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Blowup time estimates for the heat equation with a nonlocal boundary condition

Heqian Lu, Bei Hu and Zhengce Zhang

Abstract. We study the blowup time for the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ with the nonlocal boundary condition, where the normal derivative $\partial u/\partial \vec{\eta} = \int_\Omega u^p \mathrm{d}z$ on one part of boundary $\Gamma_1 \subseteq \partial\Omega$ for some $p > 1$, while $\partial u/\partial \vec{\eta} = 0$ on the rest part of the boundary. By constructing suitable auxiliary functions and analyzing the representation formula of u, we establish the finite time blowup of the solution and get both upper and lower bounds for the blowup time in terms of the parameter p, the initial value $u_0(x)$ and the volume of Γ_1. In many other studies, they require the convexity of the domain Ω and only deal with the case $\Gamma_1 = \partial\Omega$. In this article, we remove the convexity assumption and consider the problem with $\Gamma_1 \subseteq \partial\Omega$.

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Keywords. Blowup time, Heat equation, Nonlocal nonlinear boundary.

1. Introduction

In this paper, we consider the blowup time for the heat equation with the nonlocal boundary condition

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial u(x,t)}{\partial \vec{\eta}} = \displaystyle\int_\Omega u^p(z,t)\,\mathrm{d}z, & (x,t) \in \Gamma_1 \times (0,T], \\ \dfrac{\partial u(x,t)}{\partial \vec{\eta}} = 0, & (x,t) \in \Gamma_2 \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset in $\mathbb{R}^n\,(n \geqslant 2)$ with C^2 boundary $\partial\Omega$, Γ_1 and Γ_2 are two disjoint relatively open subsets of $\partial\Omega$ with $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \partial\Gamma_i \in C^1$ $(i = 1, 2)$, $\vec{\eta}(x)$ is the unit outward normal vector at $x \in \partial\Omega$ and $p > 1$. The initial value $u_0(x) \in C^1(\overline{\Omega})$ is a nonnegative and nontrivial function. The normal derivative on the boundary is given by

$$\frac{\partial u(x,t)}{\partial \vec{\eta}} = \lim\nolimits_{\varepsilon \to 0^+} Du(x_\varepsilon, t) \cdot \vec{\eta}(x), \quad (x,t) \in \partial\Omega \times (0,T], \tag{1.2}$$

where $x_\varepsilon = x - \varepsilon\vec{\eta}(x)$. The condition $\partial\Omega \in C^2$ guarantees that $x_\varepsilon \in \Omega$ when ε is positive and sufficiently small.

Various phenomena in the natural sciences and engineering lead to the nonclassical mathematical models subject to nonlocal boundary conditions, which unify the information inside of the spatial domain to define the values on the boundary. In [1], Bicadze and Samarskii introduced and systematically investigated a certain class of spatial nonlocal problems for elliptic differential equations by using the Green's function. Later on, a large amount of works have been carried out on the study of nonlocal boundary problems for different types of differential equations. To investigate the permeation pathway of single-ion channel, Levitt [15] provided a mathematical model in which the concentration of the ion satisfies a diffusion equation with a nonlocal boundary condition. In [20], McGill and Schumaker generalized Levitt's model to construct a nonlocal boundary condition with an extra flux for the steady-state concentration $C(x)$ of ion. On the other hand, parabolic equations with nonlocal boundary conditions are also discussed in many physical applications. For example, in the study of dynamics behavior of the heat conduction within linear thermoelasticity, by considering a slab $-l \leqslant x \leqslant l$ which is made of homogeneous, isotropic material and which undergoes a motion in which the displacement vector is parallel to the x-axis, Day [4,5] deduced that the entropy per unit volume, $u(x,t)$, satisfies

$$\begin{cases} u_t(x,t) = \alpha u_{xx}(x,t), & (x,t) \in (-l,l) \times (0,T], \\ u(-l,t) = \displaystyle\int_{-l}^{l} f_1(z) u(z,t)\,\mathrm{d}z, & t \in (0,T], \\ u(l,t) = \displaystyle\int_{-l}^{l} f_2(z) u(z,t)\,\mathrm{d}z, & t \in (0,T], \\ u(x,0) = u_0(x), & x \in (-l,l). \end{cases}$$

This model has been developed by Friedman [9], Deng [7] and Pao [21–23] to some more general types of reaction–diffusion equations. Friedman [9] extended Day's result to general parabolic equations in n-dimensional space subject to the following nonlocal boundary condition:

$$u(x,t) = \int_{\Omega} f(x,z) u(z,t)\,\mathrm{d}z.$$

There he used a contraction method to establish the existence and uniqueness of solutions and derived the monotonic decay property of $u(x,t)$. In [7], Deng proved the comparison principle for the same kind of problem and showed the local existence by the method of upper and lower solutions. He also discussed the decay property of the solution. In [21–23], Pao investigated a class of reaction–diffusion equations subject to the following boundary condition:

$$\alpha_0 \frac{\partial u(x,t)}{\partial \vec{\eta}} + u(x,t) = \int_{\Omega} K(x,z) u(z,t)\,\mathrm{d}z.$$

Under some assumptions on $K(x,z)$, he proved that the solution converges to the one of the corresponding steady-state problems.

After that, initial boundary value problems with nonlocal boundary conditions have been studied by many authors, and various properties, including blowup, global solvability and qualitative behavior of solutions near blowup time have attracted considerable interest. Yin [27] discussed a class of parabolic equations subject to nonlocal Neumann boundary conditions and investigated under what assumptions the solutions blow up or exist globally. The long-time behavior of solutions and convergence to a linear problem are also derived. Marras and Vernier Piro [19] considered blowup solutions to a class of reaction–diffusion equations under nonlocal Neumann boundary conditions and obtained upper bounds for the blowup time. Moreover, under the hypothesis of convexity of $\Omega \subset \mathbb{R}^3$, a lower bound for the blowup time is derived by constructing an auxiliary function and using differential inequality technique. The similar idea is also applied in more generalized problems (see [8,18,24]). For other contributions in reaction–diffusion equations with nonlocal boundary conditions, we refer to [2,10,11,16,17] and references therein. Some numerical results are provided in [6,28].

