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Boundedness and finite-time blow-up in a quasilinear parabolic–elliptic–elliptic attraction–repulsion chemotaxis system

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Abstract. This paper deals with the quasilinear attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot\big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big) + f(u),\\ 0 = \Delta v + \alpha u - \beta v,\\ 0 = \Delta w + \gamma u - \delta w \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with smooth boundary $\partial\Omega$, where $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants, and f is a function of logistic type such as $f(u) = \lambda u - \mu u^{\kappa}$ with $\lambda, \mu > 0$ and $\kappa \geq 1$, provided that the case $f(u) \equiv 0$ is included in the study of boundedness, whereas κ is sufficiently close to 1 in considering blow-up in the radially symmetric setting. In the case that $\xi = 0$ and $f(u) \equiv 0$, global existence and boundedness have already been proved under the condition $p < m + \frac{2}{n}$. Also, in the case that $m = 1$, $p = q = 2$ and f is a function of logistic type, finite-time blow-up has already been established by assuming $\chi\alpha - \xi\gamma > 0$. This paper classifies boundedness and blow-up into the cases $p < q$ and $p > q$ without any condition for the sign of $\chi\alpha - \xi\gamma$ and the case $p = q$ with $\chi\alpha - \xi\gamma < 0$ or $\chi\alpha - \xi\gamma > 0$.

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1. Introduction

Background. Chemotaxis is the property of cells to move in a directional manner in response to concentration gradients of chemical substances. One of the systems of partial differential equations describing such phenomena was proposed by Keller–Segel [23] as

$$\begin{cases} u_t = \nabla\cdot\big(\nabla u - \chi u\nabla v\big),\\ v_t = \Delta v + \alpha u - \beta v, \end{cases}$$

where $\chi, \alpha, \beta > 0$ are constants, and the functions u and v idealize the cell density and the concentration of the chemoattractant, respectively. After that, many types of chemotaxis systems have been studied (see e.g., Osaki–Yagi [35], Bellomo et al. [2], Arumugam–Tyagi [1]). From the point of view of modeling, it is significant to analyze quasilinear systems such as the system

$$\begin{cases} u_t = \nabla\cdot\big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v\big),\\ v_t = \Delta v + \alpha u - \beta v, \end{cases}$$

where $m, p \in \mathbb{R}$. This system has been proposed by Painter–Hillen [36] and has been investigated in some literatures (see e.g., Cieślak [7], Tao–Winkler [41]; cf. also [20] for the degenerate version of the system).

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In another direction, to describe the aggregation of microglial cells in Alzheimer's disease Luca et al. [29] proposed the attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u - \chi u \nabla v + \xi u \nabla w\big), \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w, \end{cases}$$

where $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants. This is also a specialized system introduced in [36, Sect. 3.3] in order to represent the quorum sensing effect that cells keep away from a repulsive chemical substance. In this system the functions u, v and w represent the cell density, the concentration of the chemoattractant and chemorepellent, respectively. The above attraction–repulsion chemotaxis system has also been actively studied as detailed in later. Here we emphasize that it is meaningful to consider the system with diffusion, attraction and repulsion terms involving nonlinearities, that is,

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big), \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w. \end{cases}$$

In the present paper, in order to gain a first insight towards a mathematical analysis of this system, we will reduce the system to the parabolic–elliptic–elliptic version. The reduction seems to be reasonable because the diffusion of chemical substances are faster than that of cells. Thus we can approximate the system by its parabolic–elliptic–elliptic version.

Problem. In this paper we consider the quasilinear parabolic–elliptic–elliptic attraction–repulsion chemotaxis system with initial and boundary conditions,

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big) + f(u), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w, \\ \nabla u \cdot \nu|_{\partial\Omega} = \nabla v \cdot \nu|_{\partial\Omega} = \nabla w \cdot \nu|_{\partial\Omega} = 0, \\ u(\cdot, 0) = u_0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$, $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants, ν is the outward normal vector to $\partial\Omega$,

$$u_0 \in C^0(\overline{\Omega}), \quad u_0 \ge 0 \text{ in } \overline{\Omega} \quad \text{and} \quad u_0 \ne 0. \tag{1.2}$$

Moreover, we assume that

- $m, p \in \mathbb{R}$, $f(u) \le \lambda_0 u - \mu_0 u^\kappa$ $(\lambda_0, \mu_0 > 0,\ \kappa \ge 1)$ in the consideration of boundedness, provided that if $\kappa = 1$, then $\lambda_0 = \mu_0$, which covers the case $f(u) \equiv 0$;
- $m \ge 1$, $p > 1$, $f(u) = \lambda(|x|)u - \mu(|x|)u^\kappa$ $(\kappa \ge 1)$ in the study of blow-up, where

$$\Omega = B_R(0) \subset \mathbb{R}^n \ (n \in \mathbb{N},\ n \ge 3) \text{ with } R > 0, \tag{1.3}$$

$$\lambda, \mu \ge 0 \text{ and } \lambda, \mu \in C^0([0, R]), \tag{1.4}$$

$$\mu(r) \le \mu_1 r^a \text{ for all } r \in [0, R] \text{ with some } \mu_1 > 0 \text{ and } a \ge 0. \tag{1.5}$$

Attraction vs. repulsion. As to the system (1.1) with $p = q = 2$, it is known that boundedness and blow-up are classified by the sign of $\chi\alpha - \xi\gamma$ (see e.g., Tao–Wang [40]). Here boundedness (including global existence), which expresses that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C$ for all $t > 0$ with some $C > 0$, implies absence of chemotactic collapse, whereas finite-time blow-up (blow-up for short), which means that $\lim_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ with some $T \in (0, \infty)$, describes the concentration of cells. On the other hand, to the best of our knowledge, no results are available for boundedness and blow-up in (1.1) with

$p \neq 2$, $q \neq 2$. Here the powers p, q imply the strengths of the effects of attraction, which promotes blow-up, and repulsion, which induces boundedness. Thus we can naturally guess as follows.

Boundedness and blow-up can be classified by the size of the powers p, q.

In the following we discuss this expectation. As will be explained later, in the case $\xi = 0$ in (1.1) it is known that boundedness holds in the case

$$p < m + \frac{2}{n}, \tag{1.6}$$

and blow-up occurs in the opposite case. In view of the first equation in (1.1), the condition (1.6) implies that the effect of diffusion "plus $\frac{2}{n}$" is stronger than the one of attraction. In the case $\xi \neq 0$ the system (1.1) involves the repulsion term which is expected to work in contrast to the attraction term. Therefore the question arises whether the repulsion term is useful for deriving boundedness, that is,

when $p < q$, does boundedness in (1.1)hold without assuming (1.6)? **(Q1)**

In the opposite case $p > q$ that the effect of attraction is more dominant than that of repulsion, we raise the following question.

When $p > q$, does blow-up in (1.1) occur? **(Q2)**

Furthermore, in the case $p = q$ that the effects of attraction and repulsion are balanced, the following question arises.

When $p = q$, are boundedness and blow-up in (1.1.) **(Q3)**
classified by conditions for the coefficients in the equations?

Overview of related works. Before giving answers to the above three questions, we summarize the previous studies related to each case.

We first focus on the reduced system without repulsion term,

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v\big) + f(u), \\ \tau v_t = \Delta v + \alpha u - \beta v, \end{cases} \tag{1.7}$$

where $m, p \in \mathbb{R}$, $\chi, \alpha, \beta > 0$, $\tau \in \{0, 1\}$ are constants and f is a function. In the case $\tau = 1$, boundedness was shown in [19,41,44,53]. More precisely, Tao–Winkler [41] derived boundedness when $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a convex domain, $f(u) \equiv 0$ and $p < m + \frac{2}{n}$ holds; after that, the convexity of Ω was removed by [19]. Conversely, when $p > m + \frac{2}{n}$ and $n \geq 2$, it is known that boundedness breaks down in some cases. Indeed, existence of unbounded solutions was shown by Winkler [45]; finite-time blow-up was proved by Winkler [47] in the case $n \geq 3$, $m = 1$, $p = 2$, and by Cieślak–Stinner [9,10] in the case $m \geq 1$ or $p \geq 2$. Also, infinite-time blow-up was shown by Cieślak–Stinner [11] under the condition $p > m + \frac{2}{n}$, $p < \frac{m}{2} + \frac{n+2}{2n}$, $m < 1 - \frac{2}{n}$ and $p < 1$; after that, Winkler [49] derived infinite-time blow-up by assuming the condition $p > m + \frac{2}{n}$, $m < 1 - \frac{2}{n}$ and $p \leq 1$. Besides, in the critical case $p = m + \frac{2}{n}$, boundedness and blow-up were classified by the condition for initial data ([4,21,25,30]). For the system in which the second equation of (1.7) is replaced by $0 = \Delta v - M + u$, where $M := \frac{1}{|\Omega|}\int_\Omega u_0$, the picture in this regard is much more complete. Indeed, in the case $p = 2$, Cieślak–Winkler [12] showed boundedness and finite-time blow-up under the condition $2 < m + \frac{2}{n}$ and $2 > m + \frac{2}{n}$, respectively; after that, Cieślak–Laurençot [8] studied finite-time blow-up in the case $0 \leq m + \frac{2}{n} \leq 2$. Also, Winkler–Djie [51] derived boundedness under the condition $p < m + \frac{2}{n}$ and $m \leq 1$ and showed finite-time blow-up under the condition $p > m + \frac{2}{n}$, $m \leq 1$ and $p > 1$. Moreover, in the case $f(u) \leq \lambda - \mu u^\kappa$ ($\lambda \geq 0$, $\mu > 0$, $\kappa > 1$), global existence of classical solutions was established by Zheng [53] under the condition that $p < \min\{\kappa - 1, m + \frac{2}{n}\}$, or that $p = \kappa$ if $\mu > 0$ is sufficiently large. On the other hand, in the case $\tau = 0$, boundedness was studied in [27,37,43,52]. Particularly, in the case $\Omega = \mathbb{R}^n$ ($n \in \mathbb{N}$), Sugiyama–Kunii [37] proved boundedness of weak solutions in the system (1.7) of a degenerate type. More precisely, in the literature the authors dealt with the case that $f(u) \equiv 0$, $m \geq 1$, $p \geq 2$ and $p < \min\{m + 1, m + \frac{2}{n}\}$. Also, in the case that $\tau = 0$, $p = 2$ and $f(u) \leq \lambda - \mu u^\kappa$ ($\lambda \geq 0$, $\mu > 0$, $\kappa > 1$), boundedness was verified by Wang et al. [43] under the condition that $m > 2 - \frac{2}{n}$ if $\kappa \in (1, 2)$, or $\mu > \mu^*$ if $\kappa \geq 2$ with some $\mu^* > 0$. In contrast, when $\tau = 0$, $m = 1$, $p = 2$ and $f(u) = \lambda u - \mu u^\kappa$ ($\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$), Winkler [48] established finite-time blow-up; after that, the result was extended to the cases $p \in (1, 2)$, $p = 2$ and $p > 1$ in [39], [3] and [38], respectively. On the other hand, in the case $\tau = 1$ some related works for the system (1.7) with signal-dependent sensitivity function $\chi(v)$, that is, the system in which the first equation of (1.7) is replaced by $u_t = \nabla \cdot \big((u + 1)^{m-1}\nabla u - u(u + 1)^{p-2}\chi(v)\nabla v\big) + f(u)$ can be found in [13,15,17,22]. For instance, when $\tau = 1$, $m = 1$, $p = 2$, $\chi(v) = \frac{\chi}{v}$ and $f(u) \equiv 0$, Fujie [15] showed boundedness in (1.7) under the condition $0 < \chi < \sqrt{\frac{2}{n}}$.

