Z. Angew. Math. Phys. (2022) 73:13

0044-2275/22/010001-18
published online November 19, 2021
https://doi.org/10.1007/s00033-021-01639-y

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global strong solution to 3D full compressible magnetohydrodynamic flows with vacuum at infinity

Xiaofeng Hou, Mina Jiang and Hongyun Peng

Abstract. In this paper, we consider the Cauchy problem of the full compressible magnetohydrodynamic equations in $\mathbb{R}^3$. When $\|\rho_0\|_{L^1} + \|H_0\|_{L^2}$ is suitably small, we establish the global existence of the strong solution, where ρ_0 and H_0 represent the initial density and magnetic field respectively. Our result shows that the strong solution may have large oscillations and can contain vacuum states.

Mathematics Subject Classification. 76D05, 35K65, 76N10.

Keywords. Compressible magnetohydrodynamic equations, Global strong solutions, Vacuum.

1. Introduction

The equations of the three-dimensional compressible, viscous and heat-conducting magnetohydrodynamic flows can be written as ([20])

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla P = (\operatorname{curl}\ H) \times H, \\ c_v \left[(\rho\theta)_t + \operatorname{div}(\rho u \theta)\right] - \kappa \Delta \theta + P \operatorname{div} u = 2\mu |\mathfrak{D}(u)|^2 + \lambda (\operatorname{div} u)^2 + \nu |\operatorname{curl}\ H|^2, \\ H_t - \operatorname{curl}\ (u \times H) = \nu \Delta H, \quad \operatorname{div} H = 0. \end{cases} \tag{1.1}$$

Here $\mathfrak{D}(u)$ is the deformation tensor given by

$$\mathfrak{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)'\right),$$

$\rho = \rho(x,t),\ u = u(x,t) = (u_1, u_2, u_3)(x,t),\ \theta = \theta(x,t),\ P = R\rho\theta (R > 0)$ and $H = (H_1, H_2, H_3)$, are unknown functions denoting the density, velocity, absolute temperature, pressure and magnetic field respectively; μ and λ are coefficients of viscosity, satisfying the following physical restrictions

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0. \tag{1.2}$$

The positive constants κ and ν are respectively the heat conduction coefficient and the magnetic diffusion coefficient.

In this paper, we are interested in the global existence of strong solutions to the Cauchy problem for (1.1) with the following initial conditions:

$$(\rho,\ u,\ \theta,\ H)|_{t=0} = (\rho_0,\ u_0,\ \theta_0,\ H_0)(x), \quad x \in \mathbb{R}^3 \tag{1.3}$$

and the far field behavior:

$$\rho(x,t) \to 0,\ u(x,t) \to 0,\ \theta(x,t) \to 0,\ H(x,t) \to 0, as \quad |x| \to \infty, \quad for \quad t \geq 0. \tag{1.4}$$

The compressible MHD system (1.1) is obtained by combining the Navier–Stokes equations of fluid dynamics and Maxwell equations of electromagnetism. There are lots of works on the well-posedness of the Cauchy problem and the initial-boundary value problem for Magnetohydrodynamics. The one-dimension problem has been studied in many papers, for example, [3,4,8,9,18,32] and the references cited therein. For the multi-dimensional case, see [6,10,12–17,22–25,27,29] and so on. To be more specific, for the 2D case, Cao and Wu [2] obtained the global existence, conditional regularity and uniqueness of a weak solution with only magnetic diffusion. Ren et al. [28] proved the global existence and the decay estimates of small smooth solution without magnetic diffusion. Yang and Sun [21] got the global existence of weak solutions for any adiabatic exponent $\gamma \geq 1$. Zhong [35] established the global existence and exponential decay of strong solutions to the initial boundary value problem of two-dimensional nonhomogeneous magnetohydrodynamic (MHD) equations with non-negative density.

For the 3D case, when the initial density is strictly positive, Vol'pert et al. [31] investigated the local strong solutions of the compressible MHD with large initial data. This result was extended by Fan et al. [7] to the case that the initial density could contain vacuum. Suen and Hoff [30] proved the global-in-time existence of weak solutions to the 3D compressible MHD equations with initial data small in L^2 and initial density positive and essentially bounded. In [34], they studied the Cauchy problem for the multi-dimensional ($N \geq 3$) non-isentropic full compressible MHD equations.They got the existence and uniqueness of a global strong solution to the system for the initial data close to a stable equilibrium state in critical Besov spaces. For more information on the MHD equations, we refer to [1,26] and the references therein. Hu and Wang [13] studied the existence of a global variational weak solution to the full MHD equations with large data. Li et al. [22] obtained the global classical solution with small initial energy but large oscillations for the isentropic flows. Later on, Hong et al. [10] got the global classical solution with large initial energy but the adiabatic exponent γ is close to 1 and the resistivity coefficient ν is suitably large. Recently, Chen et al. [5] considered the 3D MHD equations with slip boundary condition and vacuum, and obtained the global classical solutions with small energy but large oscillations.

It is worth noting that, Wen and Zhu [33] got the strong solution with vacuum and large initial data for the three-dimensional full compressible Navier–Stokes equations under the condition that the initial mass is small. One natural question is: does there exist some global solution to (1.1) in some classes of large initial data under the condition that the initial mass is small? We will answer the question in this paper. More precisely, we want to find a global strong solution to the 3D full compressible MHD equations under the small initial mass.

Before we state our main results, we would like to introduce some notations which will be used throughout this paper.

1.1. Notations

(i) $\int\limits_{\mathbb{R}^3} f = \int\limits_{\mathbb{R}^3} f \, \mathrm{d}x.$

(ii) For $1 \leq l \leq \infty$, denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$L^l = L^l(\mathbb{R}^3),\ D^{k,l} = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \|\nabla^k u\|_{L^l} < \infty\right\},\ \|u\|_{D^{k,l}} = \|\nabla^k u\|_{L^l},$$
$$W^{k,l} = L^l \cap D^{k,l},\ D^k = D^{k,2},\quad D^1_0 = \Big\{u \in L^6 :\ \|\nabla u\|_{L^2} < \infty\Big\},\quad H^k = W^{k,2}.$$

(iii) $G = (2\mu + \lambda)\mathrm{div} u - P$ is the effective viscous flux.

(iv) $\dot{h} = h_t + u \cdot \nabla h$ denotes the material derivative.

(v) $m_0 = \int\limits_{\mathbb{R}^3} \rho_0$ is the initial mass.

The rest of the paper is organized as follows. In Sect. 2, we present our main results. Section 3 focuses on obtaining the necessary *a priori* estimates for the strong solution to extend the local solution to all time. Finally, we give the proof of the main results.

2. Main results

Before stating the main results, let us make some preliminaries. Assume that μ, λ, κ and ν are constants. We assume $R = C_\nu = 1$ henceforth, since the constants R in the pressure function and C_ν in the internal energy play no role in the analysis. In this case, if the solutions are regular enough (such as strong solutions and classical solutions), (1.1) is equivalent to the following system

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \rho u_t + \rho u\cdot\nabla u + \nabla P(\rho,\theta) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + (\mathrm{curl}\ H)\times H, \\ \rho\theta_t + \rho u\cdot\nabla\theta + \rho\theta \mathrm{div} u = \frac{\mu}{2}\left|\nabla u + (\nabla u)'\right|^2 + \lambda(\mathrm{div} u)^2 + \kappa\Delta\theta + \nu|\mathrm{curl}\ H|^2, \\ H_t - \mathrm{curl}\ (u\times H) = \nu\Delta H, \quad \mathrm{div} H = 0 \ \text{ in } \mathbb{R}^3\times(0,\infty). \end{cases} \tag{2.1}$$

System (2.1) is supplemented with initial conditions

$$(\rho,u,\theta,H)|_{t=0} = (\rho_0,u_0,\theta_0,H_0)(x), \ x\in\mathbb{R}^3, \tag{2.2}$$

and the far-field conditions

$$\rho(x,t)\to 0,\ u(x,t)\to 0,\ \theta(x,t)\to 0,\ H(x,t)\to 0, \text{as } |x|\to\infty,\ \text{for } t\geq 0. \tag{2.3}$$

First, the well-known Gagliardo–Nirenberg inequality will be used frequently later (see [19]).