Despite numerous papers in this area, there have not been many articles that deal with problem (1.1). One distinct feature is that the normal derivative is not continuous along the boundary. As far as we know, the blowup time estimates, which are determined by the parameter p, the initial value $u_0(x)$ and the volume of Γ_1, have been considered rarely.

The main purpose of this paper is to study problem (1.1) in any smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geqslant 2)$. Our methods are mainly motivated by the ideas of [13,14,26]; meanwhile, we also employ some results which are developed by [3,25]. First, we prove the local existence and uniqueness of solutions by the contraction mapping principle. Next, we establish the finite time blowup of the solution and get an explicit formula of the upper bound for the blowup time. Finally, by analyzing the representation formula of the solution $u(x,t)$ and utilizing the properties of Green's function, we obtain the lower bound for the blowup time in terms of the parameter p, the initial value $u_0(x)$ and the volume of Γ_1. More specifically, we conclude that

(1) Let $u_0(x)$ and $|\Gamma_1|$ be fixed. If $p \to 1^+$, then the order of the upper and lower bounds for the blowup time is $(p-1)^{-1}$. This fact implies that the order of -1 is optimal.
(2) Let $p > 1$ and $|\Gamma_1|$ be fixed. If $\max_{x\in\overline{\Omega}} u_0(x) \to 0^+$, then the order of the lower bound for the blowup time is $\left(\max_{x\in\overline{\Omega}} u_0(x)\right)^{1-p}$. This order is sharp, because the upper bound is proved to be of order $\left(\max_{x\in\overline{\Omega}} u_0(x)\right)^{1-p}$ as long as $u_0(x)$ is comparable to $\max_{x\in\overline{\Omega}} u_0(x)$.
(3) Let $p > 1$ and $u_0(x)$ be fixed. If $|\Gamma_1| > 0$, then the upper bound for the blowup time is shown to be of order $|\Gamma_1|^{-1}$. In addition, if $|\Gamma_1| \to 0^+$, the order of the lower bound is $|\Gamma_1|^{-1}/\ln(1+1/|\Gamma_1|)$ for $n = 2$ and $|\Gamma_1|^{-\frac{1}{n-1}}$ for $n \geqslant 3$.

Remark 1.1. The set $(\overline{\Gamma}_1 \cap \overline{\Gamma}_2) \times (0,T]$ is a zero measure set. Even though there will be a discontinuity of the first order derivatives on this set, there is no impact on the definition of a weak solution of (1.1). By the standard parabolic regularity theory, the weak solution is actually C^1 on $(\overline{\Omega} \setminus \overline{\Gamma}_1 \cap \overline{\Gamma}_2) \times (0,T]$. The value of the normal derivative on the set $(\overline{\Gamma}_1 \cap \overline{\Gamma}_2) \times (0,T]$ is actually irrelevant. Nonetheless, one can show that for any $(x,t) \in (\overline{\Gamma}_1 \cap \overline{\Gamma}_2) \times (0,T]$, $\partial u/\partial \vec{\eta}$ exists and

$$\frac{\partial u(x,t)}{\partial \vec{\eta}} = \frac{1}{2} \int\limits_{\Omega} u^p(z,t)\,\mathrm{d}z; \tag{1.3}$$

its proof is an application of the Green's function. Since its value is irrelevant, we shall not include the proof of (1.3) in this paper.

A weak subsolution (and a supersolution) of (1.1) is defined in the usual manner:

Definition 1.1. For any $T > 0$, a function $u(x,t) \in C\left(\overline{\Omega} \times [0,T]\right)$ is called a weak solution (subsolution, supersolution) of problem (1.1) if for any $t \in (0,T]$ and any $\phi \in C^{2,1}\left(\overline{\Omega} \times [0,t]\right)$ (with $\phi \geqslant 0$), it satisfies

$$\begin{aligned}
&\int\limits_{\Omega} \left(u(y,t)\phi(y,t) - u_0(y)\phi(y,0)\right)\mathrm{d}y - \int\limits_0^t \int\limits_{\Omega} \left(\phi_s + \Delta\phi\right)(y,s)u(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad = (\leqslant, \geqslant) \int\limits_0^t \int\limits_{\Gamma_1} \phi(y,s) \int\limits_{\Omega} u^p(z,s)\,\mathrm{d}z\,\mathrm{d}S_y\,\mathrm{d}s - \int\limits_0^t \int\limits_{\partial\Omega} u(y,s)\frac{\partial \phi(y,s)}{\partial \vec{\eta}}\,\mathrm{d}S_y\,\mathrm{d}s.
\end{aligned}$$

Definition 1.2. We define

$$T^* = \sup\left\{T > 0 : \text{there exists a solution of (1.1) on } \overline{\Omega} \times [0,T]\right\}$$

to be the maximal existence time for (1.1). We say $T^* = 0$ if local in time existence is not valid.

This paper is organized in the following manner. In Sect. 2, we state some well-known results of Green's function and the representation formula of the solution $u(x,t)$ from the potential theory, and then, we establish the local existence and uniqueness of solutions. In Sect. 3, we establish the finite time blowup of the solution and obtain an upper bound for the blowup time. In Sect. 4, we estimate the lower bound for the blowup time, which is the primary contribution of this paper.

