periodicity should be vitally considered to study the transmission of WNv. Motivated by the previous studies, different from the ordinary differential equations and the reaction–diffusion periodic model, we investigate the following WNv model with double free boundaries in spatial heterogeneous and time almost periodic media,

$$\begin{cases} U_t = D_1 U_{xx} + \alpha_1(x,t)\beta\frac{N_1-U}{N_1}V - \gamma(x,t)U, & g(t)<x<h(t), t>0,\\ V_t = D_2 V_{xx} + \alpha_2(x,t)\beta\frac{N_2-V}{N_1}U - d(x,t)V, & g(t)<x<h(t), t>0,\\ U(x,t)=V(x,t)=0, & x=h(t) \text{ or } x=g(t), t>0,\\ h(0)=h_0, \quad h'(t)=-\mu U_x(h(t),t), & t>0,\\ g(0)=-h_0, \quad g'(t)=-\mu U_x(g(t),t), & t>0,\\ U(x,0)=U_0(x), V(x,0)=V_0(x), & -h_0\le x\le h_0, \end{cases} \tag{1.3}$$

where $\alpha_1(x,t)$, $\alpha_2(x,t)$, $\gamma(x,t)$, $d(x,t) \in C^{2+\alpha_0,1+\frac{\alpha_0}{2}}(\mathbb{R}\times[0,\infty))$ are positive bounded functions for some $\alpha_0 \in (0,1)$, and they are uniformly almost periodic in $t$. Moreover, we assume that $\alpha_1(x,t)$, $\alpha_2(x,t)$, $\gamma(x,t)$, $d(x,t)$ have positive upper and lower bounds. $(g(t),h(t))$ denotes the moving infected domain of WNv. Meanwhile, we suppose that the double free boundaries submit to classical Stefan conditions obeying the Fick's first law, that is, $g'(t)=-\mu U_x(g(t),t)$ and $h'(t)=-\mu U_x(h(t),t)$, where $\mu$ is positive. Considering that the scale of movements of birds is much larger than that of the mosquitoes, it is reasonable to assume that the infected boundary movements are driven by the infected birds. Similar free boundary condition assumptions have been applied in some ecological and epidemical models in previous studies, such as in [9,10,14].

To simplify the value of parameters in this model, denote

$$a_1(x,t) := \frac{\alpha_1(x,t)\beta}{N_1},\ a_2(x,t) := \frac{\alpha_2(x,t)\beta}{N_1},\ d_1(x,t) := \gamma(x,t),\ d_2(x,t) := d(x,t), \tag{1.4}$$

then $a_1(x,t), a_2(x,t), d_1(x,t), d_2(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}\times[0,\infty))$ for any $\alpha\in(0,\alpha_0)$. Further, in view of the assumptions about $\alpha_1(x,t), \alpha_2(x,t), \gamma(x,t), d(x,t)$ and (1.4), there are positive constants $\hat{a}_i$ $\tilde{a}_i, \hat{d}_i, \tilde{d}_i$ such that

$$\hat{a}_i \le a_i \le \tilde{a}_i,\ \hat{d}_i \le d_i \le \tilde{d}_i,\ i=1,2. \tag{1.5}$$

On the basis of the previous simplifications and assumptions, we are going to investigate the following simplified WNv system,

$$\begin{cases} U_t = D_1 U_{xx} + a_1(x,t)\,(N_1-U)\,V - d_1(x,t)U, & g(t)<x<h(t),\ t>0,\\ V_t = D_2 V_{xx} + a_2(x,t)\,(N_2-V)\,U - d_2(x,t)V, & g(t)<x<h(t),\ t>0,\\ U(x,t)=V(x,t)=0, & x=h(t) \text{ or } x=g(t),\ t>0,\\ h(0)=h_0,\ h'(t)=-\mu U_x(h(t),t), & t>0,\\ g(0)=-h_0,\ g'(t)=-\mu U_x(g(t),t), & t>0,\\ U(x,0)=U_0(x), \quad V(x,0)=V_0(x), & -h_0\le x\le h_0. \end{cases} \tag{1.6}$$

For the convenience of studying, we suppose that the initial functions $U_0$ and $V_0$ satisfy

$$\begin{cases} U_0(x) \in C^2([-h_0,h_0]),\, U_0(\pm h_0) = 0,\, 0 < U_0(x) \le N_1 \text{ in } (-h_0,h_0)\,, \\ V_0(x) \in C^2([-h_0,h_0]),\, V_0(\pm h_0) = 0,\, 0 < V_0(x) \le N_2 \text{ in } (-h_0,h_0)\,. \end{cases} \tag{1.7}$$

In this paper, our primary purpose is to research a reaction–diffusion WNv model with moving infected regions $(g(t), h(t))$ in the spatial heterogeneous and time almost periodic media, and discuss the effects of the spatial heterogeneity and time almost periodicity on the spreading and vanishing of the epidemic disease. Actually, our cooperative epidemic model (1.6) is first proposed to incorporate the spatial heterogeneity with time almost periodicity in studying the epidemic disease. We first give the global existence, uniqueness and regularity estimates of the solution, the method of which is not trivially similar to homogeneous WNv models (see Theorems 2.1, 3.1 and 3.4). Since the habitat is heterogeneous and the boundary is moving, the general basic reproduction number is difficult to be calculated as the threshold value. To overcome this obstacle, we introduce the principal Lyapunov exponent $\lambda(t)$ with respect to time $t$ (see Sect. 4) and get the initial infected domain $L^*$ as a threshold value. Moreover, we obtain the corresponding spreading–vanishing dichotomy regimes of the West Nile virus using it (see Theorem 2.2). We prove that the eventually infected domain is no more than $2L^*$ when the vanishing occurs. Importantly, we prove that the solution for system (1.6) converges to a time almost periodic function for fixed $x$ in bounded subsets of $\mathbb{R}$ when the spreading occurs. We use some new techniques in proving this result, whose asymptotic behavior is very different from other homogeneous WNv models, the solution of which converges to a positive constant equilibrium, such as [1,6,14]. Our results show that the spatial heterogeneity and temporal almost periodicity driven by spatial differences and seasonal recurrence lead to the cyclic appearance of the cases of infection. Further, we show that the initial WNv infected domain and the front expanding rate have momentous impacts on the permanence and extinction of the epidemic disease. What is more, the spreading speed for propagation of WNv is a significant index to describe the epidemic scale for disease, but many previous studies focus less on the research of the asymptotic spreading speed for WNv in the heterogeneous environment because of the difficulties in research. Here, we provide the estimates of lower and upper bound about the double free boundaries for the heterogeneous model (1.6)(see Theorem 2.4). Our techniques developed in studying almost periodic systems different from other homogeneous or periodic systems can be applied in other almost periodic systems and cooperative epidemic models.

The rest of the paper is arranged as follows. In Sect. 2, we first prepare some preliminaries and assumptions, then present the main results. In Sect. 3, we provide a detailed proof of the global existence, uniqueness and regularity estimates of the solution for problem (1.6) in the time almost periodic and spatial heterogeneous environment. In Sect. 4, considering the spatial heterogeneity and time almost periodicity, we introduce the principal Lyapunov exponent and obtain some vital properties of this threshold value. In Sect. 5, we explore the long-time asymptotic behaviors of the solution for heterogeneous system (1.6) by giving the spreading–vanishing dichotomy regimes. In Sect. 6, we present the estimates about the asymptotic spreading speeds for the double spreading fronts.

## 2. Preliminaries and main results

In the section, we make some preparations and display our main results.

### 2.1. Preliminaries

First, we recall several definitions about almost periodic function from Section 2.1 of [27] or Section 3 of [32].

**Definition 2.1.** (i) A function $f(t) \in C(\mathbb{R}, \mathbb{R}^k)(k \geq 1)$ is called an almost periodic function if for any $\epsilon > 0$, the set

$$T(f, \epsilon) := \{\tau \in \mathbb{R} \mid |f(t+\tau) - f(t)| < \epsilon \text{ for any } t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$. We say a matrix function $A(t)$ is almost periodic if every entry of it is almost periodic function.

(ii) A function $f(x,t) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is uniformly almost periodic in $t$ if $f(x, \cdot)$ is almost periodic for every $x \in \mathbb{R}$, and $f$ is uniformly continuous on $E \times \mathbb{R}$ for any compact set $E \subset \mathbb{R}$.

(iii) A function $f(x,t,u,v) \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^k)(m,n,k \geq 1)$ is uniformly almost periodic in $t$ with $x \in \mathbb{R}$ and $(u,v)$ in bounded sets if $f$ is uniformly continuous for $t \in \mathbb{R}, x \in \mathbb{R}$ and $(u,v)$ in bounded sets and $f(x,t,u,v)$ is almost periodic in $t$ for every $x \in \mathbb{R}$, $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

**Definition 2.2.** (i) The hull of a uniformly almost periodic matrix $A(x,t)$ is defined by

$$H(A) = \{B(\cdot,\cdot) \mid \exists\, t_n \to \infty, \text{such that}\, B(x, t+t_n) \to A(x,t) \,\text{uniformly for}\, t \in \mathbb{R},$$
$$x \,\text{in bounded sets}\}.$$

(ii) The hull of a uniformly almost periodic matrix $F(x,t,u,v)$ is defined by

$$H(F) = \{G(\cdot,\cdot,\cdot,\cdot) \mid \exists\, t_n \to \infty, \text{such that}\, G(x, t+t_n, u, v) \to F(x,t,u,v)$$
$$\text{uniformly for}\, t \in \mathbb{R}, (x,u,v) \,\text{in bounded sets}\}.$$

In general, the system (1.6) can be seen as the special form of the following system,

$$\begin{cases} u_t = D_1 u_{xx} + f_1(x,t,u,v), & g(t) < x < h(t),\ t > 0, \\ v_t = D_2 v_{xx} + f_2(x,t,u,v), & g(t) < x < h(t),\ t > 0, \\ u(x,t) = v(x,t) = 0, & x = h(t) \text{ or } x = g(t),\ t > 0, \\ h(0) = h_0,\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ g(0) = -h_0,\ g'(t) = -\mu u_x(g(t),t), & t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & -h_0 \leq x \leq h_0, \end{cases} \tag{2.1}$$

where initial data $(u_0, v_0)$ satisfy (1.7), and $f_i(x,t,u,v)$ satisfies the following conditions for $i = 1, 2$.

(***H1***) *$f_i(x,t,u,v) \in C^1(\mathbb{R}^4)$, $Df_i(x,t,u,v) = (\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial t}, \frac{\partial f_i}{\partial u}, \frac{\partial f_i}{\partial v})$ is bounded for $(x,t) \in \mathbb{R} \times \mathbb{R}$ and $(u,v)$ in bounded sets.*

(***H2***) *There exist positive constants $M$ and $N$ such that*

$$\sup_{\substack{t\in\mathbb{R}, x\in\mathbb{R}, \\ u \geq M, v \in \mathbb{R}}} f_1(x,t,u,v) < 0, \ \sup_{\substack{t\in\mathbb{R}, x\in\mathbb{R}, \\ u \in \mathbb{R}, v \geq N}} f_2(x,t,u,v) < 0,$$

$$\sup_{\substack{t\in\mathbb{R}, x\in\mathbb{R}, \\ u \geq 0, v \geq 0}} \frac{\partial f_1}{\partial u}(x,t,u,v) < 0, \ \sup_{\substack{t\in\mathbb{R}, x\in\mathbb{R}, \\ u \geq 0, v \geq 0}} \frac{\partial f_2}{\partial v}(x,t,u,v) < 0.$$

(***H3***) *$f_i$ and $Df_i$ are uniformly almost periodic in $t \in \mathbb{R}$ with $x \in \mathbb{R}$ and $(u,v)$ in bounded sets.*

(***H4***) *Let*

$$F(x,t,u,v) = \begin{pmatrix} f_1(x,t,u,v) \\ f_2(x,t,u,v) \end{pmatrix}. \tag{2.2}$$

For any given sequences $\{x_n\} \subset \mathbb{R}$ and $\{G_n\} \subset H(F)$, there exist subsequences $\{x_{n_k}\} \subset \{x_n\}$ *and* $\{G_{n_k}\} \subset \{G_n\}$ *such that* $\lim\limits_{k\to\infty} G_{n_k}(x + x_{n_k}, t, u, v)$ *exists for* $t \in \mathbb{R}$ *uniformly and* $(x,u,v)$ *in bounded sets.*

In this paper, we take

$$\begin{aligned} f_1(x,t,U,V) &= a_1(x,t)(N_1 - U)V - d_1(x,t)U, \\ f_2(x,t,U,V) &= a_2(x,t)(N_2 - V)U - d_2(x,t)V. \end{aligned} \tag{2.3}$$

Then the system (1.6) which satisfies the above assumptions is cooperative and monostable.

Define matrix function $A(x,t)$ by

$$A(x,t) := \begin{pmatrix} \frac{\partial f_1(x,t,U,0)}{\partial U} & \frac{\partial f_1(x,t,0,V)}{\partial V} \\ \frac{\partial f_2(x,t,U,0)}{\partial U} & \frac{\partial f_2(x,t,0,V)}{\partial V} \end{pmatrix} = \begin{pmatrix} -d_1(x,t) & a_1(x,t)N_1 \\ a_2(x,t)N_2 & -d_2(x,t) \end{pmatrix}. \quad (2.4)$$

Moreover, we assume that $A(x,t)$ satisfies

(***H5***) *There exists some* $L^* > 0$ *such that* $\inf_{\tilde{x}\in\mathbb{R}, L\geq L^*} \lambda(A(\cdot+\tilde{x},\cdot), L) > 0.$

Where $\lambda(A,L)$ is the principal Lyapunov exponent and $L^*$ is a constant dependent on $a_i(x,t), D_i, N_i, d_i(x,t)$ for $i=1,2$, which will be explicitly explained in Sect. 4.