Lemma 2.1. *For any $p\in[2,6], q\in(1,\infty)$ and $r\in(3,\infty)$, there exists some generic constant $C>0$ that may depend on q and r such that for $f\in H^1(\mathbb{R}^3)$ and $g\in L^q(\mathbb{R}^3)\cap D^{1,r}(\mathbb{R}^3)$, we have*

$$\|f\|_{L^p(\mathbb{R}^3)}^p \leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{6-p}{2}}\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{3p-6}{2}}, \tag{2.4}$$

$$\|g\|_{C(\mathbb{R}^3)} \leq C\|g\|_{L^q(\mathbb{R}^3)}^{\frac{q(r-3)}{3r+q(r-3)}}\|\nabla g\|_{L^r(\mathbb{R}^3)}^{\frac{3r}{3r+q(r-3)}}. \tag{2.5}$$

Next, we give the definition of the strong solution to (2.1)–(2.3) throughout this paper, which is similar to [11].

Definition 2.2. (*Strong solution*) For $T>0$, (ρ,u,θ,H) is called a strong solution to the compressible Magnetohydrodynamic flows (2.1)–(2.3) in $\mathbb{R}^3\times[0,T]$, if for some $q\in(3,6)$,

$$\begin{aligned} &0\leq\rho\in C([0,T];W^{1,q}\cap H^1),\ \rho_t\in C([0,T];L^2\cap L^q), \\ &(u,\theta,H)\in C([0,T];D^2\cap D_0^1)\cap L^2([0,T];D^{2,q}),\ \theta\geq 0, \\ &(u_t,\theta_t,H_t)\in L^2([0,T];D_0^1),\ (\sqrt{\rho}u_t,\sqrt{\rho}\theta_t,H_t)\in L^\infty([0,T];L^2), \end{aligned} \tag{2.6}$$

and (ρ,u,θ,H) satisfies (2.1) a.e. in $\mathbb{R}^3\times(0,T]$. In particular, the strong solution (ρ,u,θ,H) of (2.1)–(2.3) is called a global strong solution, if the strong solution satisfies (2.6) for any $T>0$, and satisfies (2.1) a.e. in $\mathbb{R}^3\times(0,\infty)$.

Then the main results in this paper can be stated as follows:

Theorem 2.3. (Global strong solution) *Assume that the initial data $(\rho_0,u_0,\theta_0,H_0)$ satisfies*

$$\rho_0\geq 0,\ \theta_0\geq 0,\ in\ \mathbb{R}^3,\ \rho_0\in H^1\cap W^{1,q}\cap L^1,\ (u_0,\theta_0,H_0)\in D^2\cap D_0^1 \tag{2.7}$$

and

$$\begin{cases} 0 \le \rho_0 \le \bar{\rho}, \quad m_0 = \int\limits_{\mathbb{R}^3} \rho_0, \quad \|H_0\|_{L^2}^2 \le m_0^{\frac{1}{2}}, \\ (1+\nu)\|\nabla H_0\|_{L^2}^2 + \|\sqrt{\rho_0}\theta_0\|_{L^2}^2 + \mu\|\nabla u_0\|_{L^2} + (\mu+\lambda)\|\mathrm{div} u_0\|_{L^2} + \frac{1}{2(2\mu+\lambda)}\|\rho_0\theta_0\|_{L^2}^2 \\ \quad + \int\limits_{\mathbb{R}^3} |\rho_0\theta_0 \mathrm{div} u_0| + \int\limits_{\mathbb{R}^3} \left(|H_0|^2|\mathrm{div} u_0| + 2|H_0 \cdot \nabla u_0 \cdot H_0|\right) \le K \end{cases} \tag{2.8}$$

for some constants $K > 1$, $\bar{\rho} > 0$ and $q \in (3, 6)$, and that the following compatibility conditions are satisfied:

$$\begin{cases} \mu\Delta u_0 + (\mu+\lambda)\nabla \mathrm{div} u_0 - \nabla P(\rho_0, \theta_0) - (\mathrm{curl} H_0) \times H_0 = \sqrt{\rho}_0 g_1, \\ \kappa\Delta\theta_0 + \frac{\mu}{2}\left|\nabla u_0 + (\nabla u_0)'\right|^2 + \lambda(\mathrm{div} u_0)^2 + \nu|\mathrm{curl} H_0|^2 = \sqrt{\rho_0} g_2, \ x \in \mathbb{R}^3, \end{cases} \tag{2.9}$$

with $g_i \in L^2$, $i = 1, 2$. Then there exists a unique global strong solution (ρ, u, θ, H) in $\mathbb{R}^3 \times [0, T]$ for any $T > 0$, provided

$$m_0 \le \varepsilon,$$

where ε is a positive constant depending on $\bar{\rho}$, K, μ and λ but independent of t.

3. Proof of Theorem 2.3

In this section, we will prove the global existence and uniqueness of the strong solution to the problem (2.1)–(2.3). The local existence and uniqueness of the strong solution has been obtained in [11] under the conditions of Theorem 2.3. In order to get the global solution, we denote

$$A(T) = \sup_{0 \le t \le T} \int\limits_{\mathbb{R}^3} |\nabla u|^2 + \int\limits_0^T \int\limits_{\mathbb{R}^3} \rho|\dot{u}|^2$$

and

$$B(T) = \sup_{0 \le t \le T} \int\limits_{\mathbb{R}^3} \rho\theta^2 + \int\limits_0^T \int\limits_{\mathbb{R}^3} |\nabla\theta|^2.$$

The following proposition plays a crucial role in the proof.

Proposition 3.1. *Assume that the initial data satisfies (2.7), (2.8) and (2.9). If the solution (ρ, u, θ, H) of (2.1)–(2.3) satisfies that for $(x, t) \in \mathbb{R}^3 \times [0, T]$,*

$$\begin{aligned} &\|H\|_{L^2}^2 \le 2m_0^{\frac{1}{2}}, \quad \|\nabla H\|_{L^2}^2 \le 2K, \quad A(T) \le 2\tilde{E}K, \\ &B(T) \le 2K, \quad 0 \le \rho \le 2\bar{\rho}, \end{aligned} \tag{3.1}$$

then

$$\begin{aligned} &\|H\|_{L^2}^2 \le \frac{3m_0^{\frac{1}{2}}}{2}, \quad \|\nabla H\|_{L^2}^2 \le \frac{3K}{2}, \quad A(T) \le \frac{3\tilde{E}K}{2}, \\ &B(T) \le \frac{3K}{2}, \quad 0 \le \rho \le \frac{3\bar{\rho}}{2}, \quad (x, t) \in \mathbb{R}^3 \times [0, T], \end{aligned} \tag{3.2}$$

provided

$$m_0 \le \min\left\{\frac{1}{CK^6}, \quad \frac{1}{C\left(\tilde{E}^2K\right)^2}, \quad \frac{1}{C\left(K^{\frac12}+\tilde{E}Km_0^{\frac{1}{12}}\right)^4}, \right.$$
$$\left. \frac{1}{C\left(\tilde{E}^{\frac32}K+\tilde{E}K^{\frac54}\right)^6}, \quad \frac{1}{C(\tilde{E}K)^{\frac{144}{45}}}, \quad \frac{1}{C\left(\tilde{E}K+m_0^{\frac14}K^{\frac32}\right)^{48}}\right\},$$

where

$$\tilde{E} = 1+\frac{4}{\mu}+\frac{2\bar{\rho}}{\mu(\mu+\lambda)}+\frac{6K}{\mu}.$$

Proof of Proposition 3.1. The Proposition 3.1 can be proved by the Lemmas 3.2–3.7 below.

Throughout the rest of the paper, we denote by C or C_i, $(i=1,2,\ldots)$ the generic positive constants which may depend on $\mu,\lambda,\kappa,\nu,\bar{\rho}$ and K but independent of time T. □

Lemma 3.2. *Under the assumptions of Proposition 3.1, it holds that*

$$\int_{\mathbb{R}^3} \rho = \int_{\mathbb{R}^3} \rho_0, \tag{3.3}$$

for any $t\in[0,T]$.

The proof of Lemma 3.2 can be found in [33], and we omit it here.

