Z. Angew. Math. Phys. (2022) 73:37

0044-2275/22/010001-30
published online January 9, 2022
https://doi.org/10.1007/s00033-021-01670-z

Zeitschrift für angewandte Mathematik und Physik ZAMP

Existence and uniqueness of solution to one-dimensional compressible biaxial nematic liquid crystal flows

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Abstract. The recent paper considers a hydrodynamic flow of compressible biaxial nematic liquid crystal in dimension one. For initial density without vacuum states, we obtain both existence and uniqueness of global classical solutions. While for initial density with possible vacuum states, both the existence and uniqueness of global strong solutions are given.

Mathematics Subject Classification. 35K55, 76N10.

Keywords. Biaxial nematic liquid crystal, Compressible flow, Global existence, Uniqueness.

1. Introduction

Let $I = [0, 1], Q_T = I \times (0, T)$ for any $T > 0$ and $\mathcal{N} = \{(n, m) \in S^2 \times S^2 |\ n \cdot m = 0\}$, here S^2 is the unit sphere in R^3. In recent paper, we will consider the following compressible hydrodynamic flow of biaxial nematic liquid crystals

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2)_x + (P(\rho))_x = \mu v_{xx} - \lambda[(|n_x|^2)_x + (|m_x|^2)_x + (2|n \cdot m_x|^2)_x], \\ n_t + v n_x - 2(n_x \cdot m)m_x = \theta(n_{xx} + |n_x|^2 n) + (m_x \cdot n_x)m + 2|n \cdot m_x|^2 n, \\ m_t + v m_x - 2(m_x \cdot n)n_x = \theta(m_{xx} + |m_x|^2 m) + (n_x \cdot m_x)n + 2|n \cdot m_x|^2 m, \end{cases} \tag{1.1}$$

with the following initial and boundary condition:

$$\begin{cases} (\rho, v, n, m)|_{t=0} = (\rho_0, v_0, n_0, m_0),\ (n_0, m_0) \in \mathcal{N}, \\ v|_{\partial I} = 0,\ n_x|_{\partial I} = m_x|_{\partial I} = 0, \end{cases} \tag{1.2}$$

where $\rho : Q_T \to R$ denotes the density, $v : Q_T \to R$ represents the velocity, $n : Q_T \to S^2$ and $m : Q_T \to S^2$ are orthogonal unit vector fields of the biaxial nematic liquid crystal molecules, here $P(\rho) = r\rho^\gamma : Q_T \to R$ denotes the pressure for some constants $\gamma > 1$ and $r > 0$. For convenient, let $\lambda = \mu = \theta = r = 1$.

The system (1.1) is a coupling between the compressible Navier–Stokes equations and a heat flow, which is a macroscopic continuum description of the development for the biaxial nematic liquid crystals. Based on the Landau–De Gennes Q-tensor theory, Govers and Vertogen proposed the elastic continuum theory of biaxial nematics in [9,10]. The Govers–Vertogen model uses a pair of orthogonal unit vector fields $(n, m) \in \mathcal{N}$, to describe the orientation field of a nematic liquid crystal, and considers the elastic energy density $\mathcal{W}(n, m, \nabla n, \nabla m)$ to be of the Oseen–Frank type. In this paper, we focus on the special elastic energy density has a simple form

$$\mathcal{W}(n, \nabla n, \nabla m) = \frac{1}{2}|\nabla n|^2 + \frac{1}{2}|\nabla m|^2 + |n \cdot \nabla m|^2. \tag{1.3}$$

Then, if we ignore ρ and v, (1.1) is a system with special elastic energy density $\mathcal{W}(n, \nabla n, \nabla m)$ in dimension one. If we ignore m, (1.1) becomes the compressible uniaxial nematic liquid crystal equations [2].

Now we first recall some previous works on the existence and uniqueness of solutions to the related systems. Ericksen [5] and Leslie [14] in the 1960s derived firstly the hydrodynamic theory of incompressible uniaxial nematic liquid crystals. This theory simplified to the incompressible uniaxial nematic liquid crystal equations, which has been successfully studied (see [6,7,13,17,18,20,26] and so on for the constant density case, and [8,15,16,27] and so on for nonconstant density case for example). For the compressible uniaxial nematic liquid crystal equations, Ding et al. [2,4] obtained the global existences of classical, strong and weak solutions in dimension one, while authors in [24] obtained the global existence and regularity of solutions in suitable Hilbert spaces in Lagrangian coordinates. In higher dimensions, authors in [23] obtained the global existence of weak solution with large initial energy and without any smallness condition on the initial density and velocity in a three-dimensional bounded domain. Lin et al. [19] established the existence of finite energy weak solutions with the large initial data in dimensions three, provided the initial orientational director field lies in the upper hemisphere. Wen et al. in [11,12] obtained the local existence of strong solution and blow-up criterion compressible nematic liquid crystal flows in dimension three. Gao et al. [8] obtained the global well-posedness of classical solution under the condition of small perturbation of constant equilibrium state in the suitable Hilbert space. Authors in [21] derived a global existence of classical solution with smooth initial data which is of small energy but possibly large oscillations in R^3. For more about the progress of mathematical researches on liquid crystals, the interested readers can consult with the review articles [1,22,28].

For the hydrodynamic flows of incompressible biaxial nematics with a constant density, Lin et al. in [18] have derived the existence of unique global weak solution in two dimensions which is smooth off at most finitely many singular times. Authors in [3] have derived the weak compactness property of solutions in two dimensions as the parameter tends to zero by Pohozaev argument.

Inspired by the work on the hydrodynamics of compressible uniaxial nematics with a nonconstant density [2], we consider the global classical and strong solutions to (1.1)–(1.2). For initial density ρ_0 without vacuum states, we obtain our first result on the existence and uniqueness of global classical solutions.

Theorem 1.1. *For $\alpha \in (0,1)$, let $\rho_0 \in C^{1+\alpha}(I)$ with $C_0^{-1} \leq \rho_0 \leq C_0$ for some positive constant C_0, $v_0 \in C^{2+\alpha}(I)$ and $(n_0, m_0) \in \mathcal{N}$ with $n_0, m_0 \in C^{2+\alpha}(I)$. Then, (1.1)–(1.2) has a unique global classical solution $(\rho, v, n, m) : I \times [0, +\infty) \to [0, +\infty) \times R \times S^2 \times S^2$, such that for any $T > 0$, there hold*

$$(\rho_x, \rho_t) \in C^{\alpha, \frac{\alpha}{2}}(Q_T), C_1^{-1} \leq \rho \leq C_1, (v, n, m) \in C^{2+\alpha, \frac{2+\alpha}{2}}(Q_T) \text{ and } (n, m) \in \mathcal{N}$$

for a positive constant C_1 depending on C_0 and T.

For initial density ρ_0 with possible vacuum states, we obtain our second result on the global existence and uniqueness of strong solutions.

Theorem 1.2. *Let $0 \leq \rho_0 \in H^1(I), v_0 \in H_0^1(I)$ and $(n_0, m_0) \in \mathcal{N}$ with $n_0, m_0 \in H^2(I)$. (1.1)–(1.2) has a unique global strong solution (ρ, v, n, m) such that for any $T > 0$, there hold $(n, m) \in \mathcal{N}$ and*

$$\rho \in L^\infty(0, T; H^1(I)), \rho_t \in L^\infty(0, T; L^2(I)),$$
$$v \in L^\infty(0, T; H_0^1(I)) \cap L^2(0, T; H^2(I)), (\rho v)_t \in L^2(0, T; L^2(I)), \sqrt{t} v_t \in L^2(0, T; H_0^1(I)),$$
$$n, m \in L^\infty(0, T; H^2(I)), n_t, m_t \in L^\infty(0, T; L^2(I)) \cap L^2(0, T; H^1(I)).$$

Our two results extend the works in [2] to biaxial nematic liquid crystals. However, because of the additional vector m and term $|n \cdot \nabla m|^2$ in elastic energy density, there are many difficulties to overcome. For example, to use the Schauder theory in constructing local existence in Sect. 2, we use some modifications in deriving the map H. To prove the global existence of solutions, we have to overcome some difficulties coming from some terms similar to gradient square like terms, for example, $(n_x \cdot m)m_x$ and

$|n \cdot m_x|^2$. In Sect. 4, in order to use the result of Theorem 1.1, we will construct a suitable approximate initial n_0^k and m_0^k such that $(n_0^k, m_0^k) \in \mathcal{N}$. Meanwhile, one observes that the system (1.1) is strongly coupled and the equations therein are strongly nonlinear. All of these suggest the main difficulties in the global estimates.

Throughout this paper, we will use the following notices for simplicity.

$$||\cdot||_{k+\alpha} = ||\cdot||_{C^{k+\alpha,\frac{k+\alpha}{2}}(Q_T)}, \alpha \in [0,1);\ ||\cdot||_p = ||\cdot||_{L^p(I)}, p \in [0,+\infty].$$

The paper is organized as follows. In Sect. 2, the existence of local classical solutions of (1.1)–(1.2) is proved. In Sect. 3, through deriving some a priori global estimates for classical solutions, we prove the global existence and uniqueness of classical solutions for initial density without vacuum states. In Sect. 4, we prove the global existence and uniqueness of strong solutions for initial density with possible vacuum states.

2. Local classical solution: existence and uniqueness

In this section, we will prove the existence and uniqueness of local classical solutions. We will assume that

$$\int_I \rho_0(\xi)\mathrm{d}\xi = 1. \tag{2.1}$$

We will rewrite (1.1)–(1.2) in Lagrangian coordinate firstly. For any $T > 0$, introduce the Lagrangian coordinate (y, τ) on $I \times (0, T)$ such that

$$y(x,t) = \int_0^x \rho(\xi,t)\mathrm{d}\xi, \quad \tau(x,t) = t.$$

Then, $(x,t) \to (y,\tau)$ is a C^1-bijective map [2]. One also has

$$\frac{\partial}{\partial t} = -\rho v\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} = \rho\frac{\partial}{\partial y}.$$

By a coordinate transformation, (1.1)–(1.2) can be changed into the following system

$$\begin{cases} \rho_\tau + \rho^2 v_y = 0, \\ v_\tau + P_y = (\rho v_y)_y - (\rho^2|n_y|^2)_y - (\rho^2|m_y|^2)_y - (2\rho^2|n\cdot m_y|^2)_y, \\ n_\tau = \rho(\rho n_y)_y + \rho^2|n_y|^2 n + \rho^2 m m_y \cdot n_y + 2\rho^2|n\cdot m_y|^2 n + 2\rho^2(n_y\cdot m)m_y, \\ m_\tau = \rho(\rho m_y)_y + \rho^2|m_y|^2 m + \rho^2 n n_y \cdot m_y + 2\rho^2|m\cdot n_y|^2 m + 2\rho^2(m_y\cdot n)n_y, \end{cases} \tag{2.2}$$

and the initial boundary conditions

$$\begin{cases} (\rho, v, n, m)|_{\tau=0} = (\rho_0, v_0, n_0, m_0),\ (n_0, m_0) \in \mathcal{N}, \\ v|_{\partial I} = 0,\ n_y|_{\partial I} = m_y|_{\partial I} = 0. \end{cases} \tag{2.3}$$

Then, we have the following result.

Theorem 2.1. *For $0 < \alpha < 1$, suppose $\rho_0 \in C^{1+\alpha}(I)$ with $0 < C_0^{-1} \le \rho_0(x,t) \le C_0$ and $v_0 \in C^{2+\alpha}(I)$, $n_0, m_0 \in C^{2+\alpha}(I)$ with $(n_0, m_0) \in \mathcal{N}$. Then, (1.1)–(1.2) has a unique local classical solution (ρ, v, n, m) such that there exists $T = T(\rho_0, v_0, n_0, m_0) > 0$ such that*

$$(\rho_x, \rho_t) \in C^{\alpha,\frac{\alpha}{2}}(Q_T), C^{-1} \le \rho(x,t) \le C, (v,n,m) \in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T) \text{ and } (n,m) \in \mathcal{N}$$

for some constant $C > 0$.

Proof. For $K>0$ large and $T>0$ small determined later, define $X=X(T,K)$ by

$$X=\{(u,z,w):Q_T\to R\times R^3\times R^3|(u,z,w)\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T),(u,z,w)|_{\tau=0}=(v_0,n_0,m_0),\\ ||(u-v_0,z-n_0,w-m_0)||_X\leqslant K\},$$

where

$$||(u,z,w)||_X=||u||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}+||z||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}+||w||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}.$$

It can be checked that X is a Banach space.

