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Infinitely many solutions for a class of critical Kirchhoff-type equations involving p-Laplacian operator

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Abstract. In this paper, we investigate the following Kirchhoff-type equation

$$-M\left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x\right)\Delta_p u = |u|^{p^*-2}u + h(x)|u|^{q-2}u,\ x \in \mathbb{R}^N,$$

where $N \geq 3,\ 1 < p < N,\ p^* = \frac{Np}{N-p},\ 0 < h \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$ with $q \in (1, p^*)$; M is a nonnegative continuous function with some growth conditions. We show that the above problem has infinitely many solutions by using variational methods.

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1. Introduction and main results

Kirchhoff equations of the type

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

are related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u),$$

which was proposed by Kirchhoff [1] in 1883 as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings. After Lions [2] proposed an abstract framework to problem (1), various kinds of Kirchhoff-type equations have been widely concerned and studied by many scholars (see [3–15] and the references therein). Among them, the critical case has been studied in [7,11,13–15]. In particular, Faraci and Farkas [15] dealt with the following Kirchhoff-type problem involving a critical term

$$\begin{cases} -M(\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = |u|^{p^*-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where Ω is an open connected set of $\mathbb{R}^N$ with smooth boundary, $N \geq 3$, $1 < p < N$, $M \in C([0,+\infty),[0,+\infty))$ and satisfies some of the following hypotheses.

(M_1) $\hat{M}(t+s) \geq \hat{M}(t) + \hat{M}(s)$, for every $t, s \in [0,+\infty)$;

(M_2) $\inf\limits_{t>0} \dfrac{\hat{M}(t)}{t^{\frac{p^*}{p}}} \geq c_p$;

(M_3) $\inf\limits_{t>0} \dfrac{M(t)}{t^{\frac{p^*}{p}-1}} > S_N^{-\frac{p^*}{p}}$,

where $\hat{M} : [0,+\infty) \to [0,+\infty)$ is the primitive of the function M, defined by

$$\hat{M}(t) = \int_0^t M(s)\mathrm{d}s;$$

c_p is a constant, defined by

$$c_p = \begin{cases} (2^{p-1}-1)^{\frac{p^*}{p}} \frac{p}{p^*} S_N^{-\frac{p^*}{p}}, \ p \geq 2; \\ 2^{2p^*-1-\frac{p^*}{p}} \frac{p}{p^*} S_N^{-\frac{p^*}{p}}, \quad 1 < p < 2, \end{cases}$$

S_N is the best Sobolev constant of $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. If (M_1) and (M_2) hold, the authors proved that the energy functional associated with problem (2) is sequentially weakly lower semicontinuous in $W_0^{1,p}(\Omega)$. When (M_3) holds, the property of Palais–Smale (for short (PS)) for the energy functional associated with problem (2) was got by using the second Concentration Compactness lemma of Lions [18] in $W_0^{1,p}(\Omega)$. Moreover, the authors provided an application to a Kirchhoff-type problem on exterior domains

$$\begin{cases} -M(\int\limits_{\Omega} |\nabla u|^p \mathrm{d}x)\Delta_p u = \lambda(u^{p^*-1} + u^{r-1}), \ x \in \Omega, \\ u \geq 0, \qquad x \in \Omega, \\ u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Omega = \mathbb{R}^N \setminus B_R(0)$, $2 \leq p < r < p^*$. Under (M_1), (M_2) and

(M_4) $\lim\limits_{t\to 0} \dfrac{\hat{M}(t)}{t^{\frac{r}{p}}} = 0$,

two nontrivial solutions of problem (3) were obtained for some $\lambda \in (0,1)$ by employing an abstract well-posedness result for a class of constrained minimization problem.

