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Small dispersion approximation of shock wave dynamics

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Abstract. We introduce a dispersion approximation model for weak, entropy solutions of multidimensional scalar conservation laws using variational kinetic representation, where equilibrium densities satisfy Gibb's entropy minimization principle for a piecewise linear, convex entropy. For such solutions, we show that small scale discontinuities, measured by the entropy increments, propagate with characteristic velocities, while the large-scale, shock-type discontinuities propagate with speeds close to the speeds of classical shock waves. In the zero-limit of the scale parameter, approximate solutions converge to a unique, entropy solution of a scalar conservation law.

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1. Introduction

Consider the Cauchy problem for a quasilinear system

$$\begin{cases} \partial_t U + \sum_{i=1}^d \partial_{x_i} F_i(U) = 0, & (x,t) \in \mathbb{R}^{d+1}_+, \\ U(x,0) = U_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $U : \mathbb{R}^{d+1}_+ \to \mathbb{R}^m$, $F_i : \mathbb{R}^m \to \mathbb{R}^m$. The main difficulty in constructing weak solutions for quasilinear systems (1) is the lack of apriori estimates on solutions in norms that control oscillations. This limits the application of such methods as viscosity or relaxation approximations of (1) for which pointwise convergence of approximate solutions is hard to establish.

The difficulty is well illustrated on an example of a shock wave. For systems with a convex entropy, weak solutions are typically restricted to verify entropy dissipation balance:

$$\partial_t \eta(U) + \mathrm{div}_x\, q(U) = r, \quad r \leq 0,$$

which provides apriori estimate on the total entropy at time t and total dissipated entropy up to time t in terms of the entropy of the initial data. This type of control is, however, too weak. For example, for a shock wave contained inside an interval $[a,b]$, the total dissipated entropy $\int\limits_0^t \int\limits_a^b r\, \mathrm{d}x$ is cubic in the strength of the shock, see theorem 8.5.1 of Dafermos [4]. Thus, in a regime of increasing number of small shock waves, the entropy does not control the oscillations as measured by the sum of all shock wave strengths.

In this paper, we explore the possibility of constructing approximate solutions of (1) for which entropy inequality implies strong compactness, at the price of distorting certain small scale details of the original solutions. More specifically, we will seek approximate, weak solutions of (1) such that

1. large shocks propagate with speeds close to the speeds computed from the original system (1);
2. the discontinuities, for which the change in the entropy is smaller than a certain threshold value ε, are transported with characteristic velocities.

Thus, the approximation of this type involves small-scale dispersion effects. The present work shows how this type of approximation can be implemented for scalar conservation laws in multi-dimensions. We consider equation

$$\partial_t\rho + \mathrm{div}_x A(\rho) = 0, \qquad (x,t)\in\mathbb{R}^{d+1}_+, \tag{2}$$

where $A:\mathbb{R}\to\mathbb{R}^d$ is a smooth vector function of fluxes. The theory of unique entropy solutions for scalar conservation laws was developed by Kruzhkov [8] using viscosity approximation. Later, different approaches have been used to build such solutions, see [1–3,6,7,10]. Our approach is based on the kinetic representation of entropy weak solutions of (2) developed by Brenier [1,2], Brenier and Corrias [3], Giga and Miyakawa [6], and Lions et al. [10]. According to the theory, an admissible $\rho(x,t)$ is represented as a moment of an "equilibrium" kinetic density f_{eq} :

$$\rho(x,t) = \int f_{\mathrm{eq}}(x,t,v)\,\mathrm{d}v, \quad f_{\mathrm{eq}}(x,t,v) = \mathbb{I}_{[0,\rho(x,t)]}(v), \tag{3}$$

with f_{eq} solving a kinetic equation

$$\partial_t f + A'(v)\cdot\nabla_x f = \partial_v m, \tag{4}$$

where m is non-negative Radon measure on $\mathbb{R}^{d+2}_+$. Here, for the simplicity of the presentation we assume that ρ is non-negative. Conversely, any solution of (4) constrained by condition (3) for some $\rho(x,t)$ defines an admissible weak solution of conservation law in (2), see [10]. Moreover, for any strictly convex function η, and a.e. (x,t), $f_{\mathrm{eq}}(x,t,v)$ is the unique minimizer of the problem

$$\min\left\{\int \eta'(v)\tilde f(v)\,\mathrm{d}v \;:\; \tilde f(v)\in[0,1],\ \int\tilde f\,\mathrm{d}v = \rho(x,t)\right\}. \tag{5}$$

