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A study on the critical Kirchhoff problem in high-dimensional space

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Abstract. In this present paper, we consider the following critical Kirchhoff problem

$$-\Big(a+\lambda\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u+mu=\mu|u|^{p-2}u+|u|^{2^*-2}u\quad\text{in }\mathbb{R}^N,$$

where $a,\lambda\in\mathbb{R}$ and $m,\mu\in\mathbb{R}^+\cup\{0\}$, $N\geq 3$ and $2<p<2^*$. In this first part, a pure critical Kirchhoff problem ($m=\mu=0$) has been considered for both $a>0$ and $a\leq 0$. We obtain a series of fairly complete existence and multiplicity results and have a clear understand the solutions of this pure critical Kirchhoff problem. In particular, if $N\geq 5$, $a>0$ and $\lambda>0$ is suitable small, we obtain two positive solutions, in which one is a mountain pass solution and another one is a global (local) minimum solution. In the second part, the original perturbation problem with $m,\mu>0$ has been considered and two positive solutions also have been obtained for $N\geq 5$, which is rather different compared with the case that $\lambda=0$.

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1. Introduction and main results

In this present paper, we investigate the following critical Kirchhoff problem

$$-\Big(a+\lambda\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u+mu=\mu|u|^{p-2}u+|u|^{2^*-2}u\quad\text{in }\mathbb{R}^N,\tag{1.1}$$

where a,m are two constants, λ,μ are two parameters, $2<p<2^*$ and $2^*=\frac{2N}{N-2}$ for $N\geq 3$ is the critical exponent. The original model comes from the following equation

$$u_{tt}-(a+b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x)\Delta u=f(x,t,u),$$

presented in Kirchhoff [23]. This kind of equations is an extension of the classical d'Alembert's wave equations because of taking into account the effects of the changes in the length of a string during vibrations. Kirchhoff type problem is concerning not only the effects of the changes in length of a string, but also the non-Newton mechanics, the physical laws of the universe, the problem of plasma, elastic theory, population dynamics models and so on.

The stationary general problem

$$-\Big(a+b\int\limits_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)\Delta u+V(x)u=g(x,u)\quad\text{in }\mathbb{R}^3,\tag{1.2}$$

has been studied by many authors under variant conditions on $V(x)$ and $g(x,u)$. Li and Ye [25] proved that problem (1.2) has a positive ground state solution for $g(x,u)=u^{p-1}$, $3<p<6$ when V satisfies some suitable conditions. For more related results, we refer the readers to the bounded domain in [31,36,39], the ground state solutions in [15,41], the nodal solutions [10,37,40,47], the periodic potential cases [3,27,48], the semi-classical and multi-peak solutions [13,21,26].

In what follows, we briefly recall some known results about the critical Kirchhoff problem. A large number of papers have been published in three-dimensional space. In Alves et al. [2], the authors studied the following Kirchhoff problem

$$\begin{cases} -M(\int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = \nu f(x,u) + u^5 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the parameter $\nu > 0$, $M(t)$ and $f(x,u)$ are continuous functions, and Ω is a bounded domain in $\mathbb{R}^3$. They obtained a positive solution if $\nu > 0$ is large enough, among other things, $f(x,u)$ satisfies the well-known Ambrosetti-Rabinowitz sup-linear condition, i.e.,

$$0 < \tau F(x,u) := \tau \int\limits_0^u f(x,s)\mathrm{d}s \leq f(x,u)u \text{ for all } x \in \Omega,\ u > 0 \text{ and some fixed } \tau.$$

After that, the authors in He and Zou [18] and Wang et al. [42] studied the following critical Kirchhoff problem

$$-(a\varepsilon^2 + b\varepsilon \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = \nu f(u) + u^5 \text{ in } \mathbb{R}^3, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are two constants and $V(x)$ is a suitable potential. By recovering a local $(PS)_c$ condition for $c < c_3'$ or c_3'', where

$$c_3' = \frac{1}{12}(aS)^{\frac{3}{2}} \quad \text{or} \quad c_3'' = \frac{1}{3}(aS)^{\frac{3}{2}} + \frac{1}{12}b^3S^6,$$

and S is the best Sobolev constant from $D^{1,2}(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$, they obtained a semi-classical solution if $\nu > 0$ large enough and $f(u)$ satisfies the sup-4-linear condition.

In Li and Ye [24] and Xie et al. [44], the authors obtained a positive solution of the critical Kirchhoff problem either in $\mathbb{R}^3$ or in a bounded domain when the perturbation term f satisfies the sup-4-linear condition by improving a local $(PS)_c$ condition for $c < c_3$, where

$$c_3 = \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{\frac{3}{2}}. \tag{1.5}$$

For more related results concerning the perturbation term f with the sup-4-linear condition, we refer the readers to Figueiredo et al. [12], He and Li [17], Liu and Guo [28,29], Naimen [33] and references therein.