## 2.2. Main results

Next we will present our main results for problem (1.6). In Sect. 3, we will prove that $h'(t)>0$ and $g'(t)<0$ in $t\in(0,+\infty)$. Therefore, we denote $g_\infty := \lim_{t\to\infty} g(t)$, $h_\infty := \lim_{t\to\infty} h(t)$ and $h_\infty - g_\infty := \lim_{t\to\infty}(h(t)-g(t))$. Further, we can obtain that $g_\infty \in [-\infty,0)$ and $h_\infty \in (0,\infty]$.

**Theorem 2.1.** (Existence and uniqueness) *Assuming any given initial functions* $(U_0,V_0)$ *satisfy* (1.7)*. For any* $\alpha\in(0,\alpha_0)$*, there exists a time* $T$ *such that the system* (1.6) *admits a unique solution* $(U,V;g,h) \in (C^{2+\alpha,1+\frac{\alpha}{2}}([g(t),h(t)]\times(0,T]))^2 \times (C^{1+\alpha/2}((0,T]))^2$*, where* $T$ *is dependent on* $\alpha, h_0, ||U_0||_{C^2([-h_0,h_0])}$ *and* $||V_0||_{C^2([-h_0,h_0])}$.

**Remark 2.1.** Actually, the solution for system (1.6) uniquely exists for all $t\in(0,\infty)$ (see Theorem 3.4).

In order to investigate the asymptotic dynamics of system (1.6), we first introduce the following system,

$$\begin{cases} U_t = D_1 U_{xx} + a_1(x,t)(N_1-U)V - d_1(x,t)U, & -\infty<x<\infty,\ t>0, \\ V_t = D_2 V_{xx} + a_2(x,t)(N_2-V)U - d_2(x,t)V, & -\infty<x<\infty,\ t>0. \end{cases} \quad (2.5)$$

The equation (2.5) admits a unique positive time almost periodic solution (see Step 2 for Theorem 5.6).

Now we have the following spreading–vanishing dichotomy result of (1.6).

**Theorem 2.2.** (Spreading–vanishing dichotomy) *Supposing* (***H1***)*–*(***H5***) *hold and the initial functions* $(U_0,V_0)$ *satisfy*(1.7)*. Let* $(U,V;U_0,V_0,h_0)$ *be the solution of* (1.6)*, for such* $L^*$ *in* (***H5***)*, the following spreading–vanishing dichotomy regimes hold:*

*Either*

*(1)* Vanishing*:* $h_\infty - g_\infty \leq 2L^*$ *and* $\lim_{t\to+\infty} U(x,t;U_0,V_0,h_0)=0$, $\lim_{t\to+\infty} V(x,t;U_0,V_0,h_0)=0$ *uniformly in* $x\in[g_\infty,h_\infty]$;

*or*

*(2)* Spreading*:* $h_\infty - g_\infty = \infty$ *and* $\lim_{t\to+\infty}(U(x,t;U_0,V_0,h_0)-U^*(x,t))=0$, $\lim_{t\to+\infty}(V(x,t;U_0,V_0,h_0)-V^*(x,t))=0$ *locally uniformly for* $x$ *in* $\mathbb{R}$*, where* $(U^*(x,t),V^*(x,t))$ *is the unique positive time almost periodic solution of (2.5).*

**Remark 2.2.** When the spreading occurs, the long-time asymptotic behavior of the solution for WNv model (1.6) is largely different from homogeneous models, which converge to a trivial constant equilibrium.

**Theorem 2.3.** (Spreading–vanishing threshold) *Suppose that* (***H1***)*–*(***H5***) *hold. For any given* $h(0), g(0)$ *and the initial functions* $(U_0,V_0)$ *satisfying* (1.7)*, let* $(U,V;U_0,V_0,g,h)$ *be the solution of* (1.6)*, for such* $L^*$ *in* (***H5***)*, the followings hold:*

*(1) If* $\lambda(0)>0$*, then* $h(0)-g(0)\geq 2L^*$*, further,* $h_\infty-g_\infty=\infty$*, thus, the spreading occurs;*

*(2) If* $\lambda(0)<0$*, then there exists a constant* $\mu^*\geq 0$ *such that the spreading occurs when* $\mu>\mu^*$ *and vanishing occurs when* $0<\mu\leq\mu^*$.

**Remark 2.3.** The explicit explanation for the principal Lyapunov exponent $\lambda(t)$ can refer to Sect. 4. The above theorem gives the sufficient conditions about the spreading and vanishing of the disease. The critical size $L^*$ determines the persistence or extinction of WNv by influencing the sign of the principal Lyapunov exponent (see Sect. 5).

The following result is about the asymptotic spreading speed estimates for the double free boundaries.

**Theorem 2.4.** (Asymptotic spreading speed) *Assume that* ***(H1)–(H5)*** *hold. Let* $(U, V; g, h)$ *be the solution of system* (1.6)*, then the asymptotic spreading speeds of the leftward front and the rightward front satisfy:*

$$c_*(\mu) \leq \liminf_{t\to\infty} \frac{-g(t)}{t} \leq \limsup_{t\to\infty} \frac{-g(t)}{t} \leq c^*(\mu) \tag{2.6}$$

*and*

$$c_*(\mu) \leq \liminf_{t\to\infty} \frac{h(t)}{t} \leq \limsup_{t\to\infty} \frac{h(t)}{t} \leq c^*(\mu), \tag{2.7}$$

*where* $c_*(\mu), c^*(\mu)$ *dependent on* $\mu$ *are the asymptotic spreading speeds of the problem* (6.1) *and* (6.2)*, respectively.*

## 3. Existence and uniqueness

In this section, we will show the existence and uniqueness of the global solution for the system (1.6). Since the system (1.6) can be regarded as a special case of the system (2.1). We only need to give an explicit proof for the system (2.1). Although there are similar results about the solution for epidemic models with constant coefficients, the proofs of the (2.1) in the heterogeneous environment can not be easily obtained by analogy. Therefore, we provide a detailed proof according to Theorem 1.1 ([33]).

**Theorem 3.1.** *Assume that* ***(H1)–(H4)*** *hold. For any* $\alpha \in (0, \alpha_0)$ *and any given* $(u_0, v_0)$ *satisfying* (1.7)*, there exists* $T > 0$ *such that the system* (2.1) *admits a unique solution* $(u, v, g, h) \in (C^{1+\alpha,(1+\alpha)/2}(D_T))^2 \times (C^{1+\alpha/2}([0, T]))^2$*, where* $D_T = \{(x, t) \in \mathbb{R}^2 \mid x \in [g(t), h(t)], t \in [0, T]\}$*, and* $T$ *is only dependent on* $\alpha, h_0, ||u_0||_{C^2([-h_0,h_0])}$ *and* $||v_0||_{C^2([-h_0,h_0])}$.

*Proof.* We divide this proof into two steps.

**Step 1** The local existence of the solution for problem (2.1).

Let

$$\begin{aligned} y &= \frac{2x}{h(t)-g(t)} - \frac{h(t)+g(t)}{h(t)-g(t)}, \\ m(y,t) &= u(x,t), n(y,t) = v(x,t), \\ \tilde{f}_1(y,t,m,n) &= f_1(x,t,u,v), \\ \tilde{f}_2(y,t,m,n) &= f_2(x,t,u,v), \end{aligned} \tag{3.1}$$

then direct calculation gives

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{2}{h(t)-g(t)} := \sqrt{A(y, g(t), h(t))}, \\ \frac{\partial y}{\partial t} &= -\frac{y(h'(t)-g'(t)) + (h'(t)+g'(t))}{h(t)-g(t)} \\ &:= B(y, g(t), g'(t), h(t), h'(t)), \end{aligned} \tag{3.2}$$

and $(m,n)$ satisfy the following system,

$$\begin{cases} m_t - D_1 A m_{yy} + B m_y = \tilde{f}_1(y,t,m,n), & y\in(-1,1), 0<t\le T,\\ n_t - D_2 A n_{yy} + B n_y = \tilde{f}_2(y,t,m,n), & y\in(-1,1), 0<t\le T,\\ m(\pm1,t)=0, n(\pm1,t)=0, & 0<t\le T,\\ m(y,0)=u_0(h_0 y), n(y,0)=v_0(h_0 y), & y\in[-1,1]. \end{cases} \tag{3.3}$$

Meanwhile, $h(t)$ and $g(t)$ satisfy

$$\begin{cases} h(0)=h_0, h'(t) = -\mu\frac{2}{h(t)-g(t)} m_y(1,t), & 0<t\le T,\\ g(0)=-h_0, g'(t) = -\mu\frac{2}{h(t)-g(t)} m_y(-1,t), & 0<t\le T. \end{cases} \tag{3.4}$$

Next, we will show the existence of the solution for (3.3) with (3.4).

Let $h^* = -\mu u_0'(h_0), g^* = -\mu u_0'(-h_0)$, $T_0 = \min\left\{1, \frac{h_0}{2(2+h^*)}, \frac{h_0}{2(2-g^*)}\right\}$,

$$\begin{aligned} \Gamma &= \left\{h_0, h^*, g^*, \|u_0\|_{C^2([-h_0,h_0])}, \|v_0\|_{C^2([-h_0,h_0])}\right\},\\ \Theta_T &= \{(g,h)\in (C^1([0,T]))^2 \mid h(0)=h_0, g(0)=-h_0, h'(0)=h^*, g'(0)=g^*,\\ &\quad \|h'-h^*\|_{L^\infty}\le 1, \|g'-g^*\|_{L^\infty}\le 1\}, \end{aligned} \tag{3.5}$$

then $h^*>0, g^*<0$ and $\Theta_T$ is a bounded closed convex subset of $(C^1([0,T_0]))^2$ for any $0<T\le T_0$.

Let

$$\Theta^*_{T_0} = \{(g,h)\in (C^1([0,T_0]))^2 \mid h(0)=h_0, h'(0)=h^*, g'(0)=g^*, \|h'-h^*\|_{L^\infty}\le 2,$$
$$\|g'-g^*\|_{L^\infty}\le 2\}.$$

For any given $(g,h)\in\Theta_T$, we can extend $h$ and $g$ such that $(g,h)\in\Theta^*_{T_0}$. Hence, if $(g,h)\in\Theta_T$, then $(g,h)\in\Theta^*_{T_0}$. And $h(t)$ and $g(t)$ satisfy

$$\begin{aligned} |h(t)-h_0| &\le T_0\|h'\|_\infty \le T_0(2+h^*) \le \frac{h_0}{2},\\ |g(t)-(-h_0)| &\le T_0\|g'\|_\infty \le T_0(2+g^*) \le T_0(2-g^*) \le \frac{h_0}{2} \end{aligned} \tag{3.6}$$

for any $t\in[0,T_0]$, then $h(t)\in[\frac{h_0}{2},\frac{3h_0}{2}]$ and $g(t)\in[-\frac{3h_0}{2},-\frac{h_0}{2}]$ in $[0,T_0]$. Hence, the transformations (3.1) and (3.2) are well defined for $t\in[0,T_0]$. Applying the standard parabolic equation theory ([34]), there exists a $T_*\in(0,T_0]$ such that there is a unique solution $(\overline{m}(y,t),\overline{n}(y,t))\in(C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_{T_*}))^2$ for problem (3.3) with $T_*$ dependent on $\Gamma, \alpha, \|u_0\|_\infty$ and $\||v_0\|_\infty$. And there exists a positive constant $C_1(\Gamma,\alpha,T_*,T_*^{-1})$ such that

$$\|\overline{m}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_{T_*})} + \|\overline{n}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_{T_*})} \le C_1(\Gamma,\alpha,T_*,T_*^{-1}),$$

where $\Delta_{T_*}=[-1,1]\times[0,T_*]$. In view of the choice of $\Gamma$ in (3.5), $T_*$ is only dependent on $\Gamma$ and $\alpha$. Hence,

$$\|\overline{m}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_{T_*})} + \|\overline{n}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_{T_*})} \le C_1,$$

with $C_1$ dependent on $\Gamma$ and $\alpha$. Moreover, for $0<T\le T_*$, it follows

$$\|\overline{m}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)} + \|\overline{n}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)} \le C_1. \tag{3.7}$$

Since $\overline{m}(y,0)$ and $\overline{n}(y,0)$ are more than but not identically equal to 0 for $y\in[-1,1]$, $\tilde{f}_i(y,t,0,0)\ge 0$, $a_iN_i\ge 0$ on $[-1,1]\times[0,T]$ and $\tilde{f}_i(y,t,m,n)$ satisfies (***H1***) for $i=1,2$, by the maximum principle (resp. Positivity Lemma, [35]), then $(\overline{m},\overline{n})>0$ for $(y,t)\in(-1,1)\times(0,T]$.

Considering that the solution $(\overline{m},\overline{n})$ depends continuously on the initial data $(g,h)\in\Theta_T$, let

$$\begin{aligned}\overline{h}(t)&=h_0-\mu\int_0^t\frac{2}{h(s)-g(s)}\overline{m}_y(1,s)ds,\\ \overline{g}(t)&=-h_0-\mu\int_0^t\frac{2}{h(s)-g(s)}\overline{m}_y(-1,s)ds\end{aligned}\tag{3.8}$$

for $t\in[0,T]$, then $(\overline{g},\overline{h})$ depends on $(g,h)\in\Theta_T$ and

$$\overline{h}(0)=h_0,\overline{h}'(0)=h^*,\overline{h}'(t)>0,\overline{g}(0)=-h_0,\overline{g}'(0)=g^*,\overline{g}'(t)<0.$$

Moreover, it follows

$$\begin{aligned}\overline{h}'(t)\in C^{\frac{\alpha}{2}}([0,T]),||\overline{h}'(t)||_{C^{\frac{\alpha}{2}}([0,T])}\le C_2,\\ \overline{g}'(t)\in C^{\frac{\alpha}{2}}([0,T]),||\overline{g}'(t)||_{C^{\frac{\alpha}{2}}([0,T])}\le C_2\end{aligned}\tag{3.9}$$

for some $C_2$ dependent on $\Gamma$ and $\alpha$.

