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Existence of positive solutions for fractional Kirchhoff equation

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Abstract. We study the following Kirchhoff equation involving fractional Laplacian in $\mathbb{R}^N$

$$\Big(a+b\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u+\mu u=g(u), \qquad (\mathcal{K})$$

where $N\geq 2$, $a\geq 0$, $b,\mu>0$, $0<s<1$, and $(-\Delta)^s$ is the fractional Laplacian with order s. By reducing $(\mathcal{K})$ to an equivalent system, we obtain the existence of a positive solution of $(\mathcal{K})$ with general nonlinearities. The positive solution is unique if $g(u)=|u|^{p-1}u$, $1<p<\frac{N+2s}{N-2s}$. Moreover, if the function g is odd, the existence of infinitely many (sign-changing) solutions is concluded. As we shall see, for the case where $0<s\leq\frac{N}{4}$, a necessary condition of existence of nontrivial solutions of $(\mathcal{K})$ is that b is small. Our method works well for the so-called degenerate case $a=0$.

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1. Introduction and main results

In this paper we are concerned with the following nonlinear nonlocal problem in $\mathbb{R}^N$

$$\Big(a+b\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u+\mu u=g(u)\,, \qquad (1.1)$$

where $N\geq 2$, $a\geq 0$, $b,\mu>0$, $0<s<1$, $(-\Delta)^s$ is the fractional Laplacian and g is a continuous function. (1.1) could be derived as a nonlocal model for the vibrating string with a fractional length in which the tension of string is related to nonlocal measurements of the modification of the string from its rest position [10].

Nonlocal problems like (1.1) have been widely studied in recent years (see, e.g., [2,3,6,8,13,14,16,17,19–21,23]). Compared to the semilinear fractional Laplacian where $b=0$ in (1.1), looking for solutions for (1.1) is more challenging due to the presence of the Kirchhoff term

$$\Big(b\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u.$$

We would like to mention that the existing results on the existence or multiplicity of solutions for Kirchhoff type problems are almost concluded by the variational methods. As far as applications of variational methods to the Kirchhoff type problems are concerned, it generally could be a complicated process, even for the case where the nonlinearity g satisfies the A-R condition (see, e.g., [12,18]). This, at the same

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time, also makes Kirchhoff type problems more attractive. In the past decade, many authors had used the direct variational methods such as the mountain pass theorem and the Nehari manifold to conclude the existence or multiplicity of solutions to the Kirchhoff type problems. However, it seems that these methods do not work for the case $0 < s \le \frac{N}{4} < 1$. There is rare study about (1.1) with $0 < s \le \frac{N}{4}$. We also mention that there are very few results about the fractional Kirchhoff type problem with sub-cubic nonlinearities which do not satisfy the A-R condition $g(t)t \ge 4\int_0^t g(s)\mathrm{d}s$.

In this paper, we introduce a new technique to study Kirchhoff type problems and obtain the existence of positive solutions and infinitely many (sign-changing) solutions to (1.1). Our arguments are quite different from the direct variational methods and do not depend on the A-R type condition. We do not need to restrict s on the range $\frac{N}{4} < s < 1$, either. Based on our simple but powerful method, we can conclude the uniqueness of positive solutions of (1.1) with power nonlinearities and deal with the so-called degenerate case $a = 0$ which has been rarely studied. To be more precise, we try to solve problem (1.1) by transforming it into the following system with respect to $(u, T) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$:

$$\begin{cases} (-\Delta)^s u + \mu u = g(u), \\ T = a + bT^{\frac{N-2s}{2s}} \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \end{cases}$$

Consider the problem (1.1) with the function g satisfying the following conditions:

(g_1) $g \in C^1(\mathbb{R}, \mathbb{R})$ and $g(t) = o(t)$ as $t \to 0^+$;

(g_2) there exists a constant $\beta \in (0, \mu)$ such that $-\infty < \liminf\limits_{t\to 0^+} \frac{g(t)}{t} < \limsup\limits_{t\to 0^+} \frac{g(t)}{t} \le \beta$;

(g_3) $-\infty \le \limsup\limits_{t\to +\infty} \frac{g(t)}{t^{2_s^*-1}} \le 0$, where $2_s^* = \frac{2N}{N-2s}$ is the Sobolev critical exponent for the fractional Laplacian;

(g_4) there exists a $\xi > 0$ such that $2G(\xi) > \mu\xi^2$, where $G(\xi) = \int_0^\xi g(t)\mathrm{d}t$.

Except for developing a new technique to study the Kirchhoff type equations like (1.1), our results are new and improve significantly the result from [1] where $b = (1-s)q$ and q depending the number of solutions is required to be small enough. The authors in [1] did not give any results about the cases neither where $a = 0$ nor where q is not infinitely small.

Our first result is the following theorem.

Theorem 1.1. *Assume that g satisfies conditions $(g_1) - (g_4)$, then we have*

(i) *if $\frac{N}{4} < s < 1$ and $a \ge 0$, (1.1) has a positive solution for any $b > 0$;*
(ii) *if $s = \frac{N}{4}$ and $a > 0$, there exist constants $b^*, b^{**} > 0$ such that (1.1) has a positive solution for any $0 < b < b^*$, and no positive solutions for any $b \ge b^{**}$.*
(iii) *if $s = \frac{N}{4}$ and $a = 0$, there exists a positive radial function $u \in H^s(\mathbb{R}^N)$ such that for any $T > 0$, $u(T\cdot)$ is a solution of (1.1) for $b = b^*$, and (1.1) has no positive solutions for any $b > b^{**}$, where b^* and b^{**} are the constants given in (ii).*
(iv) *if $0 < s < \frac{N}{4}$ and $a > 0$, there exist positive constant $\tilde{b}^*, \tilde{b}^{**} > 0$ such that (1.1) has two positive solutions u_1 and u_2 for any $0 < b < \tilde{b}^*$, satisfying $u_1(x) = u_2(Tx)$ for some $T \ne 1$ and $T > 0$, one positive solution for $b = \tilde{b}^*$, and no positive solutions for any $b > \tilde{b}^{**}$.*
(v) *if $0 < s < \frac{N}{4}$ and $a = 0$, (1.1) has one positive solution for any $b > 0$.*

Moreover, each solution u obtained in (i), (ii), (iv) *and* (v) *is radial and satisfies*

$$0 < u(x) \le \frac{C}{1 + |x|^{N+2s}}, \quad \forall\, x \in \mathbb{R}^N, \tag{1.2}$$

for some positive constant C depending on a, b, s and N.

