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Analyticity and time-decay rate of global solutions for the generalized MHD system near an equilibrium

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Abstract. In this paper, we investigate the analyticity and time-decay rate of global mild solutions to the three-dimensional generalized magnetohydrodynamics system near an equilibrium. Global existence and analyticity of solutions in the Lei–Lin space are established by energy method in Fourier space and continuous argument. Based on the analyticity of global solutions, time-decay rate of global solutions follows.

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1. Introduction

In this article, we investigate three-dimensional generalized incompressible magnetohydrodynamics (MHD) system

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha} u + u \cdot \nabla u + \nabla P - B \cdot \nabla B = 0, \\ \partial_t B + (-\Delta)^{\beta} B + B \cdot \nabla u - u \cdot \nabla B = 0, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $B = B(x,t) \in \mathbb{R}^3$, and $P = P(x,t) \in \mathbb{R}$ are the velocity, magnetic, and pressure field, respectively. Here, α and β are the parameters of the fractional dissipations corresponding to the velocity and magnetic field, respectively. It is clear that a special solution of (1.1) is given by the zero velocity field and the background magnetic field $B^{(0)} = e_1 = (1,0,0)$. The perturbation (u,b) around this equilibrium with $b = B - e_1$ obeys

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha} u + u \cdot \nabla u + \nabla P - b \cdot \nabla b - \partial_1 b = 0, \\ \partial_t b + (-\Delta)^{\beta} b + b \cdot \nabla u - u \cdot \nabla b - \partial_1 u = 0, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{1.2}$$

When $\alpha = \beta = 1$, the system (1.1) reduces to the classical MHD system, which describes the dynamics of electrically conducting fluids arising from plasmas, liquid metals, salt water or some other physical phenomena (see [2]). Due to its important physical background, rich phenomena, mathematical complexity and challenges, MHD system has attracted lots of physicists and mathematicians attention and many interesting results about stability problem near equilibrium have been established (see [1,7–10,12,16,17,24–27], [30]). We only recall the global well-posedness and asymptotic decay of solutions in Lei–Lin type spaces for our purpose. Global mild solutions in Lei–Lin space were established in [21], provided that the norms of the initial data are bounded exactly by the minimal value of the viscosity coeffcients. Later, Wang [20] obtained asymptotic behavior and stability of global mild solutions obtained in [21]. For a class of special large initial data with the critical norm of the initial velocity and magnetic field being arbitrarily large, global smooth solutions was constructed in [13]. Wang and Li [22] proved global existence of mild solutions in Lei–Lin–Gevrey space that was introduced in [6].

For the generalized MHD system (1.1), Wang et al. [19] proved the global existence and analyticity of mild solutions in Lei–Lin space. Ye [31] proved the global well-posedness of the generalized MHD equations with small initial data and proved the corresponding global solution decays to zero as time goes to infinity (see also [32]). Xiao et al. [28] obtained the temporal time-decay rate of global solution with small initial data. Melo et al. [15] established the existence of a unique global mild solution in Lei–Lin–Gevrey and Lei–Lin spaces, provided that the initial data are assumed to be small enough. Decay estimate of global solutions was also established in [18]. Very recently, Xiao and Yuan [29] proved the existence, analyticity and time-decay rate of global solutions near an equilibrium for $1/2 \leq \alpha = \beta \leq 1$. For a class of large initial value, Liu and Wang [14] obtained a class of global large solutions.

When $B = 0, \alpha = 1$, the system (1.1) reduces to the Navier–Stokes system. Lei and Lin [11] proved global well-posedness result with small initial data in the critical space χ^{-1}. Based on the work in [11], Bae [3] obtained the existence and analyticity of the solution. Long time decay to the Lei–Lin solution of 3D Navier–Stokes system was also investigated by Benameur [4]. For other related results, we refer to [5,6].

Inspired by the works [11,19,21,29,32] for three-dimensional Navier–Stokes system and MHD system, the main aim of this paper is to investigate global existence, analyticity and time-decay rate of small mild solutions to (1.2) with the initial data

$$t = 0: \quad u = u_0(x), \quad b = b_0(x), \quad x \in \mathbb{R}^3. \tag{1.3}$$

The decay structure of (1.2) plays an important role in studying the existence, analyticity and time decay-rate of global small solutions. But the decay structure of (1.2) is very complicated since the corresponding linear system is a coupled system with fractional dissipation and linear terms $\partial_1 u$ and $\partial_1 b$. One of the key observations is that the coupled system is decoupled into two same linear wave equations with structural damping by using the structure of the corresponding linear system. The pointwise estimate of solutions to the problem (1.2), (1.3) is established by applying energy method in Fourier space to the two linear wave equations with structural damping. Because α and β may take differential values, the pointwise estimate of solutions in Fourier space implies that the solutions operator acts different heat kernel in low and high frequency regions (see Lemma 2.1 and Remark 2.3 for the details), which causes difficulty in proving global existence and analyticity of solutions. Fortunately, we would overcome the difficulty by using Fourier splitting technology, energy method in Fourier space and continuous argument. Based on the analyticity of global solutions, time-decay rate of global solutions follows.

The paper is organized as follows. In Sect. 2, we firstly establish pointwise estimate of solutions to the problem (1.2), (1.3) in Fourier space by energy method in Fourier space and then prove global existence and analyticity of mild solutions by using energy method and continuous argument. In Sect. 3, time-decay rate of global solutions obtained in Sect. 2 is established.

In the last, we introduce some notations used in this paper. The Lei–Lin space χ^s is defined by

$$\chi^s := \left\{ f \in \mathbf{D}'(\mathbb{R}^3) | \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(\xi)| \mathrm{d}\xi < \infty, s \in \mathbb{R} \right\}$$

and the association norm is given by

$$\|f\|_{\chi^s} = \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(\xi)| \mathrm{d}\xi < \infty.$$

For the details, we may refer to Lei and Lin [11].

2. Global existence and analytic

In this section, our main purpose is to prove global existence and analytic of solutions to the problem (1.2), (1.3). To this end, we shall establish the following pointwise estimate of solutions to the problem (1.2), (1.3) in Fourier space by energy method in Fourier space. The method has been used in global existence and decay estimate of solutions to damped higher order wave equations (see [23]).

Lemma 2.1. *The solution to the problem* (1.2), (1.3) *in Fourier space satisfies*

$$|\hat{u}(\xi,t)| + |\hat{b}(\xi,t)| \le Ce^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}t}(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|) + C\int_0^t e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}(t-\tau)}(|\hat{F}_1| + |\hat{F}_2|)(\xi,\tau)\mathrm{d}\tau, \tag{2.1}$$

where $F_1 = (I - (-\Delta)^{-1}\nabla\nabla\cdot)(-u\cdot\nabla u + b\cdot\nabla b)$ *and* $F_2 = -b\cdot\nabla u + u\cdot\nabla b$.