Define $\mathcal{F}:\mathcal{D}_1\times\mathcal{D}_2\times\Theta_T\longrightarrow C(\Delta_T)\times C(\Delta_T)\times(C^1([0,T]))^2$ by $\mathcal{F}(m,n,g,h)=(\overline{m},\overline{n},\overline{g},\overline{h})$, where

$$\begin{aligned}\mathcal{D}_1&=\{m\in C(\Delta_T)|m(y,0)=u_0(h_0y),||m-u_0||_{C(\Delta_T)}\le 1\},\\ \mathcal{D}_2&=\{n\in C(\Delta_T)|n(y,0)=v_0(h_0y),||n-v_0||_{C(\Delta_T)}\le 1\}.\end{aligned}$$

It is easy to see that $\mathcal{F}(m,n,g,h)=(m,n,g,h)$ if and only if $(m,n,g,h)$ is the solution of (3.3) with (3.4).

Combining (3.7) and (3.9), it follows

$$\begin{aligned}&||\overline{m}-u_0||_{C(\Delta_T)}+||\overline{n}-v_0||_{C(\Delta_T)}\\ &\le||\overline{m}-u_0||_{C^{\frac{1+\alpha}{2},0}(\Delta_T)}T^{\frac{1+\alpha}{2}}+||\overline{n}-v_0||_{C^{\frac{1+\alpha}{2},0}(\Delta_T)}T^{\frac{1+\alpha}{2}}\\ &\le C_1T^{\frac{1+\alpha}{2}}\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}||\overline{h}'-h^*||_{C([0,T])}\le||\overline{h}'||_{C^{\frac{\alpha}{2}}([0,T])}T^{\frac{\alpha}{2}}\le C_2T^{\frac{\alpha}{2}},\\ ||\overline{g}'-g^*||_{C([0,T])}\le||\overline{g}'||_{C^{\frac{\alpha}{2}}([0,T])}T^{\frac{\alpha}{2}}\le C_2T^{\frac{\alpha}{2}}.\end{aligned}$$

Therefore, if we take $T=\min\left\{1,\frac{h_0}{2(2+h^*)},\frac{h_0}{2(2-g^*)},C_1^{-\frac{2}{1+\alpha}},C_2^{-\frac{2}{\alpha}}\right\}$, then $\mathcal{F}$ maps $\mathcal{D}_1\times\mathcal{D}_2\times\Theta_T$ into itself. Further, we can get that $\mathcal{F}$ is compact. Applying the Schauder fixed-point theorem to $\mathcal{F}$, there exists a fixed point $(m,n,g,h)\in\mathcal{D}_1\times\mathcal{D}_2\times\Theta_T$. Applying the Schauder estimates, $(m,n,g,h)\in(C^{1+\alpha,\frac{1+\alpha}{2}}([-1,1]\times[0,T]))^2\times(C^{1+\frac{\alpha}{2}}[0,T])^2$ *is the solution* of system (3.3) with (3.4). Hence, the problem (2.1) has a solution $(u,v,g,h)\in(C^{1+\alpha,\frac{1+\alpha}{2}}([g(t),h(t)]\times[0,T]))^2\times(C^{1+\frac{\alpha}{2}}[0,T])^2$.

**Step 2** The uniqueness of the solution for problem (2.1).

Assume that $(u_i,v_i,g,h)(i=1,2)\in\mathcal{D}_1\times\mathcal{D}_2\times\Theta_T$ are the two solutions of (2.1) for $0<T\ll 1$. Applying the strong maximum principle to $u_i$, we can get that $u_i(x,t)>0$ for $x\in(g(t),h(t))$ and $0<t<T$. In view of $u_i(t,h(t))=0,u_i(t,g(t))=0$, it follows $u_{ix}(t,h(t))<0,u_{ix}(t,g(t))>0$ for $i=1,2$, which implies $h'(t)>0,g'(t)<0$ for $t\in(0,T)$, then we can suppose that

$$h_0\le h(t)\le h_0+1,-h_0-1\le g(t)\le -h_0\tag{3.11}$$

for $t\in[0,T]$, and $u_i\le||u_0||_\infty+1,v_i\le||v_0||_\infty+1$ in $[g(t),h(t)]\times[0,T]$ for $i=1,2$.

As in transformations (3.1), take $m_i(y,t) = u_i(x,t), n_i(y,t) = v_i(x,t)$ for $i = 1,2$, then $(y,t) \in [-1,1] \times [0,T]$.

Let $m = m_1 - m_2, n = n_1 - n_2, h = h_1 - h_2, g = g_1 - g_2$, direct calculation gives the following system,

$$\begin{cases} m_t - D_1 A_1(y,t) m_{yy} + B_1(y,t) m_y - a_1(y,t) m - \tilde{a}_1(y,t) n \\ = D_1(A_1 - A_2) m_{2yy} + (B_2 - B_1) m_{2y} + b_1(y,t) \frac{y(h-g)+(h+g)}{2}, \ y \in (-1,1), 0 < t \le T, \\ n_t - D_2 A_1(y,t) n_{yy} + B_1(y,t) n_y - a_2(y,t) m - \tilde{a}_2(y,t) n \\ = D_2(A_1 - A_2) n_{2yy} + (B_2 - B_1) n_{2y} + b_2(y,t) \frac{y(h-g)+(h+g)}{2}, \quad y \in (-1,1), 0 < t \le T, \\ m(\pm 1, t) = 0, n(\pm 1, t) = 0, & 0 < t \le T, \\ m(y,0) = n(y,0) = 0, & y \in (-1,1), \end{cases} \tag{3.12}$$

with

$$\begin{aligned} h'(t) &= \mu\left(\frac{2}{h_2(t) - g_2(t)} m_{2y}(1,t) - \frac{2}{h_1(t) - g_1(t)} m_{1y}(1,t)\right), \\ g'(t) &= \mu\left(\frac{2}{h_2(t) - g_2(t)} m_{2y}(-1,t) - \frac{2}{h_1(t) - g_1(t)} m_{1y}(-1,t)\right), \end{aligned} \tag{3.13}$$

for $0 < t \le T, h(0) = 0, g(0) = 0, \ i = 1,2$, where

$$\begin{aligned} A_i(y,t) &= \frac{4}{(h_i(t) - g_i(t))^2}, \\ B_i(y,t) &= -\frac{y(h_i'(t) - g_i'(t)) + (h_i'(t) + g_i'(t))}{h_i(t) - g_i(t)}, \\ b_i(y,t) &= \int_0^1 f_{ix}(t, H(h_1, h_2, g_1, g_2, s), m_2, n_2) ds, \\ \tilde{a}_i(y,t) &= \int_0^1 f_{in}\left(t, \frac{y(h_1 - g_1) + (h_1 + g_1)}{2}, m_2, n_2 + s(m_1 - m_2)\right) ds, \\ a_i(y,t) &= \int_0^1 f_{im}\left(t, \frac{y(h_1 - g_1) + (h_1 + g_1)}{2}, m_2 + s(m_1 - m_2), n_1\right) ds, \end{aligned}$$

$$H(h_1, h_2, g_1, g_2) = \frac{y(h_2 + s(h_1 - h_2) - (g_2 + s(g_1 - g_2))) + (h_2 + s(h_1 - h_2) + (g_2 + s(g_1 - g_2)))}{2}.$$

In view of (***H1***)–(***H5***), we can get $a_i, \tilde{a}_i, b_i \in L^\infty(\Delta_T)$ for $i = 1,2$ with $||a_i(y,t)||_{L^\infty}, ||\tilde{a}_i(y,t)||_{L^\infty}$ and $||b_i(y,t)||_{L^\infty}$ dependent on $h_0, ||u_0||_{L^\infty}$ and $||v_0||_{L^\infty}$. In view of (3.7)–(3.11), applying $L^p$ theory for parabolic equations and Sobolev imbedding theorem to system (3.12), there are positive constants $C_3, C_4$

and $C_5$ which depend on $\Gamma$ and $\alpha$ such that

$$
\begin{aligned}
&||m||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)}+||n||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)}\\
&\le C_3(D_1||((h_1-g_1)^{-2}-(h_2-g_2)^{-2})m_{2yy}||_{C(\Delta_T)}+\left\|b_1\frac{y(h-g)+(h+g)}{2}\right\|_{C(\Delta_T)}\\
&\quad+\left\|\left(\frac{y(h_1'-g_1')+(h_1'+g_1')}{h_1(t)-g_1(t)}-\frac{y(h_2'-g_2')+(h_2'+g_2')}{h_2(t)-g_2(t)}\right)m_{2y}\right\|_{C(\Delta_T)})\\
&\quad+C_3(D_2||((h_1-g_1)^{-2}-(h_2-g_2)^{-2})n_{2yy}||_{C(\Delta_T)}+|b_2\frac{y(h-g)+(h+g)}{2}||_{C(\Delta_T)}\\
&\quad+\left\|\left(\frac{y(h_1'-g_1')+(h_1'+g_1')}{h_1(t)-g_1(t)}-\frac{y(h_2'-g_2')+(h_2'+g_2')}{h_2(t)-g_2(t)}\right)n_{2y}\right\|_{C(\Delta_T)})\\
&\le C_4(||h||_{C^1([0,T])}+||g||_{C^1([0,T])}+||h-g||_{C^1([0,T])}+||h+g||_{C^1([0,T])})\\
&\le C_5(||h||_{C^1([0,T])}+||g||_{C^1([0,T])}).
\end{aligned}
\tag{3.14}
$$

Applying the proofs of (5.4.3) and Theorem 5.5.4 ([36]) to $m_y(y,t)$ and $n_y(y,t)$, without needing to expand $m$ and $n$ to a larger domain, we obtain that there exists a positive constant $\tilde{C}_1$ independent of $T^{-1}$ such that

$$
\begin{aligned}
[m]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}+[m_y]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}\le\tilde{C}_1||m||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)},\\
[n]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}+[n_y]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}\le\tilde{C}_1||n||_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Delta_T)},
\end{aligned}
$$

where $[\,\cdot\,]$ is the Hölder seminorm. Therefore, according to (3.14) and the above inequalities, it follows that

$$
\begin{aligned}
[m_y]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}\le\tilde{C}_1C_5(||h||_{C^1([0,T])}+||g||_{C^1([0,T])}),\\
[n_y]_{C^{\alpha,\frac{\alpha}{2}}(\Delta_T)}\le\tilde{C}_1C_5(||h||_{C^1([0,T])}+||g||_{C^1([0,T])}).
\end{aligned}
\tag{3.15}
$$

Combining (3.13) and (3.15), there is $C_6$ dependent on $\Gamma,\alpha$ such that

$$
\begin{aligned}
[h']_{C_{\frac{\alpha}{2}}([0,T])}&\le\mu\left[\frac{2}{h_1-g_1}m_y(1,t)\right]_{C_{\frac{\alpha}{2}}([0,T])}+\mu\left[\left(\frac{2}{h_1-g_1}-\frac{2}{h_2-g_2}\right)m_{2y}(1,t)\right]_{C_{\frac{\alpha}{2}}([0,T])}\\
&\le C_6(||h||_{C^1([0,T])}+||g||_{C^1([0,T])})
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
[g']_{C_{\frac{\alpha}{2}}([0,T])}&\le\mu\left[\frac{2}{h_1-g_1}m_y(-1,t)\right]_{C_{\frac{\alpha}{2}}([0,T])}+\mu\left[\left(\frac{2}{h_1-g_1}-\frac{2}{h_2-g_2}\right)m_{2y}(-1,t)\right]_{C_{\frac{\alpha}{2}}([0,T])}\\
&\le C_6(||h||_{C^1([0,T])}++||g||_{C^1([0,T])}).
\end{aligned}
\tag{3.17}
$$

Since $h(0)=h'(0)=0$ and $g(0)=g'(0)=0$, then

$$
\begin{aligned}
||h-h(0)||_{C^1([0,T])}\le2||h'-h'(0)||_{C^{\frac{\alpha}{2}}([0,T])}T^{\frac{\alpha}{2}}\le\tilde{C}_6||h||_{C^1([0,T])}T^{\frac{\alpha}{2}},\\
||g-g(0)||_{C^1([0,T])}\le2||g'-g'(0)||_{C^{\frac{\alpha}{2}}([0,T])}T^{\frac{\alpha}{2}}\le\tilde{C}_6||g||_{C^1([0,T])}T^{\frac{\alpha}{2}}.
\end{aligned}
$$

Therefore, if $T$ is small enough, then $h=0$ and $g=0$, which implies $m=0$ and $n=0$. Thus, the local existence and uniqueness of the solution have been proved. □

*Proof of Theorem 2.1.* Let $f_i$ be defined by (2.3) for $i=1,2$. In view that $a_1(x,t),a_2(x,t),d_1(x,t),d_2(x,t)$ $\in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}\times[0,\infty))$ for any $\alpha\in(0,\alpha_0)$ and have positive upper and lower bound, it follows that $f_i(x,\cdot,U,V)\in C^{1+\frac{\alpha}{2}}([0,T])(i=1,2)$ for the $T$ in Theorem 3.1.