Solutions of (4) can be obtained as limits of solutions of a relaxation problem

$$\partial_t f + A'(v)\cdot\nabla_x f = h^{-1}(M_f - f), \tag{6}$$

where M_f is the minimizer of (5) with $\rho = \int f\,\mathrm{d}v$. Strong compactness of a family of solutions with $h\to 0$ can be obtained through the uniform L^1 continuity ([1,2,6]), or the compensated compactness method ([10]).

To obtain the approximate solutions with properties 1 and 2, described above, we will use the variational kinetic formulation (5) and (6), in which we introduce a scale parameter ε. For that purpose we replace a strictly convex function η by a continuous, piecewise linear approximate entropy η_ε. With the new entropy function, the minimization problem admits multiple solutions, with indeterminacy on small ε–scales. A particular minimizer M_f will be selected so that L^1 norm of $f-M_f$ can be estimated by the entropy increment $\int\eta'_\varepsilon(v)(f-M_f)\,\mathrm{d}v$.

Our main result, Theorem 1 describes the properties kinetic functions obtained from this kinetic relaxation approach. Such kinetic functions verify Eq. (4) where, in addition, the right-hand side is a signed Radon measure, with the total variation controlled by the entropy:

$$||\partial_v m|| \le \frac{2}{\varepsilon}\int \eta_\varepsilon(\rho_0)\,\mathrm{d}x.$$

Furthermore, we show that moments $\rho=\int f\,\mathrm{d}v$, and $\phi = \int A'(v) f\,\mathrm{d}v$, solve the balance equation

$$\partial_t\rho + \mathrm{div}_x\phi = 0,$$

and $\phi(x,t) = A(\rho(x,t)) + O(\varepsilon^2)$. In particular, if there is a co-dimension one discontinuity of ρ with values ρ^+,ρ^-, (such discontinuities do develop in the solutions), such that $|\rho^+-\rho^-|>\varepsilon$, then it propagates with the velocity

$$\sigma = \frac{A(\rho^+)-A(\rho^-)}{\rho^+-\rho^-} + O(\varepsilon).$$

The kinetic function f, as well as its moments, depends on the scale parameter ε. In Theorem 2 we show that in the limit of $\varepsilon \to 0$, $\rho = \rho^{\varepsilon}(x,t)$ converges to an admissible solutions of (2).

In summary, we describe a new type of approximation of scalar conservation laws with properties distinct from the well-known viscosity approximation of Kruzhkov [8], kinetic relaxation approximation of Brenier [1] and Giga and Miyakawa [6], or semi-linear relaxation of Katsoulakis and Tzavaras [7]. In this approximation, large shocks propagate as sharp profiles (not smoothed) with velocities approximately verifying Rankine–Hugoniot conditions, while the entropy balance controls small-scale oscillations.

2. Main result

Let $A \in C^2(\mathbb{R})^d$. Without loss of generality, we will assume that ρ_0 is non-negative and bounded, so that all kinetic functions are defined for the range of the kinetic variable $v \in [0, L]$, for some $L > 0$. Unless it is specified otherwise, in the inegrals below, the integration in v is over $[0, L]$, in x is over $\mathbb{R}^d$, and in t is over $\mathbb{R}_+$. Let $\varepsilon > 0$. Define a piecewise constant function η_ε as

$$\eta_\varepsilon(v) = k, \quad v \in [k\varepsilon, (k+1)\varepsilon),\ k = 0..\lceil L/\varepsilon \rceil.$$

η_ε approximates the derivative of the quadratic entropy function. Here, for notational convenience, we use η_ε to denote the derivative of the entropy function described in the introduction.