For any $(u,z,w)\in X$, we will firstly solve the following equation

$$\begin{cases}\rho_\tau+\rho^2u_y=0,\\ \rho|_{\tau=0}=\rho_0,\quad \rho|_{\partial I}=\rho_0|_{\partial I}.\end{cases}\tag{2.4}$$

In fact, we have

$$\rho(y,\tau)=\frac{\rho_0}{1+\rho_0\int_0^\tau u_y(y,s)\mathrm{d}s}.\tag{2.5}$$

Moreover, since $(u,z,w)\in X$, we have $||u||_X\le K$. Then if $T\le T_1:=\frac{1}{2C_0K}$, we have

$$\rho\le\frac{\rho_0}{1-|\rho_0\int\limits_0^\tau u_y(y,s)\mathrm{d}s|}\le 2C_0,\tag{2.6}$$

and

$$\rho\ge\frac{\rho_0}{1+|\rho_0\int\limits_0^\tau u_y(y,s)\mathrm{d}s|}\ge\frac{C_0^{-1}}{2}.\tag{2.7}$$

From $u\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$ and $\rho_0\in C^{1+\alpha}(I)$, we know that $\rho,\rho_y\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$ by (2.5).

Let ρ be given by (2.5). Define a map $H:X\to\ C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$ with $H(u,z,w)=(v,n,m)$, where (v,n,m) solves

$$\begin{cases}v_\tau+P_y-\rho v_{yy}=\rho_yu_y-(\rho^2|n_y|^2)_y-(\rho^2|m_y|^2)_y-(2\rho^2|n\cdot m_y|^2)_y,\\ n_\tau-\rho^2n_{yy}=\rho\rho_yz_y+\rho^2|z_y|^2z+\rho^2ww_y\cdot z_y+2\rho^2|z\cdot w_y|^2z+2\rho^2(z_y\cdot w)w_y,\\ m_\tau-\rho^2m_{yy}=\rho\rho_yw_y+\rho^2|w_y|^2w+\rho^2zz_y\cdot w_y+2\rho^2|z\cdot w_y|^2w+2\rho^2(w_y\cdot z)z_y.\end{cases}\tag{2.8}$$

with the following initial boundary conditions

$$\begin{cases}(v,n,m)|_{\tau=0}=(v_0,n_0,m_0),\ (n_0,m_0)\in\mathcal{N},\\ (v,n_y,m_y)|_{\partial I}=(0,0,0).\end{cases}\tag{2.9}$$

Now the proof of Theorem 1.2 is divided into several steps.

Step 1: To prove that H is well defined.

In fact, since $\rho,\rho_y\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$ and $z,w\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$, we know that (2.8)–(2.9) has a unique solution (v,n,m) in $C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)$ by the Schauder theory and the boundedness of ρ from (2.6) and (2.7). Hence, H is well defined.

Step 2: To prove that the image of H is in X, if K is large enough and T small enough.

Let $C_1 = ||\rho_0||_{C^{1+\alpha}(I)} + ||v_0||_{C^{2+\alpha}(I)} + ||n_0||_{C^{2+\alpha}(I)} + ||m_0||_{C^{2+\alpha}(I)}$. Differentiating (2.5) w.r.t y, we have

$$\rho_y(y,\tau) = \frac{\rho_{0y}}{1+\rho_0\int\limits_0^\tau u_y(y,s)\mathrm{d}s} - \frac{\rho_0\rho_{0y}\int\limits_0^\tau u_y(y,s)\mathrm{d}s + \rho_0^2\int\limits_0^\tau u_{yy}(y,s)\mathrm{d}s}{(1+\rho_0\int\limits_0^\tau u_y(y,s)\mathrm{d}s)^2}. \tag{2.10}$$

Then, (2.5) and (2.10) imply that if $T \le T_2 := \min\left\{T_1, (\frac{1}{K})^{\frac{2}{2-\alpha}}\right\}$, then

$$\max\left\{||\rho||_{C^{\alpha,\frac{\alpha}{2}}}, ||\rho_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\right\} \le C(C_1). \tag{2.11}$$

Applying the Schauder theory to $(2.8)_2$, one gets that for any $T \le T_2$,

$$\begin{aligned} ||n-n_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} &\le C\Big[1 + ||\rho\rho_y z_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + ||\rho^2|z_y|^2 z||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ &+ ||\rho^2 w(w_y\cdot z_y)||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + ||\rho^2|z\cdot w_y|^2 z||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + ||\rho^2(z_y\cdot w)w_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\Big]. \end{aligned} \tag{2.12}$$

Since $w - m_0 = z - n_0 = 0$ at $t = 0$, we get that

$$||z-n_0||_{C(Q_T)} \le KT,\ ||z_y - n_{0y}||_{C(Q_T)} \le KT,$$
$$||w-m_0||_{C(Q_T)} \le KT,\ ||w_y - m_{0y}||_{C(Q_T)} \le KT.$$

By the interpolation inequality, we have that for $0<\delta<1$,

$$||z-n_0||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C\left[\frac{||z-n_0||_0}{\delta} + \delta||z-n_0||_{2+\alpha}\right] \le CK\left(\delta + \frac{T}{\delta}\right),$$
$$||z_y-n_{0y}||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C\left[\frac{||z_y-n_{0y}||_0}{\delta} + \delta||z-n_0||_{2+\alpha}\right] \le CK\left(\delta + \frac{T}{\delta}\right).$$

Similarly, we also have

$$||w-m_0||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le CK\left(\delta+\frac{T}{\delta}\right),\ \ ||w_y - m_{0y}||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le CK\left(\delta+\frac{T}{\delta}\right).$$

Then, we have

$$\begin{aligned} ||\rho\rho_y z_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} &\le 3||\rho||_\alpha||\rho_y||_\alpha||z_y||_\alpha \le C(C_1)[||z_y-n_{0y}||_\alpha + ||n_{0y}||_\alpha] \\ &\le C(C_1)||z_y-n_{0y}||_\alpha + C(C_1) \\ &\le C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]. \end{aligned} \tag{2.13}$$

One also gets that

$$\begin{aligned} &||\rho^2|z_y|^2 z||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\ &\le ||\rho^2|z_y|^2 z - \rho_0^2|z_y|^2 z||_\alpha + ||\rho_0^2|z_y|^2 z - \rho_0^2|n_{0y}|^2 z||_\alpha + ||\rho_0^2|n_{0y}|^2 z - \rho_0^2|n_{0y}|^2 n_0||_\alpha + ||\rho_0^2|n_{0y}|^2 n_0||_\alpha \\ &\le 5||\rho-\rho_0||_\alpha||\rho+\rho_0||_\alpha||z_y||_\alpha^2||z||_\alpha + 5||\rho_0||_\alpha^2||z_y-n_{0y}||_\alpha||z_y+n_{0y}||_\alpha||z||_\alpha \\ &\quad + 5||\rho_0||_\alpha^2||n_{0y}||_\alpha^2||z-n_0||_\alpha + C(C_1) \\ &\le C(C_1)(||z_y-n_{0y}||_\alpha + ||n_{0y}||_\alpha)^2(||z-n_0||_\alpha + ||n_0||_\alpha) + C(C_1)(||z_y-n_{0y}||_\alpha \\ &\quad + ||n_{0y}||_\alpha)(\|z-n_0\|_\alpha + \|n_0\|_\alpha) + C(C_1)||z-n_0||_\alpha + C(C_1) \\ &\le C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^3. \end{aligned} \tag{2.14}$$

Similarly, we have

$$\begin{aligned}
&||\rho^2 w w_y z_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\\
&\quad\le ||\rho^2 w w_y z_y-\rho_0^2 w w_y z_y||_\alpha+||\rho_0^2 w w_y z_y-\rho_0^2 m_0 w_y z_y||_\alpha+||\rho_0^2 m_0 w_y z_y-\rho_0^2 m_0 m_{0y} z_y||_\alpha\\
&\qquad+||\rho_0^2 m_0 m_{0y} z_y-\rho_0^2 m_0 m_{0y} n_{0y}||_\alpha+||\rho_0^2 m_0 m_{0y} n_{0y}||_\alpha\\
&\quad\le 5||\rho-\rho_0||_\alpha||\rho+\rho_0||_\alpha||w||_\alpha||w_y||_\alpha||z_y||_\alpha+5||\rho_0||_\alpha^2||w-m_0||_\alpha||w_y||_\alpha||z_y||_\alpha\\
&\qquad+5||\rho_0||_\alpha^2||m_0||_\alpha||w_y-m_{0y}||_\alpha||z_y||_\alpha+5||\rho_0||_\alpha^2||m_0||_\alpha||m_{0y}||_\alpha||z_y-n_{0y}||_\alpha+C(C_1)\\
&\quad\le C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^3
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
&||\rho^2|z\cdot w_y|^2 z||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\\
&\quad\le ||\rho^2|z\cdot w_y|^2 z-\rho_0^2|z\cdot w_y|^2 z||_\alpha+||\rho_0^2|z\cdot w_y|^2 z-\rho_0^2|n_0\cdot w_y|^2 z||_\alpha\\
&\qquad+||\rho_0^2|n_0\cdot w_y|^2 z-\rho_0^2|n_0\cdot m_{0y}|^2 z||_\alpha+||\rho_0^2|n_0\cdot m_{0y}|^2 z-\rho_0^2|n_0\cdot m_{0y}|^2 n_0||_\alpha+C(C_1)\\
&\quad\le C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^5.
\end{aligned}\tag{2.16}$$

Finally, we also get

$$\begin{aligned}
&||\rho^2(z_y\cdot w)w_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\\
&\quad\le ||\rho^2(z_y\cdot w)w_y-\rho_0^2(z_y\cdot w)w_y||_\alpha+||\rho_0^2(z_y\cdot w)w_y-\rho_0^2(n_{0y}\cdot w)w_y||_\alpha\\
&\qquad+||\rho_0^2(n_{0y}\cdot w)w_y-\rho_0^2(n_{0y}\cdot m_0)w_y||_\alpha+||\rho_0^2(n_{0y}\cdot m_0)w_y-\rho_0^2(n_{0y}\cdot m_0)m_{0y}||_\alpha+C(C_1)\\
&\quad\le C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^3.
\end{aligned}\tag{2.17}$$

By there estimates from (2.13) to (2.17), we have

$$||n-n_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\le 5C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^5.\tag{2.18}$$

Similarly, applying the Schauder theory to $(2.8)_3$, we also have

$$||m-m_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\le 5C(C_1)\left[K\left(\delta+\frac{T}{\delta}\right)+1\right]^5.\tag{2.19}$$

Taking $T=\delta^2$, we have

$$||n-n_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}+||m-m_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\le 10C(C_1)[2KT^{\frac{1}{2}}+1]^5.\tag{2.20}$$

Then, there are $T_3>0$ small enough and $K_3>2$ large enough, such that for $0<T\le T_3$ and $K>K_3$ there holds that

$$||n-n_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}+||m-m_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\le K^{\frac{1}{8}}.\tag{2.21}$$

Now we will estimate v. Applying the Schauder theory to $(2.8)_1$, we have for $0<T\le T_3$ and $K>K_3$ that

$$\begin{aligned}
&||v-v_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}\\
&\quad\le C[1+||\rho_y u_y||_\alpha+||(\rho^2|n_y|^2)_y||_\alpha+||(\rho^2|m_y|^2)_y||_\alpha+||(\rho^2|n\cdot m_y|^2)_y||_\alpha].
\end{aligned}\tag{2.22}$$

It is not hard to see that

$$||\rho_y u_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C(C_1)[||u_y - v_{0y}||_\alpha + ||v_{0y}||_\alpha] \le C(C_1)\left[K\left(\delta + \frac{T}{\delta}\right) + 1\right].$$

Taking $\delta = \sqrt{T}$ firstly and then $0 < T \le T_4 := \min\{T_3, K^{-1}\}$, we have

$$||\rho_y u_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C(C_1)[K^{\frac{1}{2}} + 1].$$

By (2.21), we have

$$\begin{aligned}
&||(\rho^2|n_y|^2)_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \\
&\quad \le C||\rho||_\alpha ||\rho_y||_\alpha ||n_y||_\alpha^2 + C||\rho||_\alpha^2 ||n_y||_\alpha ||n_{yy}||_\alpha \\
&\quad \le C(C_1)(||n_y - n_{0y}||_\alpha + ||n_{0y}||_{\alpha(I)})^2 + C(C_1)(||n_y - n_{0y}||_\alpha + ||n_{0y}||_\alpha)(||n_{yy} - n_{0yy}||_\alpha + ||n_{0yy}||_\alpha) \\
&\quad \le C(C_1)(K^{\frac{1}{8}} + 1)^2
\end{aligned}$$

and

$$||(\rho^2|m_y|^2)_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C(C_1)(K^{\frac{1}{8}} + 1)^2$$

and

$$||(\rho^2|n \cdot m_y|^2)_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C(C_1)(K^{\frac{1}{8}} + 1)^4.$$

Putting these four estimates together and taking $K \ge K_5$ for some K_5 large enough, we have

$$||v - v_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le 7C(C_1)[K^{\frac{1}{2}} + 1] \le \frac{1}{2}K. \tag{2.23}$$

Finally, (2.20) and (2.23) imply that there are $T > 0$ small enough and $K > 0$ large enough such that

$$||v - v_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||n - n_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||m - m_0||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le K.$$

Therefore, H is a map X to X.