We next shift our focus to the attraction–repulsion system

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u - \chi u \nabla v + \xi u \nabla w\big) + f(u), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w, \end{cases} \tag{1.8}$$

where $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. In the case $f(u) = \lambda u - \mu u^\kappa$ ($\lambda \in \mathbb{R}$, $\mu > 0$, $\kappa > 1$), finite-time blow-up was recently proved in [5] via the method in [48] when κ is sufficiently closed to 1 and $\chi\alpha - \xi\gamma > 0$ holds. Moreover, some related works deriving boundedness can be found in [18,31–34], whereas finite-time blow-up was shown in [24]. Particularly, in the two-dimensional setting, Fujie–Suzuki [18] established boundedness in the fully parabolic version of (1.8) under the condition that $\beta = \delta$, $\chi\alpha - \xi\gamma > 0$ and $\|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi\alpha - \xi\gamma}$; note that the authors relaxed the condition for u_0 in the radially symmetric setting and removed the condition $\beta = \delta$. Also, Nagai–Yamada [34] proved global existence of solutions under the condition that $\alpha = \gamma = 1$, $\chi - \xi > 0$ and $\|u_0\|_{L^1(\Omega)} = \frac{8\pi}{\chi - \xi}$ in the two-dimensional setting; after that, the authors investigated boundedness of solutions in [33]. On the other hand, in the three-dimensional and radially symmetric settings, existence of solutions blowing up in finite time to the fully parabolic version of (1.8) was established by Lankeit [24] under the conditions that $\chi\alpha - \xi\gamma > 0$ and that $\|u_0\|_{L^1(\Omega)} = M$ with some $M > 0$. We can also refer to [6,26,28] for the study of (1.8) with nonlinear diffusion and signal-dependent sensitivity.

In summary, the results on boundedness and blow-up in the system (1.1) were obtained as follows: Boundedness was derived in the case $\xi = 0$ under the condition $p < m + \frac{2}{n}$; blow-up was proved only for the simplified system (1.8) under the condition $\chi\alpha - \xi\gamma > 0$. However, in previous studies, the positive and negative effects of repulsion have not been utilized, e.g., in the cases $p < q$ and $p > q$. The purpose of this paper is to classify boundedness and blow-up in the generalized system (1.1) by the size of the powers p, q. It is unknown whether the analysis of such a complex model can be applied to an example in other models, e.g., tumor invasion models of Chaplain–Anderson type [16].

Main results. Before explaining our results, we mention the expected answers to the questions (**Q1**)–(**Q3**). As to the questions (**Q1**) and (**Q2**), we can give affirmative answers. Also, regard to the question (**Q3**), we can classify boundedness and blow-up according to the sign of $\chi\alpha - \xi\gamma$. In the following we briefly state the main results which give the answers to the questions. The precise statements and their proofs will be given in Sects. 3 and 4.

(**I**) If $p < q$, then, for all initial data, the system (1.1) possesses a global bounded classical solution which is unique (Theorem 3.1).

(**II**) If $p = q$ and $\chi\alpha - \xi\gamma < 0$, then, for all initial data, the system (1.1) admits a unique global bounded classical solution (Theorem 3.5).

(**III**) If $p > q$, then there exist initial data such that the corresponding solutions blow up in finite time in the radial framework (Theorem 4.1).

(**IV**) If $p = q$ and $\chi\alpha - \xi\gamma > 0$, then there exist initial data such that the system (1.1) possesses solutions blow up in finite time in the radial framework (Theorem 4.4).

Strategies for proving boundedness and blow-up. The strategy in boundedness is to establish the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma \le -c_1\Big(\int_\Omega (u+1)^\sigma\Big)^{1+\theta_1} + c_2 \tag{1.9}$$

with some $\sigma > n, c_1, c_2, \theta_1 > 0$. The key to the derivation of (1.9) is to take advantage of the effect of repulsion. More precisely, we will estimate positive terms like $\chi\alpha\int_\Omega u^{\sigma+p-2}$ by the negative term $-\xi\gamma\int_\Omega u^{\sigma+q-2}$. On the other hand, the cornerstone of the proof of finite-time blow-up is the derivation of the differential inequality

$$\phi'(s_0,t) \ge c_3 s_0^{-\theta_2}\phi^2(s_0,t) - c_4 s_0^{\theta_3}, \tag{1.10}$$

where $c_3, c_4, \theta_2, \theta_3 > 0$ are constants. Here the moment-type functional ϕ is defined as $\phi(s_0,t) := \int_0^{s_0} s^{-b}(s_0-s)U(s,t)\,\mathrm{d}s$, where U is the mass accumulation function given by $U(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,\mathrm{d}\rho$ for $s > 0$, $t > 0$ and $b \in (0,1)$. To derive the inequality (1.10) we utilize the attraction term. More precisely, the key is to handle a term derived from the repulsion term by exploiting the effect of attraction.

Plan of the paper. This paper is organized as follows. In Sect. 2 we collect some preliminary facts about local existence in (1.1), and a lemma which asserts that an L^σ-estimate for u implies boundedness, as well as an inequality which will be used later. Section 3 is devoted to establishing results on global existence and boundedness. In Sect. 4 we give and prove results on finite-time blow-up.

Throughout this paper, we denote by c_j generic positive constants, which will be sometimes specified by $c_j(\varepsilon)$ depending on small parameters $\varepsilon > 0$.

2. Preliminaries

We first give a result on local existence in (1.1), which can be proved by standard arguments based on the contraction mapping principle (see e.g., [12,41,42]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with smooth boundary and let $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Assume that $f(u) \le \lambda u - \mu u^\kappa$ $(\kappa \ge 1)$, where $\lambda, \mu \in C^0(\overline{\Omega})$. Then for all u_0 satisfying the condition (1.2) there exists $T_{\max} \in (0,\infty]$ such that (1.1) admits a unique classical solution (u,v,w) such that*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\ v,w \in \bigcap_{\vartheta>n} C^0([0,T_{\max});W^{1,\vartheta}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})). \end{cases} \tag{2.1}$$

Moreover,

$$\text{if } T_{\max} < \infty, \quad \text{then } \lim_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

Particularly, in the case that $f(u) = \lambda(|x|)u - \mu(|x|)u^\kappa$ $(\kappa \ge 1)$ and the conditions (1.3), (1.4) hold, if u_0 is further assumed to be radially symmetric, then there exists $T_{\max} \in (0,\infty]$ such that (1.1) possesses a unique radially symmetric classical solution (u,v,w) satisfying (2.1) and (2.2).

We next give the following lemma, which provides a strategy to prove global existence and boundedness.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with smooth boundary and let $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Assume that u_0 satisfies (1.2) and that $f(u) \le \lambda_0 u - \mu_0 u^\kappa$ $(\kappa \ge 1,\ \lambda_0, \mu_0 > 0)$, provided that if $\kappa = 1$, then $\lambda_0 = \mu_0$. Denote by (u, v, w) the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max} \in (0, \infty]$ its maximal existence time. If for some $\sigma > n$,*

$$\sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^\sigma(\Omega)} < \infty, \tag{2.3}$$

then we have

$$\sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty. \tag{2.4}$$

Proof. Applying [46, Lemma 2.4 (ii) with $\theta = \sigma$ and $\mu = \infty$] along with (2.3) with $\sigma > n$ yields

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le c_1 \Big(1 + \sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^\sigma(\Omega)}\Big) \le c_2, \tag{2.5}$$

$$\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \le c_3 \Big(1 + \sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^\sigma(\Omega)}\Big) \le c_4 \tag{2.6}$$

for all $t \in (0, T_{\max})$. Thanks to [41, Lemma A.1], we can see from (2.3), (2.5) and (2.6) that (2.4) holds. □

We finally state an inequality which will be used repeatedly.

Lemma 2.3. *Let $\ell > 1$. Then for all $\varepsilon > 0$,*

$$(x+1)^\ell \le (1+\varepsilon) x^\ell + C_\varepsilon \quad (x \ge 0),$$

where $C_\varepsilon := (1+\varepsilon)\big((1+\varepsilon)^{\frac{1}{\ell-1}} - 1\big)^{-(\ell-1)}$.

Proof. Owing to the convexity of the function $y \mapsto y^\ell$ on $[1, \infty)$ we have

$$\begin{aligned}
(x+1)^\ell &= \left[\frac{1}{(1+\varepsilon)^{\frac{1}{\ell-1}}} \cdot (1+\varepsilon)^{\frac{1}{\ell-1}} x + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\ell-1}}}\right) \cdot \frac{(1+\varepsilon)^{\frac{1}{\ell-1}}}{(1+\varepsilon)^{\frac{1}{\ell-1}} - 1}\right]^\ell \\
&\le \frac{1}{(1+\varepsilon)^{\frac{1}{\ell-1}}} \cdot \left[(1+\varepsilon)^{\frac{1}{\ell-1}} x\right]^\ell + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\ell-1}}}\right) \cdot \left[\frac{(1+\varepsilon)^{\frac{1}{\ell-1}}}{(1+\varepsilon)^{\frac{1}{\ell-1}} - 1}\right]^\ell \\
&= (1+\varepsilon) x^\ell + \frac{1+\varepsilon}{\big((1+\varepsilon)^{\frac{1}{\ell-1}} - 1\big)^{\ell-1}},
\end{aligned}$$

which leads to the desired inequality. □

3. Global existence and boundedness

In this section we assume that

$$(\mathbf{A1}) \begin{cases} \Omega \subset \mathbb{R}^n (n \in \mathbb{N}) \text{ is a bounded domain with smooth boundary,} \\ m, p, q \in \mathbb{R}, \chi, \xi, \alpha, \beta, \gamma, \delta > 0, \\ f(u) \le \lambda_0 u - \mu_0 u^\kappa (\lambda_0, \mu_0 > 0, \kappa \ge 1), \text{ provided that if } \kappa = 1, \text{ then } \lambda_0 = \mu_0, \end{cases}$$

where the last condition covers the case $f(u) \equiv 0$. We will prove global existence and boundedness in (1.1) in the two cases $p < q$ and $p = q$.

3.1. The case $p < q$

In this subsection we show the following theorem, which asserts global existence and boundedness in (1.1) in the case $p < q$.