Theorem 1. *Let $f_0 \in L^1(\mathbb{R}^d \times [0, L])$ with values $\{0, 1\}$. For any $\varepsilon > 0$ there is $f \in L^1(\mathbb{R}^{d+1}_+ \times [0, L])$ with values in $[0, 1]$ and m – a non-negative Radon measure on $\mathbb{R}^{d+1}_+ \times \mathbb{R}_+$ such that $\partial_v m$ is a signed Radon measure on $\mathbb{R}^{d+1}_+ \times [0, L]$ with the following properties:*

i. (Kinetic equation) f and m verify (in distributional sense) equation

$$\partial_t f + A'(v) \cdot \nabla_x f = \partial_v m. \tag{7}$$

Moreover,

$$||\partial_v m||_{\mathbb{R}^{d+1}_+ \times [0,L]} \leq \frac{2}{\varepsilon} \iint \eta_\varepsilon f_0 \,\mathrm{d}x\mathrm{d}v; \tag{8}$$

ii. (Optimality) for a.e. (x, t), f is a minimizer of

$$\min \left\{ \int \eta_\varepsilon(v) f(v) \,\mathrm{d}v \;:\; f(v) \in [0, 1],\ \int f \,\mathrm{d}v = \rho(x, t) \right\}; \tag{9}$$

iii. (Equi-continuity) for a.e. $t > 0$, and any $\xi \in \mathbb{R}^d$,

$$\iint |f(x + \xi, t, v) - f(x, t, v)| \,\mathrm{d}x\mathrm{d}v \leq \iint |f_0(x + \xi, v) - f_0(x, v)| \,\mathrm{d}x\mathrm{d}v. \tag{10}$$

Remark 1. Estimate (10) was derived in [1,6]. We use it to show strong compactness of moments of a time discerete approximation, in the proof of Theorem 1, and to verify (9). This estimate seems to be restricted to scalar conservation laws, and does not apply to systems. However, there is an alternative way to obtain strong compactness of moments by using only entopy estimate (8), through an kinetic averaging lemma of Gérard, [5].

Kinetic functions from Theorem 1 give rise to the approximate solutions of the conservation law (2), as described in the next theorem.

Theorem 2. *For function f from the previous theorem, moments*

$$\rho(x, t) = \int f(x, t, v) \,\mathrm{d}v, \quad \phi(x, t) = \int A'(v) f(x, t, v) \,\mathrm{d}v \tag{11}$$

have the following properties.

i. $\rho, \phi \in L^\infty(\mathbb{R}^{d+1}_+)$ *and verify (in distributional sense) conservation law*

$$\partial_t \rho + \mathrm{div}_x \phi = 0. \tag{12}$$

For any $\psi \in C_0^\infty(\mathbb{R}^d)$, $\int \rho(x,t)\psi(x)\,\mathrm{d}x$ *is continuous in* t *and*

$$\lim_{t\to 0+} \int \rho(x,t)\psi(x)\,\mathrm{d}x = \iint f_0(x,v)\psi(x)\,\mathrm{d}v\mathrm{d}x;$$

ii. *for any two pairs of values* $(\rho(x,t),\phi_i(x,t))$ *and* $(\rho(y,\tau),\phi_i(y,\tau))$, *such that* $|\rho(x,t)-\rho(y,t)| \geq c_0\varepsilon$, *it holds:*

$$\frac{\phi_i(x,t)-\phi_i(y,\tau)}{\rho(x,t)-\rho(y,\tau)} = \frac{A_i(\rho(x,t))-A_i(\rho(y,\tau))}{\rho(x,t)-\rho(y,\tau)} + O(\varepsilon), \quad i=1..d; \tag{13}$$

iii. *(Limit to Kruzhkov's solution) Considered as a function of* ε, $\rho=\rho_\varepsilon$, *there is a sequence* $\varepsilon\to 0$ *on which* ρ_ε *converges to the unique, entropy solution of the conservation law* (2) *in* $L^p_{loc}(\mathbb{R}^{d+1}_+)$, *for any* $p\in[0,+\infty)$.