A typical function satisfying $(g_1)-(g_4)$ is $g(t)=|t|^{p-1}t, 1<p<2_s^*-1=\frac{N+2s}{N-2s}$. In this case, one may rewrite (1.1) as

$$\Big(a+b\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u+\mu u=|u|^{p-1}u\,,\quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N\geq 2$, $a\geq 0$, $b,\mu>0$ and $1<p<2_s^*-1$. For problem (1.3), we have the following (uniqueness) result of positive solutions.

Theorem 1.2. (i) *if* $\frac{N}{4}<s<1$ *and* $a\geq 0$, (1.3) *has a positive solution for any* $b>0$, *which is unique up to translation;*

(ii) *if* $s=\frac{N}{4}$ *and* $a>0$, *there exists constant* $b^*>0$ *such that* (1.3) *has a positive solution for any* $0<b<b^*$, *which is unique up to translation, and no positive solutions for any* $b\geq b^*$.

(iii) *if* $s=\frac{N}{4}$ *and* $a=0$, *there exists a positive radial function* $u\in H^s(\mathbb{R}^N)$ *such that for any* $T>0$, $u(T\cdot)$ *is a solution of* (1.3) *for* $b=b^*$, *and* (1.3) *has no positive solutions for any* $b\neq b^*$, *where* b^* *is the constant given in* (ii).

(iv) *if* $0<s<\frac{N}{4}$ *and* $a>0$, *there exists a positive constant* $\tilde{b}^*$ *such that* (1.3) *has exactly two positive solutions* u_1 *and* u_2 *for any* $0<b<\tilde{b}^*$, *satisfying* $u_1(x)=u_2(Tx)$ *for some* $T\neq 1$ *and* $T>0$, *one positive solution for* $b=\tilde{b}^*$, *which is unique up to translation, and no positive solutions for any* $b>\tilde{b}^*$.

(v) *if* $0<s<\frac{N}{4}$ *and* $a=0$, (1.3) *has one positive solution for any* $b>0$, *which is unique up to translation.*

Moreover, each solution u *obtained in* (i), (ii), (iv) *and* (v) *is radial,* $u\in C^\infty(\mathbb{R}^N)\cap H^{2s+1}(\mathbb{R}^N)$ *and satisfies*

$$\frac{c_1}{1+|x|^{N+2s}}\leq u(x)\leq\frac{c_2}{1+|x|^{N+2s}},\quad \forall\, x\in\mathbb{R}^N,$$

for some positive constants c_1,c_2 *depending on* a,b,s *and* N.

Remark 1.1. From the arguments in the next section, if u is a positive solution of (1.3), then there exists a $T>0$ such that $u(x)=U(Tx)$, where U is a positive solution of $(-\Delta)^s u+\mu u=|u|^{p-1}u$. We refer reader to [11] for the existence of U.

Remark 1.2. In view of Theorem 1.2, if $\{u_k\}$ is a sequence of solutions of (1.1), then u_k may be sign-changing for every integer $k\geq 2$. Moreover, for $0<s\leq\frac{N}{4}$, we know from Theorem 1.1 that, unless $a=0$, (1.3) has no nontrivial solutions if $b>0$ is large.

It is well known that the symmetry of nonlinearities may imply the multiplicity of solutions to a differential equation. For problem (1.1), we have the following results on multiplicity of solutions. For simplicity, we are just going to give the results for the case $a>0$.

Theorem 1.3. *Assume that* g *is odd and satisfies conditions* $(g_1)-(g_4)$, *then we have*

(i) *if* $\frac{N}{4}<s<1$ *and* $a>0$, (1.1) *possesses an infinite sequence of radial solutions* $\{u_k\}$ *for any* $b>0$.

(ii) *if* $s=\frac{N}{4}$ *and* $a>0$, *there exist a constant* $r_0>0$ *and a sequence* $\{b_k\}\subset\mathbb{R}^+$ *with* $b_k\downarrow 0^+$ *and* $b_k\leq r_0$ *such that, for any integer* $k\geq 1$, (1.1) *possesses a radial solution* u_k *for any* $0<b<b_k$, *no nontrivial solutions for any* $b\geq r_0$.

(iii) *if* $0<s<\frac{N}{4}$ *and* $a>0$, *there exist a constant* $\tilde{r}_0>0$ *and a sequence* $\{\tilde{b}_k\}\subset\mathbb{R}^+$ *with* $\tilde{b}_k\downarrow 0^+$ *and* $\tilde{b}_k\leq\tilde{r}_0$ *such that, for any integer* $k\geq 1$, (1.1) *possesses two radial solutions* u_k^1 *and* u_k^2 *for any* $0<b<\tilde{b}_k$, *satisfying* $u_k^1(x)=u_k^2(T_kx)$ *for some* $T_k\neq 1$ *and* $T_k>0$, *one solution* u_k *for* $b=\tilde{b}_k$, *and no nontrivial solutions for any* $b>\tilde{r}_0$.