Inspired by [15], we study the existence of infinitely many solutions for the following p-Laplacian equations of Kirchhoff type via variational methods

$$\begin{cases} -M(\int\limits_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x)\Delta_p u = |u|^{p^*-2}u + h(x)|u|^{q-2}u, \ x \in \mathbb{R}^N, \\ u \in D^{1,p}(\mathbb{R}^N), \end{cases} \tag{4}$$

where $N \geq 3$, $1 < p < N$, $p^* = \frac{Np}{N-p}$, $q \in (1,p^*)$, $D^{1,p}(\mathbb{R}^N)$ is the classic Sobolev space (the definition is given in Sect. 2), $M : [0,+\infty) \to [0,+\infty)$ is a continuous function, satisfies (M_3) with S_N is the best Sobolev constant of $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and

(M_5) $\lim\limits_{t\to 0} \dfrac{\hat{M}(t)}{t^{\frac{q}{p}}} = 0$.

h satisfies the following assumptions.

(h_1) h is positive almost everywhere;

(h_2) $h \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that assumptions (M_3), (M_5), (h_1) and (h_2) hold. Then, there are infinitely many solutions to problem (4).*

Remark. Assumption (M_3) indicates that the growth rate of $\hat{M}$ at infinity is no less than $\frac{p^*}{p}$; at the same time, the decay rate of $\hat{M}$ at zero is no more than $\frac{p^*}{p}$. It ensures that the functional associated with problem (4) is coercive in $D^{1,p}(\mathbb{R}^N)$. Assumption (M_5) indicates that the decay rate of $\hat{M}$ at zero is no less than $\frac{q}{p}$. This assumption is mainly used to ensure the functional associated with problem (4) can take a value less than near zero. Assumption (M_3) is a global constraint; we think it may be extended to some growth assumptions at zero and infinity. There are a lot of functions satisfy the assumptions (M_3) and (M_5). A class of example is

$$M(t) = \begin{cases} at^{\alpha},\ t \in [0,1]; \\ at^{\beta},\ t \in (1,\infty), \end{cases}$$

where $a > S_N^{-\frac{p^*}{p}}$, $\begin{cases} \frac{q}{p} - 1 < \alpha \leq \frac{p^*}{p} - 1,\ q > p; \\ 0 \leq \alpha \leq \frac{p^*}{p} - 1, \qquad 1 < q < p, \end{cases}$ $\beta \geq \frac{p^*}{p} - 1$. Here we can define $M(0) = a$ if $\alpha = 0$.

The rest of the paper is organized as follows. In Sect. 2, we give the variational structure of problem (4). In Sect. 3, the main result is proved by using the second Concentration Compactness lemma of Loins and a variant of Clark's theorem.

The following conventions and notations are used in this paper:
- $C, C_1, C_2, \ldots$ denote positive (possible different) constants.
- We denote weak and strong convergence by $u_n \rightharpoonup u$ and $u_n \to u$, respectively.
- $o_n(1)$ is an infinitely small quantity of 1.

2. Preliminaries

In this section, we give some preliminary results which will be used to prove our main result.

As usual, the Sobolev space $D^{1,p}(\mathbb{R}^N)$ is defined by

$$\{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

The classical Lebesgue spaces $L^q(\mathbb{R}^N)(1 \leq q \leq p^*)$ are equipped with the norms

$$\|u\|_q = \left(\int_{\mathbb{R}^N} |u|^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

S_N is the best Sobolev constant of $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, i.e.,

$$S_N = \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}.$$

As it is well known, $L^{p^*}(\mathbb{R}^N)$ is a uniformly convex Banach space.

Under our assumptions, problem (4) has a variational structure. We denote by $J : D^{1,p}(\mathbb{R}^N) \to \mathbb{R}$, the energy functional associated with problem (4), which is given by

$$J(u) = \frac{1}{p}\hat{M}(\|u\|^p) - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q \mathrm{d}x.$$

Obviously, $J \in C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$ with derivative at $u \in D^{1,p}(\mathbb{R}^N)$ given by

$$J'(u)(v) = M(\|u\|^p) \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v \mathrm{d}x - \int_{\mathbb{R}^N} |u|^{p^*-2} uv \mathrm{d}x - \int_{\mathbb{R}^N} h(x)|u|^{q-2} uv \mathrm{d}x, \; v \in D^{1,p}(\mathbb{R}^N).$$

Consequently, the critical points of J are weak solutions for problem (4).