Theorem 3.1. *Assume that* **(A1)** *is satisfied with $p < q$. Then for all u_0 satisfying (1.2) there exists a unique triplet (u, v, w) of nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v, w \in \bigcap_{\vartheta > n} C^0([0,\infty); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which solves (1.1) in the classical sense, and is bounded, that is, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for all $t > 0$ with some $C > 0$.

In the following we denote by (u, v, w) the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max} \in (0, \infty]$ its maximal existence time. To prove Theorem 3.1 it is sufficient to derive an L^σ-estimate for u with some $\sigma > n$, because Lemma 2.2 leads to an L^∞-estimate for u which together with the criterion (2.2) implies the conclusion.

As a first observation, we note that an upper bound for the mass of u can be derived quite immediately.

Lemma 3.2. *The first component of the solution satisfies that for all $t \in (0, T_{\max})$,*

$$\int_\Omega u(\cdot,t) \le M_* := \begin{cases} \displaystyle\int_\Omega u_0 & \text{when } \kappa = 1, \\ \min\Big\{ \displaystyle\int_\Omega u_0,\ \Big(\frac{\lambda_0}{\mu_0}\Big)^{\frac{1}{\kappa-1}} |\Omega| \Big\} & \text{when } \kappa > 1. \end{cases}$$

Proof. Integrating the first equation in (1.1) over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u \le \lambda_0 \int_\Omega u - \mu_0 \int_\Omega u^\kappa \le \lambda_0 \int_\Omega u - \frac{\mu_0}{|\Omega|^{\kappa-1}} \Big(\int_\Omega u \Big)^\kappa,$$

so that the conclusion results from an ODE comparison argument. □

The following lemma plays an important role in the derivation of the L^σ-estimate.

Lemma 3.3. *Let $\ell > 1$. Then the first and third components of the solution satisfy that for all $\varepsilon > 0$,*

$$\int_\Omega w^\ell \le \varepsilon \int_\Omega u^\ell + c(\varepsilon) \quad \text{on } (0, T_{\max})$$

with some $c(\varepsilon) > 0$.

Proof. Let $t \in (0, T_{\max})$ and put $u := u(\cdot,t)$, $w := w(\cdot,t)$. Multiplying the third equation in (1.1) by $w^{\ell-1}$ and integrating it over Ω, we obtain

$$\delta \int_\Omega w^\ell - \int_\Omega w^{\ell-1} \Delta w = \gamma \int_\Omega u w^{\ell-1}.$$

Since the second term on the left-hand side is rewritten as

$$-\int_\Omega w^{\ell-1} \Delta w = (\ell - 1) \int_\Omega w^{\ell-2} |\nabla w|^2 = \frac{4(\ell-1)}{\ell^2} \int_\Omega \big|\nabla w^{\frac{\ell}{2}}\big|^2,$$

we infer

$$\delta \int_\Omega w^\ell + \frac{4(\ell-1)}{\ell^2} \int_\Omega \left|\nabla w^{\frac{\ell}{2}}\right|^2 = \gamma \int_\Omega u w^{\ell-1}. \tag{3.1}$$

Now, integrating the third equation in (1.1) over Ω and invoking Lemma 3.2 entail that

$$\int_\Omega w = \frac{\gamma}{\delta} \int_\Omega u \le \frac{\gamma M_*}{\delta}. \tag{3.2}$$

Applying the Gagliardo–Nirenberg inequality to $\|w^{\frac{\ell}{2}}\|_{L^2(\Omega)}$ and using the estimate (3.2), we see that

$$\begin{aligned} \|w^{\frac{\ell}{2}}\|_{L^2(\Omega)} &\le c_1\Big(\|\nabla w^{\frac{\ell}{2}}\|^{\theta_1}_{L^2(\Omega)} \|w^{\frac{\ell}{2}}\|^{1-\theta_1}_{L^{\frac{2}{\ell}}(\Omega)} + \|w^{\frac{\ell}{2}}\|_{L^{\frac{2}{\ell}}(\Omega)}\Big) \\ &\le c_2\Big(\|\nabla w^{\frac{\ell}{2}}\|^{\theta_1}_{L^2(\Omega)} + 1\Big), \end{aligned} \tag{3.3}$$

where $\theta_1 := \frac{\frac{\ell}{2}-\frac{1}{2}}{\frac{\ell}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$. Let $\varepsilon > 0$ (fixed later). Then Young's inequality implies that

$$\|\nabla w^{\frac{\ell}{2}}\|^{\theta_1}_{L^2(\Omega)} \le \frac{1}{c_2}\sqrt{\frac{\varepsilon}{2}}\|\nabla w^{\frac{\ell}{2}}\|_{L^2(\Omega)} + c_3(\varepsilon),$$

which together with (3.3) yields that

$$\begin{aligned} \|w^{\frac{\ell}{2}}\|^2_{L^2(\Omega)} &\le \Big(\sqrt{\frac{\varepsilon}{2}}\|\nabla w^{\frac{\ell}{2}}\|_{L^2(\Omega)} + c_2(c_3(\varepsilon)+1)\Big)^2 \\ &\le \varepsilon\|\nabla w^{\frac{\ell}{2}}\|^2_{L^2(\Omega)} + c_4(\varepsilon). \end{aligned}$$

This means that

$$\int_\Omega \left|\nabla w^{\frac{\ell}{2}}\right|^2 \ge \frac{1}{\varepsilon}\int_\Omega w^\ell - c_5(\varepsilon). \tag{3.4}$$

Combining (3.1) with (3.4) and using Young's inequality, we derive that

$$\begin{aligned} \delta \int_\Omega w^\ell + \frac{c_6}{\varepsilon}\int_\Omega w^\ell &\le \gamma \int_\Omega u w^{\ell-1} + c_7(\varepsilon) \\ &\le \gamma\Big[\frac{1}{\ell}\int_\Omega u^\ell + \Big(1-\frac{1}{\ell}\Big)\int_\Omega w^\ell\Big] + c_7(\varepsilon), \end{aligned}$$

and thus infer

$$\Big(\delta + \frac{c_6}{\varepsilon} - \gamma + \frac{\gamma}{\ell}\Big)\int_\Omega w^\ell \le \frac{\gamma}{\ell}\int_\Omega u^\ell + c_7(\varepsilon). \tag{3.5}$$

We now observe that if $\varepsilon \in (0, \frac{c_6}{\gamma})$, then $\frac{c_6}{\varepsilon} - \gamma > 0$, that is,

$$\delta + \frac{c_6}{\varepsilon} - \gamma + \frac{\gamma}{\ell} > 0.$$

Therefore, picking $\varepsilon \in (0, \frac{c_6}{\gamma})$, we obtain from (3.5) that

$$\begin{aligned} \int_\Omega w^\ell &\le \frac{\frac{\gamma}{\ell}}{\delta + \frac{c_6}{\varepsilon} - \gamma + \frac{\gamma}{\ell}}\int_\Omega u^\ell + \frac{c_7(\varepsilon)}{\delta + \frac{c_6}{\varepsilon} - \gamma + \frac{\gamma}{\ell}} \\ &= \frac{\frac{\gamma}{\ell}\varepsilon}{(\delta - \gamma + \frac{\gamma}{\ell})\varepsilon + c_6}\int_\Omega u^\ell + \frac{c_7(\varepsilon)\varepsilon}{(\delta - \gamma + \frac{\gamma}{\ell})\varepsilon + c_6}. \end{aligned}$$

Noticing that for all $\overline{\varepsilon} > 0$ there exists $\varepsilon \in (0, \frac{c_6}{\gamma})$ such that $\frac{\frac{\gamma}{\ell}\varepsilon}{(\delta-\gamma+\frac{\gamma}{\ell})\varepsilon+c_6} < \overline{\varepsilon}$, we arrive at the conclusion. ☐

We now prove an L^σ-estimate for u.

Lemma 3.4. *Assume that $p < q$. Then for some $\sigma > n$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\sigma(\Omega)} \le C$$

for all $t \in (0, T_{\max})$.

Proof. Let $\sigma > \max\{n, -m+1, -p+2, -q+4\}$. Then we verify that the asserted estimate holds on $(0, T_{\max})$. We first have from the first equation in (1.1) integration by parts that

$$\begin{aligned}
&\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma \\
&= \int_\Omega (u+1)^{\sigma-1}\nabla\cdot\big((u+1)^{m-1}\nabla u\big) - \chi\int_\Omega (u+1)^{\sigma-1}\nabla\cdot\big(u(u+1)^{p-2}\nabla v\big) \\
&\quad + \xi\int_\Omega (u+1)^{\sigma-1}\nabla\cdot\big(u(u+1)^{q-2}\nabla w\big) + \int_\Omega (u+1)^{\sigma-1} f(u) \\
&\le -(\sigma-1)\int_\Omega (u+1)^{\sigma+m-3}|\nabla u|^2 + \chi(\sigma-1)\int_\Omega u(u+1)^{\sigma+p-4}\nabla u\cdot\nabla v \\
&\quad - \xi(\sigma-1)\int_\Omega u(u+1)^{\sigma+q-4}\nabla u\cdot\nabla w + \int_\Omega (u+1)^{\sigma-1}(\lambda_0 u - \mu_0 u^\kappa) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.6}$$

We estimate the terms I_1, I_2, I_3. As to the first term I_1, we rewrite it as

$$I_1 = -\frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2. \tag{3.7}$$

We next deal with the second term I_2 and third term I_3. As to the former, integration by parts and the second equation in (1.1) lead to

$$\begin{aligned}
I_2 &= \chi(\sigma-1)\int_\Omega \nabla\Big[\int_0^u s(s+1)^{\sigma+p-4}\,\mathrm{d}s\Big]\cdot\nabla v \\
&= \chi(\sigma-1)\int_\Omega \Big[\int_0^u s(s+1)^{\sigma+p-4}\,\mathrm{d}s\Big]\cdot(-\Delta v) \\
&= \chi(\sigma-1)\int_\Omega \Big[\int_0^u s(s+1)^{\sigma+p-4}\,\mathrm{d}s\Big]\cdot(\alpha u - \beta v) \\
&\le \chi\alpha(\sigma-1)\int_\Omega \Big[\int_0^u s(s+1)^{\sigma+p-4}\,\mathrm{d}s\Big]u.
\end{aligned} \tag{3.8}$$

Here, since σ satisfies $\sigma > -p+2$, we infer that

$$\Big[\int_0^u s(s+1)^{\sigma+p-4}\,\mathrm{d}s\Big]u \le \Big[\int_0^u (s+1)^{\sigma+p-3}\,\mathrm{d}s\Big]u \le \frac{1}{\sigma+p-2}(u+1)^{\sigma+p-2}u \le \frac{1}{\sigma+p-2}(u+1)^{\sigma+p-1}.$$

Combining the above estimate with (3.8) and using Lemma 2.3 with $\varepsilon = 1$ and $\sigma > -p+2$, we establish

$$I_2 \le \frac{\chi\alpha(\sigma-1)}{\sigma+p-2}\Big(2\int_\Omega u^{\sigma+p-1} + c_1\Big). \tag{3.9}$$