Moreover, for any integer $k \geq 1$, there exists a constant $C_k > 0$ such that

$$|u_k(x)| \leq \frac{C_k}{1+|x|^{N+2s}}, \ \forall\, x \in \mathbb{R}^N, \tag{1.4}$$

where $\{u_k\}$ stands for sequences of solutions obtained in (i), (ii) *and* (iii).

The paper is organized as follows. In Sect. 2 we prove the main results above. In Sect. 3 we give some basic facts which are used in Sect. 2.

2. Proofs of the main results

Let $H^s(\mathbb{R}^N)$, $0 < s < 1$, be the normal fractional Sobolev space and $H^s_r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}$. It is known that the embedding $H^s_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $2 < p < 2^*_s$, is compact. Define two functionals on $H^s(\mathbb{R}^N)$:

$$P(u) = \int\limits_{\mathbb{R}^N} G(u) - \frac{\mu}{2}\int\limits_{\mathbb{R}^N} u^2$$

and

$$Q(u) = \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

Recall that the homogeneous fractional Sobolev space $\mathcal{D} = \mathcal{D}^{s,2}(\mathbb{R}^N)$ is the closure of $C_c^\infty(\mathbb{R}^N)$ with the Gagliardo seminorm $\sqrt{Q(u)}$.

Proposition 2.1. *Problem (1.1) admits a nontrivial solution $u \in H^s(\mathbb{R}^N)$ if and only if the following system*

$$\begin{cases} (-\Delta)^s u + \mu u = g(u), \\ T = a + bT^{\frac{N-2s}{2s}} \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \end{cases} \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{2.1}$$

possesses a solution $(v,t) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ satisfying $(v,T) \neq (0,a)$.

Proof. On the one hand, $u \neq 0$ is a solution of (1.1). Let

$$T = a + b \int\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \text{ and } v(x) = u\big(T^{\frac{1}{2s}}x\big).$$

Then (u,T) is a solution of (2.1).

On the other hand, if $(v,T) \neq (0,a)$ is a solution of (2.1), then $u(\cdot) = v(T^{-\frac{1}{2s}}\cdot)$ is a solution of (1.1) and $u \neq 0$. □

Remark 2.1. In view of Proposition 2.1, problem (1.1) can be solved completely by studying system (2.1). Notice that the second equation in (2.1) is defined on the real line, problem (1.1) is almost equivalent to the following semilinear equation

$$(-\Delta)^s u + \mu u = g(u) \ \text{ in } \mathbb{R}^N. \tag{2.2}$$

Moreover, as we shall see, if u is a solution of (1.1), then exist a solution U of (2.2) and a positive constant T such that $u(x) = U(Tx)$.

Remark 2.2. A relation between problems (1.1) and (2.2) is given in the proof of Proposition 2.1. This fact could be applied to investigate behaviors of solutions of (1.1). For instance, one may obtain the Pohozaev identity associated with problem (1.1), that is, if $g \in C^1(\mathbb{R},\mathbb{R})$ satisfies conditions (g_2) and (g_3), then any solution $u \in H^s(\mathbb{R}^N)$ of (1.1) satisfies the identity

$$(N-2s)\left[aQ(u)+bQ^2(u)\right]=2NP(u). \tag{2.3}$$

Indeed, if $u \in H^s(\mathbb{R}^N)$ is a solution of (1.1), then $v=u(T^{\frac{1}{2s}}\cdot)$ is a solution of (2.2), where $T=a+bQ(u)$. Recall that g satisfies conditions (g_2) and (g_3). As the argument in [7] or [24], one has

$$(N-2s)Q(v)=2NP(v). \tag{2.4}$$

Then (2.3) follows from the inequality (2.4).

Remark 2.3. The equivalent system (2.1) to (1.1) is not unique. A general one to be equivalent to the problem (1.1) is the following

$$\begin{cases} T^m(-\Delta)^s u+\mu u=g(u), \\ T^{m+1}=a+bT^{\frac{N-2s}{2s}}\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \end{cases} \quad \text{in } \mathbb{R}^N\times\mathbb{R}^+.$$

where $m \neq -1$ is a parameter.

We state two results on the existence and multiplicity of solutions of (2.2) before we give the proofs of the main results.

Proposition 2.2. *Assume that g satisfies conditions $(g_1)-(g_4)$, then problem (2.2) possesses a positive radial solution $u \in H^s(\mathbb{R}^N)$ such that*

$$u(x) \le \frac{C}{1+|x|^{N+2s}},\ \forall\, x\in\mathbb{R}^N, \tag{2.5}$$

for some constant $C>0$.

Proposition 2.3. *Assume that g is odd and satisfies conditions $(g_1)-(g_4)$, then problem (2.2) possesses an infinite sequence of radial solutions $\{u_k\}$ such that*

(i) *for each positive integer k, there exists constant C_k such that*

$$|u_k(x)| \le \frac{C_k}{1+|x|^{N+2s}},\ \forall\, x\in\mathbb{R}^N. \tag{2.6}$$

(ii) *$Q(u_k)\ge Q(u_{k-1})$ for any $k\ge 1$, $Q(u_k)\to+\infty$ and $\frac{1}{2}Q(u_k)-P(u_k)\to+\infty$ as $k\to+\infty$.*

Proofs of propositions 2.2 and 2.3 will be given in the next section. We are turning to proofs of the main results now.

Proof. (**Proof of Theorem 1.1**). For a fixed $w\in H^s(\mathbb{R}^N)$, $w\neq 0$, define a function $f_w:\mathbb{R}^+\to\mathbb{R}$ given by $f_w(T)=T-a-bT^{\frac{N-2s}{2s}}Q(w)$. It follows from Proposition 2.2 that problem (2.2) possesses a positive radial solution $u\in H^s(\mathbb{R}^N)$ such that (2.5) holds. By Proposition 2.1, it is sufficient to discuss whether the equation $f_u(T)=0$ is solvable.