In order to prove our main result, we need the following lemma which is a variant of a result of Clark [16] and is given in [17].

Lemma 2.1. *Assume X is a Banach space, $I \in C^1(X, \mathbb{R})$ satisfying Palais–Smale condition is bounded from below and even, $I(0) = 0$. If for any $k \in \mathbb{N}$, there exist k-dimensional subspaces X^k and $\rho_k > 0$ such that*

$$\sup_{X^k \cap S_{\rho_k}} I < 0,$$

where $S_{\rho_k} = \{u \in X | \; \|u\| = \rho_k\}$; then, I has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.

3. Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, we will show the existence of infinitely many solutions for problem (4).

Lemma 3.1. *Under assumptions (M_3), (h_1) and (h_2), the functional J is coercive bounded from below in $D^{1,p}(\mathbb{R}^N)$ and satisfies Palais–Smale condition.*

Proof. From assumption (M_3), it follows that there exists a positive constant k such that $k > S_N^{-\frac{p^*}{p}}$ and $M(t) \geq kt^{\frac{p^*}{p}-1}$ for every $t \geq 0$. Then, $\hat{M}(t) \geq \frac{p}{p^*} kt^{\frac{p^*}{p}}$ for every $t \geq 0$. Since $h \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$, we have

$$\begin{aligned} J(u) &\geq \frac{1}{p^*} k\|u\|^{p^*} - \frac{1}{p^*} S_N^{-\frac{p^*}{p}} \|u\|^{p^*} - \frac{1}{q} \|h\|_{\frac{p^*}{p^*-q}} \|u\|_{p^*}^q \\ &= \frac{1}{p^*} \left(k - S_N^{-\frac{p^*}{p}} \right) \|u\|^{p^*} - \frac{1}{q} S_N^{-\frac{q}{p}} \|h\|_{\frac{p^*}{p^*-q}} \|u\|^q. \end{aligned}$$

Due to $q < p^*$, we obtain that J is coercive and bounded from below in $D^{1,p}(\mathbb{R}^N)$.

Let $\{u_n\}$ be a Palais–Smale sequence for J, that is,

$$\{J(u_n)\} \text{ is bounded}, \; J'(u_n) \to 0, \text{ as } n \to \infty.$$

Obviously, since J is coercive, $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$. Then, there exists $u \in D^{1,p}(\mathbb{R}^N)$ such that up to a subsequence

$$\begin{aligned} & u_n \rightharpoonup u \text{ in } D^{1,p}(\mathbb{R}^N), \\ & u_n \to u \text{ in } L^r_{loc}(\mathbb{R}^N), \; r \in [1, p^*), \\ & u_n \to u \text{ a.e. in } \mathbb{R}^N, \\ & |\nabla u_n|^p \rightharpoonup \eta, |u_n|^{p^*} \rightharpoonup \nu, \text{ in the sense of measures,} \end{aligned}$$

where η, ν are nonnegative and bounded measures on $\mathbb{R}^N$. By the second Concentration Compactness lemma of Lions [18] and the Concentration compactness principle at infinity of Chabrowski [19], there exist an at most countable index set Λ, a set of points $\{x_j\}_{j\in\Lambda} \subset \mathbb{R}^N$ and two families of positive numbers $\{\eta_j\}_{j\in\Lambda}$, $\{\nu_j\}_{j\in\Lambda}$ such that

$$\eta \geq |\nabla u|^p \mathrm{d}x + \sum_{j\in\Lambda} \eta_j \delta_{x_j}, \; \nu = |u|^{p^*} \mathrm{d}x + \sum_{j\in\Lambda} \nu_j \delta_{x_j},$$

and

$$S_N \nu_j^{\frac{p}{p^*}} \le \eta_j \quad \text{for every} \quad j \in \Lambda, \text{ in particular, } \sum_{j\in\Lambda} \nu_j^{\frac{p}{p^*}} < \infty,$$

where δ_{x_j} is the Dirac mass concentrated at x_j;

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} d\eta + \eta_\infty, \ \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \mathrm{d}x = \int_{\mathbb{R}^N} d\nu + \nu_\infty,$$

where

$$\eta_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{B_R^c(0)} |\nabla u_n|^p \mathrm{d}x, \ \nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{B_R^c(0)} |u_n|^{p^*} \mathrm{d}x,$$

satisfying $S_N \nu_\infty^{\frac{p}{p^*}} \le \eta_\infty$.