Remark 2.2. Lemma 2.1 implies that the decay property of the solutions operator to the problem in this paper (1.2), (1.3) is different from that in [29]. More precisely, the decay property is characterized by

$$|\hat{G}(\xi,t)| \le Ce^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}t}.$$

When $\alpha=\beta$, the above decay property reduces to

$$|\hat{G}(\xi,t)| \le Ce^{-c|\xi|^{2\alpha}t},$$

which is exactly the decay property of the solutions operator in [29]. Moreover, it seems that the proof methods used in [29] is not effective to our problem and hence we have to resort to energy method in Fourier space.

Remark 2.3. For any $R\in(0,1)$, there exists a positive constant c_1 such that

$$\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}} \ge \begin{cases} c_1|\xi|^{2\alpha}, & |\xi|\le R, \alpha\ge\beta \\ c_1|\xi|^{2\beta}, & |\xi|\ge R, \alpha\ge\beta \end{cases} \tag{2.2}$$

Proof. We consider the corresponding linear system of (1.2)

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha}u - \partial_1 b = 0, \\ \partial_t b + (-\Delta)^{\beta}b - \partial_1 u = 0. \end{cases} \tag{2.3}$$

Applying Fourier transform to (2.3) yields

$$\begin{cases} \partial_t\hat{u} + |\xi|^{2\alpha}\hat{u} + i\xi_1\hat{b} = 0, \\ \partial_t\hat{b} + |\xi|^{2\beta}\hat{b} + i\xi_1\hat{u} = 0. \end{cases} \tag{2.4}$$

It follows from (2.4) that

$$\partial_t^2\hat{u} + (|\xi|^{2\alpha} + |\xi|^{2\beta})\hat{u}_t + |\xi|^{2(\alpha+\beta)}\hat{u} + |\xi_1|^2\hat{u} = 0. \tag{2.5}$$

By using (1.3) and (2.4), the initial value becomes

$$t = 0 : \hat{u} = \hat{u}_0(\xi), \hat{u}_t = -|\xi|^{2\alpha}\hat{u}_0(\xi) - i\xi_1\hat{b}_0(\xi). \tag{2.6}$$

The characteristic equation of (2.5) is given by

$$\lambda^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})\lambda + |\xi|^{2(\alpha+\beta)} + |\xi_1|^2 = 0. \tag{2.7}$$

From (2.7), we infer that

$$\begin{aligned}\lambda_\pm(\xi) &= \frac{-(|\xi|^{2\alpha}+|\xi|^{2\beta}) \pm \sqrt{(|\xi|^{2\alpha}+|\xi|^{2\beta})^2 - 4(|\xi|^{2(\alpha+\beta)}+|\xi_1|^2)}}{2}\\ &= \frac{-(|\xi|^{2\alpha}+|\xi|^{2\beta}) \pm \sqrt{(|\xi|^{2\alpha}-|\xi|^{2\beta})^2 - 4|\xi_1|^2}}{2}.\end{aligned}$$

It is not difficult to find that the solution $(\hat{u}, \hat{b})$ to the problem (2.5), (2.6) is given by

$$\begin{aligned}\hat{u}(\xi,t) &= \frac{-(\lambda_- + |\xi|^{2\alpha})e^{\lambda_+ t} + (\lambda_+ + |\xi|^{2\alpha})e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{u}_0(\xi)\\ &\quad - i\xi_1 \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{b}_0(\xi).\end{aligned} \tag{2.8}$$

The same procedure leads to

$$\begin{aligned}\hat{b}(\xi,t) &= \frac{-(\lambda_- + |\xi|^{2\beta})e^{\lambda_+ t} + (\lambda_+ + |\xi|^{2\beta})e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{b}_0(\xi)\\ &\quad - i\xi_1 \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{u}_0(\xi).\end{aligned} \tag{2.9}$$

Then the solution to the problem (1.2), (1.3) in Fourier space may be expressed as

$$\begin{pmatrix}\hat{u}\\ \hat{b}\end{pmatrix} = \begin{pmatrix}\hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{pmatrix}\begin{pmatrix}\hat{u}_0\\ \hat{b}_0\end{pmatrix} + \int_0^t \begin{pmatrix}\hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{pmatrix}(t-\tau)\begin{pmatrix}\hat{F}_1\\ \hat{F}_2\end{pmatrix}\mathrm{d}\tau, \tag{2.10}$$

where

$$\begin{aligned}\hat{G}_{11} &= \frac{-(\lambda_- + |\xi|^{2\alpha})e^{\lambda_+ t} + (\lambda_+ + |\xi|^{2\alpha})e^{\lambda_- t}}{\lambda_+ - \lambda_-},\\ \hat{G}_{12} &= -i\xi_1 \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-},\\ \hat{G}_{21} &= -i\xi_1 \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-},\\ \hat{G}_{22} &= \frac{-(\lambda_- + |\xi|^{2\beta})e^{\lambda_+ t} + (\lambda_+ + |\xi|^{2\beta})e^{\lambda_- t}}{\lambda_+ - \lambda_-}.\end{aligned}$$

Next, multiplying (2.5) by $\hat{u}_t$ and taking the real part, it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{|\hat{u}_t|^2 + |\xi|^{2(\alpha+\beta)}|\hat{u}|^2 + |\xi_1|^2|\hat{u}|^2\Big\} + (|\xi|^{2\alpha} + |\beta|^{2\beta})|\hat{u}_t|^2 = 0. \tag{2.11}$$

We multiply (2.5) by $\overline{\hat{u}}$ and take the real part and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\mathrm{Re}(\hat{u}_t\cdot\overline{\hat{u}}) + \frac{1}{2}(|\xi|^{2\alpha}+|\xi|^{2\beta})|\hat{u}|^2\Big\} + |\xi|^{2(\alpha+\beta)}|\hat{u}|^2 + |\xi_1|^2|\hat{u}|^2 - |\hat{u}_t|^2 = 0. \tag{2.12}$$

Multiplying (2.12) by $(|\xi|^{2\alpha}+|\xi|^{2\beta})$ and then combining (2.11), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{|\hat{u}_t|^2 + |\xi|^{2(\alpha+\beta)}|\hat{u}|^2 + |\xi_1|^2|\hat{u}|^2 + \frac{1}{2}(|\xi|^{2\alpha}+|\xi|^{2\beta})^2|\hat{u}|^2\\ &\quad + (|\xi|^{2\alpha}+|\xi|^{2\beta})\mathrm{Re}(\hat{u}_t\cdot\overline{\hat{u}})\Big\} + |\xi|^{2(\alpha+\beta)}(|\xi|^{2\alpha}+|\xi|^{2\beta})|\hat{u}|^2\\ &\quad + |\xi_1|^2(|\xi|^{2\alpha}+|\xi|^{2\beta})|\hat{u}|^2 + (|\xi|^{2\alpha}+|\xi|^{2\beta})|\hat{u}_t|^2 = 0.\end{aligned} \tag{2.13}$$