(i) $\frac{N}{4}<s<1$ and $a\ge 0$. In this case, we have $0<\frac{N-2s}{2s}<1$. Thus it is easy to see, for any $b>0$, that $f_u(T)=0$ has a unique solution $T\in\mathbb{R}^+$.

(ii) $s=\frac{N}{4}$ and $a>0$. In this case, we have $f_w(T)=[1-bQ(w)]T-a$ for all $(w,T)\in H^s(\mathbb{R}^N)\times\mathbb{R}^+$. Let $b^*:=Q^{-1}(u)$. Then for any $0<b<b^*$, $T=a[1-bQ(u)]^{-1}>a$ is a solution of $f_u(T)=0$.

We now check the existence of b^{**}. Let $\mathcal{M}$ be the set of positive solutions of (2.2) and

$$Q_0:=\inf_{v\in\mathcal{M}}Q(v).$$

We claim that $Q_0 > 0$. Indeed, since g is continuous and satisfies (g_1) and (g_3), there exists a positive constant $C > 0$ such that

$$g(t) \leq \mu t + Ct^{2_s^*-1}, \ \forall t \geq 0. \tag{2.7}$$

By the Pohozaev identity (2.4), we have, for any $v \in \mathcal{M}$,

$$\frac{N-2s}{2N}Q(v) + \frac{\mu}{2}\int_{\mathbb{R}^N} v^2 = \int_{\mathbb{R}^N} G(v) \leq \frac{\mu}{2}\int_{\mathbb{R}^N} v^2 + C\int_{\mathbb{R}^N} v^{2_s^*},$$

which implies that $Q(v) \leq C\|v\|_{L^{2_s^*}}^{2_s^*}$. By the Sobolev embedding theorem, $Q(v) \leq C(Q(v))^{\frac{N}{N-2s}}$. Since $Q(v) > 0$, there is a positive constant q_0 independent of v such that $Q(v) > q_0$ for all $v \in \mathcal{M}$. Thus $Q_0 > 0$.

Let $b^{**} = Q_0^{-1}$. Then $b^{**} \geq b^*$. Moreover, for any $T \in \mathbb{R}^+$ and $b \geq b^{**}$,

$$\sup_{v\in\mathcal{M}} f_v(T) \leq (1 - bQ_0)T - a < 0.$$

As a consequence, for all $v \in \mathcal{M}$, $f_v(T) = 0$ has no solutions on $\mathbb{R}^+$.

(iii) $s = \frac{N}{4}$ and $a = 0$. It is easy to conclude the conclusion by an argument as in (ii). We omit the details here.

(iv) $0 < s < \frac{N}{4}$ and $a > 0$. In this case, for any $w \in H^s(\mathbb{R}^N)$, the function f_w has a unique maximum point

$$T_w = \left(\frac{2s}{(N-2s)bQ(w)}\right)^{\frac{2s}{N-4s}} > 0.$$

Thus,

$$\max_{T\in\mathbb{R}^+} f_u(T) = f_u(T_u) = \left(\frac{N-4s}{N-2s}\right)\left(\frac{2s}{(N-2s)bQ(w)}\right)^{\frac{2s}{N-4s}} - a.$$

Let

$$\tilde{b}^* = 2sa^{\frac{4s-N}{2s}}(N-2s)^{\frac{2s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}Q^{-1}(u).$$

Then $f_u(T_u) > 0$ if $0 < b < \tilde{b}^*$ and $f_u(T_u) = 0$ if $b = \tilde{b}^*$. This implies that $f_u(T) = 0$ possesses two solutions T_1 and T_2 satisfying $T_1 \neq T_2$ if $0 < b < \tilde{b}^*$, and a solution T_u if $b = \tilde{b}^*$.

Moreover, set

$$\tilde{b}^{**} = 2sa^{\frac{4s-N}{2s}}(N-2s)^{\frac{2s-N}{2s}}(N-4s)^{\frac{N-4s}{2s}}Q_0^{-1},$$

where Q_0 is defined as in (ii). Then $\tilde{b}^{**} \geq \tilde{b}^*$ and $\sup_{w\in\mathcal{M}} f_w(T) < 0$ for any $T \in \mathbb{R}^+$ and $b \geq \tilde{b}^{**}$. This implies that $f_u(T) = 0$ has no solutions if $b > \tilde{b}^{**}$.

(v) $0 < s < \frac{N}{4}$ and $a = 0$. It is easy to check, for any $b > 0$, that $f_u(T) = 0$ has a unique solution $T \in \mathbb{R}^+$. □

Proof. (**Proof of Theorem 1.2**). According to [11], the following problem

$$(-\Delta)^s u + \mu u = |u|^{p-1}u \tag{2.8}$$

has a unique positive solution U up to translations such that $U \in C^\infty(\mathbb{R}^N) \cap H^{2s+1}(\mathbb{R}^N)$ is radial and strictly decreasing in $r = |x|$. Moreover, there are positive constants c_1 and c_2 depending on s and N such that

$$\frac{c_1}{1+|x|^{N+2s}} \leq U(x) \leq \frac{c_2}{1+|x|^{N+2s}}, \ \forall x \in \mathbb{R}^N.$$

Since positive solutions of (2.8) are unique up to translations, for any positive solution u of (2.8), we have $Q(u) = Q(U)$. The rest arguments are similar to that in the proof of Theorem 1.1. We omit the details here. $\square$

Proof. (**Proof of Theorem 1.3**).