Make the transformations as (3.1), combining (3.7) with (3.9), then, it can be obtained that

$$\tilde{f}_i(y,t) := f_i\left(\frac{y(h(t)-g(t))+(h(t)+g(t))}{2}, t, m(y,t), n(y,t)\right) \in C^{\alpha,\frac{\alpha}{2}}\left([-1,1]\times[0,T]\right).$$

Using the Schauder theory for parabolic equations to system (3.3) and (3.4), we can get that

$$(m,n,g,h) \in \left(C^{2+\alpha,1+\frac{\alpha}{2}}([-1,1]\times(0,T])\right)^2 \times \left(C^{1+\frac{\alpha}{2}}(0,T]\right)^2.$$

Since the system (1.6) can be regarded as the special case of (2.1) and satisfies all of the assumptions in Theorem 3.1, the system (1.6) admits a unique solution $(U,V;g,h) \in (C^{2+\alpha,1+\frac{\alpha}{2}}([g(t),h(t)]\times(0,T]))^2 \times C^{1+\frac{\alpha}{2}}((0,T])^2$. Thus, the local existence and uniqueness of the solution for system (1.6) are proved. □

For the convenience of later proof, we provide the following Comparison Principle in order to estimate the boundedness of $U(x,t)$, $V(x,t)$ for system (1.6) and the free boundaries $x = g(t)$ and $x = h(t)$. The lemma is similar to Lemma 3.5 in [10].

**Lemma 3.2.** (Comparison principle) *Assume that* $T \in (0,+\infty)$, $\overline{h}(t), \overline{g}(t) \in C^1([0,T])$, $\overline{U}, \overline{V} \in C(\overline{D_T^*}) \bigcap C^{2,1}(D_T^*)$ *with* $0 < \overline{U} \le N_1, 0 < \overline{V} \le N_2$ *and* $(\overline{U}, \overline{V}; \overline{h}, \overline{g})$ *satisfies*

$$\begin{cases} \overline{U}_t - D_1\overline{U}_{xx} \ge a_1(x,t)(N_1-\overline{U})\overline{V} - d_1(x,t)\overline{U}, & \overline{g}(t) < x < \overline{h}(t),\ 0<t<T,\\ \overline{V}_t - D_2\overline{V}_{xx} \ge a_2(x,t)(N_2-\overline{V})\overline{U} - d_2(x,t)\overline{V}, & \overline{g}(t) < x < \overline{h}(t),\ 0<t<T,\\ \overline{U}(x,t)\ge 0, \overline{V}(x,t) \ge 0, & x = \overline{g}(t)\,\mathrm{or}\,\overline{\mathrm{h}}(\mathrm{t}),\ 0<\mathrm{t}<\mathrm{T},\\ \overline{h}(0)\ge h_0, \overline{h}'(t) \ge -\mu\overline{U}_x(\overline{h}(t),t), & 0<t<T,\\ \overline{g}(0) \le -h_0, \overline{g}'(t) \le -\mu\overline{U}_x(\overline{g}(t),t), & 0<t<T,\\ \overline{U}(x,0)\ge U_0(x),\ \overline{V}(x,0)\ge V_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{3.18}$$

*then the solution* $(U,V;g,h)$ *of* (1.6) *satisfies*

$$\begin{aligned} &\overline{U}(x,t) \ge U(x,t),\ \overline{V}(x,t) \ge V(x,t),\\ &\quad \overline{h}(t)\ge h(t),\ g(t) \ge \overline{g}(t),\ \mathrm{for}\, g(t)\le x\le h(t), t\in(0,T], \end{aligned} \tag{3.19}$$

*where* $D_T^* = \{(x,t) \in \mathbb{R}^2 \mid x \in (\overline{g}(t), \overline{h}(t)), t \in (0,T]\}$.

**Remark 3.1.** If $(\overline{U}, \overline{V}; \overline{g}, \overline{h})$ satisfies the conditions of Lemma 3.2, then it is called the upper solution of (1.6). The corresponding lower solution can be similarly defined by reversing the above inequalities.

In order to extend the local solution of (1.6) to all $t \in (0,\infty)$, according to Lemma 2.2 in [10] or Lemma 2.5 and Lemma 2.6 in [16], we give the rough estimates about the supper and lower bound of $U(x,t), V(x,t), g'(t)$ and $h'(t)$.

**Lemma 3.3.** *Assume that* $T \in (0,+\infty)$. *Let* $(U,V;g,h)$ *be a solution of* (1.6) *for* $t \in (0,T]$, *then there exists a positive constant* $C > 0$ *independent of* $T$ *such that*

$$\begin{aligned} &0 < U(x,t) \le N_1, 0 < V(x,t) \le N_2,\ \mathrm{for}\, g(t) < x < h(t), 0 < t \le T,\\ &\quad 0 < h'(t) \le C, -C \le g'(t) < 0,\ \mathrm{for}\, 0 < t \le T. \end{aligned} \tag{3.20}$$

Now we turn to show the global existence of the solution for the problem (1.6).

**Theorem 3.4.** *For any given initial data* $(U_0, V_0)$ *satisfying* (1.7)*, the unique solution* $(U, V; g, h)$ *of* (1.6) *exists for all* $t \in (0, \infty)$*.*

*Proof.* Now we aim to show that the solution for system (1.6) can extend to all $t \in (0, \infty)$.

If the maximal existence interval of the solution is $[0, T_{max})$, then we will show $T_{max} = +\infty$. On the contrary, assuming that $T_{max} < +\infty$. According to Lemma 3.3, we can get that $U(x,t) \le N_1, V(x,t) \le N_2$ for $(x,t)$ in $[g(t), h(t)] \times [0, T_{max})$. Moreover, for the above positive constant $C$ in Lemma 3.3 independent on $T_{max}$, it holds that $0 < h'(t), -g'(t) \le C$, follows $h_0 \le h(t) \le h_0 + CT_{max}$ and $-h_0 - CT_{max} \le g(t) \le -h_0$ for $t \in [0, T_{max})$.

As in transformation (3.1), take $m(y,t) = U(x,t), n(y,t) = V(x,t)$. For any given $T < T_{max}$, applying the $L^p$ theory to (1.6), there exists a positive constant $C_1(\Gamma, N_1, N_2, T_{max})$ independent of $T$ such that $\|m\|_{W_p^{2,1}(\Delta_T)} + \|n\|_{W_p^{2,1}(\Delta_T)} \le \tilde{C}_1(\Gamma, N_1, N_2, T_{max})$, thus, $(m,n) \in (W_p^{2,1}(\Delta_{T_{max}}))^2$ for $p > \frac{3}{1-\alpha}$ and

$$\begin{aligned} &\|m\|_{W_p^{2,1}(\Delta_{T_{max}})} + \|m\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Delta_{T_{max}})} + \|n\|_{W_p^{2,1}(\Delta_{T_{max}})} + \|n\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\Delta_{T_{max}})} \\ &\quad \le \tilde{C}_1(\Gamma, N_1, N_2, T_{max}). \end{aligned} \tag{3.21}$$

In view of (3.13), we can get $(h, g) \in (C^{1+\frac{\alpha}{2}}([0, T_{max}]))^2$ and

$$\|h\|_{C^{1+\frac{\alpha}{2}}([0,T_{max}])} \le C_2(\Gamma, N_1, N_2, T_{max}), \|g\|_{C^{1+\frac{\alpha}{2}}([0,T_{max}])} \le \tilde{C}_2(\Gamma, N_1, N_2, T_{max}). \tag{3.22}$$

Applying the Schauder theory to (1.6), we can get that $(m,n) \in (C^{2+\alpha, 1+\frac{\alpha}{2}}([-1,1] \times (0, T_{max}]))^2$, then it follows

$$\|m\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([-1,1]\times[\varepsilon,T_{max}])} + \|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([-1,1]\times[\varepsilon,T_{max}])} \le \tilde{C}_3(\varepsilon, \Gamma, N_1, N_2, T_{max})$$

for any small $0 < \varepsilon \ll T_{max}$. Therefore, $(U, V) \in (C^{2+\alpha, 1+\frac{\alpha}{2}}([g(t), h(t)] \times (0, T_{max}]))^2$ and

$$\|U\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([g(t),h(t)]\times[\varepsilon,T_{max}])} + \|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([g(t),h(t)]\times[\varepsilon,T_{max}])} \le \tilde{C}_3(\varepsilon, \Gamma, N_1, N_2, T_{max}). \tag{3.23}$$

Thus, the system (1.6) admits a solution $(U, V; g, h)$ on $(0, T_{max}]$. Take $\{T_n\} \subset (0, T_{max})$ such that $T_n \to T_{max}$ as $n \to \infty$. Let $T_n$ be the initial time and $(U(x, T_n), V(x, T_n); g(T_n), h(T_n))$ be the initial function. By Theorem 3.1, there is a constant $t_0$ small enough dependent on $g(T_n), g'(T_n), h(T_n), h'(T_n)$, $\|U(\cdot, T_n)\|_{C^2([g(T_n), h(T_n)])}$, and $\|V(\cdot, T_n)\|_{C^2([g(T_n), h(T_n)])}$ such that problem (1.6) admits a unique solution $(U_n, V_n; g_n, h_n)$ for $t \in [T_n, T_n + t_0]$. Considering the uniqueness of the solution for (1.6), it follows that the solution $(U, V; g, h) = (U_n, V_n; g_n, h_n)$ for $T_n \le t < \min\{T_n + t_0, T_{max}\}$, which implies that the solution $(U, V; g, h)$ for (1.6) can be extended to $[0, T_n + t_0)$. According to (3.22) and (3.23), $t_0$ can be taken independent of $n$ such that $T_n + t_0 > T_{max}$, which contradicts to the choice of $T_{max}$. Thus, this theorem has been proved. □

## 4. Principal Lyapunov exponent

In order to investigate the global dynamics for model (1.6), considering the spatial heterogeneity and temporal almost periodicity, we first introduce the principal Lyapunov exponent and explore several properties by skew-product semiflow methods, which will be frequently used in later studies.

For any given $L > 0$ and the uniformly almost periodic matrix function $A(x,t)$ defined by (2.4), consider the following equation,

$$\begin{cases} I_t = D(x, D)I + A(x,t)I, \ -L < x < L, \ t > 0, \\ I(-L, t) = I(L, t) = 0, \qquad t > 0, \end{cases} \tag{4.1}$$

where $-D(x, D)$ is a second-order strongly elliptic differential operator matrix of diagonal type with $D(x, D) = (D_i \partial_{ii})$ for $i = 1, 2$.

Let $X \hookrightarrow C^2([-L,L]) \times C^2([-L,L])$ be the fractional power space (Chapter 1, [37]) with respect to the sectorial operator $-D(x,D)$ with homogeneous Cauchy boundary conditions, where $\mathcal{D}(-D(x,D)) = \{(u,v) \in (C^2([-L,L]))^2 \mid u(\pm L) = v(\pm L) = 0\}$. By the standard semigroup theory ([38]), for any $I_0 \in X$, there exists a unique solution $I(t,\cdot;I_0,A)$ of (4.1) satisfying $I(0,\cdot;I_0,A) = I_0(\cdot)$.

**Definition 4.1.** (*Definition 4.3, Part II,* [39]) We define the principal Lyapunov exponent $\lambda(A,L)$ of (4.1) as

$$\lambda(A,L) = \limsup_{t\to+\infty} \frac{\ln||\Phi(A,t)||_X}{t},$$

where $\Phi(A,t)$ satisfies $\Phi(A,t)I_0 = I(t,\cdot;I_0,A)$ for $I_0 \in X$.

Assume that $f_i (i=1,2)$ satisfies (***H1***)–(***H4***), then $g_i \in H(f_i)$ satisfies (***H1***)–(***H4***). Applying the standard semigroup theory for parabolic equations, for any given $g_i \in H(f_i)$ and $(U_0,V_0) \in X^+$, there exists a unique solution $(U(\cdot,t;U_0,V_0,g_1,g_2), V(\cdot,t;U_0,V_0,g_1,g_2))$ for the following equation

$$\begin{cases} U_t = D_1 U_{xx} + g_1(x,t,U,V), & -L < x < L,\ t > 0, \\ V_t = D_2 V_{xx} + g_2(x,t,U,V), & -L < x < L,\ t > 0, \\ U(x,t) = V(x,t) = 0, & x = -L \text{ or } x = L,\ t > 0, \end{cases} \tag{4.2}$$

for all $t > 0$ with $U(\cdot,0;U_0,V_0,g_1,g_2) = U_0(x), V(\cdot,0;U_0,V_0,g_1,g_2) = V_0(x)$, where $X^+ = \{(u,v) \in X \mid (u,v) \geq 0\}, X^{++} = \text{Int}(X^+)$.

Further, the system (4.2) generates a skew-product semiflow

$$\begin{aligned} \Pi_t : X^+ \times H(f_1) \times H(f_2) &\longrightarrow X^+ \times H(f_1) \times H(f_2), \quad t \geq 0 \\ (U_0,V_0,g_1,g_2) &\mapsto (U(\cdot,t;U_0,V_0,g_1,g_2), V(\cdot,t;U_0,V_0,g_1,g_2), g_1\cdot t, g_2\cdot t), \end{aligned} \tag{4.3}$$

where $g_i \cdot t(x,\cdot,U,V) = g_i(x,\cdot+t,U,V), i = 1,2$. It can be easily seen that $\Pi_t$ is continuous and compact by Lemma 3.3.

Next, we introduce the definition of continuous separation for skew-product semiflow.