Let

$$E = |\hat{u}_t|^2 + |\xi|^{2(\alpha+\beta)}|\hat{u}|^2 + |\xi_1|^2|\hat{u}|^2 + \frac{1}{2}(|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{u}|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})\mathrm{Re}(\hat{u}_t \cdot \overline{\hat{u}})$$

and

$$F = |\xi|^{2(\alpha+\beta)}(|\xi|^{2\alpha} + |\xi|^{2\beta})|\hat{u}|^2 + |\xi_1|^2(|\xi|^{2\alpha} + |\xi|^{2\beta})|\hat{u}|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})|\hat{u}_t|^2,$$

then we obtain from (2.13)

$$\frac{\mathrm{d}E}{\mathrm{d}t} + F = 0. \tag{2.14}$$

Let

$$\widetilde{E} = |\hat{u}_t|^2 + |\xi_1|^2|\hat{u}|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{u}|^2,$$

then we have

$$C_1\widetilde{E} \le E \le C_2\widetilde{E} \tag{2.15}$$

and

$$\begin{aligned} F &= (|\xi|^{2\alpha} + |\xi|^{2\beta})\left\{|\hat{u}_t|^2 + |\xi_1|^2|\hat{u}|^2 + |\xi|^{2(\alpha+\beta)}|\hat{u}|^2\right\} \\ &\ge c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha} + |\xi|^{2\beta}}\widetilde{E} \\ &\ge c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha} + |\xi|^{2\beta}}E. \end{aligned} \tag{2.16}$$

From (2.14) and (2.16), we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} + c\omega(\xi)E \le 0, \tag{2.17}$$

where

$$\omega(\xi) = \frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha} + |\xi|^{2\beta}}.$$

(2.17) entails that

$$E \le E(\xi, 0)e^{-c\omega(\xi)t}. \tag{2.18}$$

Combining (2.15) and (2.18), we obtain

$$\begin{aligned} &|\hat{u}_t|^2 + |\xi_1|^2|\hat{u}|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{u}|^2 \\ &\quad\le (|\hat{u}_t(0)|^2 + |\xi_1|^2|\hat{u}_0|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{u}_0|^2)e^{-c\omega(\xi)t}. \end{aligned} \tag{2.19}$$

It follows from (2.19) and (2.6) that

$$\begin{aligned} &|\xi_1|^2|\hat{G}_{11}\hat{u}_0 + \hat{G}_{12}\hat{b}_0|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{G}_{11}\hat{u}_0 + \hat{G}_{12}\hat{b}_0|^2 \\ &\quad\le C(|\xi|^{4\alpha}|\hat{u}_0|^2 + |\xi_1|^2|\hat{b}_0|^2 + |\xi_1|^2|\hat{u}_0|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{u}_0|^2)e^{-c\omega(\xi)t}. \end{aligned} \tag{2.20}$$

First, we put $\hat{u}_0(\xi) = 0$. Then we have

$$|\xi_1|^2|\hat{G}_{12}\hat{b}_0|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{G}_{12}\hat{b}_0|^2 \le C|\xi_1|^2|\hat{b}_0|^2e^{-c\omega(\xi)t}. \tag{2.21}$$

From (2.21), we derive that

$$|\hat{G}_{12}|^2 \leq \frac{C|\xi_1|^2}{|\xi_1|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})} e^{-c\omega(\xi)t}. \tag{2.22}$$

Using (2.20) with $|\hat{b}_0(\xi)| = 0$, it follows that

$$\begin{aligned} &|\xi_1|^2|\hat{u}_0|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2|\hat{G}_{11}\hat{u}_0|^2 \\ &\quad \leq C\Big(|\xi|^{2\alpha} + |\xi_1|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2\Big)|\hat{u}_0|^2 e^{-c\omega(\xi)t}. \end{aligned} \tag{2.23}$$

From (2.23), we have

$$|\hat{G}_{11}|^2 \leq \frac{C\Big(|\xi|^{4\alpha} + |\xi_1|^2 + (|\xi|^{2\alpha} + |\xi|^{2\beta})^2\Big)}{(|\xi|^{2\alpha} + |\xi|^{2\beta})^2 + |\xi_1|^2} e^{-c\omega(\xi)t}. \tag{2.24}$$

Then, we obtain

$$|\hat{G}_{12}| \leq Ce^{-c\omega(\xi)t}$$

and

$$|\hat{G}_{11}| \leq Ce^{-c\omega(\xi)t}.$$

The same procedure applies to the equations for b, we arrive at

$$|\hat{G}_{21}| \leq Ce^{-c\omega(\xi)t}$$

and

$$|\hat{G}_{22}| \leq Ce^{-c\omega(\xi)t}.$$

It follows from the above four inequalities and (2.10) that (2.1) holds. We complete the proof of Lemma 2.1. □

To prove global existence and analytic of solutions to the problem (1.2), (1.3), we also need the following Lemma.

Lemma 2.4. (see [31]) *Assume that* $\frac{1}{2} \leq \gamma \leq 1$*, then the following inequality*

$$|\xi|^{2(1-\gamma)} \leq 2^{1-2\gamma}(|\xi - \eta|^{1-2\gamma}|\eta| + |\xi - \eta||\eta|^{1-2\gamma})$$

holds for any $\xi, \eta \in \mathbb{R}^n$.

With these preparations of Lemmas 2.1 and 2.4, we can establish the global well-posedness and analytic of solutions to the problem (1.2), (1.3) in the critical space $\chi^{1-2\alpha} \bigcap \chi^{1-2\beta}$ with $\frac{1}{2} \leq \alpha, \beta \leq 1$. The result can be stated as follows:

Theorem 2.5. *Let* $\frac{1}{2} \leq \beta \leq \alpha \leq 1$*. There exists a small constant* $\epsilon_1 > 0$ *such that for the initial data* $(u_0, b_0) \in \chi^{1-2\alpha} \bigcap \chi^{1-2\beta}$ *satisfies* $\|(u_0, b_0)\|_{\chi^{1-2\alpha}} + \|(u_0, b_0)\|_{\chi^{1-2\beta}} < \epsilon_1$*, then the problem* (1.2), (1.3) *admits a unique global mild solution*

$$(u, b) \in C\big(\mathbb{R}^+; \chi^{1-2\alpha} \bigcap \chi^{1-2\beta}\big) \bigcap L^1\big(\mathbb{R}^+; \chi^1\big)$$

such that

$$\begin{aligned} &\|(u, b)(t)\|_{\chi^{1-2\alpha}} + \|(u, b)(t)\|_{\chi^{1-2\beta}} + \int_0^t \|(u, b)(\tau)\|_{\chi^1} \mathrm{d}\tau \\ &\quad \leq C(\|(u_0, b_0)\|_{\chi^{1-2\alpha}} + \|(u_0, b_0)\|_{\chi^{1-2\beta}}). \end{aligned} \tag{2.25}$$

Furthermore, the solution is analytic in the sense that

$$\begin{aligned} &\|e^{c|D|^\gamma\sqrt{t}}(u,b)(t)\|_{\chi^{1-2\alpha}} + \|e^{c|D|^\gamma\sqrt{t}}(u,b)(t)\|_{\chi^{1-2\beta}} + \int_0^t \|e^{c|D|^\gamma\sqrt{t}}(u,b)(\tau)\|_{\chi^1}\mathrm{d}\tau \\ &\quad\le C(\|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \|(u_0,b_0)\|_{\chi^{1-2\beta}}), \end{aligned} \tag{2.26}$$

where $e^{c|D|^\gamma\sqrt{t}}$ *is a Fourier multiplier whose symbol is given by* $e^{c|\xi|^\gamma\sqrt{t}}$ *and* c *is a positive constant.* γ *is given by*

$$\gamma = \begin{cases} \alpha, & |\xi| \le R_0 \\ \beta, & |\xi| \ge R_0 \end{cases}$$

Remark 2.6. Compared with [32], the difficulty to obtain a priori estimate (2.25) and the analyticity (2.26) for our problem is the presence of linear terms $\partial_1 u$ and $\partial_1 b$.