Step 3: To prove that H is a contract mapping, if $T > 0$ is small enough and $K > 0$ is large enough.
Let $(u_i, z_i, w_i) \in X$ and $(v_i, n_i, m_i) = H(u_i, z_i, w_i), i = 1, 2$. Denote $\bar{u} = u_1 - u_2$, $\bar{z} = z_1 - z_2$, $\bar{w} = w_1 - w_2$, $\bar{v} = v_1 - v_2$, $\bar{n} = n_1 - n_2$, $\bar{m} = m_1 - m_2$, and $\bar{\rho} = \rho_1 - \rho_2$, where ρ_i solves the following equation

$$\rho_{i\tau} + (\rho_i u_i)_y = 0.$$

Then it is not hard to see that

$$\left(\frac{\bar{\rho}}{\rho_1\rho_2}\right)_\tau = -\bar{u}_y.$$

We get

$$\bar{\rho} = -\rho_1\rho_2 \int_0^\tau \bar{u}_y(y, s)\mathrm{d}s.$$

Because ρ_1 and ρ_2 satisfy (2.11), we get that

$$\max\left\{||\bar{\rho}||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}, ||\bar{\rho}_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\right\} \le C(C_1)T^{1-\frac{\alpha}{2}}||\bar{u}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}. \tag{2.24}$$

We also have

$$\begin{aligned}
&\bar{n}_\tau - \rho_1^2 \bar{n}_{yy} \\
&\quad = G := \bar{\rho}(\rho_1+\rho_2) z_{2yy} + \bar{\rho}\rho_{1y} z_{1y} + \rho_2 \bar{\rho}_y z_{1y} + \rho_2 \rho_{2y} \bar{z}_y + \bar{\rho}(\rho_1+\rho_2)|z_{1y}|^2 z_1 \\
&\qquad + \rho_2^2 \bar{z}_y \cdot (z_{1y}+z_{2y}) z_1 + \rho_2^2 |z_{2y}|^2 \bar{z} + \bar{\rho}(\rho_1+\rho_2) w_1 w_{1y} \cdot z_{1y} + \rho_2^2 \bar{w} w_{1y} \cdot z_{1y} + \rho_2^2 w_2 \bar{w}_y \cdot z_{1y} \\
&\qquad + \rho_2^2 w_2 w_{2y} \cdot \bar{z}_y + 2\bar{\rho}(\rho_1+\rho_2)|z_1 \cdot w_{1y}|^2 z_1 + 2\rho_2^2 \bar{z} \cdot w_{1y}(z_1 \cdot w_{1y} + z_2 \cdot w_{2y}) z_1 \\
&\qquad + 2\rho_2^2 z_2 \cdot \bar{w}_y (z_1 \cdot w_{1y} + z_2 \cdot w_{2y}) z_1 + 2\rho_2^2 |z_2 \cdot w_{2y}|^2 \bar{z} + 2\bar{\rho}(\rho_1+\rho_2)(z_{1y} \cdot w_1) w_{1y} \\
&\qquad + 2\rho_2^2 \bar{z}_y w_1 \cdot w_{1y} + 2\rho_2^2 z_{2y} \bar{w} \cdot w_{1y} + 2\rho_2^2 z_{2y} w_2 \cdot \bar{w}_y. \qquad (2.25)
\end{aligned}$$

Applying the Schauder theory to (2.25), we get

$$\begin{aligned}
||\bar{n}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} &\le C||G||_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C(C_1)K^5[||\bar{\rho}||_\alpha + ||\bar{\rho}_y||_\alpha + ||\bar{z}_y||_\alpha + ||\bar{z}||_\alpha + ||\bar{w}||_\alpha + ||\bar{w}_y||_\alpha] \\
&\le C(C_1)K^5\left(\frac{T}{\delta}+\delta\right)(||\bar{u}||_{2+\alpha} + ||\bar{z}||_{2+\alpha} + ||\bar{w}||_{2+\alpha}), \qquad (2.26)
\end{aligned}$$

where we have used (2.24) and

$$\begin{aligned}
||\bar{z}||_\alpha &\le C\left(\frac{1}{\delta}||\bar{z}||_0 + \delta||\bar{z}||_{2+\alpha}\right) \le C\left(\frac{T}{\delta}+\delta\right)||\bar{z}||_{2+\alpha}, \\
||\bar{z}_y||_\alpha &\le C\left(\frac{1}{\delta}||\bar{z}_y||_0 + \delta||\bar{z}||_{2+\alpha}\right) \le C\left(\frac{T}{\delta}+\delta\right)||\bar{z}||_{2+\alpha}, \\
||\bar{w}||_\alpha &\le C\left(\frac{1}{\delta}||\bar{w}||_0 + \delta||\bar{w}||_{2+\alpha}\right) \le C\left(\frac{T}{\delta}+\delta\right)||\bar{w}||_{2+\alpha}, \\
||\bar{w}_y||_\alpha &\le C(\frac{1}{\delta}||\bar{w}_y||_0 + \delta||\bar{w}||_{2+\alpha}) \le C(\frac{T}{\delta}+\delta)||\bar{w}||_{2+\alpha}.
\end{aligned}$$

Similarly, we have

$$||\bar{m}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le C(C_1)K^5\left(\frac{T}{\delta}+\delta\right)(||\bar{u}||_{2+\alpha} + ||\bar{z}||_{2+\alpha} + ||\bar{w}||_{2+\alpha}). \qquad (2.27)$$

Taking $\delta = \sqrt{T}$, we have

$$||\bar{n}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||\bar{m}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \le 4C(C_1)K^5 T^{\frac{1}{2}}(||\bar{u}||_{2+\alpha} + ||\bar{z}||_{2+\alpha} + ||\bar{w}||_{2+\alpha}). \qquad (2.28)$$

For $\bar{v}$, we have

$$\begin{aligned}
&||\bar{v}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \\
&\quad \le C(C_1)K^4[||\bar{\rho}||_\alpha + ||\bar{\rho}_y||_\alpha + ||\bar{n}_y||_\alpha + ||\bar{n}_{yy}||_\alpha + ||\bar{m}_y||_\alpha + ||\bar{m}_{yy}||_\alpha] \\
&\quad \le C(C_1)K^4[4C(C_1)K^5T^{\frac{1}{2}} + C(C_1)T^{\frac{1}{2}}](||\bar{u}||_{2+\alpha} + ||\bar{z}||_{2+\alpha} + ||\bar{w}||_{2+\alpha}), \qquad (2.29)
\end{aligned}$$

where we have used (2.24) and (2.28).

Therefore, there is $T > 0$ small enough and $K > 0$ large enough, such that

$$\begin{aligned}
&||\bar{v}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||\bar{n}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||\bar{m}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} \\
&\quad \le \frac{1}{2}(||\bar{u}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||\bar{z}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)} + ||\bar{w}||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_T)}), \qquad (2.30)
\end{aligned}$$

which means that H is a contract map.

Hence by the contractive fixed point theorem, we know that exists a unique $(v, n, m) \in X$, such that $H(v, n, m) = (v, n, m)$. Moreover, there is a unique ρ with $\rho_y, \rho_\tau \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$ for some small $T > 0$. Hence, (2.2)–(2.3) has a unique local classical solution, so as (1.1)–(1.2).

Step 4: To prove that $(n\cdot m)\in\mathcal{N}$.

In fact, multiplying $(1.1)_3$ by n, we have

$$\begin{aligned}&\frac{1}{2}(|n|^2-1)_t+\frac{1}{2}v(|n|^2-1)_x-\frac{1}{2}(|n|^2-1)_{xx}-2(m_x\cdot n)(n\cdot m)_x\\&\quad=(|n_x|^2+2|n\cdot m_x|^2)(|n|^2-1)+(n_x\cdot m_x)n\cdot m.\end{aligned}\tag{2.31}$$

Multiplying $(1.1)_4$ by m, we have

$$\begin{aligned}&\frac{1}{2}(|m|^2-1)_t+\frac{1}{2}v(|m|^2-1)_x-\frac{1}{2}(|m|^2-1)_{xx}-2(n_x\cdot m)(n\cdot m)_x\\&\quad=(|m_x|^2+2|n\cdot m_x|^2)(|m|^2-1)+(n_x\cdot m_x)n\cdot m.\end{aligned}\tag{2.32}$$

Multiplying $(1.1)_3$ by m and $(1.1)_4$ by n, we also have

$$\begin{aligned}&(n\cdot m)_t+v(n\cdot m)_x-(n\cdot m)_{xx}-(n_x\cdot m)(|m|^2-1)_x-(m_x\cdot n)(|n|^2-1)_x\\&\quad=(|n_x|^2+|m_x|^2+4|n\cdot m_x|^2)(n\cdot m)+(n_x\cdot m_x)[(|n|^2-1)+(|m|^2-1)].\end{aligned}\tag{2.33}$$

Denote $f_1=|n|^2-1$, $f_2=|m|^2-1$ and $f_3=n\cdot m$. In order to prove that $(n,m)\in\mathcal{N}$, we just need to prove that $f_1=f_2=f_3=0$. From (2.31) to (2.33), we have

$$f_{1t}+vf_{1x}-f_{1xx}-4(m_x\cdot n)f_{3x}=(2|n_x|^2+4|n\cdot m_x|^2)f_1+2(n_x\cdot m_x)f_3,\tag{2.34}$$

$$f_{2t}+vf_{2x}-f_{2xx}-4(n_x\cdot m)f_{3x}=(2|m_x|^2+4|n\cdot m_x|^2)f_2+2(n_x\cdot m_x)f_3,$$

$$f_{3t}+vf_{3x}-f_{3xx}-(n_x\cdot m)f_{2x}-(m_x\cdot n)f_{1x}\tag{2.35}$$

$$\quad=(|n_x|^2+|m_x|^2+4|n\cdot m_x|^2)f_3+(n_x\cdot m_x)(f_1+f_2).\tag{2.36}$$

Multiplying (2.34) with f_1 and then integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int\limits_I f_1^2\mathrm{d}x+\int\limits_I|f_{1x}|^2\mathrm{d}x\\&\quad=2\int\limits_I\left(\frac{1}{4}v_x+|n_x|^2+2|n\cdot m_x|^2\right)f_1^2\mathrm{d}x+2\int\limits_I(n_x\cdot m_x)f_1f_3\mathrm{d}x+4\int\limits_I(m_x\cdot n)f_{3x}f_1\mathrm{d}x\\&\quad\le C\int\limits_I(|v_x|+|n_x|^2+|n\cdot m_x|^2+|m\cdot n_x|^2)f_1^2\mathrm{d}x+\int\limits_I|m_x|^2f_3^2\mathrm{d}x+\frac{1}{4}\int\limits_I|f_{3x}|^2\mathrm{d}x.\end{aligned}\tag{2.37}$$

Multiplying (2.35) with f_2 and then integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int\limits_I f_2^2\mathrm{d}x+\int\limits_I|f_{2x}|^2\mathrm{d}x\\&\quad=2\int\limits_I\left(\frac{1}{4}v_x+|m_x|^2+2|n\cdot m_x|^2\right)f_2^2\mathrm{d}x+2\int\limits_I(n_x\cdot m_x)f_2f_3\mathrm{d}x+4\int\limits_I(m\cdot n_x)f_{3x}f_2\mathrm{d}x\\&\quad\le C\int\limits_I(|v_x|+|m_x|^2+|n\cdot m_x|^2+|m\cdot n_x|^2)f_2^2\mathrm{d}x+\int\limits_I|n_x|^2f_3^2\mathrm{d}x+\frac{1}{4}\int\limits_I|f_{3x}|^2\mathrm{d}x.\end{aligned}\tag{2.38}$$

Multiplying (2.36) with f_3 and then integrating by parts, we also get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I f_3^2\mathrm{d}x+\int\limits_I|f_{3x}|^2\mathrm{d}x=\int\limits_I[(n_x\cdot m)f_{2x}+(m_x\cdot n)f_{1x}]f_3\mathrm{d}x\\&\quad+\int\limits_I\left(\frac{1}{2}v_x+|n_x|^2+|m_x|^2+4|n\cdot m_x|^2\right)f_3^2\mathrm{d}x+\int\limits_I(n_x\cdot m_x)[(f_1+f_2)f_3]\mathrm{d}x\\&\le C\int\limits_I(|v_x|+|n_x|^2+|m_x|^2+|n\cdot m_x|^2+|m\cdot n_x|^2)f_3^2\mathrm{d}x+\frac{1}{2}\int\limits_I(|n_x|^2f_1^2+|m_x|^2f_2^2)\mathrm{d}x\\&\quad+\frac{1}{2}\int\limits_I(|f_{1x}|^2+|f_{2x}|^2)\mathrm{d}x.\end{aligned}\tag{2.39}$$

Putting (2.37), (2.38) and (2.39) together, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I(f_1^2+f_2^2+f_3^2)\mathrm{d}x+\int\limits_I(|f_{1x}|^2+|f_{2x}|^2+|f_{3x}|^2)\mathrm{d}x\\&\quad\le C\int\limits_I(|v_x|+|n_x|^2+|m_x|^2+|n\cdot m_x|^2+|m\cdot n_x|^2)(f_1^2+f_2^2+f_3^2)\mathrm{d}x.\end{aligned}\tag{2.40}$$

By the regularity of (v,n,m), $(n_0,m_0)\in\mathcal{N}$ and Gronwall's inequality, we get $f_1(x,t)\equiv f_2(x,t)\equiv f_3(x,t)\equiv 0$ for $(x,t)\in\overline{Q}_T$. Hence $(n,m)\in\mathcal{N}$.