2.1. Proof of Theorem 1

For a non-negative constant $\rho\in[0,L]$ consider a minimization problem

$$\min\left\{\int \eta_\varepsilon(v)f(v)\,\mathrm{d}v \,:\, f(v)\in[0,1],\ \int f\,\mathrm{d}v = \rho\right\}. \tag{14}$$

In the next lemma $\mathbb{I}_A(v)$ stands for a characteristic function of set A.

Lemma 1. *Let* $n=\lfloor \rho/\varepsilon\rfloor$. *The minimum in problem* (14) *equals*

$$\begin{cases} \varepsilon\sum_{k=0}^{n-1} k + \varepsilon n(\rho - n\varepsilon), & n\geq 1,\\ 0, & n=0.\end{cases}$$

It is achieved on minimizers

$$f_{min}(v) = \mathbb{I}_{[0,n\varepsilon]}(v) + \tilde{f}(v),$$

where $\tilde{f}$ *is an arbitrary function verifying conditions:*

$$\tilde{f}(v)\in[0,1],\quad \forall v\in[0,L];\quad \mathrm{supp}\,\tilde{f}\subset[n\varepsilon,(n+1)\varepsilon]; \tag{15}$$

$$\int \tilde{f}\,\mathrm{d}v = \rho - n\varepsilon. \tag{16}$$

Proof. $\eta_\varepsilon(v)$ is a non-decreasing function. To minimize functional $\int \eta_\varepsilon f\,\mathrm{d}v$ one needs to pick f that has all its mass as close to $v=0$ as possible, and is less than or equal 1. This shows the first statement. On interval $[n\varepsilon,(n+1)\varepsilon]$, a minimizer f_{min} can be arbitrarily re-arranged without changing the value of its η_ε moment. This leads to the second part of the lemma. □

Given a kinetic density f we select a particular minimizer of (14) with $\rho=\int f\,\mathrm{d}v$ in the following way. If $\int_{(n+1)\varepsilon}^{L} f\,\mathrm{d}v > n\varepsilon - \int_0^{n\varepsilon} f\,\mathrm{d}v$, we set

$$M_f(v) = \mathbb{I}_{[0,n\varepsilon+v_0]}(v) + f(v)\mathbb{I}_{(n\varepsilon+v_0,(n+1)\varepsilon)}(v), \tag{17}$$

where $v_0\in(0,\varepsilon)$ is determined by the relation $\int M_f\,\mathrm{d}v = \int f\,\mathrm{d}v$. It is the smallest number such that

$$\int_0^{n\varepsilon+v_0} 1-f\,\mathrm{d}v = \int_{(n+1)\varepsilon}^{L} f\,\mathrm{d}v.$$

If $\int\limits_{(n+1)\varepsilon}^{L} f\,\mathrm{d}v \geq n\varepsilon - \int\limits_{0}^{n\varepsilon} f\,\mathrm{d}v$, we set

$$M_f(v) = \mathbb{I}_{[0,n\varepsilon]}(v) + f(v)\mathbb{I}_{(n\varepsilon, n\varepsilon+v_0)}(v), \tag{18}$$

where $v_0 \in (0, \varepsilon)$ is uniquely determined as the smallest number such that

$$\int\limits_{0}^{n\varepsilon} 1 - f\,\mathrm{d}v = \int\limits_{n\varepsilon+v_0}^{L} f\,\mathrm{d}v.$$

This minimizer can be thought of as a rearrangement of mass f obtained by shifting its pieces to the locations with smaller values of $\eta_\varepsilon(v)$.

The key properties of the minimizer f_{min} are listed in the next lemma.

Lemma 2. *Let f be any function with values in $[0,1]$ and supported on $[0,L]$. For M_f, defined above*

$$\int |f - M_f|\,\mathrm{d}v \leq \frac{2}{\varepsilon}\int \eta_\varepsilon(v)(f - M_f)\,\mathrm{d}v. \tag{19}$$

For any non-decreasing function η,

$$\int \eta(v)(f(v) - M_f(v))\,\mathrm{d}v \geq 0. \tag{20}$$

For any two functions f_1, f_2 with values in $\{0,1\}$ and supported on $[0,L]$,

$$\int |M_{f_1} - M_{f_2}|\,\mathrm{d}v \leq \int |f_1 - f_2|\,\mathrm{d}v, \tag{21}$$

where M_{f_1}, M_{f_2} are the corresponding minimizers.