Next, we will prove that the index set Λ is empty. Arguing by contradiction, we may assume that there exists a j_0 such that $\nu_{j_0} \neq 0$. Consider now, for $\epsilon > 0$ a nonnegative cut-off function ϕ_ϵ such that

$$\phi_\epsilon = 1 \text{ on } B(x_{j_0}, \epsilon), \ \phi_\epsilon = 0 \text{ on } \mathbb{R}^N \setminus B(x_{j_0}, 2\epsilon), \ |\nabla \phi_\epsilon| \le \frac{2}{\epsilon}.$$

It is easy to see that the sequence $\{u_n \phi_\epsilon\}_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$. Then,

$$\lim_{n\to\infty} J'(u_n)(u_n\phi_\epsilon) = 0.$$

That is to say,

$$\begin{aligned} o_n(1) &= M(\|u_n\|^p) \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n \phi_\epsilon) \mathrm{d}x - \int_{\mathbb{R}^N} |u_n|^{p^*} \phi_\epsilon \mathrm{d}x - \int_{\mathbb{R}^N} h(x) |u_n|^q \phi_\epsilon \mathrm{d}x \\ &= M(\|u_n\|^p) \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \phi_\epsilon \mathrm{d}x + \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_\epsilon \mathrm{d}x \right) - \int_{\mathbb{R}^N} |u_n|^{p^*} \phi_\epsilon \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} h(x) |u_n|^q \phi_\epsilon \mathrm{d}x. \end{aligned} \tag{3.1}$$

Since $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$, by Hölder inequality, one has

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_\epsilon \mathrm{d}x \right| &= \left| \int_{B(x_{j_0}, 2\epsilon)} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_\epsilon \mathrm{d}x \right| \\ &\le \left(\int_{B(x_{j_0}, 2\epsilon)} |\nabla u_n|^p \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{B(x_{j_0}, 2\epsilon)} |u_n \nabla \phi_\epsilon|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\le C \left(\int_{B(x_{j_0}, 2\epsilon)} |u_n \nabla \phi_\epsilon|^p \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \int_{B(x_{j_0}, 2\epsilon)} |u_n \nabla \phi_\epsilon|^p \mathrm{d}x = \int_{B(x_{j_0}, 2\epsilon)} |u \nabla \phi_\epsilon|^p \mathrm{d}x,$$

$$\left(\int_{B(x_{j_0},2\epsilon)} |u\nabla\phi_\epsilon|^p \mathrm{d}x\right)^{\frac{1}{p}} \le \left(\int_{B(x_{j_0},2\epsilon)} |u|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} \left(\int_{B(x_{j_0},2\epsilon)} |\nabla\phi_\epsilon|^N \mathrm{d}x\right)^{\frac{1}{N}}$$
$$\le C\left(\int_{B(x_{j_0},2\epsilon)} |u|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} \to 0, \text{ as } \epsilon\to 0,$$

and the sequence $\{M(\|u_n\|^p)\}$ is bounded in $\mathbb{R}$, we can get that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} M(\|u_n\|^p)\left|\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon \mathrm{d}x\right| = 0. \tag{3.2}$$

Moreover, as $0\le\phi_\epsilon\le 1$,

$$\liminf_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x \ge k\lim_{n\to\infty}\left(\int_{B(x_{j_0},2\epsilon)} |\nabla u_n|^p\phi_\epsilon\mathrm{d}x\right)^{\frac{p^*}{p}}$$
$$\ge k\left(\int_{B(x_{j_0},2\epsilon)} |\nabla u|^p\phi_\epsilon\mathrm{d}x + \eta_{j_0}\right)^{\frac{p^*}{p}}.$$