Theorem 2.1 is proved. □

3. Global classical solution: Existence and Uniqueness

In Sect. 2, we have obtained the local existence and uniqueness of classical solution. In this section, we will derive some global estimates to get the global existence and uniqueness of solutions to (1.1)–(1.2). Let (ρ,v,n,m) be the classical solutions obtained in Sect. 2.

Lemma 3.1. *For any $t\in[0,T)$, there holds*

$$\begin{aligned}&\int\limits_I\left[|n_x|^2+|m_x|^2+2|n\cdot m_x|^2+\frac{\rho v^2}{2}+\frac{\rho^\gamma}{\gamma-1}\right](t)\mathrm{d}x\\&\quad+\int\limits_{Q_t}2|m_{xx}+|m_x|^2m+(n_x\cdot m_x)n+2|n\cdot m_x|^2m+2(m_x\cdot n)n_x|^2\mathrm{d}x\mathrm{d}t\\&\quad+\int\limits_{Q_t}|v_x|^2\mathrm{d}x\mathrm{d}t+\int\limits_{Q_t}2|n_{xx}+|n_x|^2n+(m_x\cdot n_x)m+2|n\cdot m_x|^2n+2(n_x\cdot m)m_x|^2\mathrm{d}x\mathrm{d}t\\&\quad+4\int\limits_{Q_t}(n_x\cdot m_x+n\cdot m_{xx})^2=E_0,\end{aligned}\tag{3.1}$$

where

$$E_0=\int\limits_I\left[\frac{\rho_0v_0^2}{2}+\frac{\rho_0^\gamma}{\gamma-1}+|n_{0x}|^2+|m_{0x}|^2+2|n_0\cdot m_{0x}|^2\right]\mathrm{d}x.$$

Proof. Multiplying $(1.1)_2$ by v and integrating over I, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \frac{\rho v^2}{2} - \int_I \rho^\gamma v_x = -\int_I v_x^2 + \int_I |n_x|^2 v_x + \int_I |m_x|^2 v_x + 2\int_I (|n\cdot m_x|^2) v_x.$$

Firstly, by a similar argument as in [2], we have from $(1.1)_1$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \frac{\rho^\gamma}{\gamma-1} = \int_I \rho^\gamma v_x.$$

Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \left[\left(\frac{\rho^\gamma}{\gamma-1}\right) + \frac{\rho v^2}{2}\right] + \int_I v_x^2 = \int_I (|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2) v_x. \tag{3.2}$$

Multiplying $(1.1)_3$ by $(n_{xx} + |n_x|^2 n + (m_x.n_x)m + 2|n\cdot m_x|^2 n + 2(n_x\cdot m)m_x)$ and integrating over I, we obtain

$$\int_I n_t\cdot n_{xx} + \int_I (m_x\cdot n_x)(m\cdot n_t) + \int_I v n_x\cdot n_{xx} + \int_I v(m_x\cdot n_x)(m\cdot n_x)$$
$$+ 2\int_I v(m_x\cdot n_x)(m\cdot n_x) + 2\int_I (m\cdot n_x)(m_x\cdot n_t) = \int_I A, \tag{3.3}$$

where $A = |n_{xx} + |n_x|^2 n + (m_x\cdot n_x)m + 2|n\cdot m_x|^2 n + 2(n_x\cdot m)m_x|^2$.

For the first term on the left of (3.3), we have

$$\int_I n_t\cdot n_{xx} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I |n_x|^2.$$

For the second term and fourth one on the left of (3.3), we get

$$\int_I (m_x\cdot n_x)(m\cdot n_t) + \int_I v(m\cdot n_x)(m_x\cdot n_x) = \int_I (m_x\cdot n_x)(m\cdot n_t + v n_x\cdot m).$$

For the third term on the left of (3.3), we have

$$\int_I v n_x\cdot n_{xx} = \frac{1}{2}\int_I v(|n_x|^2)_x = -\frac{1}{2}\int_I v_x|n_x|^2.$$

Then, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I |n_x|^2 + \frac{1}{2}\int_I v_x|n_x|^2 + \int_I A - \int_I (m_x\cdot n_x)m\cdot(n_t + v n_x)$$
$$- 2\int_I (m\cdot n_x)(m_x\cdot n_t) - 2\int_I v(m_x\cdot n_x)(m\cdot n_x) = 0. \tag{3.4}$$

Multiplying $(1.1)_4$ by $(m_{xx} + |m_x|^2 m + (n_x\cdot m_x)n + 2|n\cdot m_x|^2 m + 2(m_x\cdot n)n_x)$ and integrating over I, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I |m_x|^2 + \frac{1}{2}\int_I v_x|m_x|^2 + \int_I B - \int_I (n_x\cdot m_x)n\cdot(m_t + v m_x)$$
$$- 2\int_I (n\cdot m_x)(n_x\cdot m_t) - 2\int_I v(m_x\cdot n_x)(n\cdot m_x) = 0, \tag{3.5}$$

where $B = |m_{xx} + |m_x|^2 m + (n_x \cdot m_x)n + 2|n \cdot m_x|^2 m + 2(m_x \cdot n)n_x|^2$.

Combining (3.4) with (3.5), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I (|n_x|^2 + |m_x|^2) + \frac{1}{2}\int_I v_x(|n_x|^2 + |m_x|^2) + \int_I (A + B) - 2\int_I [(m \cdot n_x)(m_x \cdot n_t) + (n \cdot m_x)(n_x \cdot m_t)] = 0. \tag{3.6}$$

Now we will estimate $2\int_I [(m \cdot n_x)(m_x \cdot n_t) + (n \cdot m_x)(n_x \cdot m_t)]$.

In fact, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I |n \cdot m_x|^2 = 2\int_I [(n \cdot m_x)(m_x \cdot n_t) + (n \cdot m_x)(m_{xt} \cdot n)]$$
$$= -2\int_I [(n_x \cdot m)(m_x \cdot n_t) + (n \cdot m_x)(m_t \cdot n_x)] - 2\int_I [(n_x \cdot m_x + n \cdot m_{xx})(n \cdot m_t)].$$

Then, we have

$$-2\int_I [(n_x \cdot m)(m_x \cdot n_t) + (n \cdot m_x)(m_t \cdot n_x)] = \frac{\mathrm{d}}{\mathrm{d}t}\int_I |n \cdot m_x|^2 + 2\int_I [(n_x \cdot m_x + n \cdot m_{xx})(n \cdot m_t)]. \tag{3.7}$$

Hence combining (3.6) with (3.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I (|n_x|^2 + |m_x|^2 + 2|n \cdot m_x|^2) + \int_I v_x(|n_x|^2 + |m_x|^2) + 2\int_I (A + B) + 4\int_I [(n_x \cdot m_x + n \cdot m_{xx})(n \cdot m_t)] = 0. \tag{3.8}$$

Multiplying $(1.1)_4$ by n, we have

$$n \cdot m_t = n \cdot m_{xx} + n_x \cdot m_x - v m_x \cdot n.$$

Then, we have

$$\begin{aligned} &4\int_I [(n_x \cdot m_x + n \cdot m_{xx})(n \cdot m_t)] \\ &= 4\int_I G - 4\int_I [(n_x \cdot m_x + n \cdot m_{xx})(v m_x \cdot n)] \\ &= 4\int_I G - 2\int_I v(|n \cdot m_x|^2)_x \\ &= 4\int_I G + 2\int_I v_x |n \cdot m_x|^2, \end{aligned} \tag{3.9}$$

where $G = |n_x \cdot m_x + n \cdot m_{xx}|^2$.

Then from (3.8) and (3.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I (|n_x|^2+|m_x|^2+2|n\cdot m_x|^2) + \int_I (|n_x|^2+|m_x|^2+2|n\cdot m_x|^2)v_x + 2\int_I (A+B+2G)=0. \tag{3.10}$$

Combining (3.10) with (3.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \left[\frac{\rho v^2}{2}+\frac{\rho^\gamma}{\gamma-1}+|v_x|^2+|n_x|^2+|m_x|^2+2|n\cdot m_x|^2\right]+2\int_I (A+B+2G)=0.$$

Integrating above equality over $(0,t)$, we get (3.1). Then, Lemma 3.1 is proved. □

Lemma 3.2. *It holds that for any $T>0$,*

$$\int_0^T ||n_{xx}||_2^2+\int_0^T ||m_{xx}||_2^2 \le C. \tag{3.11}$$

Proof. Firstly, we have

$$\begin{aligned}
&|n_{xx}+|n_x|^2n+(m_x\cdot n_x)m+2|n\cdot m_x|^2n+2(n_x\cdot m)m_x|^2\\
&\quad=|n_{xx}|^2-|n_x|^4+2(n_x\cdot m_x)(m\cdot n_{xx})+4(n_x\cdot m)(n_{xx}\cdot m_x)-4|n_x|^2|n\cdot m_x|^2\\
&\qquad+|n_x\cdot m_x|^2+4|m_x|^2|n\cdot m_x|^2-4|n\cdot m_x|^4.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|m_{xx}+|m_x|^2m+(m_x\cdot n_x)n+2|m\cdot n_x|^2m+2(m_x\cdot n)n_x|^2\\
&\quad=|m_{xx}|^2-|m_x|^4+2(n_x\cdot m_x)(n\cdot m_{xx})+4(m_x\cdot n)(n_x\cdot m_{xx})-4|m_x|^2|m\cdot n_x|^2\\
&\qquad+|n_x\cdot m_x|^2+4|n_x|^2|m\cdot n_x|^2-4|m\cdot n_x|^4.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_I (|n_{xx}|^2+|m_{xx}|^2)\\
&\quad=\int_I (A+B)+\int_I (|n_x|^4+|m_x|^4+8|n\cdot m_x|^4+2|n_x\cdot m_x|^2)\\
&\qquad-4\int_I [(m\cdot n_x)(m_x\cdot n_{xx})+(n\cdot m_x)(n_x\cdot m_{xx})]\\
&\quad\le\int_I (A+B)+\frac{1}{4}\int_I (|n_{xx}|^2+|m_{xx}|^2)+C\int_I (|n_x|^4+|m_x|^4).
\end{aligned} \tag{3.12}$$

Meanwhile, we have

$$\begin{aligned}
&\int_I (|n_x|^4+|m_x|^4)\\
&\quad\le C||n_x||_2^3||n_x||_2+C||m_x||_2^3||m_x||_2\\
&\quad\le C||n_x||_2^3||n_x||_\infty+C||m_x||_2^3||m_x||_\infty\\
&\quad\le C||n_x||_2^3[||n_x||_2+||n_{xx}||_2^{\frac12}||n_x||_2^{\frac12}]+C||m_x||_2^3[||m_x||_2+||m_{xx}||_2^{\frac12}||m_x||_2^{\frac12}]\\
&\quad\le C||n_x||_2^4+C||n_x||_2^3||n_{xx}||_2+C||m_x||_2^4+C||m_x||_2^3||m_{xx}||_2\\
&\le \frac14\int_I(|n_{xx}|^2+|m_{xx}|^2)+C[\int_I(|n_x|^2+|m_x|^2)]^2. \qquad (3.13)
\end{aligned}$$

Combing (3.12) with (3.13), we get (3.11). Lemma 3.2 is proved. □

Lemma 3.3. *There holds that for any $T>0$,*

$$\begin{aligned}
&\sup_{0\le t\le T}(||n_{xx}(\cdot,t)||_2^2+||m_{xx}(\cdot,t)||_2^2)+\int_0^T(||n_{xt}||_2^2+||m_{xt}||_2^2+||n_{xxx}||_2^2+||m_{xxx}||_2^2)\\
&\le C(E_0,||n_0||_{H^2},||m_0||_{H^2},T). \qquad (3.14)
\end{aligned}$$