**Definition 4.2.** (*Definition 3.11*, [40]) The skew-product semiflow (4.3) is said to admit a continuous separation if there are subspaces $\{X_1(G)\}_{G\in H(F)}$ and $\{X_2(G)\}_{G\in H(F)}$ with the following properties:

1) $X = X_1(G) \oplus X_2(G) (G \in H(F))$ and $X_1(G), X_2(G)$ vary continuously for $G \in H(F)$;
2) $X_1(G) = \text{span}\{I(G)\}$, where $I(G) \in \mathrm{X}^{++}$ and $\|I(G)\| = 1$ for $G \in H(F)$;
3) $X_2(G) \cap X^+ = \{0\}$ for every $G \in H(F)$;
4) $\Phi(G,t)X_1(G) = X_1(G\cdot t)$ and $\Phi(t,G)X_2(G) \subset X_2(G\cdot t)$ for any $t > 0$ and $G \in H(F)$;
5) There are $K_1 > 0$ and $\sigma > 0$ such that for any $G \in H(F)$ and $w \in X_2(G)$ with $\|w\| = 1$,

$$\|\Phi(G,t)w\| \leq K_1 e^{-\sigma t} \|\Phi(G,t)I(G)\|, \quad t > 0.$$

**Lemma 4.1.** *$\lambda(A(x,t),L)$ is monotonically increasing in $L \in (0,\infty)$.*

*Proof.* According to Lemma 4.5 (Part III, [39]), the skew-product semiflow $\Pi_t$ generated by (4.3) is strongly monotone in the sense that $(U(\cdot,t,U_0,V_0,g_1,g_2), V(\cdot,t,U_0,V_0,g_1,g_2)) \in X^{++}$ for any $t > 0, (U_0,V_0) \in X^+, g_i \in H(f_i)(i=1,2)$. Thus, by Theorem 4.4 of [39], the skew-product semiflow (4.3) admits a continuous separation, then there exists $I_L : H(A) \to X^{++}$ with $I_L = (U_L,V_L)$ satisfying the following properties:

(*a*) $I_L$ is continuous and $||I_L(\tilde{A})|| = 1$ for any $\tilde{A} \in H(A)$;

(*b*) $\lambda(A,L) = \lim\limits_{t\to\infty} \frac{\ln||I(\cdot,t,I_L,\tilde{A})||}{t} = \lim\limits_{t\to\infty} \frac{\ln||\Phi(\tilde{A},t)I_L(\tilde{A})||}{t}$ for any $\tilde{A} \in H(A)$.

Assume that $I(x,t,I_{L_i},A)$ for $i=1,2$ are the solutions for (4.2) with $L = L_1, L_2$, respectively. Without loss of generation, supposing $0 < L_1 < L_2$, then there is small $\tau > 0$ such that $I_{L_2} \geq \tau I_{L_1}$ uniformly for

$x \in [-L_1, L_1]$. According to the comparison principle, $I(x,t,I_{L_2},A) \geq I(x,t,\tau I_{L_1},A)$ for $x \in [-L_1, L_1]$. In view of $(a)$ and $(b)$, for any $\tilde{A} \in H(A)$, it holds that

$$\begin{aligned}\lambda(A,L_2) &= \lim_{t\to\infty} \frac{\ln||I(\cdot,t,I_{L_2},\tilde{A})||}{t} = \lim_{t\to\infty} \frac{\ln||\Phi(\tilde{A},t)I_{L_2}(\tilde{A})||}{t} \\ &\geq \lim_{t\to\infty} \frac{\ln||I(\cdot,t,\tau I_{L_1},\tilde{A})||}{t} = \lim_{t\to\infty} \frac{\ln(\tau||\Phi(\tilde{A},t)I_{L_1}(\tilde{A})||)}{t} \\ &= \lim_{t\to\infty} \frac{\ln\tau}{t} + \frac{\ln||I(\cdot,t,I_{L_1},\tilde{A})||}{t} \\ &= \lambda(A,L_1).\end{aligned} \tag{4.4}$$

Thus, our proof is completed. □

**Remark 4.1.** In view of assumption (***H5***), $L^*$ is the minimum such that $\lambda(A,L) > 0$, then $\lambda(A,L) > 0$ for any $L \geq L^*$.

**Remark 4.2.** In this paper, we always suppose that (***H5***) holds. According to Theorem 2.3 and Lemma 5.1 (see Sect. 5), considering the meaning of biology, we can explain this assumption in the sense that the living habitat at the remote distance is in high risk of infection by the disease.

Considering that the infected domain $(g(t),h(t))$ is moving concerning time $t$, we introduce the corresponding principal Lyapunov exponent

$$\lambda(t) := \lambda\left(A, \frac{h(t)-g(t)}{2}\right),\ t \geq 0$$

for the following system

$$\begin{cases} I_t = D(x,D)I + A(x,t)I,\ g(t) < x < h(t),\ t > 0 \\ I(h(t),t) = I(g(t),t) = 0,\ t > 0, \end{cases} \tag{4.5}$$

where $-D(x,D)$ is a second-order strongly elliptic differential operator matrix of diagonal type with $D(x,D) = (D_i\partial_{ii})$ for $i = 1,2$. In view of Lemmas 3.3 and 4.1, we can easily give the following result.

**Theorem 4.2.** *$\lambda(t)$ is monotonically increasing in $t \in [0,\infty)$.*

## 5. The long-time dynamics of WNv

In this section, we will discuss the long-time dynamical behaviors of the solution for (1.6) and investigate the conditions determining the spreading permanently or vanishing eventually for this disease.

First, we give the following definitions of vanishing and spreading for WNv.

**Definition 5.1.** The disease is vanishing if $h_\infty - g_\infty < \infty$ and

$$\lim_{t\to+\infty} ||U(\cdot,t)||_{C(g(t),h(t))} = 0, \lim_{t\to+\infty} ||V(\cdot,t)||_{C(g(t),h(t))} = 0;$$

The disease is spreading if $h_\infty - g_\infty = \infty$ and

$$\liminf_{t\to+\infty} ||U(\cdot,t)||_{C(g(t),h(t))} > 0, \liminf_{t\to+\infty} ||V(\cdot,t)||_{C(g(t),h(t))} > 0.$$

Next, for system (4.2), we recall a result similar to Theorem A in [41] which will be applied in proving Theorem 2.2 and Theorem 2.3.

**Lemma 5.1.** *Let matrix function $A(x,t)$ be defined by (2.4). For any given $g_i \in H(f_i)$ for $i=1,2$. Let $(U(\cdot,t;U_0,V_0,g_1,g_2), V(\cdot,t;U_0,V_0,g_1,g_2))$ be the solution of (4.2), then the followings hold.*

***(1)*** *If $\lambda(A,L)<0$, then $\lim_{t\to\infty}||U(\cdot,t;U_0,V_0,g_1,g_2)||=0$, $\lim_{t\to\infty}||V(\cdot,t;U_0,V_0,g_1,g_2)||=0$ uniformly for $g_i\in H(f_i)$. Further, $\lim_{t\to\infty}||U(\cdot,s+t;U_0,V_0,s)||=0$ and $\lim_{t\to\infty}||U(\cdot,s+t;U_0,V_0,s)||=0$ uniformly for $s\in\mathbb{R}$.*

***(2)*** *If $\lambda(A,L)>0$, there exist $U_L: H(f_1)\times H(f_2)\longrightarrow C([-L,L])$ and $V_L: H(f_1)\times H(f_2)\longrightarrow C([-L,L])$ such that $U_L(g_1,g_2)$ and $V_L(g_1,g_2)$ are continuous for $g_i\in H(f_i)$ and $U(\cdot,t;U_L,V_L,g_1,g_2)=U_L(g_1\cdot t,g_2\cdot t)(\cdot)$, $V(\cdot,t;U_L,V_L,g_1,g_2)=V_L(g_1\cdot t,g_2\cdot t)(\cdot)$. Meanwhile, it holds that*

$$\lim_{t\to\infty}||U(\cdot,t;U_0,V_0,g_1,g_2)-U(\cdot,t;U_L(g_1,g_2),V_L(g_1,g_2),g_1,g_2)||=0,$$
$$\lim_{t\to\infty}||V(\cdot,t;U_0,V_0,g_1,g_2)-V(\cdot,t;U_L(g_1,g_2),V_L(g_1,g_2),g_1,g_2)||=0$$

*uniformly in $g_i\in H(f_i)$ for any $(U_0,V_0)\in X^+\backslash\{0\}$. Further, $U_L^*(x,t):=U_L(f_1\cdot t,f_2\cdot t)(x)$ and $V_L^*(x,t):=V_L(f_1\cdot t,f_2\cdot t)(x)$ are uniformly almost periodic in $t\in\mathbb{R}$. Moreover, for any $(U_0,V_0)\in X^+\backslash\{0\}$, it holds that*

$$\lim_{t\to\infty}||U(\cdot,s+t;U_0,V_0,s)-U_L^*(\cdot,s+t)||=0, \lim_{t\to\infty}||V(\cdot,s+t;U_0,V_0,s)-V_L^*(\cdot,s+t)||=0$$

*uniformly for $s\in\mathbb{R}$, where $U(\cdot,s+t;U_0,V_0,s)=U(\cdot,t;U_0,V_0,f_1\cdot s,f_2\cdot s), V(\cdot,s+t;U_0,V_0,s)=V(\cdot,t;U_0,V_0,f_1\cdot s,f_2\cdot s)$.*

**Lemma 5.2.** *Assume that **(H1)**–**(H5)** hold. Take $L\ge L^*$, then*

$$\inf_{\substack{x\in[-L,L],\tilde{x}\in\mathbb{R},\\ g_i\in H(f_i)}} U^*(x,0;\tilde{x},g_1,g_2,L)>0, \inf_{\substack{x\in[-L,L],\tilde{x}\in\mathbb{R},\\ g_i\in H(f_i)}} V^*(x,0;\tilde{x},g_1,g_2,L)>0,$$

*for $i=1,2$. Where $(U^*(x,t;\tilde{x},g_1,g_2,L), V^*(x,t;\tilde{x},g_1,g_2,L))$ is the unique positive almost periodic solution of the following system,*

$$\begin{cases} U_t=D_1U_{xx}+g_1(x+\tilde{x},t,U,V), & -L<x<L,\tilde{x}\in\mathbb{R},t>0,\\ V_t=D_2V_{xx}+g_2(x+\tilde{x},t,U,V), & -L<x<L,\tilde{x}\in\mathbb{R},t>0,\\ U(x,t)=V(x,t)=0, & x=-L\,\text{or}\,x=L,t>0. \end{cases} \tag{5.1}$$

Indeed, we can see that $(U^*(x,t;\tilde{x},g_1,g_2,L), V^*(x,t;\tilde{x},g_1,g_2,L)) = (U^*(x,0;\tilde{x},g_1\cdot t,g_2\cdot t,L), V^*(x,0;\tilde{x},g_1\cdot t,g_2\cdot t,L))$.

*Proof.* The proof of this lemma can refer to Lemma 4.1 in [42], it can be proved by making a minor modification, so we omit the detailed proof. □

Considering the dependence of boundary functions $g(t)$ and $h(t)$ on $\mu$, denote

$$h_\mu(t):=h(t)=h(t;U_0,V_0,h_0)\,\text{and}\,g_\mu(t):=g(t)=g(t;U_0,V_0,h_0)$$

with $h(0)=h_0, g(0)=-h_0$. Then the following result holds.

**Lemma 5.3.** *For all $t>0$, $h_\mu(t)$ is strictly monotonically increasing in $\mu$, and $g_\mu(t)$ is strictly monotonically decreasing in $\mu$.*

*Proof.* We will prove this lemma mainly by the comparison principle. Assume that $(U_1,V_1;g_{\mu_1},h_{\mu_1})$ and $(U_2,V_2;g_{\mu_2},h_{\mu_2})$ are the two solutions for problem (1.6). For simplification, we only need to compare $h_{\mu_1}(t)$ with $h_{\mu_2}(t)$, then we can similarly obtain the strict monotonicity of $g_\mu(t)$.

Without loss of generality, assume that $0<\mu_1<\mu_2$, then

$$h'_{\mu_1}(t)=-\mu_1U_{1x}(h_{\mu_1}(t),t)<-\mu_2U_{1x}(h_{\mu_1}(t),t). \tag{5.2}$$

By Lemma 3.2, it follows $h_{\mu_1}(t)\le h_{\mu_2}(t)$ for all $t\in[0,\infty)$.

Now it is our turn to prove that $h_{\mu_1}(t) < h_{\mu_2}(t)$ in $[0,\infty)$. On the contrary, assume that positive time $T^*$ is the first time such that $h_{\mu_1}(t) < h_{\mu_2}(t)$ for $t \in (0,T^*)$ and $h_{\mu_1}(T^*) = h_{\mu_2}(T^*)$, then

$$h'_{\mu_1}(T^*) \geq h'_{\mu_2}(T^*). \tag{5.3}$$

Let $\Sigma_{T^*} := \left\{(x,t) \in \mathbb{R}^2 \mid 0 \leq x < h_{\mu_1}(t), 0 < t \leq T^*\right\}$. Applying the strong maximum principle to $U_1$ and $U_2$, it follows that $U_1(x,t) < U_2(x,t)$ in $\Sigma_{T^*}$. Let $H(x,t) = U_2(x,t) - U_1(x,t)$, then $H(x,t) > 0$ for $(x,t) \in \Sigma_{T^*}$ and $H\left(h_{\mu_1}(T^*), T^*\right) = 0$. Follows, we can get that $H_x\left(h_{\mu_1}(T^*), T^*\right) < 0$. In view of $(U_i)_x\left(h_{\mu_1}(T^*), T^*\right) < 0$ and $\mu_1 < \mu_2$, then $-\mu_1 (U_1)_x\left(h_{\mu_1}(T^*), T^*\right) < -\mu_2 (U_2)_x\left(h_{\mu_2}(T^*), T^*\right)$. Therefore, $h'_{\mu_1}(T^*) < h'_{\mu_2}(T^*)$, which yields a contradiction to (5.3). Thus, $h_\mu(t)$ is strictly monotonically increasing about $\mu$ for all $t > 0$.