Together with $\int\limits_{B(x_{j_0},2\epsilon)} |\nabla u|^p\phi_\epsilon\mathrm{d}x \to 0$ as $\epsilon\to 0$, thus

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon\mathrm{d}x \ge k\eta_{j_0}^{\frac{p^*}{p}}. \tag{3.3}$$

In addition,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*}\phi_\epsilon\mathrm{d}x = \lim_{\epsilon\to 0}\left(\int_{B(x_{j_0},2\epsilon)} |u|^{p^*}\phi_\epsilon\mathrm{d}x + \langle\sum_{j\in J}\nu_j\delta_{x_j},\phi_\epsilon\rangle\right) = \nu_{j_0}. \tag{3.4}$$

By assumptions (h_1) and (h_2),

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon\mathrm{d}x = \lim_{n\to\infty}\int_{B(x_{j_0},2\epsilon)} h(x)|u_n|^q\phi_\epsilon\mathrm{d}x$$
$$= \int_{B(x_{j_0},2\epsilon)} h(x)|u|^q\phi_\epsilon\mathrm{d}x,$$

and

$$\int_{B(x_{j_0},2\epsilon)} h(x)|u|^q\phi_\epsilon \mathrm{d}x \leq \left(\int_{B(x_{j_0},2\epsilon)} |h(x)|^{\frac{p^*}{p^*-q}}\mathrm{d}x\right)^{\frac{p^*-q}{p^*}} \left(\int_{B(x_{j_0},2\epsilon)} |u|^{p^*}\mathrm{d}x\right)^{\frac{q}{p^*}}$$
$$\leq C\left(\int_{B(x_{j_0},2\epsilon)} |u|^{p^*}\mathrm{d}x\right)^{\frac{q}{p^*}}.$$

Thus,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon \mathrm{d}x = 0. \tag{3.5}$$

From (3.1),

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{p^*}\phi_\epsilon \mathrm{d}x &= M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x + M(\|u_n\|^p)\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon \mathrm{d}x + o_n(1) \\ &\geq M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x - \left|M(\|u_n\|^p)\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon \mathrm{d}x\right| \\ &\quad - \int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon \mathrm{d}x + o_n(1).\end{aligned}$$

Then,

$$\begin{aligned}\liminf_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*}\phi_\epsilon \mathrm{d}x &\geq \liminf_{n\to\infty}[M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x - \left|M(\|u_n\|^p)\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon \mathrm{d}x\right| \\ &\quad - \int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon \mathrm{d}x + o_n(1)] \\ &\geq \liminf_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x \\ &\quad + \liminf_{n\to\infty}\left(-\left|M(\|u_n\|^p)\int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon \mathrm{d}x\right|\right) \\ &\quad + \liminf_{n\to\infty}\left(-\int_{\mathbb{R}^N} h(x)|u_n|^q\phi_\epsilon \mathrm{d}x\right) \\ &= \liminf_{n\to\infty} M(\|u_n\|^p)\int_{\mathbb{R}^N} |\nabla u_n|^p\phi_\epsilon \mathrm{d}x\end{aligned}$$

$$- \limsup_{n\to\infty} \left| M(\|u_n\|^p) \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_\epsilon \mathrm{d}x \right|$$
$$- \limsup_{n\to\infty} \int_{\mathbb{R}^N} h(x)|u_n|^q \phi_\epsilon \mathrm{d}x.$$

Passing to the lim inf as $\epsilon \to 0$ in both sides of the above inequality, it follows from (3.2)-(3.5) that

$$\nu_{j_0} \geq k\eta_{j_0}^{\frac{p^*}{p}}.$$

From $S_N \nu_j^{\frac{p}{p^*}} \leq \eta_j$ for every $j \in \Lambda$, we obtain

$$kS_N^{\frac{p^*}{p}} \nu_{j_0} \leq k\eta_{j_0}^{\frac{p^*}{p}} \leq \nu_{j_0}.$$

This is a contradiction with the fact that $k > S_N^{-\frac{p^*}{p}}$. Such a conclusion implies that Λ is empty.