Proof. Differentiating $(1.1)_3$ with respect to x, multiplying by n_{xt} and integrating over $I\times(0,t)$, we have

$$\begin{aligned}
&\int_0^t ||n_{xt}||_2^2+\frac12||n_{xx}||_2^2(t)-\frac12||n_{0xx}||_2^2\\
&=\int_0^t\int_I(-v_xn_x\cdot n_{xt}-vn_{xx}\cdot n_{xt})+\int_0^t\int_I[2(n_x\cdot n_{xx})(n\cdot n_{xt})+|n_x|^2(n_x\cdot n_{xt})]\\
&\quad+\int_0^t\int_I[(m_{xx}\cdot n_x)(m\cdot n_{xt})+(m_x\cdot n_{xx})(m\cdot n_{xt})+(m_x\cdot n_x)(m_x\cdot n_{xt})]\\
&\quad+\int_0^t\int_I[4(n\cdot m_x)(n_x\cdot m_x)(n\cdot n_{xt})+4(n\cdot m_x)(n\cdot m_{xx})(n\cdot n_{xt})+2|n\cdot m_x|^2(n_x\cdot n_{xt})]\\
&\quad+2\int_0^t\int_I[(n_{xx}\cdot m)(m_x\cdot n_{xt})+(n_x\cdot m_x)(m_x\cdot n_{xt})+(n_x\cdot m)(m_{xx}\cdot n_{xt})]. \qquad (3.15)
\end{aligned}$$

For the first term of the right of (3.15), we have

$$\int_0^t\int_I(-v_xn_x\cdot n_{xt}-vn_{xx}\cdot n_{xt})\le 2\epsilon\int_0^t\int_I|n_{xt}|^2+C\int_0^t\int_I(v_x^2|n_x|^2+v^2|n_{xx}|^2).$$

For the second term of the right of (3.15), we have

$$\int_0^t\int_I[2(n_x\cdot n_{xx})(n\cdot n_{xt})+|n_x|^2(n_x\cdot n_{xt})]\le 2\epsilon\int_0^t\int_I|n_{xt}|^2+C\int_0^t\int_I(|n_x|^2|n_{xx}|^2+|n_x|^6).$$

For the third term of the right of (3.15), we have

$$\int_0^t \int_I [(m_{xx} \cdot n_x)(m \cdot n_{xt}) + (m_x \cdot n_{xx})(m \cdot n_{xt}) + (m_x \cdot n_x)(m_x \cdot n_{xt})]$$
$$\leq 3\epsilon \int_0^t \int_I |n_{xt}|^2 + C \int_0^t \int_I [|m_{xx}|^2|n_x|^2 + |m_x|^2|n_{xx}|^2 + |m_x|^4|n_x|^2].$$

For the fourth term of the right of (3.15), we have

$$\int_0^t \int_I [4(n \cdot m_x)(n_x \cdot m_x)(n \cdot n_{xt}) + 4(n \cdot m_x)(n \cdot m_{xx})(n \cdot n_{xt}) + 2|n \cdot m_x|^2(n_x \cdot n_{xt})]$$
$$\leq 3\epsilon \int_0^t \int_I |n_{xt}|^2 + C \int_0^t \int_I [|m_x|^4|n_x|^2 + |m_{xx}|^2|m_x|^2].$$

For the fifth term of the right of (3.15), we have

$$2\int_0^t \int_I [(n_{xx} \cdot m)(m_x \cdot n_{xt}) + (n_x \cdot m_x)(m_x \cdot n_{xt}) + (n_x \cdot m)(m_{xx} \cdot n_{xt})]$$
$$\leq 3\epsilon \int_0^t \int_I |n_{xt}|^2 + C \int_0^t \int_I [|n_{xx}|^2|m_x|^2 + |n_x|^2|m_x|^4 + |n_x|^2|m_{xx}|^2].$$

Then by taking $0 < \epsilon < \frac{1}{30}$, we have

$$\begin{aligned}
&\int_0^t ||n_{xt}||_2^2 + \frac{1}{2}||n_{xx}||_2^2(t) - \frac{1}{2}||n_{0xx}||_2^2 \\
&\leq \frac{1}{2}\int_0^t ||n_{xt}||_2^2 + C\int_0^t \int_I [v_x^2|n_x|^2 + v^2|n_{xx}|^2 + |n_x|^2|n_{xx}|^2 + |n_x|^6] \\
&\quad + C\int_0^t \int_I [|m_{xx}|^2|n_x|^2 + |m_x|^2|n_{xx}|^2 + |m_x|^4|n_x|^2 + |m_{xx}|^2|m_x|^2] \\
&\leq \frac{1}{2}\int_0^t ||n_{xt}||_2^2 + C\int_0^t \left[\left(\int_I v_x^2\right)\left(\int_I |n_{xx}|^2\right) + \left(\int_I |n_{xx}|^2\right)^2 + \left(\int_I |m_{xx}|^2\right)^2\right] + C,
\end{aligned} \tag{3.16}$$

where we have used $\|v\|_{L^\infty(I)}^2 \leq C(\|v\|_2^2 + \|v_x\|_2^2) \leq C\|v_x\|_2^2$ from the Poincare's inequality and

$$\|n_x\|_{L^\infty(I)}^2 \leq C(\|n_x\|_2^2 + \|n_{xx}\|_2^2).$$

Similarly, differentiating $(1.1)_4$ with respect to x, multiplying m_{xt} and integrating over $I\times(0,t)$, we also have

$$\begin{aligned}&\int_0^t ||m_{xt}||_2^2+\frac{1}{2}||m_{xx}||_2^2(t)-\frac{1}{2}||m_{0xx}||_2^2\\&\quad\le\frac{1}{2}\int_0^t ||m_{xt}||_2^2+C\int_0^t\left[\left(\int_I v_x^2\right)\left(\int_I |m_{xx}|^2\right)+\left(\int_I |n_{xx}|^2\right)^2+\left(\int_I |m_{xx}|^2\right)^2\right]+C.\end{aligned}\tag{3.17}$$

Combining (3.16) with (3.17), we get

$$\begin{aligned}&\int_0^t (\|n_{xt}\|_2^2+\|m_{xt}\|_2^2)+(\|n_{xx}\|_2^2+\|m_{xx}\|_2^2)(t)\\&\quad\le C\int_0^t\left(\int_I |m_{xx}|^2\right)^2+C\int_0^t\left(\int_I |n_{xx}|^2\right)^2+C\int_0^t\left(\int_I v_x^2\right)\left[\int_I\left(|n_{xx}|^2+\int_I |m_{xx}|^2\right)\right]+C.\end{aligned}$$

From

$$(\|v_x\|_2^2+\|n_{xx}\|_2^2+\|m_{xx}\|_2^2)(t)\in L^1(0,T)$$

and the Gronwall's inequality, we have

$$\sup_{0\le t\le T}(||n_{xx}||_2^2+||m_{xx}||_2^2)(t)+\int_0^T(\|n_{xt}\|_2^2+\|m_{xt}\|_2^2)\le C(E_0,||n_0||_{H^2},||m_0||_{H^2},T).\tag{3.18}$$

Differentiating $(1.1)_3$ and $(1.1)_4$ with respect to x, we have

$$\begin{aligned}n_{xxx}&=n_{xt}+v_xn_x+vn_{xx}-2(n_x\cdot n_{xx})n-|n_x|^2n_x-(m_{xx}\cdot n_x)m-(m_x\cdot n_{xx})m\\&\quad-(m_x\cdot n_x)m_x-4(n\cdot m_x)(n_x\cdot m_x+n\cdot m_{xx})n-2|n\cdot m_x|^2n_x-2(n_{xx}\cdot m)m_x\\&\quad-2(n_x\cdot m_x)m_x-2(n_x\cdot m)m_{xx},\\m_{xxx}&=m_{xt}+v_xm_x+vm_{xx}-2(m_x\cdot m_{xx})m-|m_x|^2m_x-(n_{xx}\cdot m_x)n-(n_x\cdot m_{xx})n\\&\quad-(n_x\cdot m_x)n_x-4(n\cdot m_x)(n_x\cdot m_x+n\cdot m_{xx})m-2|n\cdot m_x|^2m_x-2(m_{xx}\cdot n)n_x\\&\quad-2(m_x\cdot n_x)n_x-2(m_x\cdot n)n_{xx}.\end{aligned}$$

Then, (3.18) implies that

$$\int_0^T ||n_{xxx}||_2^2+\int_0^T ||m_{xxx}||_2^2\le C(E_0,||n_0||_{H^2},||m_0||_{H^2},T).$$

Hence, Lemma 3.3 is proved. □

Now we will improve the estimates of both lower and upper bounds of ρ by a similar argument as in [2].

Lemma 3.4. *There are two positive constants C_1 and C_2 depending on C_0,γ, E_0 and $\|\rho_0\|_{H^1(I)}$ such that*

$$\sup_{0\le t<T}\int_I\rho\left|\left(\frac{1}{\rho}\right)_x\right|^2+\int_0^T\int_I\rho^{\gamma-3}\rho_x^2\le C_1,\tag{3.19}$$

$$(C_2)^{-1}\le\rho(x,t)\le C_2,\ \ (x,t)\in I\times(0,T).\tag{3.20}$$

Proof. From $(1.1)_1$ and Lemma 3.4 in [2], we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2 = \int_I \rho\left(\frac{1}{\rho}\right)_x\left(\frac{1}{\rho}\right)_{xt} + \frac{1}{2}\int_I \rho_t|\left(\frac{1}{\rho}\right)_x|^2 = \int_I \left(\frac{1}{\rho}\right)_x v_{xx}. \tag{3.21}$$

On the other hand, we have

$$\begin{aligned}
\int_I \left(\frac{1}{\rho}\right)_x v_{xx} &= \int_I \left(\frac{1}{\rho}\right)_x [(\rho v^2)_x + (|n_x|^2)_x + (|m_x|^2)_x + (2|n\cdot m_x|^2)_x + (\rho^\gamma)_x + (\rho v)_t] \\
&= \int_I \left(\frac{1}{\rho}\right)_x [(\rho v^2)_x + (|n_x|^2)_x + (|m_x|^2)_x + (2|n\cdot m_x|^2)_x] - \gamma\int_I \rho^{\gamma-3}\rho_x^2 \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_I (\rho v)\left(\frac{1}{\rho}\right)_x + \int_I (\rho v)_x\left(\frac{1}{\rho}\right)_t.
\end{aligned} \tag{3.22}$$

Putting (3.22) into (3.21), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2 + 2\int_I \rho v(-\frac{1}{\rho})_x\right] + \gamma\int_I \rho^{\gamma-3}\rho_x^2 \\
&= \int_I \left(\frac{1}{\rho}\right)_x [(\rho v^2)_x + (|n_x|^2)_x + (|m_x|^2)_x + (2|n\cdot m_x|^2)_x] + \int_I (\rho v)_x\left(\frac{1}{\rho}\right)_t \\
&= 2\int_I \left(\frac{1}{\rho}\right)_x [n_x\cdot n_{xx} + m_x\cdot m_{xx} + 2(n\cdot m_x)(n_x\cdot m_x + n\cdot m_{xx})] + \int_I \frac{1}{\rho^2}[|(\rho v)_x|^2 - (\rho v^2)_x\rho_x] \\
&\le C\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2 (||n_x||_\infty^2 + ||m_x||_\infty^2) + C||\frac{1}{\rho}||_\infty \int_I (|n_{xx}|^2 + |m_{xx}|^2) + \int_I v_x^2 \\
&\le \left[C + C\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2\right]\int_I (|n_{xx}|^2 + |m_{xx}|^2) + ||v_x||_2^2 + C\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2,
\end{aligned} \tag{3.23}$$

where we have used

$$\left\|\frac{1}{\rho}\right\|_\infty \le 2 + \int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2, \tag{3.24}$$

from (3.11) in [2] and

$$||n_x||_\infty^2 \le C(||n_x||_2^2 + ||n_x||_2||n_{xx}||_2),\ ||m_x||_\infty^2 \le C(||m_x||_2^2 + ||m_x||_2||m_{xx}||_2).$$

Integrating (3.23) over $(0,t)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2+\gamma\int_0^t\int_I \rho^{\gamma-3}\rho_x^2\\
&\le \frac{1}{2}\int_I \rho_0\left|\left(\frac{1}{\rho_0}\right)_x\right|^2-\int_I \rho_0 v_0\left(\frac{1}{\rho_0}\right)_x+\int_I \rho v\left(\frac{1}{\rho}\right)_x+C\int_0^t\int_I (|n_{xx}|^2+|m_{xx}|^2)\\
&\quad+C\int_0^t\left[\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2\int_I (|n_{xx}|^2+|m_{xx}|^2)\right]+\int_0^t ||v_x||_2^2+C\int_0^t\int_I \rho|\left(\frac{1}{\rho}\right)_x|^2\\
&\le C+C\left[1+\sup_{0\le t\le T}(\|n_{xx}\|^2+\|m_{xx}\|^2)(t)\right]\int_0^t\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2.
\end{aligned}$$

Using Lemma 3.3 and the Gronwall's inequality, we get (3.19). It is easy to get (3.20) by a similar argument as [2]. We omit the details. Hence, Lemma 3.4 is proved. □

Lemma 3.5. *There holds that*

$$\sup_{0\le t\le T}||v_x(\cdot,t)||_2^2+\int_0^T(||v_t||_2^2+||v_{xx}||_2^2)\le C. \tag{3.25}$$

Proof. From $(1.1)_1$ and $(1.1)_2$, we have

$$\rho v_t+\rho v v_x+(\rho^\gamma)_x=v_{xx}-(|n_x|^2)_x-(|m_x|^2)_x-(2|n\cdot m_x|^2)_x. \tag{3.26}$$