2. Preliminaries

2.1. Green's function

The Neumann Green's function G_N is well defined from the classical single-layer potential theory. Specifically, we call a function $G_N(x,t,y,s)$ defined on $\{(x,t,y,s) : x,y \in \overline{\Omega}, t,s \in \mathbb{R}, s < t\}$ the Neumann Green's function for the heat operator

$$L_{tx} = \partial_t - \Delta_x,$$

if it satisfies

$$\begin{cases} \partial_t G_N(x,t,y,s) = \Delta_x G_N(x,t,y,s), & x \in \overline{\Omega},\ t > s, \\ \dfrac{\partial G_N(x,t,y,s)}{\partial \vec{\eta}} = 0, & x \in \partial\Omega,\ t > s, \\ \lim_{t\to s^+} G_N(x,t,y,s) = \delta(x-y), & \text{in distributional sense} \end{cases}$$

for any fixed $s \in \mathbb{R}$ and $y \in \overline{\Omega}$. Here are some well-known properties from the classical single-layer potential theory.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary $\partial\Omega$. The unique Neumann Green's function $G_N(x,t,y,s)$ for the heat operator has the following properties.*
(1) *$G_N(x,t,y,s)$ is C^2 in x and y $\big(x,y \in \overline{\Omega}\big)$, and C^1 in t and s $(s < t)$.*
(2) *For any $x,y \in \overline{\Omega}$ and $s < t$, $G(x,t,y,s)$ is nonnegative and satisfies*

$$G_N(x,t,y,s) = G_N(x,t-s,y,0) \quad \text{and} \quad G_N(x,t,y,s) = G_N(y,t,x,s).$$

(3) *For any $x \in \overline{\Omega}$ and $s < t$,*

$$\int_\Omega G_N(x,t,y,s)\,\mathrm{d}y = 1.$$

(4) *For $x,y \in \overline{\Omega}$, $0 < t-s < 1$, (here we actually only require $\partial\Omega$ to be Lipschitz), there exist $C > 0, c > 0$ such that [3, (1.3)],*

$$0 < G_N(x,t,y,s) \leqslant \frac{C}{(t-s)^{n/2}} \exp\left\{-\frac{c|x-y|^2}{t-s}\right\}. \tag{2.1}$$

(5) *Suppose that $u_0 \in L^\infty(\Omega)$, $g \in L^\infty(\partial\Omega \times (0,T))$ and is piecewise continuous. Then, the weak solution of*

$$\begin{cases} \partial_t u = \Delta_x u, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u(x,t)}{\partial \vec{\eta}} = g, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

is given by, for $x \in \overline{\Omega}$, $t > 0$,

$$u(x,t) = \int_\Omega G_N(x,t,y,0)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\partial\Omega} G_N(x,t,y,s)g(y,s)\,\mathrm{d}S_y\,\mathrm{d}s. \tag{2.2}$$

It satisfies $u \in C^\infty(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T])$. At any point $x^* \in \overline{\Omega}$ where u_0 is continuous, $u(x,t)$ is continuous at $(x^*,0)$, and at any point $(x^*,t^*) \in \partial\Omega \times (0,T)$ where g is Hölder continuous, u is C^1 in a neighborhood of this point.

Applying part (2) and part (5) of this lemma to our system (1.1), assuming $\int_\Omega u^p(z,t)\,\mathrm{d}z$ to be continuous (which will be established in the existence theorem), we find that, for $x \in \overline{\Omega}$, $t > 0$,

$$u(x,t) = \int_\Omega G_N(x,t,y,0)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\Gamma_1} G_N(x,t,y,s) \int_\Omega u^p(z,s)\,\mathrm{d}z\,\mathrm{d}S_y\,\mathrm{d}s, \tag{2.3}$$

and using T as the new initial data, for $x \in \overline{\Omega}$, $t > 0$,

$$\begin{aligned} u(x,T+t) &= \int_\Omega G_N(x,t,y,0)u(y,T)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\Gamma_1} G_N(x,t,y,s) \int_\Omega u^p(z,T+s)\,\mathrm{d}z\,\mathrm{d}S_y\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

2.2. Local existence and uniqueness

We establish the local existence and uniqueness of solutions to problem (1.1). Although the argument is more or less standard, we state it here for completeness.

Theorem 2.1. *The maximal existence time T^* for (1.1) is positive, and there exists a unique nonnegative solution $u \in C^{2,1}(\Omega \times (0,T^*)) \cap C(\overline{\Omega} \times [0,T^*))$. Moreover, if $T^* < \infty$, then*

$$\|u(\cdot,t)\|_{L^\infty} \to \infty \quad \text{as} \quad t \to T^*. \tag{2.5}$$

Proof. We prove the local existence via the contraction mapping principle. Take $T > 0$, $B > 0$ and $\mathbb{B}_T = C(\overline{\Omega} \times [0,T])$ be equipped with the maximum norm: $\|v\| = \max_{\overline{\Omega}\times[0,T]}|v|$ for any $v \in \mathbb{B}_T$, then $\mathbb{B}_T$ is a Banach space and $\mathbb{B}_{T,B} = \{v \in \mathbb{B}_T : \|v\| \leqslant B\}$ is also a Banach space. For each $v \in \mathbb{B}_{T,B}$, the linear problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial u(x,t)}{\partial \vec{\eta}} = \displaystyle\int_\Omega (v^+)^p(z,t)\,\mathrm{d}z \triangleq V(t), & (x,t) \in \Gamma_1 \times (0,T], \\ \dfrac{\partial u(x,t)}{\partial \vec{\eta}} = 0, & (x,t) \in \Gamma_2 \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{2.6}$$

admits a unique solution $u(x,t) \in C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])$, which is given by the single-layer potential (2.2), for $x \in \overline{\Omega}$, $t > 0$,

$$u(x,t) = \int_\Omega G_N(x,t,y,0)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\Gamma_1} G_N(x,t,y,s)V(s)\,\mathrm{d}S_y\,\mathrm{d}s.$$

From the fundamental estimate (2.1) on Green's function for Lipschitz domains, we have, for any $B > 0$, $0 < T_0 \leqslant 1$ and $0 < T \leqslant T_0$,

$$|u(x,t)| \leqslant C + CB^pT_0^{\frac{1}{2}}, \qquad (x,t) \in \overline{\Omega} \times [0,T].$$

If we choose B and $T_0 \leqslant 1$ such that

$$B = 4C \quad \text{and} \quad T_0 < B^{-2p},$$

then we get $\|u\| \leqslant B$. This shows that the mapping $\mathscr{M}_T$

$$\mathscr{M}_T(v) = u$$

maps $\mathbb{B}_{T,B}$ into itself.