Then, in order to get that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x,$$

it suffices to show that $\nu_\infty = 0$. Indeed, let $\psi_R \in C^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function such that

$$\psi_R(x) = 0,\ |x| < R,\ \psi_R(x) = 1,\ |x| > 2R,\ \text{and } |\nabla \psi_R| \leq \frac{2}{R}.$$

It is also easy to see that $\{u_n\psi_R\}_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$. Then,

$$\begin{aligned} o_n(1) =& M(\|u_n\|^p) \left(\int_{\mathbb{R}^N} |\nabla u_n|^p \psi_R \mathrm{d}x + \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_R \mathrm{d}x \right) \\ & - \int_{\mathbb{R}^N} |u_n|^{p^*} \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} h(x)|u_n|^q \psi_R \mathrm{d}x. \end{aligned} \tag{3.6}$$

By Hölder inequality, one has

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_R \mathrm{d}x \right| &\leq \int_{\{R\leq|x|\leq 2R\}} |u_n \nabla \psi_R| |\nabla u_n|^{p-1} \mathrm{d}x \\ &\leq \left(\int_{\{R\leq|x|\leq 2R\}} |\nabla u_n|^p \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\{R\leq|x|\leq 2R\}} |u_n \nabla \psi_R|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{\{R\leq|x|\leq 2R\}} |u_n \nabla \psi_R|^p \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Since

$$\lim_{n\to\infty} \int_{\{R\leq|x|\leq 2R\}} |u_n \nabla \psi_R|^p \mathrm{d}x = \int_{\{R\leq|x|\leq 2R\}} |u \nabla \psi_R|^p \mathrm{d}x,$$

$$\left(\int\limits_{\{R\le|x|\le 2R\}}|u\nabla\psi_R|^p\mathrm{d}x\right)^{\frac{1}{p}} \le \left(\int\limits_{\{R\le|x|\le 2R\}}|u|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}}\left(\int\limits_{\{R\le|x|\le 2R\}}|\nabla\psi_R|^N\mathrm{d}x\right)^{\frac{1}{N}}$$
$$\le C\left(\int\limits_{\{R\le|x|\le 2R\}}|u|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} \to 0,$$

as $R\to\infty$, and the sequence $\{M(\|u_n\|^p)\}_n$ is bounded, we can get that

$$\lim_{R\to\infty}\limsup_{n\to\infty} M(\|u_n\|^p)\left|\int\limits_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R\mathrm{d}x\right| = 0. \tag{3.7}$$

Moreover, by assumption (M_3) again,

$$\left(M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x\right)^{\frac{p}{p^*}} \ge k^{\frac{p}{p^*}}\int\limits_{B_{2R}^C(0)}|\nabla u_n|^p\mathrm{d}x.$$

Then,

$$\limsup_{R\to\infty}\left(\limsup_{n\to\infty} M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x\right)^{\frac{p}{p^*}} \ge \limsup_{R\to\infty}\limsup_{n\to\infty}\left(M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x\right)^{\frac{p}{p^*}}$$
$$\ge k^{\frac{p}{p^*}}\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{B_{2R}^C(0)}|\nabla u_n|^p\mathrm{d}x$$
$$= k^{\frac{p}{p^*}}\eta_\infty.$$

Thus,

$$\limsup_{R\to\infty}\limsup_{n\to\infty} M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x \ge k\eta_\infty^{\frac{p^*}{p}}. \tag{3.8}$$

In addition,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}|u_n|^{p^*}\psi_R\mathrm{d}x = \lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{B_R^C(0)}|u_n|^{p^*}\mathrm{d}x = \nu_\infty. \tag{3.9}$$