Lemma 3.3. *Under the assumptions of Proposition 3.1, it holds that*

$$\int_{\mathbb{R}^3}(\rho|u|^2+|H|^2)+\mu\int_0^T\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}s+\nu\int_0^T\int_{\mathbb{R}^3}|\nabla H|^2\mathrm{d}s \le \frac{3m_0^{\frac12}}{2}, \tag{3.4}$$

provided

$$m_0 \le \frac{1}{CK^6}.$$

Proof. Multiplying $(2.1)_2$ and $(2.1)_4$ by u and H respectively, integrating by parts over $\mathbb{R}^3$ and summing up the resulting equations, one gets that

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(\rho|u|^2+|H|^2)+\int_{\mathbb{R}^3}\mu|\nabla u|^2+\int_{\mathbb{R}^3}\left(\nu|\nabla H|^2+(\mu+\lambda)|\mathrm{div}u|^2\right)\\
&=\int_{\mathbb{R}^3}\rho\theta\mathrm{div}u+\int_{\mathbb{R}^3}(\mathrm{curl}\ H)\times H\cdot u+\int_{\mathbb{R}^3}\mathrm{curl}\ (u\times H)\cdot H\\
&:=\sum_{i=1}^3 I_i.
\end{aligned} \tag{3.5}$$

Then we estimate the terms on the right-hand side. For I_1, it follows that by using Cauchy inequality,

$$I_1 \le (\mu+\lambda)\int_{\mathbb{R}^3}|\mathrm{div}u|^2+\frac{1}{4(\mu+\lambda)}\int_{\mathbb{R}^3}\rho^2\theta^2. \tag{3.6}$$

Next, to estimate I_2 and I_3, note that

$$(\mathrm{curl}\ H)\times H = -\frac12\nabla|H|^2+H\cdot\nabla H = -\frac12\nabla|H|^2+\mathrm{div}(H\otimes H) \tag{3.7}$$

and

$$\text{curl }(u\times H)=H\cdot\nabla u-u\cdot\nabla H-\text{div}u\cdot H, \tag{3.8}$$

where we have used the fact $\text{div}H=0$. This, together with Hölder inequality, Sobolev inequality, Cauchy inequality and (3.1), deduces

$$\begin{aligned}
I_2&=\frac{1}{2}\int\limits_{\mathbb{R}^3}|H|^2\text{div}u+\int\limits_{\mathbb{R}^3}\text{div}(H\otimes H)u\\
&\le C\|\text{div}u\|_{L^2}\|H\|_{L^6}\|H\|_{L^3}+C\|H\|_{L^6}\|\nabla u\|_{L^2}\|H\|_{L^3}\\
&\le C\|\nabla H\|_{L^2}\|\nabla u\|_{L^2}\|\nabla H\|_{L^2}^{\frac12}\|H\|_{L^2}^{\frac12}\\
&\le \frac{\mu}{4}\|\nabla u\|_{L^2}^2+C\|\nabla H\|_{L^2}^2\|\nabla H\|_{L^2}\|H\|_{L^2}\\
&\le \frac{\mu}{4}\|\nabla u\|_{L^2}^2+CK^{\frac12}m_0^{\frac14}\|\nabla H\|_{L^2}^2,
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
I_3&=\int\limits_{\mathbb{R}^3}H\cdot\nabla u\cdot H-\int\limits_{\mathbb{R}^3}u\cdot\nabla H\cdot H+\int\limits_{\mathbb{R}^3}u\cdot\nabla|H|^2\\
&\le C\|H\|_{L^6}\|\nabla u\|_{L^2}\|H\|_{L^3}\\
&\le \frac{\mu}{4}\|\nabla u\|_{L^2}^2+CK^{\frac12}m_0^{\frac14}\|\nabla H\|_{L^2}^2.
\end{aligned}\tag{3.10}$$

Substituting (3.6) and (3.9)–(3.10) into (3.5) and setting $m_0\le\frac{\nu^4}{C^4K^2}$, one may arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}(\rho|u|^2+|H|^2)+\mu\int\limits_{\mathbb{R}^3}|\nabla u|^2+\nu\int\limits_{\mathbb{R}^3}|\nabla H|^2\le C\|\rho\|_{L^3}^2\|\theta\|_{L^6}^2\le Cm_0^{\frac23}\|\nabla\theta\|_{L^2}^2. \tag{3.11}$$

Integrating (3.11) over $[0,T]$, and using (3.1) again, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3}(\rho|u|^2+|H|^2)+\mu\int_0^T\int\limits_{\mathbb{R}^3}|\nabla u|^2+\nu\int_0^T\int\limits_{\mathbb{R}^3}|\nabla H|^2\\
&\quad\le\int\limits_{\mathbb{R}^3}(\rho_0|u_0|^2+|H_0|^2)+Cm_0^{\frac23}\int_0^T\|\nabla\theta\|_{L^2}^2\\
&\quad\le C\|\rho_0\|_{L^{\frac32}}\|\nabla u_0\|_{L^2}^2+m_0^{\frac12}+Cm_0^{\frac23}K\le m_0^{\frac12}\left(1+Cm_0^{\frac16}K\right)\le\frac{3m_0^{\frac12}}{2},
\end{aligned}$$

provided

$$m_0\le\frac{1}{CK^6}.$$

□

Lemma 3.4. *Under the conditions of Proposition 3.1, it holds that*

$$\|\nabla H\|_{L^2}^2+\int_0^T\left(\|H_t\|_{L^2}^2+\|\Delta H\|_{L^2}^2\right)\mathrm{d}s\le\frac{3K}{2}, \tag{3.12}$$

provided

$$m_0 \le \frac{1}{C\left(\tilde{E}^2 K\right)^2}.$$

Proof. Applying $\partial_j,\ j=1,2,3$ to $(2.1)_4$, multiplying the resulting equations by $\partial_j H$, summing with respect to j, then integrating by parts over $\mathbb{R}^3\times[0,T]$ and using Cauchy inequality, one has

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla H|^2+\nu\int_0^T\|\Delta H\|_{L^2}^2\mathrm{d}s\\
&=\frac{1}{2}\|\nabla H_0\|_{L^2}^2-\int_0^T\int_{\mathbb{R}^3}\mathrm{curl}\ (u\times H)\cdot\Delta H\mathrm{d}s\\
&\le\frac{1}{2}\|\nabla H_0\|_{L^2}^2+\frac{\nu}{4}\int_0^T\|\Delta H\|_{L^2}^2\mathrm{d}s+\frac{1}{\nu}\int_0^T\|\mathrm{curl}\ (u\times H)\|_{L^2}^2\mathrm{d}s.
\end{aligned}\tag{3.13}$$

We estimate the last term on the right-hand side of (3.13) as follows:

$$\begin{aligned}
\|\mathrm{curl}\ (u\times H)\|^2&\le\|u\cdot\nabla H\|_{L^2}^2+\|H\cdot\nabla u\|_{L^2}^2+\|\mathrm{div}uH\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^3}^2+C\|H\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}^2\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2}\\
&\le\frac{\nu^2}{4}\|\Delta H\|_{L^2}^2+\frac{C}{\nu^2}\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2.
\end{aligned}\tag{3.14}$$

On the other hand, multiplying $(2.1)_4$ by H_t, integrating by parts over $\mathbb{R}^3\times[0,T]$ and using Cauchy inequality, one has

$$\begin{aligned}
\frac{\nu}{2}\int_{\mathbb{R}^3}|\nabla H|^2+\int_0^T\|H_t\|_{L^2}^2\mathrm{d}s&\le\frac{\nu}{2}\|\nabla H_0\|_{L^2}^2+\int_0^T\|\mathrm{curl}\ (u\times H)\|^2\\
&\le\frac{\nu}{2}\|\nabla H_0\|_{L^2}^2+\frac{\nu}{4}\int_0^T\|\Delta H\|_{L^2}^2\mathrm{d}s+\frac{C}{\nu}\int_0^T\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2\mathrm{d}s.
\end{aligned}\tag{3.15}$$

Combing (3.13) with (3.15) implies

$$\begin{aligned}
\int_{\mathbb{R}^3}|\nabla H|^2+\int_0^T\left(\|H_t\|_{L^2}^2+\|\Delta H\|_{L^2}^2\right)\mathrm{d}s&\le\frac{1+\nu}{2}\|\nabla H_0\|_{L^2}^2+\frac{C}{\nu}\int_0^T\|\nabla u\|_{L^2}^4\|\nabla H\|_{L^2}^2\mathrm{d}s\\
&\le K+C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^4\int_0^T\|\nabla H\|_{L^2}^2\mathrm{d}s\\
&\le K+C\tilde{E}^2K^2m_0^{\frac{1}{2}}\le\frac{3}{2}K,
\end{aligned}$$

provided

$$m_0 \le \frac{1}{C\left(\tilde{E}^2 K\right)^2}.$$

□

Lemma 3.5. *Under the conditions of Proposition 3.1, it holds that*

$$A(T) \leq \frac{3\tilde{E}K}{2}, \tag{3.16}$$

provided

$$m_0 \leq \min\left\{\frac{1}{CK^6}, \quad \frac{1}{C\left(\tilde{E}^2K\right)^2}, \quad \frac{1}{C\left(K^{\frac{1}{2}}+\tilde{E}Km_0^{\frac{1}{12}}\right)^4}\right\}.$$