To prove that $\mathscr{M}_T$ is a contraction on $\mathbb{B}_{T,B}$, we take $v_1, v_2 \in \mathbb{B}_{T,B}$ and define by u_1, u_2 the corresponding solutions. Then, for any $(x,t) \in \overline{\Omega} \times [0,T]$, we have

$$|u_1(x,t) - u_2(x,t)| \leqslant CT^{\frac{1}{2}} \|V_1 - V_2\|_{L^\infty[0,T]} \leqslant CpB^{p-1}T^{\frac{1}{2}}\|v_1 - v_2\|,$$

which implies that $\mathscr{M}_T$ is a contraction if T is sufficiently small. Thus, $\mathscr{M}_T$ has a unique fixed point in $\mathbb{B}_{T,B}$. Now, if $u(x,t)$ is the unique fixed point of $\mathbb{B}_{T,B}$, then the standard regularity theory of parabolic PDE implies that $u(x,t) \in C^{2,1}(\Omega \times (0,T]) \cap C\left(\overline{\Omega} \times [0,T]\right)$, and by maximum principle $u(x,t) > 0$ for all $x \in \Omega$ and $t > 0$. It is therefore $\int\limits_\Omega (u^+)^p(z,t)\,\mathrm{d}z = \int\limits_\Omega u^p(z,t)\,\mathrm{d}z$, and thus, u is also a nonnegative classical solution of problem (1.1).

Since the solution is obtained by the contraction mapping principle, it is unique in $\mathbb{B}_{T,B}$. Suppose u is a solution obtained by the contraction mapping principle, then by our argument $\|u\| \leqslant \frac{1}{2}B$. Suppose $v \in C^{2,1}(\Omega \times (0,T]) \cap C\left(\overline{\Omega} \times [0,T]\right)$ is another solution. Since $\|v(\cdot,0)\|_{L^\infty} \leqslant \frac{1}{2}B$, by continuity, $\|v\|_{C(\overline{\Omega})} \leqslant B$ for $0 \leqslant t \leqslant T_1$ for some $T_1 \in (0,T]$. Repeating the above arguments, we find that $u(x,t) \equiv v(x,t)$ for $x \in \Omega, 0 \leqslant t \leqslant T_1$. But then $\|v(\cdot,T_1)\|_{L^\infty} = \|u(\cdot,T_1)\|_{L^\infty} \leqslant \frac{1}{2}B$, and the uniqueness interval can further be extended. It follows that if the maximal uniqueness interval is $[0,T_1)$, then it must coincide with $[0,T)$.

If $\limsup\limits_{t\to T^*-} \|u(\cdot,t)\|_{L^\infty} < \infty$, then by De Giorgi–Nash–Moser estimates [12, Section 3.4], $\lim\limits_{t\to T^*} u(x,t) = u(x,T^*)$ exists and is Hölder continuous on $\overline{\Omega}$. Thus, we can extend the existence time T^* to some $T' > T^*$ using the argument above, which contradicts the definition of T^*. Therefore, (2.5) holds. □

Remark 2.1. From Theorem 2.1, we conclude that if the solution is bounded all the time, it exists globally. Thus, the estimate of T^* is reduced to calculating the blowup time for the L^∞ norm of the solution.

3. Upper bound for the blowup time

In this section, we get the finite time blowup of the solution and give an upper bound for the blowup time. Our main result of this section reads as follows.

Theorem 3.1. *Let $u(x,t)$ be the solution of (1.1) and T^* be the maximal existence time, then*

$$T^* \leqslant \frac{|\Omega|^{p-1}}{(p-1)|\Gamma_1|}\left(\int\limits_\Omega u_0(z)\,\mathrm{d}z\right)^{1-p}. \tag{3.1}$$

Proof. The function $h(t) \triangleq \int\limits_\Omega u(z,t)\,\mathrm{d}z$ is continuous in t, and the Hölder's inequality implies that

$$\int\limits_\Omega u^p(z,t)\,\mathrm{d}z \geqslant |\Omega|^{1-p}h^p(t).$$

Using this estimate and taking the test function $\phi \equiv 1$ in the definition of the weak solution, we obtain

$$h(t) \geqslant h(0) + |\Gamma_1||\Omega|^{1-p}\int\limits_0^t h^p(s)\,\mathrm{d}s \triangleq H(t), \quad 0 < t < T^*.$$

It follows that $H'(t) \geqslant |\Gamma_1||\Omega|^{1-p}H^p(t)$ and $H(0) = h(0)$. Integrating over $[0, T]$ $(T < T^*)$, we get

$$0 \leqslant H^{1-p}(T) \leqslant H^{1-p}(0) - (p-1)T|\Gamma_1||\Omega|^{1-p}.$$

Hence,

$$T \leqslant \frac{|\Omega|^{p-1}}{(p-1)|\Gamma_1|}H^{1-p}(0) = \frac{|\Omega|^{p-1}}{(p-1)|\Gamma_1|}h^{1-p}(0).$$

Letting $T \to T^*-$, we complete the proof. □

4. Lower bound for the blowup time

The primary contribution of this paper is to derive the lower bound for the blowup time T^* which is based on representation formulas (2.3) and (2.4). In order to obtain a lower bound for T^*, we study how fast the solution u can grow. For this purpose, we define

$$M_0 = \max_{x\in\overline{\Omega}} u_0(x) \quad \text{and} \quad M(t) = \sup_{(y,s)\in\overline{\Omega}\times[0,t]} u(y,s). \tag{4.1}$$

The first result of the lower bound for the blowup time T^* is presented below.