Since assumptions (h_1) and (h_2) imply that

$$\int\limits_{\mathbb{R}^N} h(x)|u_n|^q\psi_R\mathrm{d}x \le \int\limits_{B_R^C(0)} h(x)|u_n|^q\mathrm{d}x$$

and

$$\lim_{n\to\infty}\int\limits_{B_R^C(0)} h(x)|u_n|^q\mathrm{d}x = \int\limits_{B_R^C(0)} h(x)|u|^q\mathrm{d}x,$$

then

$$\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} h(x)|u_n|^q\psi_R\mathrm{d}x \le \int\limits_{B_R^C(0)} h(x)|u|^q\mathrm{d}x.$$

Together with

$$\lim_{R\to\infty}\int\limits_{B_R^C(0)} h(x)|u|^q\mathrm{d}x=0,$$

we can get

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} h(x)|u_n|^q\psi_R\mathrm{d}x=0. \tag{3.10}$$

From (3.6),

$$\begin{aligned}\int\limits_{\mathbb{R}^N}|u_n|^{p^*}\psi_R\mathrm{d}x =& M(\|u_n\|^p)\left(\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x+\int\limits_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R\mathrm{d}x\right)\\ &-\int\limits_{\mathbb{R}^N}h(x)|u_n|^q\psi_R\mathrm{d}x+o_n(1)\\ \geq& M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x-\left|M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R\mathrm{d}x\right|\\ &-\int\limits_{\mathbb{R}^N}h(x)|u_n|^q\psi_R\mathrm{d}x+o_n(1);\end{aligned}$$

then,

$$\begin{aligned}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}|u_n|^{p^*}\psi_R\mathrm{d}x \geq& \limsup_{n\to\infty}[M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x\\ &-\left|M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R\mathrm{d}x\right|-\int\limits_{\mathbb{R}^N}h(x)|u_n|^q\psi_R\mathrm{d}x+o_n(1)]\\ \geq& \limsup_{n\to\infty}M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\mathrm{d}x\\ &-\limsup_{n\to\infty}\left|M(\|u_n\|^p)\int\limits_{\mathbb{R}^N}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_R\mathrm{d}x\right|\\ &-\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}h(x)|u_n|^q\psi_R\mathrm{d}x.\end{aligned}$$

Taking the $\limsup$ as $R\to\infty$ in both sides of the above inequality, it follows from (3.7)-(3.10) that

$$\nu_\infty\geq k\eta_\infty^{\frac{p^*}{p}}.$$

From $S_N\nu_\infty^{\frac{p}{p^*}}\leq\eta_\infty$, we obtain

$$kS_N^{\frac{p^*}{p}}\nu_\infty\leq k\eta_\infty^{\frac{p^*}{p}}\leq\nu_\infty.$$

If $\nu_\infty\neq 0$, it also leads to a contradiction with the fact that $k>S_N^{-\frac{p^*}{p}}$. Therefore, $\nu_\infty=0$.

The uniform convexity of $L^{p^*}(\mathbb{R}^N)$ implies that

$$u_n\to u \text{ in } L^{p^*}(\mathbb{R}^N).$$

Since $\{u_n\}$ is bounded in $D^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned}\lim_{n\to\infty} J'(u_n)(u_n-u) &= \lim_{n\to\infty}[M(\|u_n\|^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N}|u_n|^{p^*-2}u_n(u_n-u)\mathrm{d}x-\int_{\mathbb{R}^N}h(x)|u_n|^{q-2}u_n(u_n-u)\mathrm{d}x]\\ &=0.\end{aligned}$$

By Hölder inequality,

$$\left|\int_{\mathbb{R}^N}|u_n|^{p^*-2}u_n(u_n-u)\mathrm{d}x\right|\le\left(\int_{\mathbb{R}^N}|u_n|^{p^*}\mathrm{d}x\right)^{\frac{Np-N+p}{Np}}\left(\int_{\mathbb{R}^N}|u_n-u|^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}}.$$

We know from the definition of weak convergence and assumption (h_2) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x)|u_n|^{q-2}u_n(u_n-u)\mathrm{d}x=0.$$

So we deduce that

$$\lim_{n\to\infty}M(\|u_n\|^p)\left|\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\mathrm{d}x\right|=0.$$

We claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\mathrm{d}x=0. \tag{3.11}$$

In fact, if $\limsup_{n\to\infty} M(\|u_n\|^p)>0$, then, (3.11) follows at once(in the sense of subsequence). If $\lim_{n\to\infty}M(\|u_n\|^p)$ $=0$, then, by assumption (M_3) and Hölder inequality, we obtain that $u_n\to0$ in $D^{1,p}(\mathbb{R}^N)$ and (3.11) holds true also in this case.