Remark 2.7. When $\alpha = \beta$, the result in Theorem 2.5 reduces to that in Xiao and Yuan [29]. In proving analyticity and decay estimate of global solutions (see Theorem 3.2), we need to overcome the difficulty that caused by α, β taking different values.

Remark 2.8. When $\frac{1}{2} \le \alpha \le \beta \le 1$, similar result could be established.

Proof. We shall prove Theorem 2.5 by energy method in Fourier space and continuous argument. To this end, let

$$X(t) = \sup_{0\le\tau\le t} (\|(u,b)(\tau)\|_{\chi^{1-2\alpha}} + \|(u,b)(\tau)\|_{\chi^{1-2\beta}}) \tag{2.27}$$

and

$$Y(t) = \int_0^t \|(u,b)(\tau)\|_{\chi^1}\mathrm{d}\tau. \tag{2.28}$$

In what follows, we would build a closed inequality for $X(t)$ and $Y(t)$, i.e.,

$$\begin{aligned} X(t) + Y(t) \le C\Big(&\|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \|(u_0,b_0)\|_{\chi^{1-2\beta}}\Big) \\ &+ CX(t)Y(t). \end{aligned} \tag{2.29}$$

then global mild solutions follow from standard continuous argument. In fact, if (2.29) holds, it is not difficult to prove

$$X(t) + Y(t) \le C\delta \tag{2.30}$$

by standard continuous argument, provided that

$$\delta = \|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \|(u_0,b_0)\|_{\chi^{1-2\beta}}$$

is suitably small. Then the problem (1.2), (1.3) admits a unique global mild solution (u,b), which satisfies (2.25).

Next, we shall complete the proof of (2.29) by dividing it into two steps.

Step 1. The estimate for $X(t)$

Multiplying (2.1) by $|\xi|^{1-2\alpha}$ and integrating the resulting equation over $\mathbb{R}^3$ with respect to ξ, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\xi|^{1-2\alpha}|\hat{u}(t,\xi)|\mathrm{d}\xi + \int_{\mathbb{R}^3} |\xi|^{1-2\alpha}|\hat{b}(t,\xi)|\mathrm{d}\xi \\
&\quad \le C\int_{\mathbb{R}^3} e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}t}|\xi|^{1-2\alpha}(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\mathrm{d}\xi \\
&\quad\quad + C\int_0^t\int_{\mathbb{R}^3}\Big[e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}(t-\tau)}|\xi|^{2-2\alpha} \\
&\quad\quad \times (|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau.
\end{aligned}$$

By Lemma 2.4 and Young inequality, it yields

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\big[e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}(t-\tau)}|\xi|^{2-2\alpha}|\widehat{u\otimes u}|(\tau,\xi)\mathrm{d}\xi\big]\mathrm{d}\tau \\
&\quad \le C\int_0^t\left\{\int_{\mathbb{R}^3}\big[(|\eta|^{1-2\alpha}|\xi-\eta|+|\eta||\xi-\eta|^{1-2\alpha})\int_{\mathbb{R}^3}|\hat{u}(\eta)\otimes\hat{u}(\xi-\eta)|\mathrm{d}\eta\big]\mathrm{d}\xi\right\}\mathrm{d}\tau \\
&\quad \le C\int_0^t\left[\int_{\mathbb{R}^3}|\eta|^{1-2\alpha}|\hat{u}(\eta)|\mathrm{d}\eta\int_{\mathbb{R}^3}|\xi-\eta||\hat{u}(\xi-\eta)|\mathrm{d}\xi\right]\mathrm{d}\tau \\
&\quad\quad +, C\int_0^t\left[\int_{\mathbb{R}^3}|\xi-\eta|^{1-2\alpha}|\hat{u}(\xi-\eta)|\mathrm{d}\xi\int_{\mathbb{R}^3}|\eta||\hat{u}(\eta)|\mathrm{d}\eta\right]\mathrm{d}\tau \\
&\quad \le C\int_0^t\|u(t)\|_{\chi^{1-2\alpha}}\|u(t)\|_{\chi^1}\mathrm{d}\tau.
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}\left[e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}(t-\tau)}|\xi|^{2-2\alpha}(|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\right]\mathrm{d}\tau \\
&\quad \le C\int_0^t(\|b(t)\|_{\chi^{1-2\alpha}}\|b(t)\|_{\chi^1}+\|u(t)\|_{\chi^{1-2\alpha}}\|b(t)\|_{\chi^1}+\|b(t)\|_{\chi^{1-2\alpha}}\|u(t)\|_{\chi^1})\mathrm{d}\tau.
\end{aligned}$$

Combining the above three inequalities yields

$$\|(u,b)\|_{\chi^{1-2\alpha}} \le C\left(\|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \int_0^t\|(u,b)\|_{\chi^{1-2\alpha}}\|(u,b)\|_{\chi^1}\mathrm{d}\tau\right). \tag{2.31}$$

The same procedure leads to

$$\|(u,b)\|_{\chi^{1-2\beta}} \le C\left(\|(u_0,b_0)\|_{\chi^{1-2\beta}} + \int_0^t\|(u,b)\|_{\chi^{1-2\beta}}\|(u,b)\|_{\chi^1}\mathrm{d}\tau\right). \tag{2.32}$$

Combining (2.31), (2.32) yields

$$X(t) \le C\left(\|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \|(u_0,b_0)\|_{\chi^{1-2\beta}}\right) + CX(t)Y(t). \tag{2.33}$$

Step 2. The estimate for $Y(t)$

Multiplying (2.1) by $|\xi|$ and then integrating the resulting equation over $\mathbb{R}^3\times(0,t)$ about ξ and τ, we arrive at