Theorem 4.1. *Let T^* be the maximal existence time for (1.1). Then for any $\alpha \in (0, 1/(n-1))$, there exists a constant $C = C(n, \alpha, \Omega)$ such that*

$$T^* \geqslant \begin{cases} \dfrac{C}{p-1}\left((2M_0)^{p-1}|\Gamma_1|^\alpha\right)^{-\frac{2}{1+(1-n)\alpha}}, & \text{if } (2M_0)^{p-1}|\Gamma_1|^\alpha \geqslant 1, \\ \dfrac{C}{p-1}\ln\left(1+\left((2M_0)^{p-1}|\Gamma_1|^\alpha\right)^{-\frac{2}{1+(1-n)\alpha}}\right), & \text{if } (2M_0)^{p-1}|\Gamma_1|^\alpha < 1. \end{cases} \tag{4.2}$$

Remark 4.1. From Theorems 3.1 and 4.1 , we have the relationship between p and T^* when $u_0(x)$ and $|\Gamma_1|$ are fixed as $p \to 1^+$; namely, we deduce from (3.1) and (4.2) that

$$C_1(p-1)^{-1} \leqslant T^* \leqslant C_2(p-1)^{-1},$$

which implies that this order of -1 is optimal.

Remark 4.2. Note that

$$\left((2M_0)^{p-1}|\Gamma_1|^\alpha\right)^{-\frac{2}{1+(1-n)\alpha}} \geqslant \ln\left(1+\left((2M_0)^{p-1}|\Gamma_1|^\alpha\right)^{-\frac{2}{1+(1-n)\alpha}}\right),$$

thus, in both cases, we always have

$$T^* \geqslant \frac{C}{p-1}\ln\left(1+\left((2M_0)^{p-1}|\Gamma_1|^\alpha\right)^{-\frac{2}{1+(1-n)\alpha}}\right). \tag{4.3}$$

Proof of Theorem 4.1. For any $j \in \mathbb{N}$, we choose $M_j = 2^j M_0$. Define T_j to be the first time that $M(t)$ arrives at M_j and $t_j = T_j - T_{j-1}$. Since $u(x,t) \in C^{2,1}(\Omega\times(0,T^*)) \cap C\left(\overline{\Omega}\times[0,T^*)\right)$, we find

$$T_j = \min\{t \geqslant 0 : M(t) = M_j\}.$$

Applying the representation formula (2.4), we derive

$$\begin{aligned} u(x,T_j) = & \int_{\Omega} G_N(x,t_j,y,0)u(y,T_{j-1})\,\mathrm{d}y \\ & + \int_0^{t_j}\int_{\Gamma_1} G_N(x,t_j,y,s)\int_{\Omega} u^p(z,T_{j-1}+s)\,\mathrm{d}z\,\mathrm{d}S_y\,\mathrm{d}s \\ \leqslant & M_{j-1}\int_{\Omega} G_N(x,t_j,y,0)\,\mathrm{d}y + |\Omega|M_j^p \int_0^{t_j}\int_{\Gamma_1} G_N(x,t_j,y,s)\,\mathrm{d}S_y\,\mathrm{d}s \\ = & M_{j-1} + |\Omega|M_j^p \int_0^{t_j}\int_{\Gamma_1} G_N(x,t_j,y,s)\,\mathrm{d}S_y\,\mathrm{d}s, \end{aligned} \tag{4.4}$$

where the last equality is obtained by Lemma 2.1. Assume that $t_j < 1$. By virtue of the fundamental estimate (2.1) and the Hölder's inequality, for any $\alpha \in (0, 1/(n-1))$, we find

$$\begin{aligned} & \int_0^{t_j}\int_{\Gamma_1} G_N(x,t_j,y,s)\,\mathrm{d}S_y\,\mathrm{d}s \\ \leqslant & C\int_0^{t_j}\int_{\Gamma_1} (t_j-s)^{-\frac{n}{2}} \exp\left\{-\frac{c|x-y|^2}{t_j-s}\right\}\mathrm{d}S_y\,\mathrm{d}s \\ = & C\int_0^{t_j} s^{-\frac{n}{2}}\int_{\Gamma_1} \exp\left\{-\frac{c|x-y|^2}{s}\right\}\mathrm{d}S_y\,\mathrm{d}s \\ \leqslant & C|\Gamma_1|^{\alpha}\int_0^{t_j} s^{-\frac{n}{2}}\left(\int_{\Gamma_1} \exp\left\{-\frac{|x-y|^2}{s}\cdot\frac{c}{1-\alpha}\right\}\mathrm{d}S_y\right)^{1-\alpha}\mathrm{d}s \\ \leqslant & C|\Gamma_1|^{\alpha}\int_0^{t_j} s^{-\frac{n}{2}}\left(\int_{\partial\Omega} \exp\left\{-\frac{|x-y|^2}{s}\cdot\frac{c}{1-\alpha}\right\}\mathrm{d}S_y\right)^{1-\alpha}\mathrm{d}s \\ \leqslant & C|\Gamma_1|^{\alpha}\int_0^{t_j} s^{-\frac{n}{2}}\left(Cs^{\frac{n-1}{2}}\right)^{1-\alpha}\mathrm{d}s \\ \leqslant & Ct_j^{\frac{1+(1-n)\alpha}{2}}|\Gamma_1|^{\alpha}. \end{aligned} \tag{4.5}$$

Combining this with (4.4), we obtain

$$u(x,T_j) \leqslant M_{j-1} + CM_j^p t_j^{\frac{1+(1-n)\alpha}{2}}|\Gamma_1|^{\alpha},$$

which implies

$$M_j \leqslant M_{j-1} + CM_j^p t_j^{\frac{1+(1-n)\alpha}{2}}|\Gamma_1|^{\alpha},$$

i.e., (recalling that $M_j = 2^j M_0$),

$$t_j \geqslant C\left(2^{j(p-1)} M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}.$$

Consequently, for any $j \in \mathbb{N}$, we have

$$t_j \geqslant \min\left\{1,\, C\left(2^{j(p-1)} M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right\}.$$

Thus,

$$T^* = \sum_{j=1}^{\infty} t_j \geqslant C \sum_{j=1}^{\infty} \min\left\{1,\, 2^{-\frac{2j(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right\}. \tag{4.6}$$