It follows from Proposition 2.3 that equation (2.2) possesses a sequence of solutions $\{u_k\}$ satisfying $Q(u_k) \geq Q(u_{k-1})$, $k \geq 2$, and $Q(u_k) \to +\infty$.

Let $\tilde{\mathcal{M}}$ be the set of nontrivial solutions of (2.2) and $\tilde{Q}_0 = \inf_{v\in\tilde{\mathcal{M}}} Q(v)$. Since g is odd, we conclude by (2.7) that $\tilde{Q}_0 > 0$. Now, for each integer $k \geq 1$, by replacing $Q(u)$, b^*, Q_0 and $\tilde{b}^*$ with $Q(u_k)$, b_k, $\tilde{Q}_0$ and $\tilde{b}_k$, respectively, one may obtain the conclusions of Theorem 1.3. $\square$

3. Existence and multiplicity of solutions

In this section, we devote to conclude the results of Propositions 2.2 and 2.3.

3.1. Positive solutions of (2.2)

Let

$$\mathcal{V} = \{H^s(\mathbb{R}^N) : Q(u) = 1\}.$$

We will derive the existence and multiplicity of solutions of (2.2) by looking for critical points of the constrained functional $P|_{\mathcal{V}}$. The hypothesis (g_4) implies that $\int_0^{\xi}(g(t) - \mu t)\mathrm{d}t > 0$. Motivated by [4], we modify the function g as follows. If $g(t) \geq \mu t$ for $t \geq \xi$, then $\tilde{g} = g$; if there exists a $t_0 > \xi$ such that $g(t_0) \geq \mu t_0$, let $\tilde{g}(t) = g(t)$ for $0 \leq t \leq t_0$ and $\tilde{g}(t) = g(t_0)$ for $t > t_0$; $\tilde{g}$ is defined by $\tilde{g}(t) = -\tilde{g}(-t)$ for $t \leq 0$. Clearly, $\tilde{g}$ satisfies the conditions (g_2) and (g_3). Moreover, one has $\tilde{g}(0) = 0$ and

$$\lim_{t\to\infty} \frac{\tilde{g}(t)}{|t|^{2_s^*-1}} = 0. \tag{3.1.1}$$

By the maximum principle, a solution of (2.2) with $\tilde{g}$ is also a solution of (2.2) with g. We henceforth will always identify g as $\tilde{g}$ throughout this paper.

For $t \geq 0$, let $g_1(t) = (g(t) - \beta t)^+$ and $g_2(t) = g_1(t) - g(t) + \mu t$ and extend both g_1 and g_2 as odd functions for $t \leq 0$. Then $g = g_1 - g_2 - \mu$,

$$g_1(t) \geq 0, g_2(t) \geq (\mu - \beta)t,\ \ \forall t \geq 0, \tag{3.1.2}$$

and $\tilde{g}(t) = o(|t|^{2_s^*-1})$ as $t \to \infty$. Moreover, $G_i(t) = \int_0^t g_i(s)\mathrm{d}s \geq 0$, $i = 1, 2$, for any $t \in \mathbb{R}$.

Lemma 3.1. *If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then $\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} [G_i(u_n) - G_i(w_n)] = \int\limits_{\mathbb{R}^N} G_i(u)$, i=1,2, where $w_n = u_n - u$.*

Proof. By (g_2) and (3.1.1), there exists a constant $C > 0$ such that

$$|g_i(t)| \leq C(|t| + |t|^{2_s^*-1}), i = 1, 2, \tag{3.1.3}$$

for all $t \in \mathbb{R}$. We then conclude by the Young's inequality that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
|G_i(u_n) - G_i(w_n)| &= |\int_0^1 g_i(w_n + tu)u\mathrm{d}t| \\
&\leq C(|w_n||u| + |u|^2 + |w_n|^{2_s^*-1}|u| + |u|^{2_s^*}) \\
&\leq \varepsilon(|w_n|^2 + |w_n|^{2_s^*}) + C_\varepsilon(|u|^2 + |u|^{2_s^*}), i = 1, 2.
\end{aligned}$$

Let

$$f_{i,\varepsilon}^n = \left(|G_i(u_n) - G_i(w_n) - G_i(u)| - \varepsilon(|w_n|^2 + |w_n|^{2_s^*})\right)^+, i = 1, 2.$$

Then we see that $f_{i,\varepsilon}^n \to 0$ a.e. $x \in \mathbb{R}^N$, $0 \le f_{i,\varepsilon}^n + C_\varepsilon(|u|^2 + |u|^{2_s^*})$ and

$$|G_i(u_n) - G_i(w_n) - G_i(u)| \le f_{i,\varepsilon}^n + \varepsilon(|w_n|^2 + |w_n|^{2_s^*}).$$

By the Lebesgue dominated convergence theorem, one has $\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} f_{i,\varepsilon}^n = 0$ and hence

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |G_i(u_n) - G_i(w_n) - G_i(u)| = 0, i = 1, 2,$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} [G_i(u_n) - G_i(w_n)] = \int\limits_{\mathbb{R}^N} G_i(u), i = 1, 2.$$

□

Lemma 3.2. *The functional P is weakly sequentially upper semicontinuous in the $\mathcal{D}^{s,2}(\mathbb{R}^N)$-topology on $\mathcal{M} = \{u \in H_r^s(\mathbb{R}^N) : P(u) \ge 0, Q(u) = 1\}$.*

Proof. Let $\{u_n\} \subset \mathcal{M}$ be a sequence such that $u_n \rightharpoonup u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. It follows from $Q(u_n) = 1$ that $\{u_n\}$ is bounded in $L^{2_s^*}(\mathbb{R}^N)$.