Proof. Multiplying $(2.1)_2$ by u_t and integrating by parts over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \rho|\dot{u}|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P\mathrm{div}u - \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu+\lambda}\int_{\mathbb{R}^3} P_tG\\
&\quad + \int_{\mathbb{R}^3}\rho(u\cdot\nabla)u\cdot\dot{u} + \int_{\mathbb{R}^3}(\mathrm{curl}\ H)\times H\cdot u_t\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P\mathrm{div}u - \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} P^2 + \sum_{i=1}^{3} II_i,
\end{aligned}\tag{3.17}$$

where $G = (2\mu+\lambda)\mathrm{div}u - P$. Recalling $P = \rho\theta$, we obtain from $(2.1)_1$ and $(2.1)_3$

$$P_t = -\mathrm{div}(\rho\theta u) - \rho\theta\mathrm{div}u + \frac{\mu}{2}\left|\nabla u + (\nabla u)'\right|^2 + \kappa\Delta\theta + \nu|\mathrm{curl}\ H|^2. \tag{3.18}$$

Substituting (3.18) into II_1, and by integration by parts, Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
II_1 &\leq C\|\rho\theta\|_{L^3}\|u\|_{L^6}\|\nabla G\|_{L^2} + C\|\rho\theta\|_{L^3}\|\mathrm{div}u\|_{L^2}\|G\|_{L^6} + C\|G\|_{L^6}\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\\
&\quad + \frac{\kappa}{2\mu+\lambda}\|\nabla G\|_{L^2}\|\nabla\theta\|_{L^2} + C\|G\|_{L^6}\|\mathrm{curl}\ H\|_{L^2}\|\mathrm{curl}\ H\|_{L^3}\\
&\leq C\|\rho\|_{L^6}\|\theta\|_{L^6}\|\nabla u\|_{L^2}\|\nabla G\|_{L^2} + C\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\\
&\quad + \frac{\kappa}{2\mu+\lambda}\|\nabla G\|_{L^2}\|\nabla\theta\|_{L^2} + C\|\nabla G\|_{L^2}\|\nabla H\|_{L^2}^{\frac{3}{2}}\|\nabla^2 H\|_{L^2}^{\frac{1}{2}},
\end{aligned}$$

and

$$II_2 \leq C\|\sqrt{\rho}\dot{u}\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^3} \leq C\|\sqrt{\rho}\dot{u}\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}.$$

Taking div and curl, respectively, on both side of $(2.1)_2$, we get

$$\Delta G = \mathrm{div}(\rho\dot{u}) - \mathrm{div}[(\mathrm{curl}\ H)\times H], \tag{3.19}$$

$$\mu\Delta(\mathrm{curl}u) = \mathrm{curl}(\rho\dot{u}) - \mathrm{curl}[(\mathrm{curl}\ H)\times H]. \tag{3.20}$$

By (3.19), (3.20), the standard L^2-estimates, and (3.1), we get

$$\|\nabla G\|_{L^2} \leq \|\rho\dot{u}\|_{L^2} + \|(\mathrm{curl}\ H)\times H\|_{L^2} \leq \sqrt{2\bar{\rho}}\|\sqrt{\rho}\dot{u}\|_{L^2} + \|(\mathrm{curl}\ H)\times H\|_{L^2} \tag{3.21}$$

and

$$\|\nabla\mathrm{curl}u\|_{L^2} \leq \|\rho\dot{u}\|_{L^2} + \|(\mathrm{curl}\ H)\times H\|_{L^2} \leq \sqrt{2\bar{\rho}}\|\sqrt{\rho}\dot{u}\|_{L^2} + \|(\mathrm{curl}\ H)\times H\|_{L^2}. \tag{3.22}$$

Since $\nabla u = \nabla\Delta^{-1}(\nabla\mathrm{div}u - \nabla\times\mathrm{curl}u)$, we apply the Calderon-Zygmund inequality to get

$$\|\nabla u\|_{L^3} \leq C\big(\|\mathrm{curl}u\|_{L^3} + \|\mathrm{div}u\|_{L^3}\big). \tag{3.23}$$

Using (3.21)–(3.23), Sobolev inequality, Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\nabla u\|_{L^3} &\le C\|\mathrm{curl} u\|_{L^3} + C\|G\|_{L^3} + C\|\rho\theta\|_{L^3}\\ &\le C\|\mathrm{curl} u\|_{L^2}^{\frac12}\|\nabla \mathrm{curl} u\|_{L^2}^{\frac12} + C\|G\|_{L^2}^{\frac12}\|\nabla G\|_{L^2}^{\frac12} + C\|\rho\|_{L^6}\|\theta\|_{L^6}\\ &\le C\|\mathrm{curl} u\|_{L^2}^{\frac12}\left(\|\sqrt{\rho}\dot u\|_{L^2} + C\||H||\nabla H|\|_{L^2}\right)^{\frac12}\\ &\quad + C\|G\|_{L^2}^{\frac12}\left(\|\sqrt{\rho}\dot u\|_{L^2} + C\||H||\nabla H|\|_{L^2}\right)^{\frac12} + Cm_0^{\frac16}\|\nabla\theta\|_{L^2}. \end{aligned} \tag{3.24}$$

It follows from (3.21),(3.24) that

$$\begin{aligned} II_1 + II_2 &\le C\left(m_0^{\frac16}\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla u\|_{L^3} + \kappa\|\nabla\theta\|_{L^2} + \|\nabla H\|_{L^2}\|\nabla H\|_{L^3}\right)\left(\|\sqrt{\rho}\dot u\|_{L^2}\right.\\ &\quad \left.+\||H||\nabla H|\|_{L^2}\right)\\ &\le C\|\nabla u\|_{L^2}\left(\|\nabla u\|_{L^2} + \|P\|_{L^2}\right)^{\frac12}\left(\|\sqrt{\rho}\dot u\|_{L^2} + \||H||\nabla H|\|_{L^2}\right)^{\frac32}\\ &\quad + C\left(m_0^{\frac16}\|\nabla u\|_{L^2}\right)\|\nabla\theta\|_{L^2}\left(\|\sqrt{\rho}\dot u\|_{L^2} + \||H||\nabla H|\|_{L^2}\right)\\ &\quad + \frac{\kappa}{2\mu+\lambda}\|\nabla\theta\|_{L^2}\left(\sqrt{2\bar\rho}\|\sqrt{\rho}\dot u\|_{L^2} + \|(\mathrm{curl}\ H)\times H\|_{L^2}\right)\\ &\quad + C\|\nabla H\|_{L^2}^3\|\nabla^2 H\|_{L^2}\left(\|\sqrt{\rho}\dot u\|_{L^2} + \||H||\nabla H|\|_{L^2}\right)\\ &\le C\|\nabla u\|_{L^2}^6 + Cm_0\|\nabla u\|_{L^2}^4 + \frac12\|\sqrt{\rho}\dot u\|_{L^2}^2 + C\||H||\nabla H|\|_{L^2}^2 + Cm_0^{\frac13}\|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}^2\\ &\quad + \frac{\kappa^2(2\bar\rho+1)}{(2\mu+\lambda)^2}\|\nabla\theta\|_{L^2}^2 + C\|\nabla H\|_{L^2}^6 + \|\nabla^2 H\|_{L^2}^2. \end{aligned} \tag{3.25}$$

For II_3, by using (3.1), (3.7), Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} II_3 &= \frac12\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u_t - 2H\cdot\nabla u_t\cdot H\right)\\ &= \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u - 2H\cdot\nabla u\cdot H\right) - \int_{\mathbb{R}^3}\left(H\cdot H_t\mathrm{div} u - H_t\cdot\nabla u\cdot H - H\cdot\nabla u\cdot H_t\right)\\ &\le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u - 2H\cdot\nabla u\cdot H\right) + C\|H_t\|_{L^2}\|\nabla H\|_{L^2}\|\nabla u\|_{L^3}\\ &\le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u - 2H\cdot\nabla u\cdot H\right) + \|H_t\|_{L^2}^2\|\nabla H\|_{L^2}^2 + C\|\nabla u\|_{L^3}^2\\ &\le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u - 2H\cdot\nabla u\cdot H\right) + \|H_t\|_{L^2}^2\|\nabla H\|_{L^2}^2\\ &\quad + C\left(\|\nabla u\|_{L^2} + \|\rho\theta\|_{L^2}\right)\left(\|\sqrt{\rho}\dot u\|_{L^2} + \||H||\nabla H|\|_{L^2}\right)\\ &\le \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div} u - 2H\cdot\nabla u\cdot H\right) + \|H_t\|_{L^2}^2\|\nabla H\|_{L^2}^2 + \frac14\|\sqrt{\rho}\dot u\|_{L^2}^2\\ &\quad + C\left(\||H||\nabla H|\|_{L^2}^2 + \|\rho\|_{L^3}^2\|\theta\|_{L^6}^2 + \|\nabla u\|_{L^2}^2\right). \end{aligned} \tag{3.26}$$