The remaining task is to provide a lower bound for the right-hand side of (4.6). If

$$2^{-\frac{2(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} \leqslant 1, \tag{4.7}$$

then

$$\begin{aligned}
&\sum_{j=1}^{\infty} \min\left\{1,\, 2^{-\frac{2j(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right\} \\
&\quad = \sum_{j=1}^{\infty} 2^{-\frac{2j(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} \\
&\quad = \frac{2^{-\frac{2(p-1)}{1+(1-n)\alpha}}}{1 - 2^{-\frac{2(p-1)}{1+(1-n)\alpha}}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} \\
&\quad \geqslant \frac{2^{-\frac{2(p-1)}{1+(1-n)\alpha}}}{\ln\left(2^{\frac{2(p-1)}{1+(1-n)\alpha}}\right)} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} \\
&\quad = \frac{1+(1-n)\alpha}{2(p-1)\ln 2} \left((2M_0)^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}.
\end{aligned} \tag{4.8}$$

Otherwise, if (4.7) is not true, then for any $\beta \in (0, 1/2]$, there exists $J_\beta \geqslant 1$ such that

$$2^{-\frac{2J_\beta(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} > \beta \quad \text{and} \quad 2^{-\frac{2\left(J_\beta+1\right)(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} \leqslant \beta.$$

Thus, we get

$$J_\beta \geqslant \frac{1+(1-n)\alpha}{2(p-1)\ln 2} \ln\left(\beta^{-1}\left((2M_0)^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right).$$

Notice $\beta^{-1} \in [2, \infty)$ and

$$2^{-\frac{2(p-1)}{1+(1-n)\alpha}} \left(M_0^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}} > 1,$$

we have

$$J_\beta \geqslant \frac{1+(1-n)\alpha}{2(p-1)\ln 2} \ln\left(1 + \left((2M_0)^{p-1} |\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right).$$

It follows that

$$\begin{aligned}&\sum_{j=1}^{\infty}\min\left\{1,\,2^{-\frac{2j(p-1)}{1+(1-n)\alpha}}\left(M_0^{p-1}|\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right\}\\&>\sum_{j=1}^{J_\beta}\min\left\{1,\,2^{-\frac{2j(p-1)}{1+(1-n)\alpha}}\left(M_0^{p-1}|\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right\}>J_\beta\cdot\beta\\&\geqslant\frac{C}{p-1}\ln\left(1+\left((2M_0)^{p-1}|\Gamma_1|^{\alpha}\right)^{-\frac{2}{1+(1-n)\alpha}}\right).\end{aligned}\tag{4.9}$$

Substituting (4.8), (4.9) into (4.6), respectively, we get (4.2). □

Comparing (3.1) and (4.2), we notice that T^* shrinks to zero when M_0 is sufficiently large, which complies intuitively with the reality.

We next investigate the asymptotic behavior of T^* when $M_0 \to 0^+$ or $|\Gamma_1| \to 0^+$. We begin with an elementary calculus result.

Lemma 4.1. *Fix any $p > 1$ and $s_0 > 0$, set $c_p = (p-1)^{p-1}/p^p$ and define a function $q : [s_0, \infty) \to \mathbb{R}$ by $q(s) = (s - s_0)/s^p$ $(s \geqslant s_0)$, then*

(1) For any $q_0 \in [0,\, c_p s_0^{1-p}]$, there exists a unique $s \in [s_0,\, ps_0/(p-1)]$ such that $q(s) = q_0$.

(2) For any $q_0 > c_p s_0^{1-p}$, there does not exist $s > s_0$ such that $q(s) = q_0$.

Proof. A straightforward computation shows that

$$q'(s) = \frac{(1-p)s + ps_0}{s^{p+1}}.$$

It follows that $q(s)$ is strictly increasing on the interval $[s_0,\, ps_0/(p-1)]$ and strictly decreasing on the interval $[ps_0/(p-1), \infty)$. In addition, $\max_{s\geqslant s_0} q(s) = q(s)|_{s=ps_0/(p-1)} = c_p s_0^{1-p}$; therefore, conclusions (1) and (2) hold. □

The following estimate of Neumann Green's function will play an important role in establishing the lower bound for the blowup time when $M_0 \to 0^+$ or $|\Gamma_1| \to 0^+$.

Lemma 4.2. *There exists $C = C(n, \Omega)$ such that for any $x \in \overline{\Omega}$ and $t \in (0, 1]$,*

$$\int_0^t \int_{\Gamma_1} G_N(x,t,y,s)\,\mathrm{d}S_y\,\mathrm{d}s \leqslant \begin{cases} C\,|\Gamma_1|^{\frac{1}{n-1}}, & \text{if } n \geqslant 3,\\ C\,|\Gamma_1|\ln\left(1 + 1/|\Gamma_1|\right), & \text{if } n = 2.\end{cases}\tag{4.10}$$

Proof. This is a result of the fundamental estimate (2.1) on Green's function, and the detailed proof can be carried out by a similar way as in the proof of [25, Lemmas 2.7 and 2.10], so we omit it. □

Now, we are in a position to establish the lower bound for the blowup time when $M_0 \to 0^+$ or $|\Gamma_1| \to 0^+$.

Theorem 4.2. *Let $u(x, t)$ be the solution of (1.1). Suppose T^* is the maximal existence time and define*

$$Q = \begin{cases} M_0^{p-1}\,|\Gamma_1|^{\frac{1}{n-1}}, & \text{if } n \geqslant 3,\\ M_0^{p-1}\,|\Gamma_1|\ln\left(1 + 1/|\Gamma_1|\right), & \text{if } n = 2.\end{cases}\tag{4.11}$$

Then, there exist constants $Q_0 = Q_0(n, \Omega)$ and $C = C(n, \Omega)$ such that if $Q \leqslant Q_0/p$, then

$$T^* \geqslant \frac{C}{(p-1)Q}.\tag{4.12}$$

Remark 4.3. According to Theorems 3.1 and 4.2, we draw two conclusions.