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3}|\xi||\hat{u}(\tau,\xi)|\mathrm{d}\xi\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}^3}|\xi||\hat{b}(\tau,\xi)|\mathrm{d}\xi\mathrm{d}\tau\\
&\le C\int_0^t\int_{\mathbb{R}^3} e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}\tau}|\xi|(|\hat{u}_0(\xi)+|\hat{b}_0(\xi)|)\mathrm{d}\xi\mathrm{d}\tau\\
&\quad + C\int_0^t\int_0^s\int_{\mathbb{R}^3}\Big[e^{-c\frac{|\xi|^{2(\alpha+\beta)}}{|\xi|^{2\alpha}+|\xi|^{2\beta}}\tau}|\xi|^2\\
&\quad \times(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau\mathrm{d}s\\
&\le C\int_0^t|\xi|^{2\alpha}e^{-c|\xi|^{2\alpha}\tau}\mathrm{d}\tau\int_{|\xi|\le R_0}|\xi|^{1-2\alpha}(|\hat{u}_0(\xi)+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&\quad + C\int_\tau^t|\xi|^{2\alpha}e^{-c|\xi|^{2\alpha}(s-\tau)}\mathrm{d}s\int_0^t\int_{|\xi|\le R_0}|\xi|^{2-2\alpha}\\
&\quad \times(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau\\
&\quad + C\int_0^t|\xi|^{2\beta}e^{-c|\xi|^{2\beta}\tau}\mathrm{d}\tau\int_{|\xi|\ge R_0}|\xi|^{1-2\beta}(|\hat{u}_0(\xi)+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&\quad + C\int_\tau^t|\xi|^{2\beta}e^{-c|\xi|^{2\beta}(s-\tau)}\mathrm{d}s\int_0^t\int_{|\xi|\ge R_0}|\xi|^{2-2\beta}\\
&\quad \times(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau\\
&\le C\int_{\mathbb{R}^3}(|\xi|^{1-2\alpha}+|\xi|^{1-2\beta})(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&\quad + C\int_0^t\int_{\mathbb{R}^3}(|\xi|^{2-2\alpha}+|\xi|^{2-2\beta})(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\mathrm{d}\tau.
\end{aligned}$$

Along the same line of Step 1, we could obtain

$$\int_0^t\int_{\mathbb{R}^3}(|\xi|^{2-2\alpha}+|\xi|^{2-2\beta})(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau$$

$$\begin{aligned}&\le C\int_0^t(\|u(\tau)\|_{\chi^{1-2\alpha}}+\|u(\tau)\|_{\chi^{1-2\beta}})\|u(\tau)\|_{\chi^1}\mathrm{d}\tau\\&\quad+C\int_0^t(\|b(\tau)\|_{\chi^{1-2\alpha}}+\|b(\tau)\|_{\chi^{1-2\beta}})\|b(\tau)\|_{\chi^1}\mathrm{d}\tau\\&\quad+C\int_0^t(\|u(\tau)\|_{\chi^{1-2\alpha}}+\|u(\tau)\|_{\chi^{1-2\beta}})\|b(\tau)\|_{\chi^1}\mathrm{d}\tau\\&\quad+C\int_0^t(\|b(\tau)\|_{\chi^{1-2\alpha}}+\|b(\tau)\|_{\chi^{1-2\beta}})\|u(\tau)\|_{\chi^1}\mathrm{d}\tau\\&\le C\int_0^t(\|(u,b)\|_{\chi^{1-2\alpha}}+\|(u,b)\|_{\chi^{1-2\beta}})\|(u,b)\|_{\chi^1}\mathrm{d}\tau.\end{aligned}$$

We combine the above two inequalities to obtain

$$\begin{aligned}\int_0^t\|(u,b)(\tau)\|_{\chi^1}\mathrm{d}\tau\le C\Big(&\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u_0,b_0)\|_{\chi^{1-2\beta}}\\&+\int_0^t(\|(u,b)\|_{\chi^{1-2\alpha}}+\|(u,b)\|_{\chi^{1-2\beta}})\|(u,b)\|_{\chi^1}\mathrm{d}\tau\Big).\end{aligned}\tag{2.34}$$

(2.34) entails that

$$Y(t)\le C\Big(\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u_0,b_0)\|_{\chi^{1-2\beta}}\Big)+CX(t)Y(t).\tag{2.35}$$

(2.29) immediately follows from (2.33) and (2.35).

In what follows, we prove (2.26). Let

$$\hat{U}_1(t,\xi)=e^{c|\xi|^\gamma\sqrt{t}}\hat{u}(t),\ \ \hat{B}_1(t,\xi)=e^{c|\xi|^\gamma\sqrt{t}}\hat{b}(t).$$

Firstly, we estimate $\|(U_1,B_1)\|_{\chi1-2\alpha}$ and $\|(U_1,B_1)\|_{\chi1-2\beta}$. Multiplying (2.1) by $e^{c|\xi|^\gamma\sqrt{t}}$, it holds that

$$\begin{aligned}&e^{c|\xi|^\gamma\sqrt{t}}|\hat{u}(t,\xi)|+e^{c|\xi|^\gamma\sqrt{t}}|\hat{b}(t,\xi)|\\&\quad\le Ce^{-c|\xi|^{2\gamma}t+c|\xi|^\gamma\sqrt{t}}(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\\&\qquad+C\int_0^t e^{-c|\xi|^{2\gamma}(t-\tau)+c|\xi|^\gamma\sqrt{t}}|\xi|\\&\qquad\times(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u})\mathrm{d}\tau\\&\quad\le Ce^{c|\xi|^\gamma\sqrt{t}-\frac{c}{2}|\xi|^{2\gamma}t}e^{-\frac{c}{2}|\xi|^{2\gamma}t}(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\\&\qquad+C\int_0^t e^{c|\xi|^\gamma(\sqrt{t}-\sqrt{\tau})-\frac{c}{2}|\xi|^{2\gamma}(t-\tau)}e^{-\frac{c}{2}|\xi|^{2\gamma}(t-\tau)}e^{c|\xi|^\gamma\sqrt{\tau}}|\xi|\\&\qquad\times(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u})\mathrm{d}\tau\end{aligned}\tag{2.36}$$

Noting that $e^{c|\xi|^\gamma\sqrt{t}-\frac{c}{2}|\xi|^{2\gamma}t}$, and $e^{c|\xi|^\gamma(\sqrt{t}-\sqrt{\tau})-\frac{c}{2}|\xi|^{2\gamma}(t-\tau)}$ are uniformly bounded in time and using $|\xi|^\gamma\le|\xi-\eta|^\gamma+|\eta|^\gamma$, then we have

$$\begin{aligned}
&|\hat{U}_1(t,\xi)|+|\hat{B}_1(t,\xi)|\\
&\quad\le Ce^{-\frac{c}{2}|\xi|^{2\gamma}t}(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\\
&\qquad+C\int_0^t e^{-\frac{c}{2}|\xi|^{2\gamma}(t-\tau)}|\xi|\Bigg[\int_{\mathbb{R}^3}e^{c|\xi-\eta|^\gamma\sqrt{\tau}}|\hat{u}(\xi-\eta)|e^{c|\eta|^\gamma\sqrt{\tau}}|\hat{u}(\eta)|\mathrm{d}\eta\\
&\qquad+C\int_{\mathbb{R}^3}e^{c|\xi-\eta|^\gamma\sqrt{\tau}}|\hat{b}(\xi-\eta)|e^{c|\eta|^\gamma\sqrt{\tau}}|\hat{b}(\eta)|\mathrm{d}\eta\\
&\qquad+C\int_{\mathbb{R}^3}e^{c|\xi-\eta|^\gamma\sqrt{\tau}}|\hat{u}(\xi-\eta)|e^{c|\eta|^\gamma\sqrt{\tau}}|\hat{b}(\eta)|\mathrm{d}\eta\\
&\qquad+C\int_{\mathbb{R}^3}e^{c|\xi-\eta|^\gamma\sqrt{\tau}}|\hat{b}(\xi-\eta)|e^{c|\eta|^\gamma\sqrt{\tau}}|\hat{u}(\eta)|\mathrm{d}\eta\Bigg]\mathrm{d}\tau\\
&\quad\le Ce^{-\frac{c}{2}|\xi|^{2\gamma}t}(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)+C\int_0^t e^{-\frac{c}{2}|\xi|^{2\gamma}(t-\tau)}|\xi|\\
&\qquad\times(|\widehat{U_1\otimes U_1}|+|\widehat{B_1\otimes B_1}|+|\widehat{U_1\otimes B_1}|+|\widehat{B_1\otimes U_1})\mathrm{d}\tau.
\end{aligned}\tag{2.37}$$