The conditions (g_2) and (g_3) imply that, for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$0 \le G_1(t) \le \varepsilon t^2 + C_\varepsilon |t|^{2_s^*}, \ \forall t \in \mathbb{R}. \tag{3.1.4}$$

Thus, one has

$$0 \le P(u_n) \le \int\limits_{\mathbb{R}^N} G_1(u_n) - \frac{\mu}{2} \int\limits_{\mathbb{R}^N} u_n^2 \le C \int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} - \frac{\mu}{4} \int\limits_{\mathbb{R}^N} u_n^2,$$

which implies that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$ and hence $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^N)$. We may assume that $u_n \rightharpoonup u$ in $H_r^s(\mathbb{R}^N)$.

For a fixed $p \in (2, 2_s^*)$, it follows from (g_3) and the definition of g_1, for any $\varepsilon > 0$, that there exists $C_\varepsilon > 0$ such that

$$0 \le G_1(t) \le \varepsilon |t|^{2_s^*} + C_\varepsilon |t|^p, \ \forall t \in \mathbb{R}. \tag{3.1.5}$$

Thus,

$$0 \le \int\limits_{\mathbb{R}^N} G_1(u_n - u) \le \varepsilon C + C_\varepsilon \int\limits_{\mathbb{R}^N} |u_n - u|^p = \varepsilon C + o(1).$$

Since the embedding $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact, we conclude from Lemma 3.1 that

$$\int\limits_{\mathbb{R}^N} G_1(u_n) \to \int\limits_{\mathbb{R}^N} G_1(u).$$

Now by Fatou Lemma,

$$\liminf_{n\to\infty} \int\limits_{\mathbb{R}^N} \left(G_2(u_n) + \frac{\mu}{2} u_n^2\right) \ge \int\limits_{\mathbb{R}^N} \left(G_2(u) + \frac{\mu}{2} u^2\right).$$

Therefore, $P(u) \ge \limsup_{n\to\infty} P(u_n)$. □

We turn to the proof of Proposition 2.2.

Proof. (**Proof of Proposition 2.2**).

Let

$$\alpha = \sup_{u\in\mathcal{V}} P(u).$$

As in the proof of Lemma 3.2, one has $\alpha < +\infty$. Let $\{u_n\} \subset \mathcal{V}$ be a sequence such that $P(u_n) \to \alpha$. For every n, let $v_n = u_n^*$ denote the Schwarz spherically symmetric-decreasing rearrangement of $|u_n|$. Then one has $v_n \geq 0, v_n \in H_r^s(\mathbb{R}^N)$, $Q(v_n) = 1$ and $\alpha \geq P(v_n) \geq P(u_n)$. Thus $P(v_n) \to \alpha$. It follows from (g_4) that there exists $w \in H_r^s(\mathbb{R}^N)$ such that $P(w) > 0$ (see, e.g., Lemma 5.2 in [22]). Then $\alpha > 0$ and $\{v_n\} \subset \mathcal{M}$, where $\mathcal{M}$ is defined in Lemma 3.2.

Since $Q(v_n) = 1$, we may assume that $v_n \rightharpoonup v$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Thus $Q(v) \leq \liminf_{n\to\infty} Q(v_n)$. We claim that $Q(v) = 1$. Indeed, assume that by contradiction that $Q(v) < 1$. By Lemma 3.2, we have $P(v) \geq \limsup_{n\to\infty} P(v_n) = \alpha > 0$. Let $v_t = v(t\cdot)$ for $t > 0$. Then $Q(v_t) = t^{-N+2s}Q(v)$. Recalling that $N > 2s$, there exists a $t_0 \in (0,1)$ such that $Q(v_{t_0}) = 1$. Then we have

$$\alpha \geq P(v_{t_0}) = t_0^{-N}P(v) > P(v) \geq \limsup_{n\to\infty} P(v_n) = \alpha,$$

which is absurd. Thus, $Q(v) = 1$. By Lemma 3.2, α is attained by the nonnegative function v. As a result, there is a Lagrange multiplier θ such that $\frac{1}{2}Q'(v) = \theta P'(u)$. This shows that v solves equation

$$(-\Delta)^s v = \theta(g(v) - \mu v).$$

By Pohozaev identity (2.4), one has

$$\frac{N-2s}{2}P(v) = \theta N Q(v) = \theta N > 0.$$

Thus $\theta > 0$. Let $u = v(\theta^{-\frac{1}{2s}}\cdot)$. Then $u \geq 0$ is a radial solution of (2.2). Moreover, the maximum principle implies that $u > 0$.

Now we check decay property of u. Let $\omega(x,t)$ be the s-harmonic extension of $u(x)$, then ω is a solution of problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla\omega) = 0, \text{ in } \mathbb{R}_+^{N+1}, \\ -k_s\frac{\partial\omega}{\partial\nu} = g(\omega), \qquad \text{on } \mathbb{R}^N \times \{0\}, \end{cases}$$

where $\frac{\partial\omega}{\partial\nu} = \lim_{y\to0^+} y^{1-2s}\frac{\partial\omega}{\partial y}(x,y) = -\frac{1}{k_s}(-\Delta)^s u(x)$. By applying the Moser iterative argument (see, e.g., [15]), we obtain $u(x) = \omega(x,0) \in L^\infty(\mathbb{R}^N)$.