Proof. Let n be as in the previous lemma. Consider case (17).

$$\int |f - M_f|\,\mathrm{d}v = \int\limits_{0}^{n\varepsilon+v_0} 1 - f\,\mathrm{d}v + \int\limits_{(n+1)\varepsilon}^{L} f\,\mathrm{d}v = 2\int\limits_{(n+1)\varepsilon}^{L} f\,\mathrm{d}v \leq \frac{2}{\varepsilon}\int \eta_\varepsilon(v)(f - M_f)\,\mathrm{d}v, \tag{22}$$

where the last inequality holds since all mass of f on interval $[(n+1)\varepsilon, L]$ has been removed from that interval. Similarly, in case (18)

$$\int |f - M_f|\,\mathrm{d}v = \int\limits_{0}^{n\varepsilon} 1 - f\,\mathrm{d}v + \int\limits_{n\varepsilon+v_0}^{L} f\,\mathrm{d}v = 2\int\limits_{n\varepsilon+v_0}^{L} f\,\mathrm{d}v \leq \frac{2}{\varepsilon}\int \eta_\varepsilon(v)(f - M_f)\,\mathrm{d}v. \tag{23}$$

For a non-decreasing function η, (20) follows from the definition of M_f.

To prove (21) it suffices to show that

$$\int f_1 f_2\,\mathrm{d}v \leq \int M_{f_1} M_{f_2}\,\mathrm{d}v, \tag{24}$$

since functions take only values 0 or 1. Let $n_1, v_{1,0}$ and $n_2, v_{2,0}$ be the corresponding values of n and v_0 from (17), (18) for functions f_1 and f_2.

Consider the case $n_1 > n_2$ first. Here

$$\int M_{f_1} M_{f_2}\,\mathrm{d}v = \int\limits_{0}^{(n_2+1)\varepsilon} M_{f_2}\,\mathrm{d}v = \int f_2\,\mathrm{d}v \geq \int f_1 f_2\,\mathrm{d}v.$$

Next, consider the case $n_1 = n_2$ $(= n)$. Suppose that representation (17) applies to both functions f_1, f_2, and assume $v_{1,0} \geq v_{2,0}$. Then,

$$\int M_{f_1} M_{f_2}\,\mathrm{d}v \geq \int_{n\varepsilon+v_{1,0}}^{(n+1)\varepsilon} f_1 f_2\,\mathrm{d}v + \int_0^{n\varepsilon+v_{1,0}} f_2\,\mathrm{d}v + \int_{(n+1)\varepsilon}^{L} f_2\,\mathrm{d}v \geq \int f_1 f_2\,\mathrm{d}v.$$

Suppose that representation (18) applies to both functions f_1, f_2, and assume $v_{1,0} \geq v_{2,0}$. Then,

$$\int M_{f_1} M_{f_2}\,\mathrm{d}v \geq \int_0^{n\varepsilon} f_2\,\mathrm{d}v + \int_{n\varepsilon+v_{2,0}}^{L} f_2\,\mathrm{d}v + \int_{n\varepsilon}^{n\varepsilon+v_{2,0}} f_1 f_2\,\mathrm{d}v \geq \int f_1 f_2\,\mathrm{d}v.$$

Suppose that (18) applies to function f_1 and (17) to f_2. If $v_{1,0} \geq v_{2,0}$ then

$$\int M_{f_1} M_{f_2}\,\mathrm{d}v \geq \int_0^{n\varepsilon+v_{2,0}} f_1\,\mathrm{d}v + \int_{n\varepsilon+v_{1,0}}^{L} f_1\,\mathrm{d}v + \int_{n\varepsilon+v_{2,0}}^{n\varepsilon+v_{1,0}} f_1 f_2\,\mathrm{d}v \geq \int f_1 f_2\,\mathrm{d}v.$$