Multiplying (2.37) by $|\xi|^{1-2\alpha}$ and $|\xi|^{1-2\beta}$, respectively, and then integrating the resulting equation over $\mathbb{R}^3$ with respect to ξ and using Lemma 2.4, we arrive at

$$\begin{aligned}
\|(U_1,B_1)\|_{\chi^{1-2\alpha}}\le C\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+C\Bigg(&\int_0^t\|U_1\|_{\chi^{1-2\alpha}}\|U_1\|_{\chi^1}\mathrm{d}\tau\\
&+\int_0^t\|B_1\|_{\chi^{1-2\alpha}}\|B_1\|_{\chi^1}\mathrm{d}\tau+\int_0^t\|U_1\|_{\chi^{1-2\alpha}}\|B_1\|_{\chi^1}\mathrm{d}\tau\\
&+\int_0^t\|B_1\|_{\chi^{1-2\alpha}}\|U_1\|_{\chi^1}\mathrm{d}\tau\Bigg)
\end{aligned}\tag{2.38}$$

and

$$\begin{aligned}
\|(U_1,B_1)\|_{\chi^{1-2\beta}}\le C\|(u_0,b_0)\|_{\chi^{1-2\beta}}+C\Bigg(&\int_0^t\|U_1\|_{\chi^{1-2\beta}}\|U_1\|_{\chi^1}\mathrm{d}\tau\\
&+\int_0^t\|B_1\|_{\chi^{1-2\beta}}\|B_1\|_{\chi^1}\mathrm{d}\tau+\int_0^t\|U_1\|_{\chi^{1-2\beta}}\|B_1\|_{\chi^1}\mathrm{d}\tau\\
&+\int_0^t\|B_1\|_{\chi^{1-2\beta}}\|U_1\|_{\chi^1}\mathrm{d}\tau\Bigg).
\end{aligned}\tag{2.39}$$

Finally, we estimate $\int_0^t \|(U_1,B_1)\|_{\chi^1}\mathrm{d}\tau$. We multiply (2.37) by $|\xi|$ and integrating the resulting equation over $\mathbb{R}^3$ with respect to ξ, useing Lemma 2.4 to obtain

$$\begin{aligned}
\int_0^t \|(U_1,B_1)\|_{\chi^1}\mathrm{d}\tau \le{}& C\int_0^t |\xi|^{2\alpha}e^{-c|\xi|^{2\alpha}\tau}\mathrm{d}\tau \int_{|\xi|\le R_0} |\xi|^{1-2\alpha}(|\hat{u}_0(\xi)+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&+C\int_\tau^t |\xi|^{2\alpha}e^{-c|\xi|^{2\alpha}(s-\tau)}\mathrm{d}s\int_0^t\int_{|\xi|\le R_0}|\xi|^{2-2\alpha}\\
&\times(|\widehat{U_1\otimes U_1}|+|\widehat{B_1\otimes B_1}|+|\widehat{U_1\otimes B_1}|+|\widehat{B_1\otimes U_1}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau\\
&+C\int_0^t |\xi|^{2\beta}e^{-c|\xi|^{2\beta}\tau}\mathrm{d}\tau \int_{|\xi|\ge R_0} |\xi|^{1-2\beta}(|\hat{u}_0(\xi)+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&+C\int_\tau^t |\xi|^{2\beta}e^{-c|\xi|^{2\beta}(s-\tau)}\mathrm{d}s\int_0^t\int_{|\xi|\ge R_0}|\xi|^{2-2\beta}\\
&\times(|\widehat{U_1\otimes U_1}|+|\widehat{B_1\otimes B_1}|+|\widehat{U_1\otimes B_1}|+|\widehat{B_1\otimes U_1}|)(\tau,\xi)\mathrm{d}\xi\Big]\mathrm{d}\tau\\
\le{}& C\int_{\mathbb{R}^3}(|\xi|^{1-2\alpha}+|\xi|^{1-2\beta})(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|)\mathrm{d}\xi\\
&+C\int_0^t\int_{\mathbb{R}^3}(|\xi|^{2-2\alpha}+|\xi|^{2-2\beta})(|\widehat{U_1\otimes U_1}|+|\widehat{B_1\otimes B_1}|\\
&+|\widehat{U_1\otimes B_1}|+|\widehat{B_1\otimes U_1}|)(\tau,\xi)\mathrm{d}\xi\mathrm{d}\tau\\
\le{}& C(\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u_0,b_0)\|_{\chi^{1-2\beta}})\\
&+C\int_0^t(\|(U_1,B_1)\|_{\chi^{1-2\alpha}}+\|(U_1,B_1)\|_{\chi^{1-2\beta}})\|(U_1,B_1)\|_{\chi^1}\mathrm{d}\tau. \qquad (2.40)
\end{aligned}$$

Therefore, collecting (2.38), (2.39) and (2.40), we arrive at

$$\begin{aligned}
&\|(U_1,B_1)\|_{\chi^{1-2\alpha}}+\|(U_1,B_1)\|_{\chi^{1-2\beta}}+\int_0^t\|(U_1,B_1)\|_{\chi^1}\mathrm{d}\tau\\
&\le C(\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u_0,b_0)\|_{\chi^{1-2\beta}}) \qquad (2.41)\\
&+C\int_0^t(\|(U_1,B_1)\|_{\chi^{1-2\alpha}}+\|(U_1,B_1)\|_{\chi^{1-2\beta}})\|(U_1,B_1)\|_{\chi^1}\mathrm{d}\tau.
\end{aligned}$$

As the proof of (2.25), then we can deduce by the standard continuous argument

$$\begin{aligned}&\|e^{c|D|^\gamma\sqrt{t}}(u,b)(t)\|_{\chi^{1-2\alpha}}+\|e^{c|D|^\gamma\sqrt{t}}(u,b)(t)\|_{\chi^{1-2\beta}}+\int_0^t\|e^{c|D|^\gamma\sqrt{\tau}}(u,b)(\tau)\|_{\chi^1}\mathrm{d}\tau\\&\le C(\|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u_0,b_0)\|_{\chi^{1-2\beta}}).\end{aligned}$$

We immediately complete the proof of Theorem 2.5 eventually. □

3. Time-decay rate

In this section, our main purpose is to obtain the time-decay rate of global solutions established in Theorem 2.5. To do this, we need the following Lemma.