Similarly, we can easily get that $-g_{\mu_1}(t) < -g_{\mu_2}(t)$ for all $t > 0$. Therefore, our proof is completed. □

In the rest of this section, for any given $(U_0, V_0)$ satisfying (1.7), let $(U(x,t;U_0,V_0,h_0), V(x,t;U_0, V_0,h_0))$ denote the solution of system (1.6) with $U(x,0;U_0,V_0,h_0) = U_0, V(x,0;U_0,V_0,h_0) = V_0, h(0) = h_0, g(0) = -h_0$.

**Theorem 5.4.** *If $h_\infty - g_\infty < \infty$, then $\lim_{t\to\infty} h'(t,U_0,V_0,h_0) = 0$, $\lim_{t\to\infty} g'(t,U_0,V_0,h_0) = 0$.*

*Proof.* Now we only necessarily prove the case of $h'(t,U_0,V_0,h_0)$. On the contrary, assume that there exists a positive sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} h'(t_n,U_0,V_0,h_0) > 0. \tag{5.4}$$

Let $h_n(t) = h(t+t_n,U_0,V_0,h_0)$, for $t \geq 0$, then $\lim_{n\to\infty} h_n(t) = h_\infty$ uniformly for $t \geq 0$. According to Lemma 3.3, we can get that $\{h'_n(t)\}$ is uniformly bounded and equicontinuous on $[0,\infty)$. By Arzela–Ascoli theorem, there exists $h^*(t)$ such that $\lim_{n\to\infty} h'_n(t) = h^*(t)$ uniformly in any bounded sets of $[0,\infty)$. Since $\lim_{n\to\infty} h_n(t) = h_\infty < \infty$, then $h^*(t) \equiv 0$, which implies that $\lim_{n\to\infty} h'(t_n,U_0,V_0,h_0) = 0$. It is a contradiction to (5.4). Thus, $\lim_{t\to\infty} h'(t,U_0,V_0,h_0) = 0$. Similarly, we can prove $\lim_{t\to\infty} g'(t,U_0,V_0,h_0) = 0$. □

**Theorem 5.5.** *Assume that **(H1)**–**(H5)** hold. If $h_\infty - g_\infty < \infty$, then*

$$\lim_{t\to+\infty} U(x,t;U_0,V_0,h_0) = 0 \text{ and } \lim_{t\to+\infty} V(x,t;U_0,V_0,h_0) = 0$$

*uniformly in $x \in [g_\infty, h_\infty]$. That is, the disease will vanish.*

*Proof.* Let $f_i$ be defined as in (2.3) for $i = 1,2$. then $f_1$ and $f_2$ satisfy (***H1***)–(***H4***) and $A(x,t)$ defined by (2.4) satisfies (***H5***).

If $h_\infty - g_\infty < \infty$, it is easy to obtain that $h_\infty < \infty$ and $g_\infty > -\infty$.

According to regularity and the prior estimates about parabolic equations ([37]), considering the system (1.6), for any given sequence $\{t_n\}$ satisfying $t_n \to \infty$ as $n \to \infty$, there exist a subsequence $\{t_{n_k}\}$ satisfying $t_{n_k} \to \infty$ as $k \to \infty$, $(\hat{U}^*(x,t), \hat{V}^*(x,t)) \in (C([g_\infty,h_\infty]\times\mathbb{R}))^2$ and $g_i^* \in H(f_i)$ for $i = 1,2$ such that $f_i \cdot t_{n_k} \to g_i^*$,

$$\lim_{k\to\infty} ||U(\cdot,t+t_{n_k};U_0,V_0,h_0) - \hat{U}^*(\cdot,t)||_{C^1([g(t+t_{n_k}),h(t+t_{n_k})])} = 0 \tag{5.5}$$

and

$$\lim_{k\to\infty} ||V(\cdot,t+t_{n_k};U_0,V_0,h_0) - \hat{V}^*(\cdot,t)||_{C^1([g(t+t_{n_k}),h(t+t_{n_k})])} = 0, \tag{5.6}$$

where $(\hat{U}^*(x,t), \hat{V}^*(x,t))$ is the entire solution for the following system,

$$\begin{cases} U_t = D_1 U_{xx} + g_1^*(x,t,U,V), & g_\infty < x < h_\infty, \\ V_t = D_2 V_{xx} + g_2^*(x,t,U,V), & g_\infty < x < h_\infty, \\ U(x,t) = V(x,t) = 0, & x = g_\infty \text{ or } x = h_\infty. \end{cases} \tag{5.7}$$

Next, we accomplish the proof of this theorem in two steps.

**Step 1** To show $h_\infty - g_\infty \leq 2L^*$ following from $h_\infty - g_\infty < \infty$.

On the contrary, assume that $h_\infty - g_\infty \in (2L^*, \infty)$, then there exist $t^* > 0$ and $\epsilon > 0$ such that $h(t) - g(t) > h_\infty - g_\infty - 2\epsilon > 2L^*$ for $t \geq t^*$, thus, by (***H5***) and Theorem 4.2, $\lambda(t) > 0$. For the following system

$$\begin{cases} U_t = D_1 U_{xx} + f_1(x,t,U,V), & g_\infty + \epsilon < x < h_\infty - \epsilon,\ t > 0, \\ V_t = D_2 V_{xx} + f_2(x,t,U,V), & g_\infty + \epsilon < x < h_\infty - \epsilon,\ t > 0, \\ U(x,t) = V(x,t) = 0, & x = g_\infty + \epsilon \text{ or } x = h_\infty - \epsilon,\ t > 0, \end{cases} \tag{5.8}$$

by comparison principle, we can get that

$$\begin{aligned} U(\cdot, t+t^*; U_0, U_0, h_0) &\geq \tilde{U}(\cdot, t+t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*), \\ V(\cdot, t+t^*; U_0, U_0, h_0) &\geq \tilde{V}(\cdot, t+t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*). \end{aligned} \tag{5.9}$$

where $(\tilde{U}(\cdot, t+t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*), \tilde{V}(\cdot, t+t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*))$ is the solution of (5.8) with

$$\begin{aligned} \tilde{U}(\cdot, t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*) &= U(\cdot, t^*; U_0, U_0, h_0), \\ \tilde{V}(\cdot, t^*; U(\cdot, t^*; U_0, U_0, h_0), V(\cdot, t^*; U_0, U_0, h_0), t^*) &= V(\cdot, t^*; U_0, U_0, h_0). \end{aligned}$$

In view of Lemma 5.1, the system (5.8) admits a positive almost time periodic solution $(U_\epsilon(x,t), V_\epsilon(x,t))$. Moreover, for any $(U_0, V_0) \in X^{++}$, it holds that

$$\lim_{t\to\infty} ||\tilde{U}(\cdot, t+t^*; U_0, V_0, t^*) - U_\epsilon(\cdot, t+t^*)|| = 0 \tag{5.10}$$

and

$$\lim_{t\to\infty} ||\tilde{V}(\cdot, t+t^*; U_0, V_0, t^*) - V_\epsilon(\cdot, t+t^*)|| = 0. \tag{5.11}$$

By comparison principle, combining (5.10) and (5.11), we get

$$\hat{U}^*(x,t) > 0, \hat{V}^*(x,t) > 0, x \in (g_\infty, h_\infty), t \in \mathbb{R},$$

which implies $\hat{U}^*_x(h_\infty, t) < 0, \hat{V}^*_x(h_\infty, t) < 0$. Therefore, $\limsup_{t\to\infty} U_x(h(t), t; U_0, V_0, h_0) < 0$, it implies $\liminf_{t\to\infty} h'(t) = \liminf_{t\to\infty} -\mu U_x(h(t), t; U_0, V_0, h_0) > 0$, which contradicts to Theorem 5.4. Thus, we can obtain that $h_\infty - g_\infty < \infty$ gives $h_\infty - g_\infty \leq 2L^*$.

**Step 2** To show that if $h_\infty - g_\infty < \infty$, then

$$\lim_{t\to\infty} \|U(\cdot, t; U_0, V_0, h_0)\|_{C([g(t),h(t)])} = 0, \lim_{t\to\infty} \|V(\cdot, t; U_0, V_0, h_0)\|_{C([g(t),h(t)])} = 0. \tag{5.12}$$

Let

$$\tilde{u}_0(x) = \begin{cases} U_0(x), & \text{for } -h_0 \leq x \leq h_0, \\ 0, & \text{for } |x| > h_0. \end{cases} \quad \tilde{v}_0(x) = \begin{cases} V_0(x), & \text{for } -h_0 \leq x \leq h_0, \\ 0, & \text{for } |x| > h_0. \end{cases}$$

Assume that $(\bar{u}(x,t), \bar{v}(x,t))$ is the solution of the problem

$$\begin{cases} \bar{u}_t = D_1 \bar{u}_{xx} + f_1(x,t,\bar{u},\bar{v}), & g_\infty < x < h_\infty, t > 0, \\ \bar{v}_t = D_2 \bar{v}_{xx} + f_2(x,t,\bar{u},\bar{v}), & g_\infty < x < h_\infty, t > 0, \\ \bar{u}(g_\infty, t) = \bar{u}(h_\infty, t) = 0, & t > 0, \\ \bar{v}(g_\infty, t) = \bar{v}(h_\infty, t) = 0, & t > 0, \\ \bar{u}(x,0) = \tilde{u}_0(x), \bar{v}(x,0) = \tilde{v}_0(x), & g_\infty \leq x \leq h_\infty. \end{cases}$$

Applying Lemma 3.2, we can get that $\bar{u}(x,t) \geq U(x,t; U_0, V_0, h_0) \geq 0, \bar{v}(x,t) \geq V(x,t; U_0, V_0, h_0) \geq 0$ for $x \in [g(t), h(t)]$, $t > 0$. If $h_\infty - g_\infty < 2L^*$, assuming (***H5***), then $\lambda\left(A, \frac{h_\infty - g_\infty}{2}\right) < 0$. By Lemma 5.1, $\lim_{t\to\infty} (\bar{u}, \bar{v}) = (0,0)$ uniformly for $x \in [g_\infty, h_\infty]$. Hence,

$$\lim_{t\to\infty} \|U(\cdot, t; U_0, V_0, h_0)\|_{C([g(t),h(t)])} = 0, \lim_{t\to\infty} \|V(\cdot, t; U_0, V_0, h_0)\|_{C([g(t),h(t)])} = 0.$$

If $h_\infty - g_\infty = 2L^*$, without loss of generality, assume that $\lim_{t\to\infty} \|U(\cdot,t;U_0,V_0,h_0)\|_{C([g(t),h(t)])} \neq 0$, then there exist a sequence $\{\breve{s}_n\}$ with $\breve{s}_n \longrightarrow \infty$ as $n \to \infty$, $(\breve{U}^*(x), \breve{V}^*(x))$ with $\breve{U}^*, \breve{V}^* \geq, \not\equiv 0$ and $\breve{g}_i^* \in H(f_i)$ such that $\lim_{n\to\infty} f_i \cdot \breve{s}_n = \breve{g}_i^*, i = 1,2$ and $\lim_{n\to\infty} \left\| U(\cdot, \breve{s}_n; U_0, V_0, h_0) - \breve{U}^*(\cdot) \right\|_{C([g(\breve{s}_n),h(\breve{s}_n)])} = 0$, $\lim_{n\to\infty} \left\| V(\cdot, \breve{s}_n; U_0, V_0, h_0) - \breve{V}^*(\cdot) \right\|_{C([g(\breve{s}_n),h(\breve{s}_n)])} = 0$. It follows that $\left(U\left(\cdot, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*\right), V\left(\cdot, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*\right)\right)$ is the entire solution for the following equation,

$$\begin{cases} u_t = D_1 u_{xx} + \breve{g}_1^*(x,t,u,v), & g_\infty < x < h_\infty, \\ v_t = D_2 v_{xx} + \breve{g}_2^*(x,t,u,v), & g_\infty < x < h_\infty, \\ u(g_\infty,t) = u(h_\infty,t) = 0, \\ v(g_\infty,t) = v(h_\infty,t) = 0. \end{cases} \tag{5.13}$$

Applying Hopf lemma to $U(h_\infty, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*)$ and $U(g_\infty, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*)$, we can get that

$$U_x\left(h_\infty, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*\right) < 0, U_x\left(g_\infty, t; \breve{U}^*, \breve{V}^*, \breve{g}_1^*, \breve{g}_2^*\right) > 0,$$

which implies

$$\lim_{n\to\infty} h'(\breve{s}_n) = -\lim_{n\to\infty} \mu U_x(h(\breve{s}_n), \breve{s}_n; U_0, V_0, h_0) > 0, \lim_{n\to\infty} g'(\breve{s}_n) = -\lim_{n\to\infty} \mu U_x(g(\breve{s}_n), \breve{s}_n; U_0, V_0, h_0) < 0.$$

This is a contradiction to Theorem 5.4. Thus, our proof is completed. □

**Remark 5.1.** From the proof of the above theorem, we can obtain that the densities of infected populations will decay to 0 and the eventually infected domain is no more than the critical size $2L^*$ when the disease vanishes.