Multiplying (3.26) by v_t and integrating over I, we have

$$\begin{aligned}
&\int_I \rho v_t^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I v_x^2\\
&=-\int_I \rho v v_x v_t-\int_I (\rho^\gamma)_x v_t-\int_I (|n_x|^2)_x v_t-\int_I (|m_x|^2)_x v_t-2\int_I (|n\cdot m_x|^2)_x v_t\\
&\le \frac{1}{2}\int_I \rho v_t^2+C\int_I \frac{1}{\rho}(|n_x|^2|n_{xx}|^2+2|m_x|^2|m_{xx}|^2+|m_x|^4|n_x|^2)+C\int_I \rho v^2 v_x^2+C\int_I \rho^{2\gamma-3}\rho_x^2\\
&\le \frac{1}{2}\int_I \rho v_t^2+C\int_I (|n_x|^2|n_{xx}|^2+2|m_x|^2|m_{xx}|^2+|m_x|^4|n_x|^2)+C||v_x||_2^4+C\int_I \rho\left|\left(\frac{1}{\rho}\right)_x\right|^2.
\end{aligned} \tag{3.27}$$

Combining (3.19) with (3.20) and using the Gronwall's inequality, we get

$$\sup_{0\le t\le T}||v_x(\cdot,t)||_2^2+\int_0^T ||v_t||_2^2\le C. \tag{3.28}$$

From (3.26), it is easy to get that

$$\int_0^T\int_I |v_{xx}|^2\le C. \tag{3.29}$$

Therefore Lemma 3.5 is proved. □

Lemma 3.6. *There holds that*

$$\sup_{0\le t\le T}(||v_t||_2^2+||v_{xx}||_2^2)(t)+\int_0^T ||v_{xt}||_2^2 \le C. \tag{3.30}$$

Proof. Differentiating (3.26) with respect to t, multiplying v_t and integrating over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho v_t^2+||v_{xt}||_2^2\\
&=2\int_I [n_x\cdot n_{xt}v_{xt}+m_x\cdot m_{xt}v_{xt}+2(n\cdot m_x)(n_t\cdot m_x)v_{xt}+2(n\cdot m_x)(n\cdot m_{xt})v_{xt}]\\
&\quad+\int_I [(\rho v)_x v_t^2+(\rho v)_x v v_x v_t-\rho v_t^2 v_x-\gamma\rho^{\gamma-1}(\rho v)_x v_{xt}]\\
&\le \frac{1}{2}||v_{xt}||_2^2+C\int_I (|n_x|^2|n_{xt}|^2+|m_x|^2|m_{xt}|^2+|m_x|^4|n_t|^2+\rho^2v^2v_t^2+\rho^2v^4v_x^2)\\
&\quad+C\int_I(\rho^{2\gamma-2}v^2\rho_x^2+\rho^{2\gamma}v_x^2)+C\int_I(\rho v^2v_x^4+\rho v^4 v_{xx}^2)+C(1+\|v_x\|_\infty)\int_I \rho v_t^2.
\end{aligned} \tag{3.31}$$

Hence, one gets

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho v_t^2+||v_{xt}||_2^2 &\le C(\|n_x\|_\infty^2+\|m_x\|_\infty^2+\|m_x\|_\infty^4)(||n_{xt}||_2^2+||m_{xt}||_2^2+\|n_t\|_2^2)\\
&\quad+C(1+\|v_x\|_\infty+\|\rho\|_\infty\|v\|_\infty^2)\int_I \rho v_t^2\\
&\quad+C(\|\rho\|_\infty^{2\gamma}+\|\rho\|_\infty^2\|v\|_\infty^4+\|\rho\|_\infty\|v\|_\infty^2\|v_x\|_\infty^2)\int_I v_x^2\\
&\quad+C\|\rho\|_\infty\|v\|_\infty^4\int_I v_{xx}^2+C\|\rho\|_\infty^{\gamma+1}\|v\|_\infty^2\int_I \rho^{\gamma-3}\rho_x^2\\
&\le C\int_I \rho v_t^2+C,
\end{aligned}$$

where we have used the following estimate,

$$\|\rho\|_\infty+\|v\|_\infty+\|v_x\|_\infty+\|n_x\|_\infty+\|m_x\|_\infty\le C,$$

which comes from Lemma 3.1 to Lemma 3.5.

Hence, we get (3.30) from the Gronwall's inequality. Therefore, Lemma 3.6 is proved. □

Lemma 3.7. [2] *Suppose that*

$$\sup_{0\le t\le T}|v(x,t_1)-v(x,t_2)|\le\theta_1|t_1-t_2|^\alpha,\forall t_1,t_2\in[0,T]$$

and

$$\sup_{0\le t\le T}|v_x(x_1,t)-v_x(x_2,t)|\le\theta_2|x_1-x_2|^\beta,\forall x_1,x_2\in I$$

then

$$\sup_{0\le t\le T}|v_x(x,t_1)-v_x(x,t_2)|\le\theta|t_1-t_2|^{\delta},\forall t_1,t_2\in[0,T],$$

where $\delta=\frac{\alpha\beta}{1+\beta}$, *and* θ *depends only on* $\alpha,\beta,\theta_1,\theta_2$.

Proof of Theorem 1.1. We will use a proof of contradiction to prove this theorem, which is similar as in [2].

Suppose there exists a maximal finite time interval $T^*>0$, such that there is a unique classical solution $(\rho,v,n,m):I\times[0,T^*]\to R_+\times R\times S^2\times S^2$ to (1.1)–(1.2), but at least one of the following properties fails:

(i) $(\rho_x,\rho_t)\in C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})$;
(ii) $0<C_2^{-1}\le\rho(x,t)\le C_2<+\infty,\quad\forall(x,t)\in Q_{T^*}$;
(iii) $(v,n,m)\in C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{T^*})$.

It is easy to see that (ii) holds from (3.20) in Lemma 3.4. Hence, either (i) or (iii) fails.

From Lemma 3.1 to Lemma 3.6, we have

$$\sup_{0\le t\le T^*}(||v(\cdot,t)||_2^2+||v_x(\cdot,t)||_2^2+||v_{xx}(\cdot,t)||_2^2)\le C,$$
$$\sup_{0\le t\le T^*}(||n(\cdot,t)||_2^2+||n_x(\cdot,t)||_2^2+||n_{xx}(\cdot,t)||_2^2)\le C,$$
$$\sup_{0\le t\le T^*}(||m(\cdot,t)||_2^2+||m_x(\cdot,t)||_2^2+||m_{xx}(\cdot,t)||_2^2)\le C.$$

By the Sobolev embedding theorem, we have

$$\max\left\{||v||_{C^{1,\frac12}(Q_{T^*})},||n||_{C^{1,\frac12}(Q_{T^*})},||m||_{C^{1,\frac12}(Q_{T^*})}\right\}<+\infty.$$

Using Lemma 3.7 for $\alpha=\frac12$, $\beta=\frac12$ and $\delta=\frac16$, we have

$$\max\{||v_x||_{C^{\frac13,\frac16}(Q_{T^*})},||n_x||_{C^{\frac13,\frac16}(Q_{T^*})},||m_x||_{C^{\frac13,\frac16}(Q_{T^*})}\}<+\infty.$$

Using the Schauder theory to $(1.1)_3$ and $(1.1)_4$, we have

$$||n||_{C^{2+\frac13,1+\frac16}(Q_{T^*})}<+\infty,\ ||m||_{C^{2+\frac13,1+\frac16}(Q_{T^*})}<+\infty.$$

Hence,

$$||n_x||_{C^{1,\frac12}(Q_{T^*})}<+\infty,\ ||m_x||_{C^{1,\frac12}(Q_{T^*})}<+\infty.$$

Using the Schauder theory to $(1.1)_3$ and $(1.1)_4$ again, we also get

$$||n||_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T^*})}<+\infty,\ ||m||_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_{T^*})}<+\infty.$$

For ρ and v, denote $G(x,t)=-(|n_x|^2+|m_x|^2+2|n\cdot m_x|^2)_x$. Then, $||G||_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})}<+\infty$.

In Lagrangian coordinate, $(1.1)_1$ and $(1.1)_2$ are changed to

$$\begin{cases}\rho_\tau+\rho^2v_y=0,\\ v_\tau+(\rho^\gamma)_y=(\rho v_y)_y+G.\end{cases}\tag{3.32}$$

From Lemma 3.4 to Lemma 3.6 in the Lagrangian coordinate, we have

$$0<C_2\le\rho\le C_2<+\infty,\tag{3.33}$$
$$\sup_{0\le t\le T^*}||\rho_y(\cdot,t)||_2^2\le C<+\infty,\tag{3.34}$$
$$\sup_{0\le t\le T^*}(||v_y(\cdot,t)||_2^2+||v_{yy}(\cdot,t)||_2^2)\le C<+\infty.\tag{3.35}$$

By a similar argument as in [2], we get that

$$\max\{||v||_{C^{2+\alpha,\frac{2+\alpha}{2}}(Q_{T^*})}, ||\rho_y||_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})}, ||\rho_\tau||_{C^{\alpha,\frac{\alpha}{2}}(Q_{T^*})}\} < +\infty.$$

Hence, both (i) and (ii) are right if T^* is finite. This is a contradiction. Then, T^* is infinite.

Theorem 1.1 is proved. □

4. Global strong solution: Existence and uniqueness

In this section, we will establish global existence and uniqueness of strong solution for initial density with possible vacuum states. In order to use the result of Theorem 1.1, we will construct approximate solutions firstly. For any large $k > 0$, define a family of approximate initial datas

$$\rho_0^k = \eta_k * \rho_0 + \frac{1}{k},\ v_0^k = \eta_k * v_0,\ n_0^k = \frac{\eta_k * n_0}{|\eta_k * n_0|} \text{ and } m_0^k = \frac{\widetilde{m}_0^k}{|\widetilde{m}_0^k|},$$

where $\widetilde{m}_0^k = \widehat{m}_0^k - (\widehat{m}_0^k \cdot n_0^k)n_0^k$ and $\widehat{m}_0^k = \eta_k * m_0$. Then, we have $\rho_0^k \geq k^{-1}$, $n_0^k \cdot m_0^k = 0$ and

$$\lim_{k\to+\infty}\{\|\rho_0^k - \rho_0\|_{H^1(I)} + \|v_0^k - v_0\|_{H^1(I)} + \|n_0^k - n_0\|_{H^2(I)} + \|m_0^k - m_0\|_{H^2(I)}\} = 0.$$

Let (ρ^k, v^k, n^k, m^k) be the unique global classical solution to (1.1) with initial data $(\rho_0^k, v_0^k, n_0^k, m_0^k)$ and boundary condition $(v^k, n_x^k, m_x^k) = (0, 0, 0)$ constructed by Theorem 1.1.

In order to prove Theorem 1.2, we will establish several new estimates for (ρ^k, v^k, n^k, m^k) that are independent of k. We will omit the superscripts of (ρ^k, v^k, n^k, m^k) for simplicity.

By a same argument in Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have the following lemma.

Lemma 4.1. *For any $T > 0$, there is a constant $C > 0$ independent of k, such that*

$$\sup_{0\leq t\leq T}\int_I (\rho v^2 + \rho^\gamma + |n_x|_2^2 + |m_x|_2^2 + |n_{xx}|_2^2 + |m_{xx}|_2^2) + \int_0^T (||v_x||_2^2 + ||n_{xt}||_2^2 + ||m_{xt}||_2^2 + ||n_{xxx}||_2^2 + ||m_{xxx}||_2^2) \leq C. \tag{4.1}$$

Lemma 4.2. *For any $T > 0$, there is a positive constant C independent of k, such that*

$$||\rho||_{L^\infty(I\times(0,T))} \leq C. \tag{4.2}$$

Proof. Let

$$f(x,t) = \int_0^t (v_x - |n_x|^2 - |m_x|^2 - 2|n\cdot m_x|^2 - \rho v^2 - \rho^\gamma) + \int_0^x (\rho_0 v_0).$$

Then, we have

$$f_t = v_x - |n_x|^2 - |m_x|^2 - 2|n\cdot m_x|^2 - \rho v^2 - \rho^\gamma \text{ and } f_x = \rho v.$$

Then,

$$||f||_\infty \leq C\int_I (|f| + |f_x|) \leq C,$$

here we have used Lemma 4.1.

Let $x(z,t)$ solve

$$\begin{cases} \frac{\mathrm{d}x(z,t)}{\mathrm{d}t} = v(x(z,t),t), & 0 \le t < \tau, \\ x(z,\tau) = z, & 0 \le z \le 1. \end{cases}$$

Let $g = e^f$. Then, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\rho g(x(z,t),t)) &= (\rho_t + \rho_x v)g + \rho g(f_t + vf_x) \\ &= [-\rho(|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2) - \rho^{\gamma+1}]g \le 0. \end{aligned}$$

Then,

$$\rho g(z,\tau) = \rho g(x(z,\tau),\tau) \le \rho g(x(z,0),0) \le C.$$

Hence, we get (4.2) from the definition of g.