By (2.4), Lemmas 3.3 and 3.4 we have

$$\||H||\nabla H|\|_{L^2}^2 \le C\|H\|_{L^2}\|\nabla H\|_{L^2}\|\nabla^2 H\|_{L^2}^2 \le Cm_0^{\frac14}K^{\frac12}\|\nabla^2 H\|_{L^2}^2. \tag{3.27}$$

Substituting (3.25), (3.26) into (3.17), and integrating the resulting equation over $[0,T]$, using Cauchy inequality, we have

$$\begin{aligned}
&\frac{1}{4}\int_0^T\int_{\mathbb{R}^3}\rho|\dot{u}|^2\mathrm{d}s+\mu\int_{\mathbb{R}^3}|\nabla u|^2+(\mu+\lambda)\int_{\mathbb{R}^3}|\mathrm{div}u|^2+\frac{1}{2(2\mu+\lambda)}\int_{\mathbb{R}^3}\rho^2\theta^2\\
&\le \mu\|\nabla u_0\|_{L^2}+(\mu+\lambda)\|\mathrm{div}u_0\|_{L^2}+\frac{1}{2(2\mu+\lambda)}\|\rho_0\theta_0\|_{L^2}^2-\int_{\mathbb{R}^3}\rho_0\theta_0\mathrm{div}u_0-\int_{\mathbb{R}^3}\left(|H_0|^2\mathrm{div}u_0-2H_0\cdot\nabla u_0\cdot H_0\right)\\
&\quad+\int_{\mathbb{R}^3}P\mathrm{div}u+\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div}u-2H\cdot\nabla u\cdot H\right)+\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^4+m_0\|\nabla u\|_{L^2}^2+1\right)\int_0^T\|\nabla u\|_{L^2}^2\mathrm{d}s\\
&\quad+\sup_{0\le t\le T}\left(m_0^{\frac{1}{3}}\|\nabla u\|_{L^2}^2+m_0^{\frac{2}{3}}+\frac{\kappa^2(2\bar{\rho}+1)}{(2\mu+\lambda)^2}\right)\int_0^T\|\nabla\theta\|_{L^2}^2\mathrm{d}s+\sup_{0\le t\le T}\|\nabla H\|_{L^2}^2\int_0^T\|H_t\|_{L^2}^2\mathrm{d}s\\
&\quad+\left(Cm_0^{\frac{1}{4}}K_0^{\frac{1}{2}}+1\right)\int_0^T\|\nabla^2H\|_{L^2}^2\mathrm{d}s+C\sup_{0\le t\le T}\|\nabla H\|_{L^2}^4\int_0^T\|\nabla H\|_{L^2}^2\mathrm{d}s.
\end{aligned}\tag{3.28}$$

It follows by Hölder inequality and (2.4) that

$$\int_{\mathbb{R}^3}P\mathrm{div}u\le\|\rho\theta\|_{L^2}\|\mathrm{div}u\|_{L^2}\le\frac{\bar{\rho}}{4(\mu+\lambda)}\|\sqrt{\rho}\theta\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}u\|_{L^2}^2,\tag{3.29}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(|H|^2\mathrm{div}u-2H\cdot\nabla u\cdot H\right)\le C\|H\|_{L^4}^2\|\nabla u\|_{L^2}\\
&\le C\|H\|_{L^2}^{\frac{1}{2}}\|\nabla H\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\le C\|H\|_{L^2}\|\nabla H\|_{L^2}^3+\frac{\mu}{2}\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.30}$$

Then (3.28) together with (3.29), (3.30) leads to

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}\rho|\dot{u}|^2\mathrm{d}s+\int_{\mathbb{R}^3}|\nabla u|^2\\
&\quad\le\frac{2K}{\mu}+\frac{2\bar{\rho}}{\mu(\mu+\lambda)}\|\sqrt{\rho}\theta\|_{L^2}^2+C\|H\|_{L^2}\|\nabla H\|_{L^2}^3+C\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^4+m_0\|\nabla u\|_{L^2}^2+1\right)\int_0^T\|\nabla u\|_{L^2}^2\mathrm{d}s\\
&\qquad+\sup_{0\le t\le T}\left(m_0^{\frac{1}{3}}\|\nabla u\|_{L^2}^2+m_0^{\frac{2}{3}}+\frac{\kappa^2(2\bar{\rho}+1)}{(2\mu+\lambda)^2}\right)\int_0^T\|\nabla\theta\|_{L^2}^2\mathrm{d}s+\frac{2}{\mu}\sup_{0\le t\le T}\|\nabla H\|_{L^2}^2\int_0^T\|H_t\|_{L^2}^2\mathrm{d}s\\
&\qquad+\left(Cm_0^{\frac{1}{4}}K_0^{\frac{1}{2}}+\frac{2}{\mu}\right)\int_0^T\|\nabla^2H\|_{L^2}^2\mathrm{d}s+C\sup_{0\le t\le T}\|\nabla H\|_{L^2}^4\int_0^T\|\nabla H\|_{L^2}^2\mathrm{d}s\\
&\quad\le\left(\frac{2}{\mu}+\frac{2\bar{\rho}}{\mu(\mu+\lambda)}\right)K+Cm_0^{\frac{1}{4}}K^{\frac{3}{2}}+C\left(4\tilde{E}^2K^2+m_0\tilde{E}K+1\right)m_0^{\frac{1}{2}}+C\left(2m_0^{\frac{1}{3}}\tilde{E}K+m_0^{\frac{2}{3}}\right)K\\
&\qquad+4K^2+\left(Cm_0^{\frac{1}{4}}K^{\frac{1}{2}}+C_1\right)K+CK^2m_0^{\frac{1}{2}}
\end{aligned}$$

$$\leq \left(\frac{4}{\mu}+\frac{2\bar{\rho}}{\mu(\mu+\lambda)}+\frac{6K}{\mu}\right)K + C\left(K^{\frac12}+\tilde{E}Km_0^{\frac14}+\tilde{E}m_0^{\frac54}+\frac{m_0^{\frac14}}{K}+Km_0^{\frac1{12}}+m_0^{\frac5{12}}+Km_0^{\frac14}\right)Km_0^{\frac14}$$
$$\leq \tilde{E}K + C\left(K^{\frac12}+\tilde{E}Km_0^{\frac1{12}}\right) \leq \frac{3\tilde{E}K}{2},$$

provided

$$m_0 \leq \frac{1}{C\left(K^{\frac12}+\tilde{E}Km_0^{\frac1{12}}\right)^4}$$

with

$$\tilde{E} = 1+\frac{4}{\mu}+\frac{2\bar{\rho}}{\mu(\mu+\lambda)}+\frac{6K}{\mu}.$$

Thus, the proof of Lemma 3.5 is completed. □

Lemma 3.6. *Under the conditions of Proposition 3.1, it holds that*

$$B(T) \leq \frac{3K}{2},$$

provided

$$m_0 \leq \min\left\{\frac{1}{CK^6}, \frac{1}{C\left(\tilde{E}^2K\right)^2}, \frac{1}{C\left(K^{\frac12}+\tilde{E}Km_0^{\frac1{12}}\right)^4}, \frac{1}{C\left(\tilde{E}^{\frac32}K+\tilde{E}K^{\frac54}\right)^6}\right\}.$$

Proof. Multiplying $(2.1)_3$ by θ, and integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|\theta|^2+\kappa\int_{\mathbb{R}^3}|\nabla\theta|^2\\
&=-\int_{\mathbb{R}^3}\rho\theta^2\mathrm{div}u+\int_{\mathbb{R}^3}\frac{\mu}{2}|\nabla u+(\nabla u)'|^2\theta+\int_{\mathbb{R}^3}\lambda(\mathrm{div}u)^2\theta+\nu\int_{\mathbb{R}^3}|\mathrm{curl}\ H|^2\theta\\
&:=\sum_{i=1}^4 III_i.
\end{aligned}\tag{3.31}$$