The following theorem gives the long-time asymptotic behavior as the spreading happens, which is the sharp distinction for our spatial heterogeneous and time almost periodic WNv model.

**Theorem 5.6.** *Assume that **(H1)**–**(H5)** hold. For any given $h_0$ and $(U_0, V_0)$ satisfying (1.7), let*

$$(U(x,t;U_0,V_0,h_0), V(x,t;U_0,V_0,h_0))$$

*be the solution for (1.6). If $h_\infty - g_\infty = \infty$, then*

$$\lim_{t\to+\infty}(U(x,t;U_0,V_0,h_0) - U^*(x,t)) = 0, \lim_{t\to+\infty}(V(x,t;U_0,V_0,h_0) - V^*(x,t)) = 0 \tag{5.14}$$

*locally uniformly for $x \in \mathbb{R}$, where $(U^*(x,t), V^*(x,t))$ is the unique positive almost periodic solution of the system (2.5). That is, the disease will spread.*

*Proof.* Now we complete the proof of this theorem in three steps.

**Step 1**: To show that $h_\infty = \infty$ and $g_\infty = -\infty$ when $h_\infty - g_\infty = \infty$.

We give a proof by contradiction for this argument. Without loss of generality, assume that $g_\infty = -\infty$ and $h_\infty < \infty$. According to Theorem 5.4, we have $\lim_{t\to\infty} h'(t) = 0$. Choose $t_0^* > 0$ large enough such that $h(t_0^*) - g(t_0^*) > 2L^*$. Given (***H5***) and Lemma 5.2, it follows

$$\inf_{t>t_0^*, x\in[g(t_0^*),h(t_0^*)]} U(t,x;U_0,V_0,h_0) > 0,$$
$$\inf_{t>t_0^*, x\in[g(t_0^*),h(t_0^*)]} V(t,x;U_0,V_0,h_0) > 0.$$

Take $t_n \to \infty$ such that $f(x,t+t_n,U,V) \to f^*(x,t,U,V)$ and

$$U(x,t+t_n;U_0,V_0,h_0) \to U^*(x,t), \ V(x,t+t_n;U_0,V_0,h_0) \to V^*(x,t).$$

We thereby obtain that $(U^*(x,t), V^*(x,t)$ is the solution of

$$\begin{cases} u_t = D_1 u_{xx} + f^*(x,t,u,v),\ -\infty < x < h_\infty, \\ v_t = D_2 v_{xx} + f^*(x,t,u,v),\ -\infty < x < h_\infty, \\ u\,(h_\infty,t) = 0,\ v\,(h_\infty,t) = 0, \end{cases}$$

and $\inf_{t\in\mathbb{R},x\in[g(t_0^*),h(t_0^*)]} U^*(x,t) > 0$, $\inf_{t\in\mathbb{R},x\in[g(t_0^*),h(t_0^*)]} V^*(x,t) > 0$. Apply Hopf lemma to get that $U_x^*\,(h_\infty,t) < 0$, which implies that

$$h'\,(t+t_n) \to -\mu U_x^*\,(h_\infty,t) > 0.$$

It is contradict to the fact that $\lim_{t\to\infty} h'(t) = 0$. Therefore, $g_\infty = -\infty$ and $h_\infty = \infty$.

**Step 2**: To prove the existence and uniqueness of the positive time almost periodic solution for (2.5).

Let $U_0 := N_1, V_0 := N_2$, then by comparison principle, $U(x,t;N_1,N_2,f_1\cdot(-t),f_2\cdot(-t))$ and $V(x,t;N_1, N_2,f_1\cdot(-t),f_2\cdot(-t))$ decrease in $t\in\mathbb{R}$. Take

$$U^*(f_1,f_2)(x) := \lim_{t\to\infty} U(x,t;N_1,N_2,f_1\cdot(-t),f_2\cdot(-t)),$$
$$V^*(f_1,f_2)(x) := \lim_{t\to\infty} V(x,t;N_1,N_2,f_1\cdot(-t),f_2\cdot(-t))$$

for $x\in\mathbb{R}$. Then it follows

$$U(\cdot,t;U^*(f_1,f_2)(x),V^*(f_1,f_2)(x),f_1,f_2) = U^*(f_1\cdot t,f_2\cdot t)(\cdot),$$
$$V(\cdot,t;U^*(f_1,f_2)(x),V^*(f_1,f_2)(x),f_1,f_2) = V^*(f_1\cdot t,f_2\cdot t)(\cdot),$$

where $(U(x,t;N_1,N_2,f_1,f_2),V(x,t;N_1,N_2,f_1,f_2))$ is the solution for (4.2) for $U_0 = N_1, V_0 = N_2$ and $L=\infty$. Applying the similar methods in proving Proposition 6.1 of [42], we can show that $(U^*(f_1\cdot t,f_2\cdot t)(x),V^*(f_1\cdot t,f_2\cdot t)(x))$ is the unique solution for (2.5). Moreover, by Lemma 5.2, it follows

$$\inf_{x\in\mathbb{R},t\in\mathbb{R}^+} U^*(f_1\cdot t,f_2\cdot t)(x) > 0, \qquad \inf_{x\in\mathbb{R},t\in\mathbb{R}^+} V^*(f_1\cdot t,f_2\cdot t)(x) > 0. \tag{5.15}$$

Now it is only necessary to prove that $U^*(f_1\cdot t,f_2\cdot t)(x)$ and $V^*(f_1\cdot t,f_2\cdot t)(x)$ are uniformly almost periodic in $t\in\mathbb{R}$ with $x$ in bounded subsets of $\mathbb{R}$. Since $f_i(x,t,U,V)$ is uniformly almost periodic in $t$ with $x\in\mathbb{R}$ and $(U,V)$ in bounded subsets of $\mathbb{R}^2$ for $i=1,2$, according to Theorems 1.17 and 2.10 ([43]), for any sequences $\{a_n\}\subset\mathbb{R}$ and $\{b_n\}\subset\mathbb{R}$, there exist $\{t_n\}\subset\{a_n\}$ and $\{s_n\}\subset\{b_n\}$ such that $\lim_{n\to\infty} f_i(x,t+t_n+s_n,U,V) = \lim_{n\to\infty}\lim_{m\to\infty} f_i(x,t+t_n+s_m,U,V)$ for $(x,t,U,V)\in\mathbb{R}^4, i=1,2$. Assume that

$$\lim_{n\to\infty} f_i(x,t+t_n+s_n,U,V) = f_i^*(x,t,U,V),\ \lim_{m\to\infty} f_i(x,t+s_m,U,V) = f_i^{**}(x,t,U,V).$$

Then we can get that

$$\lim_{m\to\infty} U(x,t+s_m,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2) = U^*(f_1^{**}\cdot t,f_2^{**}\cdot t)(x),$$
$$\lim_{m\to\infty} V(x,t+s_m,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2) = V^*(f_1^{**}\cdot t,f_2^{**}\cdot t)(x)$$

uniformly for $x$ in bounded sets of $\mathbb{R}$. Further, it follows that

$$\begin{aligned} &\lim_{n\to\infty}\lim_{m\to\infty} U(x,t+t_n+s_m,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2) \\ &\quad= \lim_{n\to\infty} U(x,t_n,U^*(f_1^{**}\cdot t,f_2^{**}\cdot t),V^*(f_1^{**}\cdot t,f_2^{**}\cdot t),f_1^{**}\cdot t,f_2^{**}\cdot t) \\ &\quad= \lim_{n\to\infty} U(x,t,U^*(f_1^{**}\cdot t_n,f_2^{**}\cdot t_n),V^*(f_1^{**}\cdot t_n,f_2^{**}\cdot t_n),f_1^{**}\cdot t_n,f_2^{**}\cdot t_n) \\ &\quad= U^*(f_1^*\cdot t,f_2^*\cdot t)(x), \\ &\lim_{n\to\infty}\lim_{m\to\infty} V(x,t+t_n+s_m,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2) \\ &\quad= \lim_{n\to\infty} V(x,t_n,U^*(f_1^{**}\cdot t,f_2^{**}\cdot t),V^*(f_1^{**}\cdot t,f_2^{**}\cdot t),f_1^{**}\cdot t,f_2^{**}\cdot t) \end{aligned} \tag{5.16}$$

$$= \lim_{n\to\infty} V(x,t,U^*(f_1^{**}\cdot t_n, f_2^{**}\cdot t_n), V^*(f_1^{**}\cdot t_n, f_2^{**}\cdot t_n), f_1^{**}\cdot t_n, f_2^{**}\cdot t_n)$$
$$= V^*(f_1^*\cdot t, f_2^*\cdot t)(x) \tag{5.17}$$

uniformly for $x$ in bounded sets of $\mathbb{R}$. Moreover,

$$\lim_{n\to\infty} U(x,t+t_n+s_n, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2) = U^*(f_1^*\cdot t, f_2^*\cdot t)(x)$$
$$\lim_{n\to\infty} V(x,t+t_n+s_n, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2) = V^*(f_1^*\cdot t, f_2^*\cdot t)(x)$$

uniformly for $x$ in bounded sets of $\mathbb{R}$. Thus,

$$\lim_{n\to\infty}\lim_{m\to\infty} U(x,t+t_n+s_m, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2)$$
$$= \lim_{n\to\infty} U(x,t+t_n+s_n, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2),$$
$$\lim_{n\to\infty}\lim_{m\to\infty} V(x,t+t_n+s_m, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2)$$
$$= \lim_{n\to\infty} V(x,t+t_n+s_n, U^*(f_1,f_2), V^*(f_1,f_2), f_1, f_2).$$

According to the regularity and prior estimates for parabolic differential equations,

$$U(x,t,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2) \quad \text{and} \quad V(x,t,U^*(f_1,f_2),V^*(f_1,f_2),f_1,f_2)$$

are uniformly continuous for $(x,t)\in\mathbb{R}^2$, applying Theorems 1.17 and 2.10 ([43]), it follows that $U^*(f_1\cdot t, f_2\cdot t)(x)$ and $V^*(f_1\cdot t, f_2\cdot t)(x)$ are almost periodic in $t\in\mathbb{R}$ uniformly with $x$ in bounded sets of $\mathbb{R}$.

**Step 3**: To prove the convergence result (5.14).

Let $U_L(f_1,f_2)(x)$ and $V_L(f_1,f_2)(x)$ be in Lemma 5.1, then for any fixed $x, U_L(f_1,f_2)(x)$ and $V_L(f_1,f_2)(x)$ are increasing in $L$. Applying the Comparison Principle and Lemma 5.2, we can obtain that

$$\lim_{L\to\infty} U_L(f_1,f_2)(x) = U^*(f_1,f_2)(x), \lim_{L\to\infty} V_L(f_1,f_2)(x) = V^*(f_1,f_2)(x) \tag{5.18}$$

locally uniformly for $x\in\mathbb{R}$.

For any $T>0$ satisfying $h(T)-g(T)>2L^*$, denote $U(\cdot,T;U_0,V_0,h_0) := U(\cdot,T)$ and $V(\cdot,T;U_0,V_0,h_0) := V(\cdot,T)$, we can get

$$U(x,t+T;U_0,V_0,h_0) \geq U_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) \quad \text{for} \quad t\geq 0,$$
$$V(x,t+T;U_0,V_0,h_0) \geq V_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) \quad \text{for} \quad t\geq 0,$$

where $(U_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T), V_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T))$ is the solution of following system

$$\begin{cases} U_t = D_1U_{xx} + f_1\cdot T(x,t,U,V), & g(T;U_0,V_0,h_0) < x < h(T;U_0,V_0,h_0),\ t>0,\\ V_t = D_2V_{xx} + f_2\cdot T(x,t,U,V), & g(T;U_0,V_0,h_0) < x < h(T;U_0,V_0,h_0),\ t>0,\\ U(x,t)=V(x,t)=0, & x=g(T;U_0,V_0,h_0) \text{ or } h(T;U_0,V_0,h_0),\ t>0 \end{cases} \tag{5.19}$$

with $L = \dfrac{h(T;U_0,V_0,h_0) - g(T;U_0,V_0,h_0)}{2}$,

$$U_L(x,0;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) = U(x,T;U_0,V_0,h_0)$$

and

$$V_L(x,0;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) = V(x,T;U_0,V_0,h_0).$$

According to Lemma 5.1,

$$\lim_{t\to\infty}(U_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) - U_L(f_1\cdot(t+T), f_2\cdot(t+T)))(x) = 0,$$
$$\lim_{t\to\infty}(V_L(x,t;U(\cdot,T),V(\cdot,T),f_1\cdot T,f_2\cdot T) - V_L(f_1\cdot(t+T), f_2\cdot(t+T)))(x) = 0$$

uniformly for $x$ in $[g(T; U_0, V_0, h_0), h(T; U_0, V_0, h_0)]$. In view of (5.18)

$$\lim_{L\to\infty}\left(U_L(f_1\cdot(t+T), f_2\cdot(t+T)) - U^*(f_1\cdot(t+T), f_2\cdot(t+T))\right)(x) = 0,$$
$$\lim_{L\to\infty}\left(V_L(f_1\cdot(t+T), f_2\cdot(t+T)) - V^*(f_1\cdot(t+T), f_2\cdot(t+T))\right)(x) = 0$$

uniformly for $x$ in any bounded sets of $\mathbb{R}$. By Comparison Principle,

$$(U(x,t;U_0,V_0,h_0), V(x,t;U_0,V_0,h_0)) \geq (U_L(x,t;U_0,V_0,h_0), V_L(x,t;U_0,V_0,h_0))$$

uniformly for $(x,t) \in [g(t), h(t)] \times [0,\infty)$. Then we can get

$$\lim_{t\to\infty}(U(x,t;U_0,V_0,h_0) - U^*(f_1\cdot t, f_2\cdot t)(x)) = 0, \lim_{t\to\infty}(V(x,t;U_0,V_0,h_0) - V^*(f_1\cdot t, f_2\cdot t)(x)) = 0$$

locally uniformly for $x \in \mathbb{R}$. Take $U^*(x,t) = U^*(f_1\cdot t, f_2\cdot t)(x)$, $V^*(x,t) = V^*(f_1\cdot t, f_2\cdot t)(x)$. Therefore, our proof is completed. □

Combining Theorem 5.5 with Theorem 5.6, we can give the following proof.