Similarly, we have

$$I_3 = \xi(\sigma-1)\int_\Omega \Big[\int_0^u s(s+1)^{\sigma+q-4}\,\mathrm{d}s\Big]\cdot \Delta w = \xi(\sigma-1)\int_\Omega \Big[\int_0^u s(s+1)^{\sigma+q-4}\,\mathrm{d}s\Big]\cdot (\delta w - \gamma u). \tag{3.10}$$

Here, noting that $s^{\sigma+q-3} \le s(s+1)^{\sigma+q-4} \le (s+1)^{\sigma+q-3}$ because $\sigma \ge -q+4$, we see that

$$\frac{1}{\sigma+q-2}u^{\sigma+q-2} \le \int_0^u s(s+1)^{\sigma+q-4}\,\mathrm{d}s \le \frac{1}{\sigma+q-2}(u+1)^{\sigma+q-2}, \tag{3.11}$$

where we neglected the term $-\frac{1}{\sigma+q-2}$ on the most right-hand side. Due to Lemma 2.3 with $\varepsilon = 1$ we obtain that

$$\Big[\int_0^u s(s+1)^{\sigma+q-4}\,\mathrm{d}s\Big]w \le \frac{1}{\sigma+q-2}(u+1)^{\sigma+q-2}w \le \frac{1}{\sigma+q-2}\Big(2u^{\sigma+q-2}w + c_2 w\Big). \tag{3.12}$$

Therefore a combination of the above estimates (3.10)–(3.12) yields that

$$I_3 \le \frac{\xi(\sigma-1)}{\sigma+q-2}\Big(2\delta\int_\Omega u^{\sigma+q-2}w + \delta c_2\int_\Omega w - \gamma\int_\Omega u^{\sigma+q-1}\Big). \tag{3.13}$$

Collecting (3.7), (3.9) and (3.13) in (3.6), we derive

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma \le{}& -\frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \big|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\big|^2 \\ &+ \frac{\chi\alpha(\sigma-1)}{\sigma+p-2}\Big(2\int_\Omega u^{\sigma+p-1} + c_1\Big) \\ &+ \frac{\xi(\sigma-1)}{\sigma+q-2}\Big(2\delta\int_\Omega u^{\sigma+q-2}w + \delta c_2\int_\Omega w - \gamma\int_\Omega u^{\sigma+q-1}\Big) + I_4. \end{aligned} \tag{3.14}$$

Moreover, taking $\varepsilon_1 > 0$ which will be fixed later and applying Young's inequality to $u^{\sigma+p-1}$, we observe that $u^{\sigma+p-1} \le \varepsilon_1 u^{\sigma+q-1} + c_3(\varepsilon_1)$. Additionally, again by the estimate (3.2) we deduce that

$$\begin{aligned}
&\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \\
&\le \frac{\chi\alpha(\sigma-1)}{\sigma+p-2}\Big[2\Big(\varepsilon_1\int_\Omega u^{\sigma+q-1} + c_3(\varepsilon_1)\Big) + c_1\Big] \\
&\quad + \frac{\xi(\sigma-1)}{\sigma+q-2}\Big(2\delta\int_\Omega u^{\sigma+q-2}w + c_4 - \gamma\int_\Omega u^{\sigma+q-1}\Big) + I_4.
\end{aligned} \tag{3.15}$$

We next estimate the term $\int_\Omega u^{\sigma+q-2}w$. Due to Hölder's inequality, we infer

$$\int_\Omega u^{\sigma+q-2}w \le \Big(\int_\Omega u^{\sigma+q-1}\Big)^{\frac{\sigma+q-2}{\sigma+q-1}}\Big(\int_\Omega w^{\sigma+q-1}\Big)^{\frac{1}{\sigma+q-1}}.$$

We now take $\varepsilon_2 > 0$ which will be fixed later. Firstly applying Lemma 3.3 with $\ell = \sigma+q-1$ and $\varepsilon = (\frac{\varepsilon_2}{2})^{\sigma+q-1}$ to $\int_\Omega w^{\sigma+q-1}$ and secondly using the fact $(A+B)^{\frac{1}{\sigma+q-1}} \le A^{\frac{1}{\sigma+q-1}} + B^{\frac{1}{\sigma+q-1}}$ for $A, B > 0$ and thirdly employing Young's inequality, we establish

$$\begin{aligned}
\int_\Omega u^{\sigma+q-2}w &\le \Big(\int_\Omega u^{\sigma+q-1}\Big)^{\frac{\sigma+q-2}{\sigma+q-1}}\Big[\Big(\frac{\varepsilon_2}{2}\Big)^{\sigma+q-1}\int_\Omega u^{\sigma+q-1} + c_5(\varepsilon_2)\Big]^{\frac{1}{\sigma+q-1}} \\
&\le \frac{\varepsilon_2}{2}\int_\Omega u^{\sigma+q-1} + c_5(\varepsilon_2)^{\frac{1}{\sigma+q-1}}\Big(\int_\Omega u^{\sigma+q-1}\Big)^{\frac{\sigma+q-2}{\sigma+q-1}} \\
&\le \frac{\varepsilon_2}{2}\int_\Omega u^{\sigma+q-1} + c_5(\varepsilon_2)^{\frac{1}{\sigma+q-1}}\Big(\frac{\varepsilon_2}{2c_5(\varepsilon_2)^{\frac{1}{\sigma+q-1}}}\int_\Omega u^{\sigma+q-1} + c_6(\varepsilon_2)\Big) \\
&= \varepsilon_2\int_\Omega u^{\sigma+q-1} + c_7(\varepsilon_2),
\end{aligned} \tag{3.16}$$

which in conjunction with (3.15) implies

$$\begin{aligned}
&\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \\
&\le c_8\varepsilon_1\int_\Omega u^{\sigma+q-1} + c_9\Big[2\delta\Big(\varepsilon_2\int_\Omega u^{\sigma+q-1} + c_7(\varepsilon_2)\Big) - \gamma\int_\Omega u^{\sigma+q-1}\Big] + c_{10}(\varepsilon_1) \\
&= c_8\varepsilon_1\int_\Omega u^{\sigma+q-1} + c_9(2\delta\varepsilon_2 - \gamma)\int_\Omega u^{\sigma+q-1} + c_{11}(\varepsilon_1,\varepsilon_2) + I_4.
\end{aligned} \tag{3.17}$$

Here we choose $\varepsilon_2 > 0$ satisfying $\varepsilon_2 < \frac{\gamma}{2\delta}$, that is, $2\delta\varepsilon_2 - \gamma < 0$. Then we have from (3.17) that

$$\begin{aligned}
&\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \\
&\le \big(c_8\varepsilon_1 - c_9(\gamma - 2\delta\varepsilon_2)\big)\int_\Omega u^{\sigma+q-1} + c_{11}(\varepsilon_1) + I_4,
\end{aligned} \tag{3.18}$$

and we thereby let

$$\varepsilon_1 := \frac{c_9(\gamma - 2\delta\varepsilon_2)}{c_8} > 0.$$

Then it follows from (3.18) that

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \le c_{11} + I_4, \tag{3.19}$$

where due to Young's inequality, I_4 appearing in (3.6) can be estimated independently of the other terms as

$$\begin{aligned} I_4 &= \lambda_0 \int_\Omega u(u+1)^{\sigma-1} - \mu_0 \int_\Omega u^\kappa (u+1)^{\sigma-1} \\ &\le \lambda_0 \int_\Omega (u+1)^\sigma - \mu_0 \int_\Omega u^{\sigma+\kappa-1} \\ &\le 2^{\sigma-1}\lambda_0 \int_\Omega u^\sigma - \mu_0 \int_\Omega u^{\sigma+\kappa-1} + 2^{\sigma-1}\lambda_0|\Omega| \\ &\le c_0. \end{aligned}$$

This entails that

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \le \widetilde{c_{11}}. \tag{3.20}$$

We finally estimate the second term on the left-hand side of this inequality to derive a differential inequality for $\int_\Omega (u+1)^\sigma$. Again using the Gagliardo–Nirenberg inequality and Lemma 3.2, we see that

$$\begin{aligned} &\|u(\cdot,t)+1\|_{L^\sigma(\Omega)} \\ &\quad = \left\|(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^{\frac{2\sigma}{\sigma+m-1}}(\Omega)}^{\frac{2}{\sigma+m-1}} \\ &\quad \le c_{12}\Big(\left\|\nabla(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^2(\Omega)}^{\theta_2}\left\|(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^{\frac{2}{\sigma+m-1}}(\Omega)}^{1-\theta_2} \\ &\qquad + \left\|(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^{\frac{2}{\sigma+m-1}}(\Omega)}\Big)^{\frac{2}{\sigma+m-1}} \\ &\quad \le c_{12}\Big(\left\|\nabla(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^2(\Omega)}^{\frac{2}{\sigma+m-1}\theta_2}\|u(\cdot,t)+1\|_{L^1(\Omega)}^{1-\theta_2} + \|u(\cdot,t)+1\|_{L^1(\Omega)}\Big) \\ &\quad \le c_{13}\Big(\left\|\nabla(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^2(\Omega)}^{\frac{2}{\sigma+m-1}\theta_2} + 1\Big) \end{aligned}$$

with $\theta_2 := \frac{\frac{\sigma+m-1}{2} - \frac{\sigma+m-1}{2\sigma}}{\frac{\sigma+m-1}{2} + \frac{1}{n} - \frac{1}{2}} \in (0,1)$, which implies

$$\begin{aligned} \left\|\nabla(u(\cdot,t)+1)^{\frac{\sigma+m-1}{2}}\right\|_{L^2(\Omega)}^2 &\ge \Big(\frac{1}{c_{13}}\|u(\cdot,t)+1\|_{L^\sigma(\Omega)} - 1\Big)^{\frac{\sigma+m-1}{\theta_2}} \\ &\ge c_{14}\|u(\cdot,t)+1\|_{L^\sigma(\Omega)}^{\frac{\sigma+m-1}{\theta_2}} - 1. \end{aligned} \tag{3.21}$$

A combination of (3.20) and (3.21) yields that

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + c_{15}\Big(\int_\Omega (u+1)^\sigma\Big)^{\frac{\sigma+m-1}{\sigma\theta_2}} \le c_{16},$$

where $\frac{\sigma+m-1}{\sigma\theta_2} > 0$, because $\sigma > -m+1$. Upon an ODE comparison argument this inequality warrants that

$$\int_\Omega (u+1)^\sigma \le \max\Big\{\Big(\frac{c_{16}}{c_{15}}\Big)^{\frac{\sigma\theta_2}{\sigma+m-1}},\ \int_\Omega (u_0+1)^\sigma\Big\}$$

for all $t \in (0, T_{\max})$. This proves the conclusion. □

We are now in a position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. A combination of Lemmas 3.6 and 2.2 along with the criterion (2.2) leads to the conclusion of Theorem 3.1. □

3.2. The case $p = q$

In this subsection we prove the following theorem, which guarantees global existence and boundedness in (1.1) in the case $p = q$.