It follows from the definition of weak convergence that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\mathrm{d}x=0.$$

Then,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\nabla(u_n-u)\mathrm{d}x=0.$$

By the boundedness of $\{u_n\}$ in $D^{1,p}(\mathbb{R}^N)$ and the well-known Simon inequalities

$$|\xi-\eta|^p\le\begin{cases}c_p(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta), & p\ge2;\\ C_p[(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)]^{\frac{p}{2}}\times(|\xi|^p+|\eta|^p)^{\frac{2-p}{2}}, & 1<p<2,\end{cases}$$

for all $\xi,\eta\in\mathbb{R}^N$, where c_p and C_p are positive constants depending only on p, we can obtain

$$\|u_n-u\|^p=\int_{\mathbb{R}^N}|\nabla u_n-\nabla u|^p\mathrm{d}x\to0,\ n\to\infty.$$

Therefore, $u_n\to u$ in $D^{1,p}(\mathbb{R}^N)$. □

Lemma 3.2. *For any $m \in \mathbb{N}$, there exists a m-dimensional subspace X^m of $D^{1,p}(\mathbb{R}^N)$ and $\rho_m > 0$ such that* $\sup\limits_{u\in X^m\cap S_{\rho_m}} J(u) < 0$, *where* $S_{\rho_m} := \{u \in D^{1,p}(\mathbb{R}^N) : \|u\| = \rho_m\}$.

Proof. For any $m \in \mathbb{N}$, we can find m functions $e_1, e_2, \ldots, e_m \in C_0^\infty(\mathbb{R}^N)$ of linearly independent. The m-dimensional subspace X^m is defined by $X^m = \text{span}\{e_1, e_2, \ldots, e_m\}$ equipped with the norm of $D^{1,p}(\mathbb{R}^N)$. While $\|u\|_{q,h} := \left(\int\limits_{\mathbb{R}^N} h(x)|u|^q \mathrm{d}x\right)^{\frac{1}{q}}$ is also a norm of X^m. Because all norms are equivalent in X^m, there exists $C_m > 0$ such that

$$\|u\| \leq C_m\|u\|_{q,h},\ u \in X^m.$$

We know from (M_5) that for some $C_0 \in (0, \frac{p}{qC_m^q})$, there exists $\delta > 0$ such that

$$\hat{M}(t) \leq C_0 t^{\frac{q}{p}},\ |t| \leq \delta.$$

Let $\rho_m \in (0, \delta^{\frac{1}{p}})$ be small sufficiently, we have that

$$\begin{aligned} J(u) &= \frac{1}{p}\hat{M}(\|u\|^p) - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - \frac{1}{q}\int\limits_{\mathbb{R}^N} h(x)|u|^q \mathrm{d}x \\ &\leq \frac{1}{p}C_0\|u\|^q - \frac{1}{qC_m^q}\|u\|^q \\ &= \left(\frac{1}{p}C_0 - \frac{1}{qC_m^q}\right)\|u\|^q \\ &< 0, \end{aligned}$$

when $u \in X^m \cap S_{\rho_m}$. The proof of Lemma 3.2 is completed. □

Proof of Theorem 1.1. Assumption (M_5) implies that $\hat{M}(0) = 0$. By the definition of J, we can get that $J(0) = 0$ and $J \in C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$ is even. According to Lemma 3.1 and Lemma 3.2, J satisfies all the conditions of Lemma 2.1. Therefore, there are infinitely many solutions to problem (4).

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