As in Lemma 4.2 in [9], there exist an $R_1 > 0$ and a function w satisfying

$$0 < w(x) \leq \frac{C}{1+|x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N, \tag{3.1.6}$$

and

$$(-\Delta)^s w + (\mu - \beta)w \geq 0, \quad \text{in } \mathbb{R}^N\backslash B_{R_1}(0), \tag{3.1.7}$$

where β appears in the condition (g_2). Notice that $u \in L^2(\mathbb{R}^N)$ is spherically symmetric and decreasing with respect to $r = |x|$, there exists a constant $C > 0$ such that $|u(x)| \leq C|x|^{-\frac{N}{2}}$. By the definition of g_1, there exists an $R_2 > 0$ such that

$$(-\Delta)^s u + (\mu-\beta)u \leq (-\Delta)^s u + g_2(u) = g_1(u) = 0, \quad \text{in } \mathbb{R}^N\backslash B_{R_2}(0). \tag{3.1.8}$$

Taking $R = \max\{R_1, R_2\}$, $\gamma = \inf_{B_R(0)} w$ and $W = (\|u\|_{L^\infty}+1)w - \gamma u$, we see that $\gamma > 0$ and $W \geq w > 0$ in $B_R(0)$. Furthermore, (3.1.7) and (3.1.8) imply that

$$(-\Delta)^s W + (\mu-\beta)W \geq 0, \quad \text{in } \mathbb{R}^N\backslash B_R(0). \tag{3.1.9}$$

We claim that $W \geq 0$ in $\mathbb{R}^N$. Indeed, assume by contradiction that there exists an $x_0 \in \mathbb{R}^N$ such that $W(x_0) < 0$. Note that $\lim_{|x|\to\infty} W(x) = 0$. We assume without loss of generality that $W(x_0) = \inf_{\mathbb{R}^N} W < 0$. Then one has

$$(-\Delta)^s W(x_0) = \frac{1}{2} C_s \int\limits_{\mathbb{R}^N} \frac{2W(x_0) - W(x_0+y) - W(x_0-y)}{|y|^{N+2s}} \mathrm{d}y \leq 0,$$

and therefore $(-\Delta)^s W(x_0) + (\mu-\beta)W(x_0) < 0$. It follows from (3.1.9) that $x_0 \in B_R(0)$, which contradicts $W > 0$ in $B_R(0)$. Thus $W \geq 0$ in $\mathbb{R}^N$ and we conclude from (3.1.6) that

$$u(x) \leq \frac{C}{1+|x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N.$$

□

3.2. Infinitely many solutions of (2.2)

In this part, we shall seek infinitely many solutions of (2.2) which are spherically symmetric.

Define a set of $\mathcal{V}$

$$\mathcal{V}_r = \{u \in \mathcal{V} : u(x) = u(|x|)\}.$$

Let $\Sigma(\mathcal{V})$ denote the set of compact and symmetric subsets of $\mathcal{V}$. Recall that the genus $\gamma(A)$ of a set $A \in \Sigma(\mathcal{V})$ is the least integer $n \geq 1$ such that there exists an odd continuous mapping $\varphi : A \to S^{n-1} = \{z \in \mathbb{R}^N : |z| = 1\}$. We set $\gamma(A) = +\infty$ is such an integer does not exist.

For any integer $k \geq 1$, let

$$c_k = \sup_{A\in\Gamma_k} \inf_{u\in A} P(u),$$

where $\Gamma_k = \{A \in \Sigma(\mathcal{V}_r) : \gamma(A) \geq 1\}$. To obtain Proposition 2.3, we are going to show each c_k is a positive critical value of $P|_{\mathcal{V}_r}$ for any $k \geq 1$.

Lemma 3.3. *$c_k > 0$ for any positive integer k.*

Proof. Consider the k-dimensional polyhedron

$$V^{k-1} = \{r = (r_1, r_2, \cdots, r_k) \in \mathbb{R}^k : \Sigma_{i=1}^k |r_i| = 1\}, \ k \geq 1.$$

By the argument as Theorem 10 in [5], for any $k \geq 1$, there exists a constant $R = R(k) > 0$ and an odd continuous mapping $\tau : V^{k-1} \to H_0^s(B_R)$ such that

(i) $\tau(r)$ is a radial function for all $r \in V^{k-1}$ and $0 \notin \tau(V^{k-1})$;

(ii) There exist constants $d_1, d_2 > 0$ such that $d_1 \leq \|(-\Delta)^{\frac{s}{2}}\big(\tau(r)\big)\|_{L^2(B_R)} \leq d_2$;

(iii) $\int\limits_{B_R} \big(G(\tau(r) - \frac{\mu}{2}\tau^2(r))\big) dr \geq 1$ for any $r \in V^{k-1}$.

Let $W^{k-1} = \tau(V^{k-1})$. For ahy $u \in W^{k-1}$, by (ii), there exists a unique constant $t = t(u) > 0$ such that $u(t\cdot) \in \mathcal{V}_r$ and $\sup_{u\in W^{k-1}} t(u) < \infty$. Thus we may define a mapping $\pi : W^{k-1} \to \mathcal{V}_r$ satisfying $\pi(u) = u(t\cdot)$ for any $u \in W^{k-1}$. From (iii), we then have

$$P(\pi(u)) = t^{-N} \int\limits_{\mathbb{R}^N} \Big(G(u) - \frac{\mu}{2}u^2\Big) \geq t^{-N} \geq \Big(\sup_{u\in W^{k-1}} t(u)\Big)^{-N} > 0.$$

Set $A_k = \pi(W^{k-1})$. Then $A_k \in \Sigma(\mathcal{V}_r)$ and $\gamma(A_k) \geq k$. Hence $A_k \in \Gamma_k$ and $\inf_{u\in A_k} P(u) > 0$. This implies that $c_k > 0$. □

Lemma 3.4. *For any $c > 0$, the functional $P|_{\mathcal{V}_r}$ satisfies the $(PS)_c$ condition, that is, any sequence $\{u_n\} \subset \mathcal{V}_r$ satisfying $P(u_n) \to c > 0$ and $P'|_{\mathcal{V}_r}(u_n) \to 0$ has a convergent subsequence.*