Theorem 3.5. *Assume that* **(A1)** *is satisfied with $p = q$ and $\chi\alpha - \xi\gamma < 0$. Then for all u_0 satisfying (1.2) there exists a unique triplet (u, v, w) of nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\ v, w \in \bigcap_{\vartheta>n} C^0([0,\infty); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\end{cases}$$

which solves (1.1) in the classical sense, and is bounded, that is, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$ for all $t > 0$ with some $C > 0$.

As in the previous subsection, we denote by (u, v, w) the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max} \in (0,\infty]$ its maximal existence time. The proof of Theorem 3.5 relies also on an L^σ-estimate for u.

Lemma 3.6. *Suppose that $p = q$. Then for some $\sigma > n$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\sigma(\Omega)} \le C$$

for all $t \in (0, T_{\max})$.

Proof. Let $\sigma > \max\{n, -m+1,\ -p+4\}$. Then we prove that the asserted estimate holds on $(0, T_{\max})$. Let $\varepsilon_1 > 0$ which will be fixed later. Proceeding similarly in the proof of Lemma 3.6, we see that (3.14) with $p = q$ holds, that is,

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma \le{}& -\frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \big|\nabla(u+1)^{\frac{\sigma+m-1}{2}}\big|^2\\ &+\frac{\chi\alpha(\sigma-1)}{\sigma+p-2}\Big((1+\varepsilon_1)\int_\Omega u^{\sigma+p-1} + c_1(\varepsilon_1)\Big)\\ &+\frac{\xi(\sigma-1)}{\sigma+p-2}\Big(2\delta\int_\Omega u^{\sigma+p-2}w + \delta c_2\int_\Omega w - \gamma\int_\Omega u^{\sigma+p-1}\Big) + I_4,\end{aligned}$$

where I_4 is the term appearing in (3.6). Also, recalling the estimate (3.2), we have

$$\begin{aligned}\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \\ \le \chi\alpha c_3\Big((1+\varepsilon_1)\int_\Omega u^{\sigma+p-1} + c_1(\varepsilon_1)\Big) \\ + \xi c_3\Big(2\delta\int_\Omega u^{\sigma+p-2}w + c_4 - \gamma\int_\Omega u^{\sigma+p-1}\Big) + I_4. \end{aligned} \tag{3.22}$$

We now take $\varepsilon_2 > 0$ which will be fixed later. Then, an argument similar to that in the derivation of (3.16) implies

$$\int_\Omega u^{\sigma+p-2}w \le \frac{\varepsilon_2}{2\xi\delta}\int_\Omega u^{\sigma+p-1} + c_5(\varepsilon_2).$$

Thus we obtain

$$\begin{aligned}&\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \\ &\le \chi\alpha c_3\Big((1+\varepsilon_1)\int_\Omega u^{\sigma+p-1} + c_1(\varepsilon_1)\Big) \\ &\quad + \xi c_3\Big(2\delta\int_\Omega u^{\sigma+p-2}w + c_4 - \gamma\int_\Omega u^{\sigma+p-1}\Big) \\ &\le c_3\Big[\chi\alpha(1+\varepsilon_1)\int_\Omega u^{\sigma+p-1} + 2\xi\delta\Big(\frac{\varepsilon_2}{2\xi\delta}\int_\Omega u^{\sigma+p-1} + c_5(\varepsilon_2)\Big) - \xi\gamma\int_\Omega u^{\sigma+p-1}\Big] \\ &\quad + c_6(\varepsilon_1) \\ &= c_3\Big[\big(\chi\alpha(1+\varepsilon_1) - \xi\gamma\big) + \varepsilon_2\Big]\int_\Omega u^{\sigma+p-1} + c_7(\varepsilon_1,\varepsilon_2) + I_4. \end{aligned} \tag{3.23}$$

Here since $\chi\alpha - \xi\gamma < 0$ by assumption, we can pick $\varepsilon_1 > 0$ satisfying $\chi\alpha(1+\varepsilon_1) - \xi\gamma < 0$. Then, taking

$$\varepsilon_2 := \xi\gamma - \chi\alpha(1+\varepsilon_1) > 0,$$

we have from (3.22) and (3.23) that

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^\sigma + \frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{\sigma+m-1}{2}}\right|^2 \le c_7 + I_4,$$

which corresponds to (3.19). Therefore the conclusion results from an argument similar to that in the proof of Lemma 3.6. □

Employing Lemma 3.4, we can prove Theorem 3.5.

Proof of Theorem 3.5. In view of Lemmas 3.4 and 2.2 along with the criterion (2.2) we immediately arrive at the conclusion of Theorem 3.5. □

4. Finite-time blow-up

In the following we suppose that

$$(\mathbf{A2})\begin{cases}\Omega = B_R(0)\subset\mathbb{R}^n (n\in\mathbb{N}, n\geq 3) \text{ with } R>0,\\ m\geq 1, p>1, q\in\mathbb{R}, \chi,\xi,\alpha,\beta,\gamma,\delta>0,\\ f(u)=\lambda(|x|)u-\mu(|x|)u^\kappa(\kappa\geq 1), \text{ where } \lambda,\mu \text{ satisfy (1.4) and (1.5)},\\ u_0 \text{ is radially symmetric and fulfills (1.2).}\end{cases}$$

Then we denote by $(u, v, w) = (u(r,t), v(r,t), w(r,t))$ the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max}\in(0,\infty]$ its maximal existence time.

In order to state the main theorems we give the conditions **(C1)**–**(C3)** as follows:

$$(\mathbf{C1})\begin{cases} n\in\{3,4\};\quad p<\dfrac{2}{n+1}m+\dfrac{2(n^2+1)}{n(n+1)},\\ p<-\dfrac{1}{n-2}m+\dfrac{2(n^2-n-1)}{n(n-2)},\quad m-p<-\dfrac{2}{n};\end{cases}$$

$$(\mathbf{C2})\begin{cases} n\geq 5;\quad -\dfrac{2}{n-3}m+\dfrac{2(n^2-2n-1)}{n(n-3)}<p<\dfrac{2}{n+1}m+\dfrac{2(n^2+1)}{n(n+1)},\\ p<-\dfrac{n+2}{n-4}m+\dfrac{3n^2-5n-4}{n(n-4)},\quad p\leq\dfrac{n+2}{3}m-\dfrac{n^2-3n-4}{3n};\end{cases}$$

$$(\mathbf{C3})\begin{cases} n\geq 5;\quad -\dfrac{2}{n-3}m+\dfrac{2(n^2-2n-1)}{n(n-3)}<p<\dfrac{2}{n+1}m+\dfrac{2(n^2+1)}{n(n+1)},\\ -\dfrac{n+2}{n-4}m+\dfrac{3n^2-5n-4}{n(n-4)}\leq p<-\dfrac{1}{n-2}m+\dfrac{2(n^2-n-1)}{n(n-2)},\\ m-p<-\dfrac{2}{n}.\end{cases}$$

4.1. The case $p>q$

In this subsection we establish finite-time blow-up in (1.1) in the case $p>q$.

Theorem 4.1. *Assume that* **(A2)** *is satisfied with $p>q$. Also, suppose that m,κ fulfill the following conditions*:

(i) *In the case* **(C1)**,

$$\kappa<1+\frac{(n-2)\big((m-p+1)n+1\big)}{n(n-1)}+\frac{a\big((m-p+1)n+1\big)}{n(n-1)}-(m-1)-(2-p)_+;$$

(ii) *In the case* **(C2)**,

$$\kappa<1+\frac{(n-2)\big((m-p+1)n+1\big)}{n(n-1)}+\frac{a\big((m-p+1)n+1\big)}{n(n-1)}-(m-1)-(2-p)_+;$$

(iii) *In the case* **(C3)**,

$$\kappa<1+\frac{(m-p+1)n+1}{2(n-1)}+\frac{a\big((m-p+1)n+1\big)}{n(n-1)}-\frac{(2-p)_+}{2},$$

where $a\geq 0$ is given in (1.5) and $y_+:=\max\{0,y\}$. Then for all $M_0>0$, $M_1\in(0,M_0)$ and $L>0$, one can find $\varepsilon_0>0$ and $r_1\in(0,R)$ with the following property: If u_0 satisfies $u_0(x)\leq L|x|^{-\sigma}$, where

$\sigma = \frac{n(n-1)}{(m-p+1)n+1} + \varepsilon_0$ *as well as* $\int_\Omega u_0 = M_0$ *and* $\int_{B_{r_1}(0)} u_0 \ge M_1$*, then the solution* (u, v, w) *to* (1.1) *blows up at* $t = T^* \in (0, \infty)$ *in the sense that*

$$\lim_{t \nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

We first show the following lemma giving the profile of u, in which we include the case $p = q$ toward the next subsection. For the proof we rely on [14, Theorem 1.1], which is useful because it generalizes a precedent for the case of linear diffusion in [50, Theorem 1.1] and covers the case of nonlinear diffusion.

Lemma 4.2. *Assume that* $p \ge q$*. Also, suppose that* m, p *fulfill*

$$m \ge 1, \quad m - p \in \Big(-1 - \frac{1}{n},\ -\frac{2}{n}\Big].$$

Let $M_0 > 0$*,* $L > 0$ *and take any finite time* T *such that* $T \in (0, T_{\max})$*. Let* $\varepsilon > 0$ *and set* $\sigma := \frac{n(n-1)}{(m-p+1)n+1} + \varepsilon$*. Then there exists* $C = C(M_0, L, T) > 0$ *such that the following property holds: If* u_0 *satisfies* $\int_\Omega u_0 = M_0$ *and*

$$u_0(x) \le L|x|^{-\sigma}$$

for all $x \in \Omega$*, then* u *has the estimate*

$$u(x, t) \le C|x|^{-\sigma} \tag{4.1}$$

for all $x \in \Omega$ *and all* $t \in (0, T)$*.*

Proof. In view of the condition for the function λ (see (1.4)) we can find $\lambda_1 > 0$ such that $\lambda(|x|) \le \lambda_1$ for all $x \in \Omega$. We next set

$$\widetilde{u}(x, t) := e^{-\lambda_1 t} u(x, t), \quad D(x, t, \rho) := (e^{\lambda_1 t}\rho + 1)^{m-1},$$
$$S_1(x, t, \rho) := -\chi(e^{\lambda_1 t}\rho + 1)^{p-2}\rho, \quad S_2(x, t, \rho) := \xi(e^{\lambda_1 t}\rho + 1)^{q-2}\rho$$

for $x \in \Omega$, $t \in (0, T)$ and $\rho > 0$. Since $S_1(\cdot, \cdot, \cdot) < 0$ on $\Omega \times (0, T) \times (0, \infty)$, we have

$$S_1(x, t, \rho)\nabla v(x, t) + S_2(x, t, \rho)\nabla w(x, t) = S_1(x, t, \rho)\Big[\nabla v(x, t) + \frac{S_2(x, t, \rho)}{S_1(x, t, \rho)}\nabla w(x, t)\Big]$$

for all $x \in \Omega$, $t \in (0, T)$ and all $\rho > 0$. Putting

$$\mathbf{f}(x, t) := \nabla v(x, t) + \frac{S_2(x, t, \rho)}{S_1(x, t, \rho)}\nabla w(x, t),$$

we obtain from (1.1) that

$$\begin{cases} \widetilde{u}_t \le \nabla \cdot (D(x, t, \widetilde{u})\nabla \widetilde{u} + S_1(x, t, \widetilde{u})\,\mathbf{f}(x, t)) & \text{in } \Omega \times (0, T), \\ (D(x, t, \widetilde{u})\nabla \widetilde{u} + S_1(x, t, \widetilde{u})\,\mathbf{f}(x, t)) \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T), \\ \widetilde{u}(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