For III_1, it follows by Hölder inequality and Sobolev inequality that

$$III_1 \leq C\|\mathrm{div}u\|_{L^2}\|\theta\|_{L^6}^2\|\rho\|_{L^6} \leq Cm_0^{\frac16}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2.\tag{3.32}$$

For III_2 and III_3, by virtue of Hölder inequality and Sobolev inequality, together with (3.24), one has

$$\begin{aligned}
III_2+III_3 &\leq C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\|\theta\|_{L^6}\\
&\leq C\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}\|\nabla\theta\|_{L^2}\\
&\leq C\|\nabla u\|_{L^2}^{\frac32}\|\nabla\theta\|_{L^2}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)^{\frac12}\\
&\quad + C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\rho\theta\|_{L^2}^{\frac12}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)^{\frac12}\\
&\quad + Cm_0^{\frac16}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2.
\end{aligned}\tag{3.33}$$

Then we estimate III_4 as

$$III_4 \leq C\|\nabla H\|_{L^2}\|\theta\|_{L^6}\|\nabla H\|_{L^3} \leq C\|\nabla\theta\|_{L^2}\|\nabla H\|_{L^2}^{\frac32}\|\nabla^2 H\|_{L^2}^{\frac12}.\tag{3.34}$$

We substitute (3.32)–(3.33) into (3.31) to obtain

$$
\begin{aligned}
\kappa\int_{\mathbb{R}^3}|\nabla\theta|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\rho|\theta|^2 &\le C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla\theta\|_{L^2}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)^{\frac{1}{2}}\\
&\quad+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}\|\rho\theta\|_{L^2}^{\frac{1}{2}}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)_{L^2}^{\frac{1}{2}}\\
&\quad+C\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2\|\rho\|_{L^6}+C\|\nabla\theta\|_{L^2}\|\nabla H\|_{L^2}^{\frac{3}{2}}\|\nabla^2 H\|_{L^2}^{\frac{1}{2}}.\\
&\le\frac{\kappa}{2}\|\theta\|_{L^2}^2+C\|\nabla u\|_{L^2}^3\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)\\
&\quad+C\|\nabla u\|_{L^2}^2\|\rho\theta\|_{L^2}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)\\
&\quad+Cm_0^{\frac{1}{6}}\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2+C\|\nabla H\|_{L^2}^3\|\nabla^2 H\|_{L^2}.
\end{aligned}\tag{3.35}
$$

Integrating (3.35) over $[0,T]$, and using (3.1), (3.3), (3.4) and (3.27), we have

$$
\begin{aligned}
B(T)&\le\int_{\mathbb{R}^3}\rho_0|\theta_0|^2+C\int_0^T\|\nabla u\|_{L^2}^3\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)\mathrm{d}s+C\int_0^T\|\nabla H\|_{L^2}^3\|\nabla^2 H\|_{L^2}\mathrm{d}s\\
&\quad+C\int_0^T\|\nabla u\|_{L^2}^2\|\rho\theta\|_{L^2}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}+\||H||\nabla H|\|_{L^2}\right)\mathrm{d}s+C\int_0^T\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2}^2\|\rho\|_{L^6}\mathrm{d}s\\
&\le K+C\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\left(\int_0^T\|\nabla u\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\left(\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}+K^{\frac{1}{4}}m_0^{\frac{1}{8}}\left(\int_0^T\|\nabla^2 H\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\right)\\
&\quad+C\sup_{0\le t\le T}\|\nabla H\|_{L^2}^2\left(\int_0^T\|\nabla H\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^T\|\nabla^2 H\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+Cm_0^{\frac{1}{3}}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\left(\int_0^T\|\nabla\theta\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\left(\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}+m_0^{\frac{1}{8}}K^{\frac{1}{4}}\left(\int_0^T\|\nabla H^2\|_{L^2}^2\mathrm{d}s\right)^{\frac{1}{2}}\right)\\
&\quad+Cm_0^{\frac{1}{6}}\left(\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2\right)^{\frac{1}{2}}\int_0^T\|\nabla\theta\|_{L^2}^2\mathrm{d}s\\
&\le K+Cm_0^{\frac{1}{6}}K\left(\tilde{E}^{\frac{3}{2}}K+\tilde{E}K^{\frac{5}{4}}+\tilde{E}^{\frac{1}{2}}K^{\frac{1}{2}}\right)\le\frac{3K}{2},
\end{aligned}\tag{3.36}
$$

provided

$$
m_0\le\frac{1}{C\left(\tilde{E}^{\frac{3}{2}}K+\tilde{E}K^{\frac{5}{4}}\right)^6}.
$$

□

Lemma 3.7. *Under the assumptions of Proposition 3.1, it holds that*

$$
0\le\rho\le\frac{3\bar{\rho}}{2},\tag{3.37}
$$

for any $(x,t)\in\mathbb{R}^3\times[0,T]$, *provided*

$$m_0\le\min\left\{\frac{1}{CK^6},\quad \frac{1}{C\left(\tilde{E}^2K\right)^2},\quad \frac{1}{C\left(K^{\frac12}+\tilde{E}Km_0^{\frac{1}{12}}\right)^4},\right.$$
$$\left.\frac{1}{C\left(\tilde{E}^{\frac32}K+\tilde{E}K^{\frac54}\right)^6},\quad \frac{1}{C(\tilde{E}K)^{\frac{144}{45}}},\quad \frac{1}{C\left(\tilde{E}K+K^{\frac32}m_0^{\frac14}\right)^{48}}\right\}.$$

Proof. It follows from $(2.1)_1$ formally that

$$Y'(s)=g(Y)+b'(s), \tag{3.38}$$

where

$$Y(s)=\log\rho\left(X(t;x,s),s\right),\ g(Y)=-\frac{P\left(X(t;x,s),s\right)}{2\mu+\lambda},\ b(s)=-\frac{1}{2\mu+\lambda}\int_0^s G\left(X(t;x,\tau),\tau\right)\,\mathrm{d}\tau,$$

and $X(t;x,s)$ is given by

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}s}X(t;x,s)=u\left(X(t;x,s),s\right),\ 0\le s<t,\\ X(t;x,t)=x.\end{cases}$$

By $(2.1)_1$, we have

$$\begin{aligned}G\left(X(t;x,\tau),\tau\right)&=\Delta^{-1}\mathrm{div}\left((\rho u)_\tau+u\cdot\nabla(\rho u)+\rho u\mathrm{div}u\right)+\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H)\times H]\\&=\Delta^{-1}\mathrm{div}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}(\rho u)+\rho u\mathrm{div}u\right)+\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H)\times H].\end{aligned}$$

This deduces

$$\begin{aligned}b(t)-b(0)&=-\frac{1}{2\mu+\lambda}\int_0^t\Delta^{-1}\mathrm{div}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}(\rho u)+\rho u\mathrm{div}u\right)\,\mathrm{d}\tau-\frac{1}{2\mu+\lambda}\int_0^t\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H)\times H]\mathrm{d}\tau\\&=-\frac{1}{2\mu+\lambda}\Delta^{-1}\mathrm{div}(\rho u)+\frac{1}{2\mu+\lambda}\Delta^{-1}\mathrm{div}(\rho_0u_0)-\frac{1}{2\mu+\lambda}\int_0^t\Delta^{-1}\mathrm{div}\left(\rho u\mathrm{div}u\right)\,\mathrm{d}\tau\\&\quad-\frac{1}{2\mu+\lambda}\int_0^t\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H)\times H]\mathrm{d}\tau\\&\le C\|\Delta^{-1}\mathrm{div}(\rho u)\|_{L^\infty}+C\|\Delta^{-1}\mathrm{div}(\rho_0u_0)\|_{L^\infty}+C\int_0^t\|\Delta^{-1}\mathrm{div}\left(\rho u\mathrm{div}u\right)\|_{L^\infty}\,\mathrm{d}\tau\\&\quad+C\int_0^t\|\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H)\times H]\|_{L^\infty}\mathrm{d}\tau=\sum_{i=1}^4 IV_i.\end{aligned}$$