*Proof of Theorem 2.2.* Assume that (***H1***)–(***H5***) hold. For any given $g(0), h(0)$ and initial functions $(U_0, V_0)$ satisfying (1.7). Let $(U(x,t,U_0,V_0,g,h), V(x,t,U_0,V_0,g,h))$ be the solution of system (1.6), It is easy to see that either $h_\infty - g_\infty < \infty$ or $h_\infty - g_\infty = \infty$ holds. According to Theorem 5.5, if $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \leq 2L^*$. And $\lim_{t\to\infty}(U(x,t;U_0,V_0,h_0), V(x,t;U_0,V_0,h_0)) = 0$ uniformly for $x \in [g_\infty, h_\infty]$. According to Theorem 5.6, if $h_\infty - g_\infty = \infty$, then $\lim_{t\to+\infty}(U(x,t;U_0,V_0,h_0) - U^*(x,t)) = 0$, $\lim_{t\to+\infty}(V(x,t;U_0,V_0,h_0) - V^*(x,t)) = 0$ locally uniformly for $x$ in $\mathbb{R}$. Thus, the spreading–vanishing dichotomy for system (1.6) with (1.7) holds. □

Then we can get the sufficient conditions for spreading and vanishing of the disease.

**Corollary 5.2.** *According to the above theorems, assume that (**H1**)–(**H5**) hold, it is natural to obtain that if $\lambda(t) < 0$ for any $t > 0$, then the disease will vanish and the trivial equilibrium $(0,0)$ is globally asymptotically stable. If $\lambda(T) > 0$ for some $T > 0$, then $h(T) - g(T) \geq 2L^*$. Taking $T$ as the initial time, we can get that the disease will spread and the trivial equilibrium $(0,0)$ is unstable.*

**Corollary 5.3.** *According to the above arguments and the positivity of $(U^*(x,t), V^*(x,t))$ in (5.15), by the persistence theory in Section 3 in Smith and Zhao [44] or the upper and lower solution method, we can get that when the spreading happens, there is a $\rho > 0$ such that $\lim_{t\to\infty} U(x,t;U_0,V_0,h_0) \geq \rho, \lim_{t\to\infty} V(x,t;U_0,V_0,h_0) \geq \rho$ locally uniformly for $x \in \mathbb{R}$.*

Finally, we turn to prove Theorem 2.3.

*Proof of Theorem 2.3.* **(1)** Assume that (***H5***) holds, considering that $h(t)$ is increasing and $g(t)$ is decreasing, if $\lambda(0) > 0$, then $h(0) - g(0) \geq 2L^*$, and

$$h_\infty - g_\infty > h(0) - g(0) \geq 2L^*. \tag{5.20}$$

Further, we can get that $\lambda\left(A, \frac{h_\infty - g_\infty}{2}\right) > 0$. According to Theorem 2.2, we can obtain that $h_\infty - g_\infty = \infty$. Therefore, the disease is spreading.

**(2)** Assume that $h(0) - g(0) < 2L^*$. Denote $h_\mu(\infty) := \lim_{t\to\infty} h_\mu(t), g_\mu(\infty) := \lim_{t\to\infty} g_\mu(t)$, $h_\mu(\infty) - g_\mu(\infty) := \lim_{t\to\infty}(h_\mu(t) - g_\mu(t))$. Let

$$\Lambda := \{\mu \mid h_\mu(\infty) - g_\mu(\infty) < \infty\}, \nu := \sup\Lambda. \tag{5.21}$$

If $\Lambda$ is an empty set, then $h_\mu(\infty) - g_\mu(\infty) = \infty$ for all $\mu > 0$. In this case, $\mu^* = 0$ satisfies the conditions. If $\Lambda$ is a nonempty set, we first prove that $\nu \in \Lambda$. On the contrary, assume that $h_\nu(\infty) - g_\nu(\infty) = \infty$. Then there exists a $T > 0$ such that $h_\nu(T) - g_\nu(T) > 2L^*$. In view of the continuous dependence of $h_\mu$

and $g_\mu$ on $\mu$, there is a $\varepsilon > 0$ small enough such that $h_\mu(T) - g_\mu(T) > 2L^*$ for any $\mu \in [\nu - \varepsilon, \nu + \varepsilon]$. Therefore, we have

$$h_\mu(\infty) - g_\mu(\infty) = \lim_{t\to\infty}(h_\mu(t) - g_\mu(t)) > h_\mu(T) - g_\mu(T) > 2L^*,\ \mu \in [\nu - \varepsilon, \nu + \varepsilon]\,.$$

According to (5.20), we obtain that $h_\mu(\infty) - g_\mu(\infty) = \infty$, which implies that $\Lambda \cap [\nu - \varepsilon, \nu + \varepsilon]$ is an empty set. It contradicts to (5.21). Thus, we have proved that $h_\nu(\infty) - g_\nu(\infty) < \infty$.

When $\mu > \nu$, we claim that $h_\mu(\infty) - g_\mu(\infty) = \infty$. On the contrary, assume that $h_\mu(\infty) - g_\mu(\infty) < \infty$, then $\mu \le \nu$, which is a contradiction. Therefore, by Theorem 2.2, the spreading happens.

When $\mu \le \nu$, by Lemma 5.3, we can obtain $h_\mu(t) - g_\mu(t) \le h_\nu(t) - g_\nu(t)$ for all $t \in (0, +\infty)$. Moreover, $h_\mu(\infty) - g_\mu(\infty) \le h_\nu(\infty) - g_\nu(\infty) < \infty$, thus, by Theorem 2.2, the vanishing happens. In this case, we can take $\mu^* = \nu$. Therefore, our proof is completed. □

**Remark 5.4.** When the initial infected domain is smaller than $2L^*$, for any given initial functions $(U_0, V_0)$, the spreading or vanishing of the epidemic disease mainly depends on the front expanding rate $\mu$.

## 6. Asymptotic spreading speeds

Spreading speed is a significant index to describe the propagation scale of the epidemic disease. In this section, we will give the lower and the upper bound estimates about the asymptotic spreading speeds of the left front and the right front when the disease is spreading.

*Proof of Theorem 2.4.* We will prove the case of asymptotic spreading speed for the rightward front, the case for the leftward front can be similarly estimated.

According to (1.5), let $(\underline{u}, \underline{v}; \underline{g}, \underline{h})$ be the solution of the following problem

$$\begin{cases} U_t = D_1 U_{xx} + \hat{a}_1\,(N_1 - U)\,V - \tilde{d}_1 U, & g(t) < x < h(t),\ t > 0,\\ V_t = D_2 V_{xx} + \hat{a}_2\,(N_2 - V)\,U - \tilde{d}_2 V, & g(t) < x < h(t),\ t > 0,\\ U(x,t) = V(x,t) = 0, & x = h(t) \text{ or } x = g(t),\ t > 0,\\ h(0) = h_0,\ h'(t) = -\mu U_x(h(t),t), & t > 0,\\ g(0) = -h_0,\ g'(t) = -\mu U_x(g(t),t), & t > 0,\\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{6.1}$$

By comparison principle, we can obtain

$$g(t) \le \underline{g},\ \underline{h} \le h, \text{ for } t > 0.$$

Apply Theorem 3.15 in [45] to (6.1), there exists a $c_*(\mu) > 0$ such that

$$\lim_{t\to\infty} \frac{-\underline{g}(t)}{t} = \lim_{t\to\infty} \frac{\underline{h}(t)}{t} = c_*(\mu).$$

Thus, we have proved $\liminf\limits_{t\to\infty} \frac{h(t)}{t} \ge c_*(\mu)$.

For the following system

$$\begin{cases} U_t = D_1 U_{xx} + \tilde{a}_1\,(N_1 - U)\,V - \hat{d}_1 U, & g(t) < x < h(t),\ t > 0,\\ V_t = D_2 V_{xx} + \tilde{a}_2\,(N_2 - V)\,U - \hat{d}_2 V, & g(t) < x < h(t),\ t > 0,\\ U(x,t) = V(x,t) = 0, & x = h(t) \text{ or } x = g(t),\ t > 0,\\ h(0) = h_0,\ h'(t) = -\mu U_x(h(t),t), & t > 0,\\ g(0) = -h_0,\ g'(t) = -\mu U_x(g(t),t), & t > 0,\\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{6.2}$$

Let $(\overline{u}, \overline{v}; \overline{g}, \overline{h})$ be the solution of (6.2), applying the similar arguments, we can get

$$\overline{g}(t) \le g,\ h \le \overline{h}, \text{ for } t > 0.$$

And there exists a $c^*(\mu) > 0$ such that

$$\lim_{t\to\infty} \frac{-\overline{g}(t)}{t} = \lim_{t\to\infty} \frac{\overline{h}(t)}{t} = c^*(\mu).$$

Thus, we have proved $\limsup\limits_{t\to\infty} \frac{h(t)}{t} \le c^*(\mu)$. □

**Remark 6.1.** Due to our limited knowledge, we may not give more explicit estimates about the asymptotic spreading speeds for the double free boundaries of (1.6). However, according to the arguments in Li et al. [46], we could make a reasonable conjecture: assume that (***H1***)–(***H5***) hold and $a_i(x,t) \equiv a_i^*(t), d_i(x,t) \equiv d_i^*(t)$ for $i = 1,2$ are almost periodic in $t \in [0,\infty)$. For any given $(U_0, V_0)$ satisfying (1.7), let $(U, V; g, h)$ be the solution for problem (1.6), there exists a $c_\mu > 0$ such that $\lim\limits_{t\to\infty} \frac{-g(t)}{t} = \lim\limits_{t\to\infty} \frac{h(t)}{t} = c_\mu$ as the spreading occurs. Here, $c_\mu$ is called the asymptotic spreading speed of system(1.6).

## 7. Discussion

In this paper, unlike the previous ordinary differential equations and constant coefficient periodic reaction–diffusion equations, we mainly propose a new reaction–diffusion WNv model (1.6) with moving infected domains $(g(t), h(t))$ in the spatial heterogeneous and time almost periodic environment and explore the long-time asymptotic dynamical behaviors of the solution for this model.

First, considering the spatial heterogeneity and time almost periodicity, we prove the global existence, uniqueness and get the regularity estimates of solution for (1.6), which is not trivial to obtain. Next, we define the principal Lyapunov exponent $\lambda(A, L)$ and $\lambda(t)$ concerning time $t$ and get some analytic properties of it. Moreover, we give the initial infected domain critical size $L^*$ using the principal Lyapunov exponent. In this paper, under the assumption of $\lambda(A, L) > 0$ for $L \ge L^*$, we obtain the following results: if $\lambda(t_0) > 0$ for some $t_0 \ge 0$, that is $h(t_0) - g(t_0) \ge 2L^*$, then $h_\infty - g_\infty = \infty$ and the disease will spread no matter how big the diffusion rates and the initial data are; if $h(0) - g(0) < 2L^*$, there exists a threshold value $\mu^* \ge 0$ which represents the infected region expanding capacity. When $\mu > \mu^*$, the disease will spread and the disease will vanish when $\mu \le \mu^*$. What is most important, assuming (***H1***)–(***H5***), we obtain the long-time dynamical behaviors of WNv model by giving the spreading–vanishing dichotomy regimes of (1.6). When the disease is vanishing, the densities $(U(x,t,g,h), V(x,t,g,h))$ of infected birds and mosquitoes will asymptotically converge to 0 uniformly for $x \in [g_\infty, h_\infty]$ and the eventually infected domain is no more than $2L^*$. When the disease is spreading, the densities $(U(x,t,g,h), V(x,t,g,h))$ of infected birds and mosquitoes will converge to a positive almost periodic solution $(U^*(x,t), V^*(x,t))$ of (2.5) uniformly for $x$ in any compact subsets of $\mathbb{R}$. When the spreading occurs, the asymptotic behavior of the solution is largely different from the other homogeneous WNv models. This result indicates that if we only consider the long-time behavior of WNv in a homogeneous environment, the spreading dynamics will be misjudged. What is more, it is well known that spreading speed is an important standard to predict the propagation rate of the epidemic disease. We give the following simple estimates about the lower and the upper bound of the asymptotic spreading speeds for the leftward and rightward fronts for (1.6):

$$\begin{aligned} c_*(\mu) \le \liminf_{t\to\infty} \frac{-g(t)}{t} \le \limsup_{t\to\infty} \frac{-g(t)}{t} \le c^*(\mu), \\ c_*(\mu) \le \liminf_{t\to\infty} \frac{h(t)}{t} \le \limsup_{t\to\infty} \frac{h(t)}{t} \le c^*(\mu). \end{aligned} \tag{7.1}$$

Moreover, our techniques in studying almost periodic systems different from other homogeneous and periodic systems can be applied in other almost periodic equations. Our methods using the principal Lyapunov exponent can also be applied to investigate other epidemic models. In order to better analyze the spreading dynamics of the epidemic model, we will try to give sharp calculation results about the asymptotic spreading speeds of the double free fronts for cooperative and competition systems in the next work. These are interesting and valuable researches.

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