Also, it can be checked that for all $x \in \Omega$, $t \in (0, T)$ and all $\rho > 0$,

$$\begin{aligned} D(x, t, \rho) &\ge \rho^{m-1}, \\ D(x, t, \rho) &\le (e^{\lambda_1 T}\rho + 1)^{m-1} \le (e^{\lambda_1 T} + 1)^{m-1} \max\{\rho, 1\}^{m-1}, \\ |S_1(x, t, \rho)| &\le \chi(e^{\lambda_1 T} + 1)^{p-1} \max\{\rho, 1\}^{p-1}. \end{aligned}$$

Moreover, the initial condition in (4.2) implies that $\int_\Omega \widetilde{u}(\cdot, 0) = \int_\Omega u_0 = M_0$. We now choose $\theta > n$ satisfying

$$m - p \in \Big(\frac{1}{\theta} - 1 - \frac{1}{n},\ \frac{1}{\theta} - \frac{2}{n}\Big]$$

and

$$\begin{aligned}\sigma &= \frac{n(n-1)}{(m-p+1)n+1} + \varepsilon \\ &> \frac{n(n-1)}{(m-p+1)n+1-\frac{n}{\theta}} = \frac{n-1}{(m-p)+1+\frac{1}{n}-\frac{1}{\theta}}.\end{aligned}$$

Since $p \geq q$ and

$$\Big|\frac{S_2(x,t,\rho)}{S_1(x,t,\rho)}\Big| = \frac{\xi(e^{\lambda_1 t}\rho+1)^{q-2}\rho}{\chi(e^{\lambda_1 t}\rho+1)^{p-2}\rho} = \frac{\xi}{\chi}(e^{\lambda_1 t}\rho+1)^{q-p} \leq \frac{\xi}{\chi},$$

for all $x \in \Omega$, $t \in (0,T)$ and all $\rho > 0$, following the steps in the proof of [3, Lemma 5.2], we establish

$$\int_\Omega |x|^{(n-1)\theta}|\mathbf{f}(x,t)|^\theta \,\mathrm{d}x \leq c_1\Big(\frac{\alpha}{\beta} + \frac{\xi}{\chi}\cdot\frac{\gamma}{\delta}\Big)^\theta\Big(\frac{2e^{\lambda_1 T}M_0}{\omega_{n-1}}\Big)^\theta|\Omega|$$

for all $t \in (0,T)$ with some $c_1 > 0$, where ω_{n-1} denotes the $(n-2)$-dimensional surface area of the unit sphere in $\mathbb{R}^{n-1}$. Thanks to [14, Theorem 1.1], we derive that there exists $c_2 > 0$ such that $\widetilde{u}(x,t) \leq c_2|x|^{-\sigma}$ for all $x \in \Omega$ and all $t \in (0,T)$, which leads to the end of the proof. □

We now introduce the mass accumulation functions $U = U(s,t), V = V(s,t)$ and $W = W(s,t)$ as

$$U(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,\mathrm{d}\rho, \tag{4.3}$$

$$V(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}v(\rho,t)\,\mathrm{d}\rho \tag{4.4}$$

and

$$W(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1}w(\rho,t)\,\mathrm{d}\rho, \tag{4.5}$$

where $s := r^n$ for $r \in [0,R]$ and $t \in [0,T_{\max})$. We next define the moment-type functional

$$\phi(s_0,t) := \int_0^{s_0} s^{-b}(s_0-s)U(s,t)\,\mathrm{d}s \tag{4.6}$$

for $s_0 \in (0,R^n)$, $t \in [0,T_{\max})$ and $b \in (0,1)$.

Lemma 4.3. *Assume that $p > q$. Let $\mu_1 > 0$, $\kappa \ge 1$ and $a \ge 0$. Then there exist $C_1, C_2 > 0$ such that for any $b \in (0,1)$ and $s_0 \in (0, R^n)$, the function $\phi(s_0, \cdot)$ belongs to $C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ and satisfies*

$$\begin{aligned}
\phi'(s_0,t) &\ge \left[C_1 \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}U(s,t)U_s(s,t)\,\mathrm{d}s - C_2\phi(s_0,t)\right] \\
&\quad + n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0-s)(nU_s(s,t)+1)^{m-1}U_{ss}(s,t)\,\mathrm{d}s \\
&\quad - \chi\beta n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}V(s,t)U_s(s,t)\,\mathrm{d}s \\
&\quad - n^{\kappa-1}\mu_1 \int_0^{s_0} s^{-b}(s_0-s)\Big[\int_0^{s_0} \eta^{\frac{a}{n}} U_s^\kappa(\eta,t)\,\mathrm{d}\eta\Big]\,\mathrm{d}s \\
&=: J_1(s_0,t)+J_2(s_0,t)+J_3(s_0,t)+J_4(s_0,t)
\end{aligned} \tag{4.7}$$

for all $t \in (0, T_{\max})$.

Proof. We first note that $\phi(s_0,\cdot) \in C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$ for all $b \in (0,1)$ and $s_0 \in (0,R^n)$ by the proof of [48, Lemma 4.1]. The first equation in (1.1) implies that $u = u(r,t)$, $v = v(r,t)$, $w = w(r,t)$ satisfy

$$\begin{aligned}
u_t &= \frac{1}{r^{n-1}}\big((u+1)^{m-1}r^{n-1}u_r\big)_r - \chi\frac{1}{r^{n-1}}\big(u(u+1)^{p-2}r^{n-1}v_r\big)_r \\
&\quad + \xi\frac{1}{r^{n-1}}\big(u(u+1)^{q-2}r^{n-1}w_r\big)_r + \lambda u - \mu u^\kappa.
\end{aligned} \tag{4.8}$$

Moreover, the second and third equations in (1.1) yield that

$$r^{n-1}v_r(r,t) = \beta V(r^n,t) - \alpha U(r^n,t)$$

and

$$r^{n-1}w_r(r,t) = \delta W(r^n,t) - \gamma U(r^n,t).$$

Integrating (4.8) combined with these relations with respect to r over $[0, s^{\frac{1}{n}}]$, we see from the nonnegativity of λ and (1.5) that

$$\begin{aligned}
U_t &\ge n^2 s^{2-\frac{2}{n}}(nU_s+1)^{m-1}U_{ss} + \chi nU_s(nU_s+1)^{p-2}(\alpha U - \beta V) \\
&\quad - \xi nU_s(nU_s+1)^{q-2}(\gamma U - \delta W) - n^{\kappa-1}\mu_1 \int_0^s \eta^{\frac{a}{n}}U_s^\kappa(\eta,t)\,\mathrm{d}\eta \\
&= \chi nU_s(nU_s+1)^{p-2}\cdot\alpha U - \xi nU_s(nU_s+1)^{q-2}\cdot\gamma U \\
&\quad + n^2 s^{2-\frac{2}{n}}(nU_s+1)^{m-1}U_{ss} \\
&\quad - \chi nU_s(nU_s+1)^{p-2}\cdot\beta V + \xi nU_s(nU_s+1)^{q-2}\cdot\delta W \\
&\quad - n^{\kappa-1}\mu_1 \int_0^s \eta^{\frac{a}{n}}U_s^\kappa(\eta,t)\,\mathrm{d}\eta,
\end{aligned}$$

which together with (4.6) entails

$$\begin{aligned}
\phi'(s_0,t) \geq{}& \chi\alpha n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}U(s,t)U_s(s,t)\,\mathrm{d}s \\
&- \xi\gamma n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{q-2}U(s,t)U_s(s,t)\,\mathrm{d}s \\
&+ n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0-s)(nU_s(s,t)+1)^{m-1}U_{ss}(s,t)\,\mathrm{d}s \\
&- \chi\beta n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}V(s,t)U_s(s,t)\,\mathrm{d}s \\
&+ \xi\delta n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{q-2}W(s,t)U_s(s,t)\,\mathrm{d}s \\
&- n^{\kappa-1}\mu_1 \int_0^{s_0} s^{-b}(s_0-s)\Big[\int_0^{s_0} \eta^{\frac{a}{n}}U_s^{\kappa}(\eta,t)\,\mathrm{d}\eta\Big]\,\mathrm{d}s \\
=:{}& \widetilde{J_1}(s_0,t)+E_1(s_0,t)+\widetilde{J_2}(s_0,t)+\widetilde{J_3}(s_0,t)+E_2(s_0,t)+\widetilde{J_4}(s_0,t)
\end{aligned} \tag{4.9}$$

for all $s_0 \in (0,R^n)$ and all $t \in (0,T_{\max})$. Here we estimate the term $E_1(s_0,t)$. We first consider the case $q>1$. In this case, using Young's inequality, we see that for all $\varepsilon_1>0$,

$$\begin{aligned}
(nU_s(s,t)+1)^{q-2}U_s(s,t) &\leq \varepsilon_1\Big[(nU_s(s,t)+1)^{(q-1)-1}U_s(s,t)\Big]^{\frac{p-1}{q-1}} + c_1(\varepsilon_1) \\
&= \varepsilon_1(nU_s(s,t)+1)^{p-1-\frac{p-1}{q-1}}U_s^{\frac{p-1}{q-1}}(s,t)+c_1(\varepsilon_1).
\end{aligned} \tag{4.10}$$

Here we notice from the relation $\frac{p-1}{q-1}>1$ by $p>q>1$ that

$$U_s^{\frac{p-1}{q-1}}(s,t) = U_s^{\frac{p-1}{q-1}-1}(s,t)U_s(s,t) \leq (nU_s(s,t)+1)^{\frac{p-1}{q-1}-1}U_s(s,t). \tag{4.11}$$

A combination of (4.10) and (4.11) implies that

$$(nU_s(s,t)+1)^{q-2}U_s(s,t) \leq \varepsilon_1(nU_s(s,t)+1)^{p-2}U_s(s,t)+c_1(\varepsilon_1). \tag{4.12}$$