Lemma 4.2 is proved. □

Lemma 4.3. *For any $T > 0$, there is a positive constant C independent on of k, such that*

$$\sup_{0\le t\le T} ||v_x(\cdot,t)||_2^2 + \int\limits_{Q_T} \rho v_t^2 \le C. \tag{4.3}$$

Proof. As Lemma 3.5, multiplying (3.26) by v_t and integrating over I, we have

$$\begin{aligned} &\int\limits_I \rho v_t^2 + \frac{\mathrm{d}}{\mathrm{d}t}||v_x(\cdot,t)||_2^2 \\ &\le 2\left[\int\limits_I \rho v_t^2 + \|\rho\|_\infty \|v\|_\infty^2 \int\limits_I v_x^2 + \int\limits_I \rho^\gamma v_{xt} + \int\limits_I (|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2) v_{xt}\right] \\ &\le C\left[\left(\int\limits_I v_x^2\right)^2 + \int\limits_I \rho^\gamma v_{xt} + \int\limits_I \left(|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2\right) v_{xt}\right], \end{aligned} \tag{4.4}$$

where we have used Lemma 4.2 and $\|v\|_\infty^2 \le C\|v\|_2^2$. For the second term on the right of (4.4), we have

$$\begin{aligned} \int\limits_I \rho^\gamma v_{xt} &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I \rho^\gamma v_x + \int\limits_I \gamma\rho^{\gamma-1}(\rho v)_x v_x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I \rho^\gamma v_x + \int\limits_I (\rho^\gamma)_x v v_x + \gamma\int\limits_I \rho^\gamma v_x^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I \rho^\gamma v_x + (\gamma-1)\int\limits_I \rho^\gamma v_x^2 - \int\limits_I \rho^\gamma v v_{xx} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_I \rho^\gamma v_x + (\gamma-1)\int\limits_I \rho^\gamma v_x^2 - \int\limits_I \rho^\gamma v[\rho v_t + \rho v v_x + (\rho^\gamma)_x + (|n_x|^2)_x + (|m_x|^2)_x] \\ &\quad - \int\limits_I \rho^\gamma v(2|n\cdot m_x|^2)_x \end{aligned}$$

$$\leq \frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho^\gamma v_x + \frac{1}{2}\int_I \rho v_t^2 + C(1+\int_I v_x^2)\int_I v_x^2 + C\int_I [v^2(|n_x|^2+|m_x|^2) + (|n_{xx}|^2+|m_{xx}|^2)]$$
$$\leq \frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho^\gamma v_x + \frac{1}{2}\int_I \rho v_t^2 + C(\int_I v_x^2)^2 + C,$$

where we have used (4.1) in the last inequality.

For the third term on the right of (4.4), we have

$$\begin{aligned}
&\int_I (|n_x|^2+|m_x|^2+2|n\cdot m_x|^2)v_{xt}\\
&\quad = -2\int_I n_x\cdot n_{xt}v_x + \frac{\mathrm{d}}{\mathrm{d}t}\int_I |n_x|^2 v_x - 2\int_I m_x\cdot m_{xt}v_x + \frac{\mathrm{d}}{\mathrm{d}t}\int_I |m_x|^2 v_x\\
&\qquad - 4\int_I (n\cdot m_x)(n_t\cdot m_x + n\cdot m_{xt})v_x + \frac{\mathrm{d}}{\mathrm{d}t}\int_I 2|n\cdot m_x|^2 v_x\\
&\quad \leq C\int_I v_x^2 + C\int_I (|n_{xt}|^2+|m_{xt}|^2+|n_t|^2) + \frac{\mathrm{d}}{\mathrm{d}t}\int_I (|n_x|^2+|m_x|^2+2|n\cdot m_x|^2)v_x,
\end{aligned}$$

where we have used $\|n_x\|_\infty \leq C\|n_{xx}\|_2 < C$ and $\|m_x\|_\infty \leq C\|m_{xx}\|_2 < C$ by Lemma 4.1 and Poincare's inequality.

Putting above two estimates into (4.4), we have

$$\begin{aligned}
&\int_I \rho v_t^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_I |v_x|^2\\
&\quad \leq C(1+||v_x||_2^4) + \frac{\mathrm{d}}{\mathrm{d}t}\int_I (\rho^\gamma v_x + |n_x|^2 v_x + |m_x|^2 v_x + 2|n\cdot m_x|^2 v_x) + C||n_{xt}||_2^2 + C||m_{xt}||_2^2.
\end{aligned}\tag{4.5}$$

Then integrating (4.5) over $(0,t)$ and using Lemma 4.1, we have

$$\begin{aligned}
&\int_{Q_t} \rho v_t^2 + ||v_x||_2^2(t)\\
&\quad \leq C\int_0^t ||v_x||_2^4 + \frac{1}{2}||v_x||_2^2(t) + C(||n_x||_4^4 + ||m_x||_4^4) + C\\
&\quad \leq C\int_0^t ||v_x||_2^4 + \frac{1}{2}||v_x||_2^2(t) + C(||n_x||_2^2\|n_{xx}\|_2^2 + ||m_x||_2^2\|m_{xx}\|_2^2) + C\\
&\quad \leq C\int_0^t ||v_x||_2^4 + \frac{1}{2}||v_x||_2^2(t) + C.
\end{aligned}$$

Then, we have

$$\int_{Q_t} \rho v_t^2 + ||v_x||_2^2(t) \leq C + C\int_0^t ||v_x||_2^4.$$

From $\|v\|_2^2(t) \in L^1(0,T)$ and the Gronwall's inequality, we obtain (4.3).

Hence, Lemma 4.3 is proved. □

Lemma 4.4. *For any $T>0$, there has a positive constant C independent of k, such that*

$$\sup_{0\le t\le T}||\rho_x(\cdot,t)||_2^2 + \int_0^T (||v_x||_\infty^2 + ||v_{xx}||_2^2) \le C. \tag{4.6}$$

Proof. From $(1.1)_1$ and $(1.1)_2$, we have

$$\begin{aligned}
||v_x||_\infty^2 &\le 2||v_x - \rho^\gamma - |n_x|^2 - |m_x|^2 - 2|n\cdot m_x|^2||_\infty^2 + 2||\rho^\gamma + |n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2||_\infty^2\\
&\le C[||v_x - \rho^\gamma - |n_x|^2 - |m_x|^2 - 2|n\cdot m_x|^2||_2^2 + ||v_{xx} - (\rho^\gamma)_x - (|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2)_x||_2^2]\\
&\quad + C||n_{xx}||_2^4 + C||m_{xx}||_2^4 + C\|\rho^\gamma\|_\infty^2\\
&\le C + C||n_x||_\infty^2||n_x||_2^2 + C||m_x||_\infty^2||m_x||_2^2 + C||\rho v_t + \rho v v_x||_2^2\\
&\le C(1 + ||v_x||_2^4 + ||n_x||_2^2||n_{xx}||_2^2 + ||m_x||_2^2||m_{xx}||_2^2) + C||\rho v_t^2||_1\\
&\le C + C(||n_{xx}||_2^2 + ||m_{xx}||_2^2 + ||\rho v_t^2||_1).
\end{aligned}$$

Lemmas 4.1 and 4.3 imply that

$$\int_0^T ||v_x||_\infty^2 \le C. \tag{4.7}$$

Next we are going to estimate ρ_x.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_x^2 &= \int_I 2\rho_x\rho_{xt} = -2(\rho v)_x\rho_x|_{x=0}^{x=l} + 2\int_I (\rho v)_x\rho_{xx}\\
&= -2\rho v_x\rho_x|_{x=0}^{x=l} + 2\int_I \rho_x v\rho_{xx} + 2\int_I \rho v_x\rho_{xx}\\
&= -2\rho v_x\rho_x|_{x=0}^{x=l} - \int_I \rho_x^2 v_x + 2\rho v_x\rho_x|_{x=0}^{x=l} - 2\int_I \rho_x^2 v_x - 2\int_I \rho_x\rho v_{xx}\\
&= -3\int_I \rho_x^2 v_x - 2\int_I \rho\rho_x^2 v_{xx}\\
&\le C||v_x||_\infty\int_I \rho_x^2 - 2\int_I \rho\rho_x[\rho v_t + \rho v v_x + (\rho^\gamma)_x + (|n_x|^2)_x + (|m_x|^2)_x + (2|n\cdot m_x|^2)_x]\\
&\le C(1 + ||v_x||_\infty)\int_I \rho_x^2 + C\int_I \rho v_t^2 + C(||v_x||_2^4 + \|n_{xx}\|_2^2 + \|m_{xx}\|_2^2)\\
&\le C + C\int_I \rho_x^2 + C\int_I \rho v_t^2.
\end{aligned} \tag{4.8}$$

From Lemma 4.3 and Gronwall's inequality, we get

$$\sup_{0\le t\le T}||\rho_x(\cdot,t)||_2^2 \le C.$$

Finally, we estimate $\|v_{xx}\|_{L^2(Q_T)}$. In fact, (3.26) implies that

$$v_{xx} = \rho v_t + \rho v v_x + (\rho^\gamma)_x + (|n_x|^2 + |m_x|^2 + 2|n\cdot m_x|^2)_x.$$

Then, it is easy to get

$$\int_0^T ||v_{xx}||_2^2 \leq C. \tag{4.9}$$

Hence, Lemma 4.4 is proved. □

By a similar argument as in [2], we also have an important estimate as follows.

Lemma 4.5. *For any $T > 0$, there is a positive constant C, independent of k, such that*

$$\int_0^T t||v_{xt}(\cdot,t)||_2^2 \leq C.$$

Proof. Differentiating (3.26) w.r.t. t, multiplying v_t and integrating over I, it is not hard to get that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho v_t^2 + \int_I |v_{xt}|^2 \\ &\quad\leq C(||n_{xt}||_2^2 + ||m_{xt}||_2^2 + ||v_x||_\infty^2 + \|v_{xx}\|_2^2) + C(1+||v_x||_\infty^2)\int_I \rho v_t^2 + C. \end{aligned} \tag{4.10}$$

Multiplying (4.10) by $t > 0$, one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int_I \rho v_t^2\right) + \int_I |v_{xt}|^2 \\ &\leq \int_I \rho v_t^2 + Ct(||n_{xt}||_2^2 + ||m_{xt}||_2^2 + ||v_x||_\infty^2 + \|v_{xx}\|_2^2) + C(1+||v_x||_\infty^2)t\int_I \rho v_t^2 + C. \end{aligned} \tag{4.11}$$

By Lemma 4.3, we have

$$\lim_{t_i\to 0} t_i \int_I \rho v_t^2(x,t_i)\mathrm{d}x = 0. \tag{4.12}$$

Integrating (4.11) from t_i to t and using (4.12), we obtain the result of Lemma 4.5 according to Lemma 4.1, Lemma 4.3 and Lemma 4.4.

Therefore Lemma 4.5 is proved. □

The following Aubin–Lions's lemma is needed in proving Theorem 1.2.

Lemma 4.6. [25] *Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then, the following embedding is compact:*

$$(i)\left\{\varphi : \varphi \in L^q(0,T;X), \frac{\partial\varphi}{\partial t} \in L^q(0,T;Y)\right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \text{ if } q \in [1,+\infty];$$

$$(ii)\left\{\varphi : \varphi \in L^\infty(0,T;X), \frac{\partial\varphi}{\partial t} \in L^r(0,T;Y)\right\} \hookrightarrow\hookrightarrow C([0,T];E), \text{ if } r \in (1,+\infty].$$

Now we are going to prove Theorem 1.2.

Proof of Theorem 1.2. Let (ρ^k, v^k, n^k, m^k) be the unique global classical solution to (1.1) with the initial data $(\rho_0^k, v_0^k, n_0^k, m_0^k)$ and boundary condition $(v^k, n_x^k, m_x^k) = (0,0,0)$ constructed by Theorem 1.1. From Lemma 4.1 to Lemma 4.5, we get that

$$\sup_{0\le t\le T}(||\rho^k||_{H^1(I)} + ||\rho_t^k||_2 + ||v^k||_{H^1(I)} + ||n^k||_{H^2(I)} + ||n_t^k||_2 + ||m^k||_{H^2(I)} + ||m_t^k||_2)(t) + \int_0^T [t||v_{xt}^k||_2 + ||(\rho v^k)_t||_2^2 + ||v_{xx}^k||_2^2 + ||n_t^k||_{H^1(I)}^2 + ||m_t^k||_{H^1(I)}^2] \le C,$$

where the positive constant C is independent of k.