If $v_{1,0} < v_{2,0}$ then

$$\int M_{f_1} M_{f_2}\,\mathrm{d}v \geq \int_0^{L} f_1\,\mathrm{d}v \geq \int f_1 f_2\,\mathrm{d}v.$$

The contraction property (21) is proved now. □

Now we consider a discrete-time approximation, with time step $h > 0$ and $t_n = nh$, $n = 0, 1, 2...$ Given $f_{n-1}(x, v)$ the next period kinetic function

$$f_n(x, v) = M_{\hat{f_n}}, \quad \hat{f}_n(x, v) = f_{n-1}(x - A'(v)h),$$

with f_0 being the initial data. A continuous time approximate is defined as

$$f^h(x, v, t) = \begin{cases} f_{n-1}(x - A'(v)(t - nh)), & t \in [(n-1)h, nh), \\ f_n(x, v), & t = nh. \end{cases} \tag{25}$$

Remark 2. It can be easily seen that in dimension one, if initial data f_0 is such that $f_0(x, v) = 1$, for $0 \leq v \leq k\varepsilon$ and $f_0(x, v) = 0$ for $v > ((k + 1)\varepsilon$, then f_n is evolved by simple translation with kinetic velocities v, leading to dispersion effect. On the other hand if initial data, for example, has a form

$$f_0(x, v) = \begin{cases} \mathbb{I}_{[0,v_1]}(v), & x < 0, \\ \mathbb{I}_{[0,v_2]}(v), & x > 0, \end{cases}$$

with $v_1 - v_2 > \varepsilon$ and $A(v) = v$ (corresponding to Burger's equation) then f_n evolves as a classical shock wave in a discrete-time approximation.

The following properties of f^h follow from its definition and properties established in Lemma 2.

Lemma 3. *It holds:*

i. for any (x, v, t), $f^h \in \{0, 1\}$;
ii. for any (x, t), $\mathrm{supp} f^h \subset [0, L]$;
iii. for any $t > 0$,

$$\iint f^h(x, v, t)\,\mathrm{d}v\mathrm{d}x \leq f_0(x, v)\,\mathrm{d}v\mathrm{d}x; \tag{26}$$

$$\iint \eta_\varepsilon(v) f^h(x, v, t)\,\mathrm{d}v\mathrm{d}x \leq \eta_\varepsilon(v) f_0(x, v)\,\mathrm{d}v\mathrm{d}x; \tag{27}$$

iv. f^h *is a weak solution of the equation*

$$\partial_t f^h + A'(v) \cdot \nabla_x f^h = R^h, \tag{28}$$

where

$$R^h = \sum_{n=1}^{\infty} \delta(t - nh)(f_n(x, v) - f_{n-1}(x - A'(v)h)); \tag{29}$$

v. for any $t > 0$ *and any* $\xi \in \mathbb{R}^d$,

$$\iint |f^h(x + \xi, t, v) - f^h(x, t)| \, \mathrm{d}x\mathrm{d}v \le \iint |f_0(x + \xi, v) - f_0(x, v)| \, \mathrm{d}x\mathrm{d}v.$$

Next, we estimate the interaction term R^h in Eq. (28)

Lemma 4. *For any* $t > 0$,

$$\iint R^h \, \mathrm{d}v\mathrm{d}x \le \sum_{n=1}^{\infty} \delta(t - nh) \iint |(f_n(x, v) - f_{n-1}(x - A'(v)h))| \, \mathrm{d}x\mathrm{d}v;$$

and

$$\int_0^{\infty} \iint |R^h| \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \le \frac{2}{\varepsilon} \iint \eta_\varepsilon(v) f_0(x, v) \, \mathrm{d}v\mathrm{d}x.$$

Proof. The first inequality is obvious. Using Eq. (28) we find that

$$\sum_{n=1}^{\infty} \iint \eta_\varepsilon(v)(f_n(x, v) - f_{n-1}(x - A'(v)h, v)) \, \mathrm{d}x\mathrm{d}v \le \iint \eta_\varepsilon(v) f_0(x, v) \, \mathrm{d}x\mathrm{d}v.$$