Lemma 3.1. (see [5]) *Let $T>0$ and $f:[0,T]\to\mathbb{R}^+$ be a continuous function such that*

$$f(t)\le M_0+\theta_1 f(\theta_2 t);\quad \forall 0\le t\le T. \tag{3.1}$$

with $M_0\ge 0$ and $\theta_1,\theta_2\in(0,1)$. Then

$$f(t)\le\frac{M_0}{1-\theta_1};\quad \forall 0\le t\le T.$$

We state the time-decay rate of global solutions as follows:

Theorem 3.2. *Assume that the condition of Theorem 2.5 hold. Furthermore, we assume $u_0,b_0\in L^2$. Then the solution (u,b) to the problem (1.2), (1.3) satisfies*

$$\|(u,b)(t)\|_{\chi^{1-2\alpha}}+\|(u,b)(t)\|_{\chi^{1-2\beta}}\le C(1+t)^{-(\frac{5}{4\alpha}-1)}. \tag{3.2}$$

Remark 3.3. If the condition $\frac{1}{2}\le\beta\le\alpha\le 1$ in Theorem 3.2 is replaced by $\frac{1}{2}\le\alpha\le\beta\le 1$, then we have

$$\|(u,b)(t)\|_{\chi^{1-2\alpha}}+\|(u,b)(t)\|_{\chi^{1-2\beta}}\le C(1+t)^{-(\frac{5}{4\beta}-1)}. \tag{3.3}$$

Remark 3.4. [32] obtained global solutions and proved the solution decay to 0 in the Lei–Lin space as $t\to\infty$. In this paper, we not only prove analyticity of global solutions, but also obtain the accurate time-decay rate of global solutions.

Proof. To prove (3.2), it is suffice to prove that (3.2) holds when time is sufficiently large. Owing to the definition of Lei–Lin spaces, we obtain

$$\begin{aligned}\|(u,b)(t)\|_{\chi^{1-2\alpha}}&=\int_{|\xi|\le\varepsilon}|\xi|^{1-2\alpha}|(\hat{u},\hat{b})(\xi,t)|\mathrm{d}\xi+\int_{|\xi|>\varepsilon}|\xi|^{1-2\alpha}|(\hat{u},\hat{b})(\xi,t)|\mathrm{d}\xi\\&=I_1+I_2\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|(u,b)(t)\|_{\chi^{1-2\beta}}&=\int_{|\xi|\le\varepsilon}|\xi|^{1-2\beta}|(\hat{u},\hat{b})(\xi,t)|\mathrm{d}\xi+\int_{|\xi|>\varepsilon}|\xi|^{1-2\beta}|(\hat{u},\hat{b})(\xi,t)|\mathrm{d}\xi.\\&=J_1+J_2.\end{aligned} \tag{3.5}$$

It follows from Hölder inequality and the basic energy equality that

$$\begin{aligned} I_1 &\le \left(\int\limits_{|\xi|\le\varepsilon} |\xi|^{2(1-2\alpha)} \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int\limits_{|\xi|\le\varepsilon} |(\hat{u},\hat{b})(\xi,t)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\le C\varepsilon^{\frac{5-4\alpha}{2}} \|(u,b)(t)\|_{L^2} \\ &\le C\varepsilon^{\frac{5-4\alpha}{2}} \|(u_0,b_0)\|_{L^2} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} J_1 &\le \left(\int\limits_{|\xi|\le\varepsilon} |\xi|^{2(1-2\beta)} \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int\limits_{|\xi|\le\varepsilon} |(\hat{u},\hat{b})(\xi,t)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\le C\varepsilon^{\frac{5-4\beta}{2}} \|(u,b)(t)\|_{L^2} \\ &\le C\varepsilon^{\frac{5-4\beta}{2}} \|(u_0,b_0)\|_{L^2}. \end{aligned} \tag{3.7}$$

By direct computation, we have

$$\begin{aligned} I_2 &\le \int\limits_{|\xi|>\varepsilon} e^{-c\sqrt{\frac{t}{2}}|\xi|^\beta} e^{c\sqrt{\frac{t}{2}}|\xi|^\beta} |\xi|^{1-2\alpha} |(\hat{u},\hat{b})(\xi,t)| \mathrm{d}\xi \\ &\le Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^\beta} \int\limits_{|\xi|>\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^\beta} |\xi|^{1-2\alpha} |(\hat{u},\hat{b})(\xi,t)| \mathrm{d}\xi \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} J_2 &\le \int\limits_{|\xi|>\varepsilon} e^{-c\sqrt{\frac{t}{2}}|\xi|^\beta} e^{c\sqrt{\frac{t}{2}}|\xi|^\beta} |\xi|^{1-2\beta} |(\hat{u},\hat{b})(\xi,t)| \mathrm{d}\xi \\ &\le Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^\beta} \int\limits_{|\xi|>\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^\beta} |\xi|^{1-2\beta} |(\hat{u},\hat{b})(\xi,t)| \mathrm{d}\xi. \end{aligned} \tag{3.9}$$

For fixed time $t > 0$, setting $U(z,x) = u(z+\frac{t}{2},x)$, $H(z,x) = b(z+\frac{t}{2},x)$ and $Q(z,x) = P(z+\frac{t}{2},x)$. Let $\epsilon > 0$ such that $\epsilon < \epsilon_1$, according to the assumption of Theorem 2.5, if

$$\|(U,H)(x,0)\|_{\chi^{1-2\alpha}} + \|(U,H)(x,0)\|_{\chi^{1-2\beta}} \le \epsilon.$$

Without loss of generality, (U,H) is the unique global solution to the following initial value problem

$$\begin{cases} \partial_z U + (-\Delta)^\alpha U + U\cdot\nabla U + \nabla Q - H\cdot\nabla H - \partial_1 H = 0, \\ \partial_z H + (-\Delta)^\beta H + U\cdot\nabla H - H\cdot\nabla U - \partial_1 U = 0, \\ \nabla\cdot U = \nabla\cdot H = 0, \\ U(0,x) = u\left(\dfrac{t}{2},x\right), H(0,x) = b\left(\dfrac{t}{2},x\right). \end{cases} \tag{3.10}$$

By (2.26), we obtain

$$\begin{aligned} &\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^\alpha} |\xi|^{1-2\alpha} |\hat{U}(\xi,s)| \mathrm{d}\xi + \int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^\alpha} |\xi|^{1-2\alpha} |\hat{H}(\xi,s)| \mathrm{d}\xi \\ &+ \int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^\beta} |\xi|^{1-2\alpha} |\hat{U}(\xi,s)| \mathrm{d}\xi + \int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^\beta} |\xi|^{1-2\alpha} |\hat{H}(\xi,s)| \mathrm{d}\xi \end{aligned}$$

$$
\begin{aligned}
&+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\beta}|\hat{U}(\xi,s)|\mathrm{d}\xi+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\beta}|\hat{H}(\xi,s)|\mathrm{d}\xi\\
&+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\beta}|\hat{U}(\xi,s)|\mathrm{d}\xi+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\beta}|\hat{H}(\xi,s)|\mathrm{d}\xi\\
&\le C\left(\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\alpha}}+\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\beta}}\right),
\end{aligned}\tag{3.11}
$$

which gives

$$
\begin{aligned}
&\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\alpha}\left|\hat{u}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\alpha}\left|\hat{b}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\alpha}\left|\hat{u}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\alpha}\left|\hat{b}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\beta}\left|\hat{u}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{s}|\xi|^{\alpha}}|\xi|^{1-2\beta}\left|\hat{b}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\beta}\left|\hat{u}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{s}|\xi|^{\beta}}|\xi|^{1-2\beta}\left|\hat{b}\left(\xi,s+\frac{t}{2}\right)\right|\mathrm{d}\xi\\
&\le C\left(\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\alpha}}+\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\beta}}\right).
\end{aligned}\tag{3.12}
$$