Then, there is a subsequences of (ρ^k, v^k, n^k, m^k) (still denoted by (ρ^k, v^k, n^k, m^k)) and (ρ, v, n, m), such that

$$(\rho^k, \rho_x^k, \rho_t^k, v^k, v_x^k) \rightharpoonup (\rho, \rho_x, \rho_t, v, v_x) \ \text{ weakly star in } L^\infty(0,T; L^2(I)),$$

$$(v_{xx}^k, \sqrt{t} v_{xx}^k) \rightharpoonup (v_{xx}, \sqrt{t} v_{xx}) \ \text{ weakly in } L^2(0,T; L^2(I)),$$

$$(n^k, n_x^k, n_t^k, n_{xx}^k, m^k, m_x^k, m_t^k, m_{xx}^k) \rightharpoonup (n, n_x, n_t, n_{xx}, m, m_x, m_t, m_{xx}) \ \text{ weakly star in } L^\infty(0,T; L^2(I)),$$

$$(n_{xt}^k, n_{xxx}^k, m_{xt}^k, m_{xxx}^k) \rightharpoonup (n_{xt}, n_{xxx}, m_{xt}, m_{xxx}) \ \text{ weakly in } L^2(0,T; L^2(I))$$

and

$$(\rho^k v^k)_t \rightharpoonup (\rho v)_t \ \text{ weakly in } L^2(0,T; L^2(I)).$$

Moreover, because ρ^k is bounded in $L^\infty(0,T; H^1(I))$ and ρ_t^k bounded in $L^\infty(0,T; L^2(I))$, we have from Lemma 4.6 that

$$\rho^k \to \rho \ \text{ strongly in } \ C(Q_T).$$

Similarly, because $\rho^k v^k$ and $(\rho^k v^k)_t$ are bounded in $L^1(0,T; H^1(I))$ and $L^2(0,T; L^2(I))$, respectively, we know that

$$\rho^k v^k \to \rho v \ \text{ strongly in } C(Q_T)$$

by Lemma 4.6.

It is easy to see that

$$\rho^k (v^k)^2 \rightharpoonup \rho v^2, \quad [\rho^k (v^k)^2]_x \rightharpoonup (\rho v^2)_x \ \textit{ and } \ ((\rho^k)^\gamma)_x \rightharpoonup (\rho^\gamma)_x \quad \text{weakly star in } L^\infty(0,T; L^2(I)),$$

since $[\rho^k (v^k)^2]_x$ is bounded in $L^\infty(0,T; L^2(I))$.

Lemma 4.6 also implies

$$(n^k, (n^k)_x) \to (n, n_x), \quad (m^k, (m^k)_x) \to (m, m_x) \ \text{ strongly in } \ C(Q_T).$$

Therefore, we get

$$\begin{aligned}
(|n_x^k|^2 n^k, |m_x^k|^2 m^k) &\to (|n_x|^2 n, |m_x|^2 m) \ \text{ strongly in } \ C(Q_T),\\
(|n^k \cdot m_x^k|^2 n^k, |n^k \cdot m_x^k|^2 m^k) &\to (|n \cdot m_x|^2 n, |n \cdot m_x|^2 m) \ \text{ strongly in } \ C(Q_T),\\
((n_x^k \cdot m^k) m_x^k, (m_x^k \cdot n^k) n_x^k) &\to ((n_x \cdot m) m_x, (m_x \cdot n) n_x) \ \text{ strongly in } \ C(Q_T),\\
((m_x^k \cdot n_x^k) m^k, (n_x^k \cdot m_x^k) n^k) &\to ((m_x \cdot n_x) m, (n_x \cdot m_x) n) \ \text{ strongly in } \ C(Q_T),\\
v^k n_x^k \rightharpoonup v n_x, \quad (|n^k \cdot m_x^k|^2)_x &\rightharpoonup (|n \cdot m_x|^2)_x \ \text{ weakly star in } L^\infty(0,T; L^2(I)).
\end{aligned}$$

Therefore, we know that (ρ, v, n, m) is a strong solution to (1.1)–(1.2).

Finally, we will prove the uniqueness of the global strong solutions.

Denote $\bar{\rho} = \rho_1 - \rho_2$, $\bar{v} = v_1 - v_2$, $\bar{n} = n_1 - n_2$, $\bar{m} = m_1 - m_2$, where $(\rho_i, v_i, n_i, m_i)(i = 1, 2)$ are two strong solutions to (1.1)–(1.2). Hence, $(\bar{\rho}, \bar{v}, \bar{n}, \bar{m})$ solves

$$\begin{cases} \bar{\rho}_t + (\bar{\rho} v_1)_x + (\rho_2 \bar{v})_x = 0, \\ \rho_1 \bar{v}_t - \bar{v}_{xx} = -\bar{\rho} v_{2t} - \bar{\rho} v_2 v_{2x} - \rho_1 \bar{v} v_{2x} - \rho_1 v_1 \bar{v}_x - (\rho_1^\gamma - \rho_2^\gamma)_x - 2n_{1x} \cdot \bar{n}_{xx} \\ \qquad -2\bar{n}_x \cdot n_{2xx} - 2m_{1x} \cdot \bar{m}_{xx} - 2\bar{m}_x \cdot m_{2xx} - 4(\bar{n} \cdot m_{1x})(n_{1x} \cdot m_{1x}) - 4(n_2 \cdot \bar{m}_x)(n_{1x} \cdot m_{1x}) \\ \qquad -4(n_2 \cdot m_{2x})(\bar{n}_x \cdot m_{1x}) - 4(n_2 \cdot m_{2x})(n_{2x} \cdot \bar{m}_x) - 4(\bar{n} \cdot m_{1x})(n_1 \cdot m_{1xx}) \\ \qquad -4(n_2 \cdot \bar{m}_x)(n_1 \cdot m_{1xx}) - 4(n_2 \cdot m_{2x})(\bar{n} \cdot m_{1xx}) - 4(n_2 \cdot m_{2x})(n_2 \cdot \bar{m}_{xx}), \\ \bar{n}_t + v_1 \bar{n}_x + \bar{v} n_{2x} = \bar{n}_{xx} + |n_{1x}|^2 \bar{n} + \bar{n}_x \cdot (n_{1x} + n_{2x}) n_2 + \bar{m}_x \cdot n_{1x} m_1 + m_{2x} \cdot \bar{n}_x m_1 \\ \qquad + m_{2x} \cdot n_{2x} \bar{m} + 2|\bar{n} \cdot m_{1x}|^2 n_1 + 2|n_2 \cdot \bar{m}_x|^2 n_1 + 2|n_2 \cdot m_{2x}|^2 \bar{n} \\ \qquad + 2(\bar{n}_x \cdot m_1) m_{1x} + 2(n_{2x} \cdot \bar{m}) m_{1x} + 2(n_{2x} \cdot m_2) \bar{m}_x, \\ \bar{m}_t + v_1 \bar{m}_x + \bar{v} m_{2x} = \bar{m}_{xx} + |m_{1x}|^2 \bar{m} + \bar{m}_x \cdot (m_{1x} + m_{2x}) m_2 + \bar{n}_x \cdot m_{1x} n_1 + n_{2x} \cdot \bar{m}_x n_1 \\ \qquad + n_{2x} \cdot m_{2x} \bar{n} + 2|\bar{n} \cdot m_{1x}|^2 m_1 + 2|n_2 \cdot \bar{m}_x|^2 m_1 + 2|n_2 \cdot m_{2x}|^2 \bar{m} \\ \qquad + 2(\bar{m}_x \cdot n_1) n_{1x} + 2(m_{2x} \cdot \bar{n}) n_{1x} + 2(m_{2x} \cdot n_2) \bar{n}_x, \end{cases} \tag{4.13}$$

with the following initial and boundary conditions

$$\begin{cases} (\bar{\rho}, \bar{v}, \bar{n}, \bar{m})|_{t=0} = (0, 0, 0, 0), \\ (\bar{v}, \bar{n}_x, \bar{m}_x)|_{\partial I} = (0, 0, 0). \end{cases} \tag{4.14}$$

Multiplying $(4.13)_1$ by $\bar{\rho}$ and integrating over I, we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_I |\bar{\rho}|^2 &= -\int_I (\bar{\rho} \bar{\rho}_x v_1 + \bar{\rho}^2 v_{1x} + \bar{\rho} \rho_{2x} \bar{v} + \bar{\rho} \rho_2 \bar{v}_x) \\ &= -\frac{1}{2} \int_I |\bar{\rho}|^2 v_{1x} - \int_I (\rho_{2x} \bar{v} + \rho_2 \bar{v}_x) \bar{\rho} \\ &\le \frac{1}{2} ||v_{1x}||_\infty \int_I |\bar{\rho}|^2 + ||\bar{v}||_\infty ||\rho_{2x}||_2 ||\bar{\rho}||_2 + ||\rho_2||_\infty ||\bar{v}_x||_2 ||\bar{\rho}||_2 \\ &\le \frac{1}{2} ||v_1||_{H^2(I)} \int_I |\bar{\rho}|^2 + C ||\bar{v}_x||_2 ||\bar{\rho}||_2 + ||\rho_2||_\infty ||\bar{v}_x||_2 ||\bar{\rho}||_2 \\ &\le C ||v_1||_{H^2(I)} \int_I |\bar{\rho}|^2 + C ||\bar{v}_x||_2 ||\bar{\rho}||_2 \\ &\le C(||v_1||_{H^2(I)} + 1) \int_I |\bar{\rho}|^2 + C \int_I |\bar{v}_x|^2, \end{aligned}$$

where we have used the regularities of ρ_i and v_i for $i = 1, 2$. The Gronwall's inequality implies that for $t \in [0, T]$,

$$||\bar{\rho}(\cdot, t)||_2 \le Ct \int_{Q_t} |\bar{v}_x|^2. \tag{4.15}$$

Multiplying $(4.13)_2$ by $\bar{v}$, integrating over I and using the Cauchy's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1|\bar{v}|^2+||\bar{v}_x||_2^2\\&\quad\le C||\bar{\rho}||_2^2||v_{2t}||_2^2+C||\bar{\rho}||_2^2+C||v_{2x}||_\infty\int_I \rho_1|\bar{v}|^2+C||\bar{n}_x||_2^2+C||\bar{m}_x||_2^2+C||\bar{n}||_2^2.\end{aligned}\tag{4.16}$$

(4.15) and (4.16) imply that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_I \rho_1|\bar{v}|^2+\int_{Q_t}|\bar{v}_x|^2\right]\\&\le Ct||v_{2t}||_2^2(t)\int_{Q_t}|\bar{v}_x|^2+Ct\int_{Q_t}|\bar{v}_x|^2+C||v_{2x}||_\infty(t)\int_I \rho_1|\bar{v}|^2+C(||\bar{n}_x||_2^2+||\bar{m}_x||_2^2+||\bar{n}||_2^2).\end{aligned}$$

Multiplying $(4.13)_3$ by $\bar{n}$, $(4.13)_4$ by $\bar{m}$ and integrating over I, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I(|\bar{n}|^2+|\bar{m}|^2)+||\bar{n}_x||_2^2+||\bar{m}_x||_2^2\\&\le C||\bar{n}_x||_2||\bar{n}||_2+C||\bar{v}_x||_2||\bar{n}||_2+C||\bar{m}_x||_2||\bar{n}||_2+C||\bar{m}_x||_2||\bar{m}||_2+C||\bar{v}_x||_2||\bar{m}||_2\\&\le\frac{1}{2}(||\bar{n}_x||_2^2+||\bar{m}_x||_2^2)+\epsilon||\bar{v}_x||_2^2+C(||\bar{n}||_2^2+||\bar{m}||_2^2),\end{aligned}$$

where we have used the Cauchy's inequality and ϵ is small enough to be chosen later.

Hence, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I(|\bar{n}|^2+|\bar{m}|^2)+||\bar{n}_x||_2^2+||\bar{m}_x||_2^2\le 2\epsilon||\bar{v}_x||_2^2+C(||\bar{n}||_2^2+||\bar{m}||_2^2).\tag{4.17}$$

Then by taking $\epsilon=\frac{1}{8C}$, we get from (4.16) and (4.17) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_I(\rho_1|\bar{v}|^2+2C|\bar{n}|^2+2C|\bar{m}|^2)+\frac{1}{2}\int_{Q_t}|\bar{v}_x|^2\right]+C\int_I(|\bar{n}_x|^2+|\bar{m}_x|^2)\\&\quad\le Ct||v_{2t}||_2^2\int_{Q_t}|\bar{v}_x|^2+Ct\int_{Q_t}|\bar{v}_x|^2+C||v_{2x}||_\infty\int_I\rho_1|\bar{v}|^2+C\int_I(|\bar{n}|^2+|\bar{m}|^2).\end{aligned}$$

From the initial data of $(\bar{\rho},\bar{v},\bar{n},\bar{m})$ and the Gronwall's inequality, we have

$$\bar{\rho}=0,\bar{v}=0,\bar{n}=0,\bar{m}=0.$$

Therefore, the uniqueness of global strong solutions is proved.

Theorem 1.2 is proved. □

Acknowledgements

The second author is partially supported by NSF of China (No.11571117).

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(Received: August 19, 2021; revised: December 8, 2021; accepted: December 12, 2021)