For IV_1, using (2.5), Sobolev inequality, Calderon–Zygmund inequality, Hölder inequality, (3.1) and (3.3), we get

$$IV_1\le C\|\Delta^{-1}\mathrm{div}(\rho u)\|_{L^6}^{\frac13}\|\nabla\Delta^{-1}\mathrm{div}(\rho u)\|_{L^4}^{\frac23}\le C\|\rho u\|_{L^2}^{\frac13}\|\rho u\|_{L^4}^{\frac23}\le C\|\rho\|_{L^3}^{\frac13}\|u\|_{L^6}^{\frac13}\|\rho\|_{L^{12}}^{\frac23}\|u\|_{L^6}^{\frac23}\le Cm_0^{\frac{7}{18}}\tilde{E}K.$$

Similarly, it holds

$$IV_2 \leq Cm_0^{\frac{7}{18}} K,$$

and then it implies

$$IV_1 + IV_2 \leq Cm_0^{\frac{7}{18}} \tilde{E}K \leq m_0^{\frac{1}{16}}, \tag{3.39}$$

provided

$$m_0 \leq \frac{1}{C(\tilde{E}K)^{\frac{144}{45}}}.$$

(2.4) and (2.5) give

$$\begin{aligned} IV_3 &\leq C \int_0^t \|\Delta^{-1}\mathrm{div}\,(\rho u \mathrm{div} u)\,\|_{L^6}^{\frac{1}{3}} \|\nabla\Delta^{-1}\mathrm{div}\,(\rho u \mathrm{div} u)\,\|_{L^4}^{\frac{2}{3}}\, \mathrm{d}\tau \\ &\leq C \int_0^t \|\rho u \mathrm{div} u\|_{L^2}^{\frac{1}{3}} \|\rho u \mathrm{div} u\|_{L^4}^{\frac{2}{3}}\, \mathrm{d}\tau \\ &\leq C \int_0^t \|\rho u \mathrm{div} u\|_{L^2}\, \mathrm{d}\tau + C \int_0^t \|\rho u \mathrm{div} u\|_{L^4}\, \mathrm{d}\tau. \end{aligned} \tag{3.40}$$

By the definition of G, it reads

$$\rho u \mathrm{div} u = \frac{1}{2\mu+\lambda}\rho u G + \frac{1}{2\mu+\lambda}\rho^2 u\theta.$$

This, together with Hölder inequality, Sobolev inequality, (3.22), (3.21), (3.3) and (3.1) deduces

$$\begin{aligned} \|\rho u \mathrm{div} u\|_{L^2} &\leq C\|\rho u G\|_{L^2} + C\|\rho^2 u\theta\|_{L^2} \\ &\leq C\|\rho\|_{L^6}\|u\|_{L^6}\|G\|_{L^6} + \|\rho\|_{L^{12}}^2\|u\|_{L^6}\|\theta\|_{L^6} \\ &\leq C\|\rho\|_{L^6}\|\nabla u\|_{L^2}\|\nabla G\|_{L^2} + C\|\rho\|_{L^{12}}^2\|\nabla u\|_{L^2}\|\nabla\theta\|_{L^2} \\ &\leq m_0^{\frac{1}{6}}\|\nabla u\|_{L^2}\left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2}\right) \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} &\|\rho u \mathrm{div} u\|_{L^4} \\ &\quad\leq C\|\rho u G\|_{L^4} + C\|\rho^2 u\theta\|_{L^4} \\ &\quad\leq \|u\|_{L^\infty}\Big(\|\rho\|_{L^{12}}\|G\|_{L^6} + \|\rho\|_{L^{24}}^2\|\theta\|_{L^6}\Big) \\ &\quad\leq C(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^6})\Big(\|\rho\|_{L^{12}}\left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2}\right) + \|\rho\|_{L^{24}}^2\|\nabla\theta\|_{L^2}\Big) \\ &\quad\leq Cm_0^{\frac{1}{12}}\left(\|\nabla u\|_{L^2} + \|\nabla\times u\|_{L^6} + \|\mathrm{div} u\|_{L^6}\right)\Big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2}\Big) \\ &\quad\leq Cm_0^{\frac{1}{12}}\Big(\|\nabla u\|_{L^2} + \left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2}\right)\Big)\Big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2}\Big). \end{aligned} \tag{3.42}$$

Putting (3.41) and (3.42) into (3.40), and using Cauchy inequality, (3.1) and (3.4), we obtain

$$\begin{aligned}
IV_3 \le{}& m_0^{\frac{1}{6}} \int_0^t \|\nabla u\|_{L^2} (\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2})\, \mathrm{d}\tau \\
&+ m_0^{\frac{1}{12}} \int_0^t \Big(\|\nabla u\|_{L^2} + (\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2})\Big) \\
&\times \Big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \||H||\nabla H|\|_{L^2} + \|\nabla\theta\|_{L^2}\Big)\, \mathrm{d}\tau \\
\le{}& C(m_0^{\frac{1}{6}} + m_0^{\frac{1}{12}}) \int_0^t \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau + (m_0^{\frac{1}{6}} + m_0^{\frac{1}{12}}) \int_0^t (\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \||H||\nabla H|\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\, \mathrm{d}\tau \\
\le{}& C(m_0^{\frac{1}{6}} + m_0^{\frac{1}{12}}) m_0^{\frac{1}{2}} + (m_0^{\frac{1}{6}} + m_0^{\frac{1}{12}}) \Big(\tilde{E}K + m_0^{\frac{1}{4}} K^{\frac{3}{2}} + K\Big) \le m_0^{\frac{1}{16}},
\end{aligned}$$

provided

$$m_0 \le \frac{1}{C\left(\tilde{E}K + K^{\frac{3}{2}} m_0^{\frac{1}{4}}\right)^{48}}.$$

For IV_4, it follows that by using the above inequality,

$$\begin{aligned}
IV_4 \le{}& C \int_0^t \|\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H) \times H]\|_{L^6}^{\frac{1}{3}} \|\nabla\Delta^{-1}\mathrm{div}[(\mathrm{curl}\ H) \times H]\|_{L^4}^{\frac{2}{3}} \,\mathrm{d}\tau \\
\le{}& C \int_0^t \|(\mathrm{curl}\ H) \times H\|_{L^2}^{\frac{1}{3}} \|(\mathrm{curl}\ H) \times H\|_{L^4}^{\frac{2}{3}} \,\mathrm{d}\tau \\
\le{}& C \int_0^t \|(\mathrm{curl}\ H) \times H\|_{L^2} \,\mathrm{d}\tau + \int_0^T \|(\mathrm{curl}\ H) \times H\|_{L^4} \,\mathrm{d}\tau \\
\le{}& C \int_0^t \|H\|_{L^6} \|\nabla H\|_{L^3} \,\mathrm{d}\tau + \int_0^t \|H\|_{L^\infty} \|\nabla H\|_{L^4} \,\mathrm{d}\tau \\
\le{}& C \left(\int_0^T \|\nabla H\|_{L^2}^2 \,\mathrm{d}\tau\right)^{\frac{3}{4}} \left(\int_0^t \|\nabla^2 H\|_{L^2}^2 \,\mathrm{d}\tau\right)^{\frac{1}{4}} + \left(\int_0^t \|\nabla H\|_{L^2}^2 \,\mathrm{d}\tau\right)^{\frac{3}{8}} \left(\int_0^t \|\nabla^2 H\|_{L^2}^2 \,\mathrm{d}\tau\right)^{\frac{5}{8}} \\
\le{}& C m_0^{\frac{3}{8}} K^{\frac{1}{4}} + C m_0^{\frac{3}{16}} K^{\frac{5}{8}} \le m_0^{\frac{1}{16}},
\end{aligned}$$

provided

$$m_0 \le \frac{1}{CK^5}.$$

By the estimates of $IV_i, i = 1, 2, 3, 4$, it yields

$$b(t) - b(0) \le m_0^{\frac{1}{16}} \log \frac{3}{2} \le \log \frac{3}{2},$$

provided

$$m_0 \le \min\left\{\frac{1}{C(\tilde{E}K)^{\frac{144}{45}}},\quad \frac{1}{C\left(\tilde{E}K + K^{\frac{3}{2}}m_0^{\frac{1}{4}}\right)^{48}},\quad \frac{1}{CK^5}\right\}.$$

Integrating (3.38) with respect to s over $[0,t]$, we get

$$\log\rho(x,t) = \log\rho_0\left(X(t;x,0)\right) + \int_0^T g(Y)\,\mathrm{d}t + b(t) - b(0) \le \log\bar{\rho} + \log\frac{3}{2},$$

which deduces

$$\rho \le \frac{3\bar{\rho}}{2}.$$

□

Remark 3.8. The proof of the upper bound of ρ is not rigorous, since ρ may vanish. In fact, this could be handled by constructing an approximate solution $\rho^\delta > 0$ to $(2.1)_1$ with initial data $\rho_0 + \delta > 0$ for the constant $\delta > 0$. Then one can replace $\log\rho$ by $\log\rho^\delta$ in the above process and finally pass to the limits $\delta \to 0^+$ to obtain the desire estimate.