In the case $q \leq 1$, noting that

$$(nU_s(s,t)+1)^{q-2}U_s(s,t) \leq (nU_s(s,t)+1)^{-1}U_s(s,t) \leq n^{-1},$$

we can choose $\varepsilon_1 = 0$ and $c_1(\varepsilon_1) = n^{-1}$ in the estimate (4.12). In view of (4.12) we obtain

$$\begin{aligned}
E_1(s_0,t) &= -\xi\gamma n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{q-2}U(s,t)U_s(s,t)\,\mathrm{d}s \\
&\ge -\varepsilon_1\xi\gamma n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}U(s,t)U_s(s,t)\,\mathrm{d}s \\
&\quad - c_1(\varepsilon_1)\int_0^{s_0} s^{-b}(s_0-s)U(s,t)\,\mathrm{d}s \\
&= -\varepsilon_1\xi\gamma n \int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}U(s,t)U_s(s,t)\,\mathrm{d}s + c_1(\varepsilon_1)\phi(s_0,t).
\end{aligned} \tag{4.13}$$

Combining (4.13) with (4.9) and noting that $E_2(s_0,t) \ge 0$, we establish

$$\begin{aligned}
\phi'(s_0,t) \ge (\chi\alpha - \varepsilon_1\xi\gamma)n &\int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}U(s,t)U_s(s,t)\,\mathrm{d}s \\
&- c_1(\varepsilon_1)\phi(s_0,t) \\
&+ n^2\int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0-s)(nU_s(s,t)+1)^{m-1}U_{ss}(s,t)\,\mathrm{d}s \\
&- \chi\beta n\int_0^{s_0} s^{-b}(s_0-s)(nU_s(s,t)+1)^{p-2}V(s,t)U_s(s,t)\,\mathrm{d}s \\
&- n^{\kappa-1}\mu_1\int_0^{s_0} s^{-b}(s_0-s)\Big[\int_0^{s_0}\eta^{\frac{a}{n}}U_s^{\kappa}(\eta,t)\,\mathrm{d}\eta\Big]\,\mathrm{d}s.
\end{aligned}$$

Here, choosing $\varepsilon_1 := \frac{\chi\alpha}{2\xi\gamma}$ when $q > 1$ and recalling that $\varepsilon_1 = 0$ when $q \le 1$, we see that $\chi\alpha - \varepsilon_1\xi\gamma > 0$, which means that the desired inequality (4.7) holds. □

Proof of Theorem 4.1. In order to prove the assertion, assume on the contrary that the conclusion does not hold, that is, suppose that there exist $M_0 > 0$, $M_1 \in (0, M_0)$ and $L > 0$ such that given $T > 0$, $\varepsilon > 0$ and $r_1 \in (0,R)$ one can take u_0 fulfilling $u_0(x) \le L|x|^{-\sigma}$ for all $x \in \Omega$, where $\sigma := \frac{n(n-1)}{(m-p+1)n+1} + \varepsilon$, $\int_\Omega u_0 = M_0$ and $\int_{B_{r_1}(0)} u_0 \ge M_1$, and the corresponding solution u of (1.1) does not blow up at T, that is, $T < T_{\max}$. Then it follows from (4.1) that $u(x,t) \le K|x|^{-\sigma}$ for all $x \in \Omega$ and all $t \in (0,T)$ with some $K = K(M_0,L,T) > 0$, which is rewritten as

$$nU_s(s,t) \le Ks^{-\frac{\sigma}{n}} \quad \text{for all } s \in (0,R^n] \text{ and all } t \in (0,T). \tag{4.14}$$

As in the proofs of [38, Lemmas 3.3 and 3.9], in the case $p > 2$, this estimate provides $(nU_s+1)^{p-2} \le (Ks^{-\frac{\sigma}{n}}+1)^{p-2} \le (K+R^\sigma)^{p-2}s^{-\frac{\sigma}{n}(p-2)}$, whereas in the case $p \in (1,2]$, we have $(nU_s+1)^{p-2} \le 1$, which can be unified such that

$$(nU_s+1)^{p-2} \le c_1 s^{-\frac{\sigma}{n}(p-2)_+} \tag{4.15}$$

for all $s \in (0, R^n]$ and all $t \in (0,T)$, where c_1 depends on T when $p > 2$. Let $s_0 \in (0, R^n)$. Invoking (4.15) with t fixed in $(0,T)$, we see that $J_1(s_0,t)$ is estimated as

$$\begin{aligned} J_1(s_0,t) &= C_1 \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} U(s,t)U_s(s,t)\,\mathrm{d}s - C_2\phi(s_0,t) \\ &\geq c_1 C_1 \psi_p(s_0,t) - C_2\phi(s_0,t), \end{aligned}$$

where

$$\psi_p(s_0,t) := \int_0^{s_0} s^{-b+\frac{\sigma}{n}(2-p)_+}(s_0 - s)U(s,t)U_s(s,t)\,\mathrm{d}s.$$

Using (4.15) again, we deduce from integration by parts that

$$J_3(s_0,t) \geq -\chi\beta n c_1 \Big(b + \frac{\sigma}{n}(p-2) + 1\Big) s_0 \int_0^{s_0} s^{-b-1-\frac{\sigma}{n}(p-2)} V(s,t)U(s,t)\,\mathrm{d}s,$$

which can be further estimated as

$$J_3(s_0,t) \geq -c_2 s_0^{\frac{2}{n}+\frac{1-b}{2}-\frac{\sigma}{2n}[(2-p)_+ + 2(p-2)_+]}\sqrt{\psi_p(s_0,t)} - c_2 s_0^{\frac{2}{n}-\frac{\sigma}{n}[(2-p)_+ + (p-2)_+]}\psi_p(s_0,t)$$

in light of

$$V(s,t) \leq \frac{c_3}{n} s_0^{\frac{2}{n}-1} s + c_4 s_0^{-\frac{1}{2}+\frac{b-\widetilde{b}}{2}} s^{\frac{2}{n}+\frac{\widetilde{b}}{2}}\sqrt{\psi_p(s_0,t)}$$

with some $\widetilde{b} > 0$, where c_2 depends on T when $p > 2$ (for details, see the proof of [38, Lemma 3.9]). Next, as in the proof of [3, Lemma 3.6 (i)], integration by parts yields

$$\begin{aligned} J_2(s_0,t) &= \frac{n}{m} \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s)((nU_s(s,t)+1)^m)_s\,\mathrm{d}s \\ &\geq -\frac{n}{m}\Big(2 - \frac{2}{n} - b\Big) \int_0^{s_0} s^{1-\frac{2}{n}-b}(s_0 - s)(nU_s(s,t)+1)^m\,\mathrm{d}s \end{aligned}$$

and hence, applying

$$(nU_s + 1)^m \leq c_5 s^{-\frac{\sigma}{n}(m-1)} U_s + c_5$$

for all $s \in (0, R^n]$ and all $t \in (0,T)$, where $c_5 := \max\{n, 2^{m-1}, 2^{m-1}nK^{m-1}\}$, we can see from integration by parts and an estimate for U in [3, Lemma 3.4] that

$$J_2(s_0,t) \geq -c_6 s_0^{\frac{3-b}{2}-\frac{2}{n}-\frac{\sigma}{2n}[2(m-1)+(2-p)_+]}\sqrt{\psi_p(s_0,t)} - c_6 s_0^{3-\frac{2}{n}-b}.$$

Also, from an argument similar to that in the proof of [3, Lemma 3.5] it follows that

$$J_4(s_0,t) \geq -\frac{n^{\kappa-1}\mu_1}{1-b} s_0^{1-b} \int_0^{s_0} s^{\frac{a}{n}}(s_0 - s)U_s^{\kappa}(s,t)\,\mathrm{d}s$$

and by (4.14),

$$\int_0^{s_0} s^{\frac{a}{n}}(s_0 - s)U_s^{\kappa}(s,t)\,\mathrm{d}s \leq c_7 s_0 \int_0^{s_0} s^{\frac{a}{n}-\frac{\sigma}{n}(\kappa-1)-1} U(s,t)\,\mathrm{d}s,$$

where $c_7 := \frac{K^{\kappa-1}}{n^{\kappa-1}}\big[(\frac{\sigma}{n}(\kappa-1)-\frac{a}{n})_+ + 1\big]$. Again by using an estimate for U, we have

$$J_4(s_0,t) \geq -c_8 s_0^{\frac{3-b}{2}+\frac{a}{n}-\frac{\sigma}{2n}[2(\kappa-1)+(2-p)_+]}\sqrt{\psi_p(s_0,t)}.$$

Collecting the estimates for $J_1(s_0,t), J_2(s_0,t), J_3(s_0,t), J_4(s_0,t)$, we infer

$$\begin{aligned}\phi'(s_0,t) &\geq c_9\psi_p(s_0,t) - c_{10}\phi(s_0,t)\\ &\quad - c_{11}s_0^{\frac{3-b}{2}-\frac{2}{n}-\frac{\sigma}{2n}[2(m-1)+(2-p)_+]}\sqrt{\psi_p(s_0,t)} - c_{11}s_0^{3-\frac{2}{n}-b}\\ &\quad - c_{11}s_0^{\frac{2}{n}+\frac{1-b}{2}-\frac{\sigma}{2n}[(2-p)_++2(p-2)_+]}\sqrt{\psi_p(s_0,t)} - c_{11}s_0^{\frac{2}{n}-\frac{\sigma}{n}[(2-p)_++(p-2)_+]}\psi_p(s_0,t)\\ &\quad - c_{11}s_0^{\frac{3-b}{2}+\frac{a}{n}-\frac{\sigma}{2n}[2(\kappa-1)+(2-p)_+]}\sqrt{\psi_p(s_0,t)},\end{aligned}$$

where c_9, c_{11} depend on T when $p>2$ or $m>1$ or $\kappa>1$. Let $\varepsilon_1>0$ which will be fixed later. Using Young's inequality, we can see that

$$\begin{aligned}\phi'(s_0,t) &\geq c_9\psi_p(s_0,t) - \varepsilon_1\psi_p(s_0,t) - c_{11}s_0^{\frac{2}{n}-\frac{\sigma}{n}[(2-p)_++(p-2)_+]}\psi_p(s_0,t)\\ &\quad - c_{12}(\varepsilon_1)\Big(s_0^{3-b-\frac{4}{n}-\frac{\sigma}{n}[2(m-1)+(2-p)_+]} + s_0^{2-\frac{2}{n}-b}\\ &\quad + s_0^{\frac{4}{n}+1-b-\frac{\sigma}{n}[(2-p)_++2(p-2)_+]} + s_0^{3-b+\frac{2a}{n}-\frac{\sigma}{n}[2(\kappa-1)+(2-p)_+]}\Big)\\ &\quad - c_{10}\phi(s_0,t)\end{aligned} \tag{4.16}$$

for all $s_0\in(0,R^n)$ and all $t\in(0,T)$. We now choose $s_1=s_1(T)$ small enough such that $s_1\in(0,R^n)$ and

$$c_{11}s_0^{\frac{2}{n}-\frac{\sigma}{n}[(2-p)_++(p-2)_+]}\psi_p(s_0,t) \leq \frac{1}{4}c_9\psi_p(s_0,t)$$

for all $s_0\in(0,s_1)$ and all $t\in(0,T)$. From now to the end of this proof, we suppose that s_1, s_0, t are in these regions. Setting $\varepsilon_1 := \frac{c_9}{4}$, we have from (4.16) that

$$\begin{aligned}\phi'(s_0,t) &\geq \frac{1}{2}c_9\psi_p(s_0,t)\\ &\quad - c_{12}\Big(s_0^{3-b-\frac{4}{n}-\frac{\sigma}{n}[2(m-1)+(2-p)_+]} + s_0^{2-\frac{2}{n}-b}\\ &\qquad + s_0^{\frac{4}{n}+1-b-\frac{\sigma}{n}[(2-p)_++2(p-2)_+]} + s_0^{3-b+\frac{2a}{n}-\frac{\sigma}{n}[2(\kappa-1)+(2-p)_+]}\Big)\\ &\quad - c_{10}\phi(s_0,t).\end{aligned}$$