(3.12) with $s=\frac{t}{2}$ implies that

$$
\begin{aligned}
&\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\alpha}}|\xi|^{1-2\alpha}|\hat{u}(\xi,t)|\mathrm{d}\xi+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\alpha}}|\xi|^{1-2\alpha}|\hat{b}(\xi,t)|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\beta}}|\xi|^{1-2\alpha}|\hat{u}(\xi,t)|\mathrm{d}\xi+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\beta}}|\xi|^{1-2\alpha}|\hat{b}(\xi,t)|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\alpha}}|\xi|^{1-2\beta}|\hat{u}(\xi,t)|\mathrm{d}\xi+\int\limits_{|\xi|\le\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\alpha}}|\xi|^{1-2\beta}|\hat{b}(\xi,t)|\mathrm{d}\xi\\
&\quad+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\beta}}|\xi|^{1-2\beta}|\hat{u}(\xi,t)|\mathrm{d}\xi+\int\limits_{|\xi|\ge\varepsilon} e^{c\sqrt{\frac{t}{2}}|\xi|^{\beta}}|\xi|^{1-2\beta}|\hat{b}(\xi,t)|\mathrm{d}\xi\\
&\le C\left(\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\alpha}}+\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\beta}}\right).
\end{aligned}\tag{3.13}
$$

Then combining (3.8), (3.9) and (3.13) yields

$$
I_2\le Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\beta}}\left(\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\alpha}}+\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\beta}}\right)\tag{3.14}
$$

and

$$
J_2\le Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\beta}}\left(\left\|(u,b)\left(\frac{t}{2}\right)\right\|_{\chi^{1-2\alpha}}+\left\|(u,b)\left(\frac{t}{2}\right)\right|_{\chi^{1-2\beta}}\right).\tag{3.15}
$$

Inserting (3.6), (3.14) and (3.7), (3.15) into (3.4) and (3.5) yields, respectively

$$\begin{aligned}\|(u,b)(t)\|_{\chi^{1-2\alpha}} \leq & C\varepsilon^{\frac{5-4\alpha}{2}}\|(u_0,b_0)\|_{L^2} + Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\beta}}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} \\ & + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|(u,b)(t)\|_{\chi^{1-2\beta}} \leq & C\varepsilon^{\frac{5-4\beta}{2}}\|(u_0,b_0)\|_{L^2} + Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\beta}}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} \\ & + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big).\end{aligned} \tag{3.17}$$

Adding up (3.16) and (3.17), we obtain

$$\begin{aligned}&\|(u,b)(t)\|_{\chi^{1-2\alpha}} + \|(u,b)(t)\|_{\chi^{1-2\beta}} \\ &\leq C(\varepsilon^{\frac{5-4\alpha}{2}} + \varepsilon^{\frac{5-4\beta}{2}})\|(u_0,b_0)\|_{L^2} \\ &+ Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\beta}}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big).\end{aligned} \tag{3.18}$$

Let $\varepsilon = \varepsilon(t)$ satisfy $0 < \varepsilon \ll 1$ when time goes to infinity, then (3.18) can be write as

$$\begin{aligned}&\|(u,b)(t)\|_{\chi^{1-2\alpha}} + \|(u,b)(t)\|_{\chi^{1-2\beta}} \\ &\leq C\varepsilon^{\frac{5-4\alpha}{2}}\|(u_0,b_0)\|_{L^2} \\ &+ Ce^{-\sqrt{\frac{t}{2}}\varepsilon^{\alpha}}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big).\end{aligned} \tag{3.19}$$

Multiplying (3.19) by $t^{\frac{5}{4\alpha}-1}$, it holds that

$$\begin{aligned}&t^{\frac{5}{4\alpha}-1}(\|(u,b)(t)\|_{\chi^{1-2\alpha}} + \|(u,b)(t)\|_{\chi^{1-2\beta}}) \\ \leq & C\varepsilon^{\frac{5-4\alpha}{2}}t^{\frac{5}{4\alpha}-1}\|(u_0,b_0)\|_{L^2} \\ &+ Ce^{-c\sqrt{\frac{t}{2}}\varepsilon^{\alpha}}t^{\frac{5}{4\alpha}-1}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big) \\ = & C\varepsilon^{\frac{5-4\alpha}{2}}t^{\frac{5}{4\alpha}-1}\|(u_0,b_0)\|_{L^2} \\ &+ C2^{\frac{5}{4\alpha}-1}e^{-c\sqrt{\frac{t}{2}}\varepsilon^{\alpha}}\Big(\frac{t}{2}\Big)^{\frac{5}{4\alpha}-1}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big).\end{aligned} \tag{3.20}$$

Taking $\varepsilon = \Big(\frac{5\sqrt{2}\log 2}{4\alpha\sqrt{t}}\Big)^{\frac{1}{\alpha}}$, then

$$\begin{aligned}&t^{\frac{5}{4\alpha}-1}\big(\|(u,b)(t)\|_{\chi^{1-2\alpha}} + \|(u,b)(t)\|_{\chi^{1-2\beta}}\big) \\ \leq & C\Big(\frac{5\sqrt{2}\log 2}{4\alpha}\Big)^{\frac{5-4\alpha}{2\alpha}}\|(u_0,b_0)\|_{L^2} \\ &+ \frac{1}{2}\Big(\frac{t}{2}\Big)^{\frac{5}{4\alpha}-1}\Big(\Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\alpha}} + \Big\|(u,b)\Big(\frac{t}{2}\Big)\Big\|_{\chi^{1-2\beta}}\Big).\end{aligned} \tag{3.21}$$

Let

$$K_0 = C\Big(\frac{5\sqrt{2}\log 2}{4\alpha}\Big)^{\frac{5-4\alpha}{2\alpha}}\|(u_0,b_0)\|_{L^2} \tag{3.22}$$

and

$$f(t) = t^{\frac{5}{4\alpha}-1}\Big(\|(u,b)(t)\|_{\chi^{1-2\alpha}} + \|(u,b)(t)\|_{\chi^{1-2\beta}}\Big), \tag{3.23}$$

then (3.21) becomes

$$f(t) \leq K_0 + \frac{1}{2} f\left(\frac{t}{2}\right) \tag{3.24}$$

with $\theta_1 = \theta_2 = \frac{1}{2}$. It follows from Lemma 3.1 that

$$f(t) \leq \frac{K_0}{1-\theta_1} = 2K_0.$$

Then (3.2) follows. Thus we complete the proof of Theorem 3.2. □

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