Now we are in the position to prove Theorem 2.3. Recall that in [11], the Serrin-Type blowup criterion for the strong solution to the problem (2.1)–(2.3) is obtained, which is

$$\limsup_{T\nearrow T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)}\right) = \infty,\quad \frac{2}{s} + \frac{3}{r} \le 1. \tag{3.43}$$

Then combining Proposition 3.1 with (3.43), we can get the global existence and uniqueness of the strong solution to (2.1)–(2.3). Indeed, the boundedness of ρ is given in Lemma 3.7. On the other hand, by choosing $s = 6,\ r = 6$ in (3.43), and using Lemmas 3.3 and 3.5, one has

$$\int_0^T \|u\|_{L^6}^6 \le \sup_{0\le t\le T}\|\nabla u\|_{L^2}^4 \int_0^T \|\nabla u\|_{L^2}^2 \le C.$$

Thus, we obtain that $T^* = \infty$ and complete the proof of Theorem 2.3.

Acknowledgements

The authors are grateful to the referees whose suggestions improve the exposition of the paper. The research of X.F. Hou was supported by the National Natural Science Foundation of China (Grant No.11701100). H.Y. Peng was supported from the National Natural Science Foundation of China (Grant No. 11901115), Natural Science Foundation of Guangdong Province (Grant No. 2019A1515010706) and Grant from GDUT (Grant No. 220413228).

References

[1] Cabannes, H.: Theoretical Magneto uiddynamics. Academic Press, New York (1970)

[2] Cao, C.S., Wu, J.H.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803–1822 (2011)

[3] Chen, G.Q., Wang, D.H.: Global solution of nonlinear magnetohydrodynamics with large initial data. J. Differ. Equ. **182**, 344–376 (2002)

[4] Chen, G.Q., Wang, D.H.: Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. Z. Angew. Math. Phys. **54**, 608–632 (2003)

[5] Chen, Y.Z., Huang, B., Shi, X.D.: Global strong solutions to the compressible magnetohydrodynamic equations with slip boundary conditions in 3D bounded domains. arxiv:org/pdf/2102.07341.pdf

[6] Ducomet, B., Feireisl, E.: The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. Commun. Math. Phys. **266**, 595–629 (2006)

[7] Fan, J.S., Yu, W.H.: Strong solution to the compressible magnetohydrodynamic equations with vacuum. Nonlinear Anal. Real World Appl. **10**, 392–409 (2009)

[8] Fan, J.S., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**, 691–708 (2007)

[9] Hoff, D., Tsyganov, E.: Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z. Angew. Math. Phys. **56**, 791–804 (2005)

[10] Hong, G.Y., Hou, X.F., Peng, H.Y., Zhu, C.J.: Global existence for a class of large solutions to three-dimensional compressible magnetohydrodynamic equations with vacuum. SIAM J. Math. Anal. **49**, 2409–2441 (2017)

[11] Huang, X.D., Li, J.: Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier–Stokes and magnetohydrodynamic flows. Commun. Math. Phys. **324**, 147–171 (2013)

[12] Hu, X.P., Wang, D.H.: Compactness of weak solutions to the three-dimensional compressible magnetohydrodynamic equations. J. Differ. Equ. **245**, 2176–2198 (2008)

[13] Hu, X.P., Wang, D.H.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. **283**, 255–284 (2008)

[14] Hu, X.P., Wang, D.H.: Low Mach number limit of viscous compressible magnetohydrodynamic flows. SIAM J. Math. Anal. **41**, 1272–1294 (2009)

[15] Hu, X.P., Wang, D.H.: Global existence and large-time behavior of solutions to the three dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)

[16] Huang, Y.T., Liu, C.-J., Yang, T.: Local-in-time well-posedness for compressible MHD boundary layer. J. Differ. Equ. **266**, 2978–3013 (2019)

[17] Jiang, S., Ju, Q.C., Li, F.C.: Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. Commun. Math. Phys. **297**, 371–400 (2010)

[18] Kawashima, S., Okada, M.: Smooth global solutions for the one-dimensional equations in magnetohydrodynamics. Proc. Jpn. Acad. Ser. A Math. Sci. **58**, 384–387 (1982)

[19] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and quasilinear equations of parabolic type. In: Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence (1968)

[20] Laudau, L.D., Lifshitz, E.M.: Electrodynamics of Continuous Media, 2nd edn. Pergamon press, New York (1984)

[21] Li, Y., Sun, Y.Z.: Global weak solutions to a two-dimensional compressible MHD equations of viscous non-resistive fluids. J. Differ. Equ. **267**, 3827–3851 (2019)

[22] Li, H.L., Xu, X.Y., Zhang, J.W.: Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum. SIAM J. Math. Anal. **45**, 1356–1387 (2013)

[23] Liu, C.-J., Xie, F., Yang, T.: MHD boundary layers theory in Sobolev spaces without monotonicity I: well-posedness theory. Commun. Pure Appl. Math. **72**, 63–121 (2019)

[24] Liu, C.-J., Xie, F., Yang, T.: Justification of Prandtl ansatz for MHD boundary layer. SIAM J. Math. Anal. **51**, 2748–2791 (2019)

[25] Liu, C.-J., Wang, D.H., Xie, F., Yang, T.: Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces. J. Funct. Anal. **279**, 108637 (2020)

[26] Liu, S.Q., Yu, H.B., Zhang, J.W.: Global weak solutions of 3D compressible MHD with discontinuous initial data and vacuum. J. Differ. Equ. **254**, 229–255 (2013)

[27] Liu, T.P., Zeng, Y.: Large Time Behavior of Solutions for General Quasilinear Hyperbolic-Parabolic Systems of Conservation Laws, vol. 125, Memoirs of the American Mathematical Society (1997)

[28] Ren, X.X., Wu, J.H., Xiang, Z.Y., Zhang, Z.F.: Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion. J. Funct. Anal. **267**, 503–541 (2014)

[29] Rozanova, O.: Blow up of smooth solutions to the barotropic compressible magnetohydrodynamic equations with finite mass and energy. In: Proceedings of Symposia in Applied Mathematics 2009, Hyperbolic Problems: Theory, Numerics and Applications, vol. 67, AMS, Providence, pp. 911–918 (2009)
[30] Suen, A., Hoff, D.: Global low-energy weak solutions of the equations of three-dimensional compressible magnetohydrodynamics. Arch. Ration. Mech. Anal. **205**, 27–58 (2012)
[31] Vol'pert, A.I., Hudjaev, S.I.: The Cauchy problem for composite systems of nonlinear differential equations. Mat. Sb. **87**, 504–528 (1972)
[32] Wang, D.H.: Large solutions to the initial-boundary value problem for planar magnetohydrodynamics. SIAM J. Appl. Math. **63**, 1424–1441 (2003)
[33] Wen, H.Y., Zhu, C.J.: Global solutions to the three-dimensional full compressible Navier–Stokes equations with vacuum at infinity in some classes of large data. SIAM J. Math. Anal. **49**, 162–221 (2017)
[34] Xu, F.Y., Zhang, X.G., Wu, Y.H., Caccetta, L.: Global well-posedness of the non-isentropic full compressible magnetohydrodynamic equations. Acta Math. Sin. (Engl. Ser.) **32**, 227–250 (2016)
[35] Zhong, X.: Global strong solution and exponential decay for nonhomogeneous Navier–Stokes and magnetohydrodynamic equations. Discrete Contin. Dyn. Syst. Ser. B **26**, 3563–3578 (2021)

Xiaofeng Hou
Department of Mathematics
Shanghai University
Shanghai 200444
China
e-mail: xiaofengh@shu.edu.cn

Mina Jiang
School of Mathematics and Statistics
Central China Normal University
Wuhan 430079
China
e-mail: jmn3911@mail.ccnu.edu.cn

Hongyun Peng
School of Mathematics and Statistics
Guangdong University of Technology
Guangzhou 510006
China
e-mail: penghy010@163.com

(Received: May 28, 2021; revised: October 7, 2021; accepted: October 14, 2021)