(1) The relationship between M_0 and T^*: let $p>1$ and $|\Gamma_1|$ be fixed, if $M_0 \to 0^+$, then we deduce from (4.12) that

$$T^* \geqslant C_1 M_0^{1-p}.$$

This order is sharp because (3.1) implies $T^* \leqslant C_2 M_0^{1-p}$ provided the initial value satisfies, for example, $cM_0 \leqslant u_0(x) \leqslant M_0$ for some $c>0$.

(2) The relationship between $|\Gamma_1|$ and T^*: let $p>1$ and $u_0(x)$ be fixed, if $|\Gamma_1| \to 0^+$, then it follows from (3.1) that

$$T^* \leqslant C_3 |\Gamma_1|^{-1}.$$

In addition, (4.12) implies that T^* is at least of order $|\Gamma_1|^{-1}/\ln(1+1/|\Gamma_1|)$ for $n=2$ and $|\Gamma_1|^{-\frac{1}{n-1}}$ for $n \geqslant 3$.

Proof of Theorem 4.2. We shall prove Theorem 4.2 in three steps below. **Step 1.** We claim that for any $T \in [0,T^*)$ and $0 \leqslant t < \min\{1, T^*-T\}$, there exists a fixed constant $C^* = C^*(n,\Omega)$ such that

$$\frac{M(T+t)-M(T)}{M^p(T+t)} \leqslant \gamma \triangleq \begin{cases} C^*\,|\Gamma_1|^{\frac{1}{n-1}}, & \text{if} \quad n \geqslant 3, \\ C^*\,|\Gamma_1| \ln\left(1+1/\,|\Gamma_1|\right), & \text{if} \quad n=2. \end{cases} \tag{4.13}$$

Indeed, by virtue of the representation formula (2.4), for any $T \in [0,T^*)$ and $0 \leqslant t < \min\{1, T^*-T\}$, we have

$$\begin{aligned} u(x,T+t) = & \int_\Omega G_N(x,t,y,0) u(y,T)\,\mathrm{d}y \\ & + \int_0^t \int_{\Gamma_1} G_N(x,t,y,s) \int_\Omega u^p(z,T+s)\,\mathrm{d}z\,\mathrm{d}S_y\,\mathrm{d}s \\ \leqslant & M(T) \int_\Omega G_N(x,t,y,0)\,\mathrm{d}y + |\Omega| M^p(T+t) \int_0^t \int_{\Gamma_1} G_N(x,t,y,s)\,\mathrm{d}S_y\,\mathrm{d}s \\ = & M(T) + |\Omega| M^p(T+t) \int_0^t \int_{\Gamma_1} G_N(x,t,y,s)\,\mathrm{d}S_y\,\mathrm{d}s, \end{aligned}$$

where the last equality is derived by Lemma 2.1. Combining the above inequality with Lemma 4.2, we get (4.13).

Step 2. Based on Lemma 4.1, we construct a strictly increasing sequence $\{M_j\}_{j=0}^\infty$.

◇ Define M_0 as in (4.1). Given that $pc_p > 1/e$ for all $p>1$ (see (4.16)), we first assume

$$Q \leqslant \frac{c_p}{2C^*}, \tag{4.14}$$

i.e., $2M_0^{p-1}\gamma \leqslant c_p$, then we construct a sequence by induction. Suppose M_{j-1} has been constructed for some $j \geqslant 1$.

◇ If $2M_{j-1}^{p-1}\gamma \leqslant c_p$, then we define $M_j \in (M_{j-1},\, pM_{j-1}/(p-1)]$ to be the unique solution such that

$$\frac{M_j - M_{j-1}}{M_j^p} = 2\gamma.$$

◇ If $2M_{j-1}^{p-1}\gamma > c_p$, we stop this construction and denote the term to be M_L, namely, M_L represents the first term with $2M_L^{p-1}\gamma > c_p$.

⋄ Define

$$M_j = \left(\frac{p}{p-1}\right)^{j-L} M_L, \qquad \forall\, j \geqslant L+1.$$

For any $j \in \mathbb{N}$, we define $T_j = \min\{t \geqslant 0 : M(t) = M_j\}$ and denote $t_j = T_j - T_{j-1}$ to be the time spent in the jth step. In particular, $T_0 = 0$. In light of (4.14), we have $L \geqslant 1$. We next show

$$t_j \geqslant 1, \qquad \forall\, 1 \leqslant j \leqslant L. \tag{4.15}$$

In fact, for any $1 \leqslant j \leqslant L$, if $t_j < 1$, using (4.13) with $T = T_{j-1}$ and $t = t_j$, we have

$$\frac{M_j - M_{j-1}}{M_j^p} \leqslant \gamma,$$

which contradicts to the choice of M_j $(1 \leqslant j \leqslant L)$. Thus, (4.15) holds.

Step 3. We first show that the integer L is finite. From the above construction, $\{M_j\}_{j=0}^{L}$ is a strictly increasing sequence and

$$M_j = M_{j-1} + 2M_j^p\gamma \geqslant M_{j-1}(1 + 2M_0^{p-1}\gamma) \geqslant M_0(1 + 2M_0^{p-1}\gamma)^j \to \infty \quad \text{as} \quad j \to \infty,$$

which implies $2M_j^{p-1}\gamma > c_p$ when j is sufficiently large. Therefore, the integer L is finite.