By an argument similar to that in the proof of [38, Lemma 4.3], due to the conditions **(C1)**–**(C3)**, we can pick $\varepsilon_0>0$ and then for $\sigma = \frac{n(n-1)}{(m-p+1)n+1}+\varepsilon_0$ there exists $\theta\in(0,\ 2-\frac{\sigma}{n}(2-p)_+)$ such that

$$\phi'(s_0,t) \geq \frac{1}{2}c_9\psi_p(s_0,t) - c_{13}s_0^{3-b-\theta} - c_{14}\phi(s_0,t). \tag{4.17}$$

Applying the estimate $\sqrt{\psi_p(s_0,t)} \geq c_{15}s_0^{\frac{b-3}{2}+\frac{\sigma}{2n}(2-p)_+}\phi(s_0,t)$ (see [38, Lemma 3.10]) to the first term on the right-hand side of (4.17), we have

$$\phi'(s_0,t) \geq c_{15}s_0^{b-3+\frac{\sigma}{n}(2-p)_+}\phi^2(s_0,t) - c_{13}s_0^{3-b-\theta} - c_{14}\phi(s_0,t). \tag{4.18}$$

Again by Young's inequality, we derive that

$$c_{14}\phi(s_0,t) \leq \frac{1}{2}c_{15}s_0^{b-3+\frac{\sigma}{n}(2-p)_+}\phi^2(s_0,t) + c_{16}s_0^{3-b-\frac{\sigma}{n}(2-p)_+},$$

which along with (4.18) yields

$$\begin{aligned}\phi'(s_0,t) &\geq \frac{1}{2}c_{15}s_0^{b-3+\frac{\sigma}{n}(2-p)_+}\phi^2(s_0,t) - c_{13}s_0^{3-b-\theta} - c_{16}s_0^{3-b-\frac{\sigma}{n}(2-p)_+}\\ &\geq \frac{1}{2}c_{15}s_0^{b-3+\frac{\sigma}{n}(2-p)_+}\phi^2(s_0,t) - c_{17}s_0^{\widetilde{\theta}}\end{aligned} \tag{4.19}$$

with $\widetilde{\theta} = \min\{3-b-\theta,\ 3-b-\frac{\sigma}{n}(2-p)_+\}$. Here, in view of the conditions **(C1)**–**(C3)** we can take $b \in (0,1)$ satisfying

$$b < 2 - \frac{4}{n} - \frac{\sigma}{n}[2(m-1) + (2-p)_+]$$

(see [38, Lemma 4.1]). This entails that

$$b - 3 + \frac{\sigma}{n}(2-p)_+ < \left\{2 - \frac{4}{n} - \frac{\sigma}{n}[2(m-1)+(2-p)_+]\right\} - 3 + \frac{\sigma}{n}(2-p)_+ \quad = -1 - \frac{4}{n} - \frac{2\sigma}{n}(m-1) < 0$$

and moreover, recalling the choice that $\theta \in (0,\ 2-\frac{\sigma}{n}(2-p)_+)$, we have

$$\begin{aligned}3-b-\theta &> 3-b-\left[2-\frac{\sigma}{n}(2-p)_+\right]\\ &= 1-b+\frac{\sigma}{n}(2-p)_+ > 0,\end{aligned}$$

which lead to $\widetilde{\theta} > 0$. We now set $k_1 = k_1(s_0) := \frac{1}{2}c_{15}s_0^{b-3+\frac{\sigma}{n}(2-p)_+}$, $k_2 = k_2(s_0) := c_{17}s_0^{\widetilde{\theta}}$. Then (4.19) is rewritten as

$$\phi'(s_0,t) \geq k_1\phi^2(s_0,t) - k_2.$$

As in the proof of [3, Theorem 1.1], we appropriately select $\phi(s_0,0)$ later and employ the solution

$$y(t) = \sqrt{\frac{k_1}{k_2}} \cdot \frac{1 + \frac{\sqrt{\frac{k_1}{k_2}}y_0 - 1}{\sqrt{\frac{k_1}{k_2}}y_0+1}e^{2\sqrt{k_1k_2}t}}{1 - \frac{\sqrt{\frac{k_1}{k_2}}y_0 - 1}{\sqrt{\frac{k_1}{k_2}}y_0+1}e^{2\sqrt{k_1k_2}t}} \qquad \text{of} \quad \begin{cases} y' = k_1y^2 - k_2,\\ y(0) = y_0,\end{cases}$$

where $y_0 > 0$ and we pick $s_0 \in (0,s_1)$ further small such that $\sqrt{\frac{k_1}{k_2}}y_0 > 1$. The solution $y(t)$ blows up at $t = T_0$ with $e^{2\sqrt{k_1k_2}T_0} = \frac{\sqrt{\frac{k_1}{k_2}}y_0+1}{\sqrt{\frac{k_1}{k_2}}y_0-1}$. Here we note that if $\sqrt{\frac{k_1}{k_2}}y_0 > 2$, then $\frac{\sqrt{\frac{k_1}{k_2}}y_0+1}{\sqrt{\frac{k_1}{k_2}}y_0-1} < \frac{\sqrt{\frac{k_1}{k_2}}y_0+\frac{1}{2}\sqrt{\frac{k_1}{k_2}}y_0}{\sqrt{\frac{k_1}{k_2}}y_0-\frac{1}{2}\sqrt{\frac{k_1}{k_2}}y_0} = 3$, which means $T_0 < \frac{1}{2\sqrt{k_1k_2}}\log 3$. Thus, again taking $s_0 \in (0,s_1)$ sufficiently small such that $\frac{1}{2\sqrt{k_1k_2}}\log 3 < T$, $\sqrt{\frac{k_1}{k_2}}y_0 > 2$ and $\phi(s_0,0) \geq y_0$ (use [3, Lemma 4.1]), we obtain from an ODE comparison argument that $\phi(s_0,t) \geq y(t)$ for all $t \in (0,T_0)$, which implies a contradiction, because $\phi(s_0,\cdot)$ is bounded on $(0,T_0)$ $(\subset (0,T))$, whereas $y(t)$ blows up at $t = T_0$. Therefore we arrive at the desired conclusion. □

4.2. The case $p = q$

In this subsection we show the following theorem, which guarantees finite-time blow-up in (1.1) in the case $p = q$.

Theorem 4.4. *Assume that* **(A2)** *is satisfied with $p = q$ and $\chi\alpha - \xi\gamma > 0$. Moreover, suppose that m, p and κ fulfill the same conditions as in Theorem 4.1. Then the conclusion of Theorem 4.1 holds.*

In order to prove the above theorem we show the following lemma giving the pointwise lower estimate for ϕ', where U, V, W and ϕ are defined as in (4.3)–(4.5) and (4.6), respectively.

Lemma 4.5. *Suppose that $p = q$. Let $\mu_1 > 0$, $\kappa \geq 1$ and $a \geq 0$. Then for any $b \in (0,1)$ and $s_0 \in (0, R^n)$, the function $\phi(s_0, \cdot)$ belongs to $C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ and satisfies*

$$\begin{aligned}
\phi'(s_0,t) \geq (\chi\alpha - \xi\gamma) n \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} U(s,t) U_s(s,t)\,\mathrm{d}s \\
+ n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s)(nU_s(s,t)+1)^{m-1} U_{ss}(s,t)\,\mathrm{d}s \\
- \chi\beta n \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} V(s,t) U_s(s,t)\,\mathrm{d}s \\
- n^{\kappa-1}\mu_1 \int_0^{s_0} s^{-b}(s_0 - s)\Big[\int_0^{s_0} \eta^{\frac{a}{n}} U_s^{\kappa}(\eta,t)\,\mathrm{d}\eta\Big]\,\mathrm{d}s
\end{aligned}$$

for all $t \in (0, T_{\max})$.

Proof. Arguing as in Lemma 4.3, we see that $\phi(s_0, \cdot) \in C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ for all $b \in (0,1)$ and $s_0 \in (0, R^n)$, and have (4.9) with $q = p$. We then rearrange it as

$$\begin{aligned}
\phi'(s_0,t) \geq (\chi\alpha - \xi\gamma) n \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} U(s,t) U_s(s,t)\,\mathrm{d}s \\
+ n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s)(nU_s(s,t)+1)^{m-1} U_{ss}(s,t)\,\mathrm{d}s \\
- \chi\beta n \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} V(s,t) U_s(s,t)\,\mathrm{d}s \\
+ \xi\delta n \int_0^{s_0} s^{-b}(s_0 - s)(nU_s(s,t)+1)^{p-2} W(s,t) U_s(s,t)\,\mathrm{d}s \\
- n^{\kappa-1}\mu_1 \int_0^{s_0} s^{-b}(s_0 - s)\Big[\int_0^{s_0} \eta^{\frac{a}{n}} U_s^{\kappa}(\eta,t)\,\mathrm{d}\eta\Big]\,\mathrm{d}s.
\end{aligned}$$

Here, compared with (4.9), the terms corresponding to $E_1(s_0,t)$ and $\widetilde{J_1}(s_0,t)$ are arranged into the first term on the right-hand side of the above inequality, and the other terms are the same as those in (4.9), provided that the fourth term is equal to $E_2(s_0,t)$ with $q = p$, which is nonnegative. In light of this observation, we obtain the desired inequality. □

Proof of Theorem 4.4. Let $T \in (0, T_{\max})$. In view of Lemma 4.5, proceeding as in the proof of Theorem 4.1 and taking σ properly, we can find $c_1, c_2 > 0$ and $\theta \in (0,\ 2 - \frac{\sigma}{n}(2-p)_+)$ such that

$$\phi'(s_0,t) \geq c_1 s_0^{b-3+\frac{\sigma}{n}(2-p)_+} \phi^2(s_0,t) - c_2 s_0^{3-b-\theta}$$

for all $s_0 \in (0, s_1)$ and all $t \in (0, T)$ for some small $s_1 > 0$. This inequality corresponds to (4.19) and proves Theorem 4.4. □

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