Since $f_n = M_{f_{n-1}(x - A'(v)h, v)}$, using inequality (19) we get

$$\sum_{n=1}^{\infty} \iint |f_n(x, v) - f_{n-1}(x - A'(v)h, v))|, \mathrm{d}x\mathrm{d}v \le \frac{2}{\varepsilon} \iint \eta_\varepsilon(v) f_0(x, v) \, \mathrm{d}x\mathrm{d}v,$$

from which the second inequality of the lemma follows. □

With the information from the last two lemma, we consider compactness properties of f^h as $h \to 0$. There is f with a.e. values in $[0, 1]$ and a signed Radon measure $\tilde{m}$ such that on a suitable subsequence $h_k \to 0$,

$$f^{h_k} \to f \quad \text{*-weakly in } L^\infty(\mathbb{R}_+^{d+1} \times [0, L]),$$
$$R^{h_k} \to \tilde{m} \quad \text{*-weakly in } \mathcal{M}_{loc}(\mathbb{R}_+^{d+1} \times [0, L]),$$

for a.e. $t > 0$,

$$\iint f(x, v, t) \, \mathrm{d}v\mathrm{d}x \le \iint f_0(x, v) \, \mathrm{d}v\mathrm{d}x,$$
$$\iint \eta_\varepsilon(v) f(x, v, t) \, \mathrm{d}v\mathrm{d}x \le \iint \eta_\varepsilon(v) f_0(x, v) \, \mathrm{d}v\mathrm{d}x,$$

and inequalities (8) and (10) hold.

Inequality (20) implies that $\langle \tilde{m}, \eta(v)\psi(x, t) \rangle \le 0$ for any continuously non-decreasing function η, and any non-negative $\psi \in C_0^\infty(\mathbb{R}_+^{d+1})$. Thus, $\tilde{m} = \partial_v m$ for a non-negative Radon measure.

Now we show that v–moments of f^h are compact in L^p norms.

Lemma 5. *Let $\omega(v)$ be a measurable, bounded function on $[0, L]$. Then, the set of moments*

$$\left\{ \int \omega(v) f^h(x, v, t) \, \mathrm{d}v \right\} \quad \text{is pre-compact in } L^p_{loc}(\mathbb{R}^{d+1}_+),\ p \in [0, +\infty).$$

Proof. Denote by $\rho^h_\omega = \int \omega(v) f^h(x, v, t) \, \mathrm{d}v$. ρ^h_ω is bounded in $L^\infty(\mathbb{R}^{d+1}_+)$. It follows from part v. of Lemma 3 that for any $\xi \in \mathbb{R}^d$, and any $T > 0$, and $p \in [1, +\infty)$,

$$\|\rho^h_\omega(x + \xi, t) - \rho^h_\omega(x, t)\|_{L^\infty((0,T);L^p(\mathbb{R}^d))} \to 0, \quad |\xi| \to 0,$$

uniformly in h. It follows from Eq. (28) that for any $T > 0$ and $p \in [0, +\infty)$,

$$\left\{\partial_t \rho^h_\omega\right\} \quad \text{is bounded in } \mathcal{M}((0, T); L^p(\mathbb{R}^d)) + L^\infty((0, T); W^{-1,p}_{loc}(\mathbb{R}^d)).$$

Under these conditions, compactness lemma 5.1 of Lions [9] ensures that on a suitable sequence of values of $h \to 0$, $(\rho^h_\omega)^2 \to (\rho_\omega)^2$ in distributional sense, where ρ_ω is a limiting point of ρ^h_ω in *-weak topology of $L^\infty(\mathbb{R}^{d+1}_+)$. This implies the statement of the lemma. □

A little bit more can be said about moments $\rho^h = \int f^h(x, v, t) \, \mathrm{d}v$. Indeed,

$$\{\partial_t \rho^h\} \quad \text{is bounded in } L^\infty((0, T); W^{-1,p}_{loc}(\mathbb{R}^d)),\ p \in [1, \infty).$$

Thus, ρ^h converges for a limiting point ρ, in $C([0, T]; W^{-1,p}_{loc}(\mathbb{R}^d))$. This shows, in particular, that $\rho(x, 0) = \int f_0(x, v) \, \mathrm{d}v$.