Now, we proceed to derive a lower bound for L. The above construction in Step 2 implies,

$$2M_{L-1}^{p-1}\gamma \leqslant c_p \quad \text{and} \quad 2M_L^{p-1}\gamma > c_p.$$

In addition, for any $1 \leqslant j \leqslant L$, we have

$$M_{j-1} = M_j - 2M_j^p\gamma = M_j\left(1 - 2M_j^{p-1}\gamma\right),$$

which implies

$$2M_{j-1}^{p-1}\gamma = 2M_j^{p-1}\gamma\left(1 - 2M_j^{p-1}\gamma\right)^{p-1}.$$

The function pc_p is decreasing for $p > 1$ and thus

$$\frac{1}{e} = \lim_{p\to\infty} pc_p < pc_p < \lim_{p\to 1+} pc_p = 1, \quad 1 < p < \infty. \tag{4.16}$$

Moreover, the function $(p-1)c_p$ is increasing for $p > 1$, then

$$0 = \lim_{p\to 1+}(p-1)c_p < (p-1)c_p < \lim_{p\to\infty}(p-1)c_p = \frac{1}{e}, \quad 1 < p < \infty.$$

Therefore, if we set $\beta = \min\{1/2,\, c_p\}$, and assume that $2M_0^{p-1}\gamma \leqslant 1/2$, namely,

$$Q \leqslant \frac{1}{4C^*}, \tag{4.17}$$

then there exists $1 \leqslant L_0 \leqslant L$ such that

$$2M_{L_0-1}^{p-1}\gamma \leqslant \beta \quad \text{and} \quad 2M_{L_0}^{p-1}\gamma > \beta.$$

Moreover, for any $0 \leqslant j \leqslant L_0$, we define $x_j = 2M_{L_0-j}^{p-1}\gamma$, then

$$x_0 = 2M_{L_0}^{p-1}\gamma > \beta \quad \text{and} \quad x_1 = 2M_{L_0-1}^{p-1}\gamma \leqslant \beta,$$

and

$$x_j = x_{j-1}\left(1 - x_{j-1}\right)^{p-1}, \qquad \forall\, 1 \leqslant j \leqslant L_0.$$

Notice that $\{x_j\}_{j=1}^{L_0}$ is a decreasing positive sequence and $x_j \leqslant \beta \leqslant 1/2$ $(1 \leqslant j \leqslant L_0)$, then

$$x_j = x_{j-1}\left(1 - x_{j-1}\right)^{p-1} \geqslant x_{j-1}\left[1 - 2(p-1)x_{j-1}\right], \qquad \forall\, 2 \leqslant j \leqslant L_0.$$

Notice that

$$1-2(p-1)x_{j-1}\geqslant 1-2(p-1)\beta\geqslant 1-2(p-1)c_p>1-\frac{2}{e}>\frac{1}{4},$$

we derive

$$\frac{1}{x_j}\leqslant\frac{1}{x_{j-1}\left[1-2(p-1)x_{j-1}\right]}=\frac{1}{x_{j-1}}+\frac{2(p-1)}{1-2(p-1)x_{j-1}}\leqslant\frac{1}{x_{j-1}}+8(p-1) \tag{4.18}$$

for any $2\leqslant j\leqslant L_0$. Therefore,

$$\frac{1}{x_{L_0}}\leqslant\frac{1}{x_1}+8(p-1)\left(L_0-1\right). \tag{4.19}$$

Since (noticing that $e\approx 2.718<3$)

$$\frac{1}{3p}<\beta<x_0=\left(\frac{M_{L_0}}{M_{L_0-1}}\right)^{p-1}x_1\leqslant\left(\frac{p}{p-1}\right)^{p-1}\beta\leqslant\left(\frac{p}{p-1}\right)^{p-1}c_p=\frac{1}{p},$$

we deduce

$$x_1=x_0\left(1-x_0\right)^{p-1}\geqslant\frac{1}{3p}\left(1-\frac{1}{p}\right)^{p-1}=\frac{c_p}{3}.$$

Substituting the above inequality and $x_{L_0}=2M_0^{p-1}\gamma$ into (4.19), we get

$$\frac{1}{2M_0^{p-1}\gamma}\leqslant\frac{3}{c_p}+8(p-1)\left(L_0-1\right)<9p+8(p-1)\left(L_0-1\right),$$

i.e.,

$$L_0\geqslant\frac{1}{8(p-1)}\left(\frac{1}{2M_0^{p-1}\gamma}-9p\right)+1.$$

It follows from (4.15) and $L\geqslant L_0$ that

$$T^*=\sum_{j=1}^{\infty}t_j>\sum_{j=1}^{L}t_j\geqslant L\geqslant\frac{1}{8(p-1)}\left(\frac{1}{2M_0^{p-1}\gamma}-9p\right)+1.$$

By virtue of the definition of Q and γ, we have $M_0^{p-1}\gamma=C^*Q$. If we choose

$$Q\leqslant\frac{1}{36C^*p}, \tag{4.20}$$

then

$$T^*\geqslant\frac{1}{8(p-1)}\left(\frac{1}{2C^*Q}-\frac{1}{4C^*Q}\right)+1=\frac{1}{32C^*Q(p-1)}+1\geqslant\frac{C}{Q(p-1)}. \tag{4.21}$$

Therefore, in view of (4.14), (4.17), (4.20), taking

$$Q_0=\min\left\{\frac{c_p}{2C^*},\frac{1}{4C^*},\frac{1}{36C^*p}\right\}\cdot p=\frac{1}{36C^*},$$

we complete the proof of Theorem 4.2. □

The heat equation in a bounded domain with a nonlocal boundary condition is considered in this paper. As an immediate consequence of the potential theory and the contraction mapping principle, the local existence and uniqueness of the solution are established. The primary results of this paper are on the blowup behavior of the system. Based on the representation formula of the solution and the properties of the Neumann Green's function, both upper and lower bounds for the blowup time T^* (also called the maximal existence time) are obtained. Some interesting asymptotic behavior of the blowup time T^* is characterized as $p \to 1^+$, $\max_{x\in\overline{\Omega}} u_0(x) \to 0^+$ or $|\Gamma_1| \to 0^+$. In contrast to some results in this area in the literature, our lower bound estimate for the blowup time does not require the convexity of the domain.

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Heqian Lu and Zhengce Zhang
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an 710049
People's Republic of China
e-mail: zhangzc@mail.xjtu.edu.cn

Heqian Lu
e-mail: look3114054014@stu.xjtu.edu.cn

Bei Hu
Department of Applied and Computational Mathematics and Statistics
University of Notre Dame
Notre Dame IN 46556
USA
e-mail: b1hu@nd.edu

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