Proof. Let $c > 0$ and $\{u_n\} \subset \mathcal{V}_r$ be a a (PS) sequence of $P|_{\mathcal{V}_r}$. Then $P(u_n) \to c$ and $P'|_{\mathcal{V}_r}(u_n) \to 0 \to 0$. Without loss of generality, we assume that $P(u_n) > 0$ for all n. As in the proof of Lemma 3.2, $\{u_n\}$ is bounded in $H^s_r(\mathbb{R}^N)$. Thus we have $P'(u_n) - \langle P'(u_n), u_n\rangle u_n \to 0$ in $H^{-s}_r(\mathbb{R}^N)$. This implies that

$$d_n(-\Delta)^s u_n - g(u_n) + \mu u_n \to 0 \ in \ D^{-s,2}(\mathbb{R}^N), \tag{3.2.1}$$

where $d_n = \langle P'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \big(g(u_n) - \mu u_n\big)u_n$. From (3.1.3) and the embedding theorem, $\{d_n\}$ is bounded. We may suppose that $d_n \to d$. By the boundedness of $\{u_n\}$, $u_n \rightharpoonup u$ up to a subsequence. Applying Theorem A.1 in [4], for any bounded Borel set $B \subset \mathbb{R}^N$, we have

$$\int_B |g(u_n) - \mu u_n - g(u) + \mu u| \to 0.$$

Thus $g(u_n) - \mu u_n \to g(u) - \mu u$ in $L^1_{loc}(\mathbb{R}^N)$ and $g(u_n) - \mu u_n \to g(u) - \mu u$ in $\mathcal{D}'(\mathbb{R}^N)$. Therefore, we know that from (3.2.1) that

$$d(-\Delta)^s u + \mu u = g(u), \ u \in H^s_r(\mathbb{R}^N). \tag{3.2.2}$$

Moreover, by Pohozaev identity and Lemma 3.2, we have

$$dQ(u) = \frac{2N}{N-2s}P(u) \geq \frac{2N}{N-2s}\limsup_{n\to\infty} P(u_n) = \frac{2Nc}{N-2s} > 0.$$

Thus $d > 0$ and $Q(u) > 0$. Using the argument in Lemma 3.2, it is easy to check that

$$\int_{\mathbb{R}^N} g_1(u_n)u_n \to \int_{\mathbb{R}^N} g_1(u)u \tag{3.2.3}$$

and

$$d = \limsup_{n\to\infty} \int_{\mathbb{R}^N} \big(g(u_n)u_n - \mu u_n^2\big) \leq \int_{\mathbb{R}^N} \big(g(u)u - \mu u^2\big)$$

It then follows from (3.2.2) that $dQ(u) = \int_{\mathbb{R}^N} \big(g(u)u - \mu u^2\big) \geq d$. Thus $Q(u) \geq 1$ and

$$\int_{\mathbb{R}^N} \big(g(u_n)u_n - \mu u_n^2\big) \to \int_{\mathbb{R}^N} \big(g(u)u - \mu u^2\big). \tag{3.2.4}$$

Note that Q is weakly lower semicontinuous. We have $Q(u) = 1$ and $u_n \to u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Since $g_2 = g_1 - g + \mu$, we conclude from (3.2.3) and (3.2.4) that

$$\int_{\mathbb{R}^N} g_2(u_n)u_n \to \int_{\mathbb{R}^N} g_2(u)u.$$

Let $g_2(t) = g_2'(t) + \frac{1}{2}(\mu - \beta)t$. Then $g_2'(t)t \geq \frac{1}{2}(\mu - \beta)t^2$ for any $t \in \mathbb{R}$. By Fatou Lemma, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} g_2'(u_n)u_n \geq \int_{\mathbb{R}^N} g_2'(u)u, \ \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 \geq \int_{\mathbb{R}^N} u^2.$$

Consequently, we get $u_n \to u$ in $L^2(\mathbb{R}^N)$ and $u_n \to u$ in $H^s_r(\mathbb{R}^N)$. □

Proof. (**Proof of Proposition** 2.3.) In view of Lemmas 3.3 and 3.4, a standard argument shows that c_k is a positive critical value of $P|_{\mathcal{V}_r}$ for any integer $k \geq 1$. In particular, there exist $\theta_k \in \mathbb{R}$ and $v_k \in \mathcal{V}_r$ such that $Q(v_k) = 1$, $P(v_k) = c_k$ and

$$(-\Delta)^s v_k = \theta_k\big(g(v_k) - \mu v_k\big).$$

It follows from Pohozaev identity that

$$\frac{N-2s}{2N}Q(v_k) = \theta_k P(v_k) = \theta_k c_k,$$

which implies $\theta_k > 0$. Let $u_k = v_k(\theta_k^{-\frac{1}{2s}}\cdot)$. Then u_k is a solution of (2.2) for any integer $k \geq 1$. Moreover, by Theorem 9 in [5], one has $c_k \to 0^+$. Thus, we have

$$\frac{1}{2}Q(u_k) - P(u_k) = \frac{1}{2}\theta_k^{\frac{N-2s}{2s}}Q(v_k) - \theta_k^{\frac{N}{2s}}P(v_k) = \left[\frac{1}{2}\left(\frac{N-2s}{2s}\right)^{\frac{N-2s}{2s}} - \left(\frac{N-2s}{2s}\right)^{\frac{N}{2s}}\right]c_k^{\frac{2s-N}{2s}} \to +\infty.$$

By the definition of c_k, it is easy to check that $c_{k-1} \geq c_k$ for $k \geq 2$. This shows $Q(u_k) \geq Q(u_{k-1})$ and $Q(u_k) \to +\infty$. Finally, as in the proof of Lemma 2.2, we may obtain the decay inequality (2.6). □

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