We consider the moments of f^h from the set $\omega \in \{1, \eta_\varepsilon(v), A_1(v), .., A_d(v)\}$ and select a sequence $h = h_k \to 0$ on which f^h and ρ^h_ω converge in the topologies described above to their limiting values.

To finish the proof of Theorem 1 it remains to establish (9). Let $\hat{\rho}^h = \int \hat{f}^h \, \mathrm{d}v$. For each (x, t), $\hat{f}^h(x, t, v)$ is a minimizer of the problem (9) with $\rho = \hat{\rho}^h(x, t)$. Since this problem depends continuously on the value of the constraint $\hat{\rho}^h$ and the latter converges a.e. (x, t) to $\rho(x, t)$, then the limit of the minimizers $\tilde{f}^h$ is a minimizer corresponding to ρ.

2.2. Proof of Theorem 2

Part i. of the Theorem 2 was established in proving Theorem 1. Part ii. follows from from (9) and Lemma 1. Indeed, let ρ, and ϕ be given by (11), and (x, t) is such that $f(x, t, \cdot)$ is the minimizer of (9). Let n and $\tilde{f}$ be as in Lemma 1. We can write for any $i = 1..d$,

$$\begin{aligned}
\phi_i(x, t) = \int A_i'(v) f(x, t, v) \, \mathrm{d}v &= A_i(\rho(x, t)) + \int_{n\varepsilon}^{(n+1)\varepsilon} A_i'(v) \left(\tilde{f} - \mathbb{I}_{[0,\rho]}(v)\right) \mathrm{d}v \\
&= A_i(\rho(x, t)) + \int_{n\varepsilon}^{(n+1)\varepsilon} (A_i'(v) - A_i'(n\varepsilon)) \left(\tilde{f} - \mathbb{I}_{[0,\rho]}(v)\right) \mathrm{d}v \\
&= A_i(\rho(x, t)) + O(\varepsilon^2),
\end{aligned}$$

which establishes (13).

To show part iii of the theorem, we consider the sequence of kinetic functions f^ε and their moments $\rho^\varepsilon = \int f^\varepsilon \, \mathrm{d}v$, $\phi^\varepsilon_i = \int A_i'(v) f^\varepsilon \, \mathrm{d}v$ from Theorem 1 in the limit $\varepsilon \to 0$.

Given the uniform bounds on the sequence f^ε, continuity estimate (10) and Eq. (7), one can repeat the arguments of the proof of Theorem 1 to establish that v–moments of f^ε are pre-compact in $L^p_{loc}(\mathbb{R}^{d+1}_+)$. In particular, $(1, A_1(v), .., A_d(v))$ moments of f^ε converge (on a subsequence) to a pair (ρ, ϕ) – a solution of (12). f^ε itself converges weakly to a function that f that verifies the kinetic equation (7), Moreover, a.e. (x,t), f is a minimizer of problem (9) with function $\eta(v)=v$, in place of η_ε. This means that f has a structure of an equilibrium density $f(x,\cdot,t)=\mathbb{I}_{\rho(x,t)}(\cdot)$ and, thus, $\phi(x,t)=A(\rho(x,t)$ a.e. (x,t).

This new problem

$$\min\left\{\int \eta(v)f(v)\,\mathrm{d}v\;:\;f(v)\in[0,1],\;\int f\,\mathrm{d}v=\rho(x,t)\right\}$$

has a unique minimizer in the form $f(x,v,t)=\mathbb{I}_{[0,\rho(x,t)]}(v)$. Thus, $\phi(x,t)=A(\rho(x,t))$ a.e. (x,t) and ρ is a unique entropy solution of the conservation law (2). The uniqueness implies that the sequence ρ^ε converges to ρ in the limit of $\varepsilon\to 0$.

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