In He and Li [16], the authors considered the following critical Kirchhoff problem with a sub-4-linear perturbation term

$$-(a\varepsilon^2 + b\varepsilon \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = \nu u^{p-1} + u^5, \quad u > 0 \text{ in } \mathbb{R}^3. \tag{1.6}$$

They obtained some existence results of the semi-classical solutions for $2 < p < 4$ and $\nu > 0$ large enough. Sun and Liu [38] obtained a positive solution for $1 < p < 2$ and $\nu > 0$ small enough. Some results concerning the critical Kirchhoff problem with a nonhomogeneous term can be found in [8,22].

Little results have been obtained for the high-dimensional cases $N \geq 4$. The authors in Xie et al. [45,46] studied the following critical Kirchhoff problem in $D^{1,2}(\mathbb{R}^N)$,

$$-\Big(a + b\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)\Delta u + V(x)u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N.$$

With some suitable assumptions on b and the potential V, they obtained a bound stated solution for $N = 3, 4$ and two solutions for $N \geq 5$ by recovering a local $(PS)_c$ condition with

$$c \in (c_N, 2c_N) \quad \text{for } N = 3 \text{ or } 4 \text{ and } c \neq c_{N,\pm} \text{ for } N \geq 5, \tag{1.7}$$

where $c_{N,\pm}$ are two fixed levels depending on N, a, b for $N \geq 5$ and c_N is defined in (1.5) for $N = 3$ and

$$c_4 = \frac{(aS)^2}{4(1 - bS^2)} \quad \text{for N = 4.} \tag{1.8}$$

In Naimen [32], the author considered the following problem in high-dimensional space

$$\begin{cases} -(a + b\int\limits_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = \nu u^{p-1} + \mu u^{2^*-1}, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $p \in [2, 2^*)$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 4$. With some suitable assumptions on the positive constants a, b and the parameters ν, μ, the author obtained a positive solution in four-dimensional case by recovering a local $(PS)_c$ condition for $c < c_{4,\mu}$, where $c_{4,\mu} = \frac{(aS)^2}{4(\mu - bS^2)}$. After that, Naimen and Shibata [34] obtained two positive solutions for problem (1.9) in high-dimensional case $N \geq 5$ by using the critical levels $c_{N,\pm}$, which coincides with (1.7). A related result can be found in Naimen and Shibata [35] for four-dimensional case. We also refer the readers to Hebey [19,20] for the critical Kirchhoff problem in manifolds.

The rescaling argument is an effective method dealing with a kind of autonomous Kirchhoff problem. In Azzollini [6], the author considered the following Kirchhoff problem by a rescaling argument

$$-\Big(a + b\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = g(u) \quad \text{in } \mathbb{R}^N,$$

where g satisfies some suitable assumptions. The author proved that there exist some positive constants t such that $u(t\cdot)$ solving the Kirchhoff problem, where u is a solution of $-\Delta u = g(u)$ in $\mathbb{R}^N$. More related results by using this similar method can be found in Azzollini [5], Lu [30] and Wu et al. [43].

From the above-mentioned works, we can find that it is necessary to make a clear and complete study on the solutions and corresponding critical levels of the pure critical Kirchhoff problem, which are interesting and basic works and heavily affect the compactness results or the local $(PS)_c$ condition. The levels c_N, which are defined by (1.5) and (1.8) for $N = 3$ and 4, respectively, are the first and optimal threshold for the lack of compactness result. It should be mentioned that these are not clear for the high-dimensional cases. From this point, we try to give a complete research on the pure critical Kirchhoff problem, i.e., problem (1.1) with $m = \mu = 0$, especially for the high-dimensional cases $N \geq 5$.

In the first part of this paper, we investigate the following pure critical Kirchhoff problem

$$\begin{cases} -\Big(a + \lambda \int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{K_λ^*}$$

where a is a constant, λ is a parameter and $N \geq 3$. Roughly speaking, we try to find solutions of (K_λ^*) from the solutions of the following classical critical elliptic equation by a multiplying argument

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{S^*}$$

It should be mentioned that this method has no difference comparing with the rescaling argument used by Azzollini [6].

To state our results, we need some basic and suitable variational setting. It is well-known that the solutions of (K_λ^*) correspond to the critical points of C^2 functional I: $D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{a}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{\lambda}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{2^*}\int\limits_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x. \tag{1.10}$$

For each $u \in D^{1,2}(\mathbb{R}^N)$, we denote the fibering map $Q_u(t) : [0, +\infty) \to \mathbb{R}$,

$$Q_u(t) := I(tu) = \frac{at^2}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{\lambda t^4}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{t^{2^*}}{2^*}\int\limits_{\mathbb{R}^N} |u|^{2^*} \mathrm{d}x.$$

A good candidate for appropriate subset of $D^{1,2}(\mathbb{R}^N)$ is so-called Nehari manifold

$$\mathcal{N}^N := \left\{u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\} : Q_u'(1) = 0\right\}. \tag{1.11}$$

With the help of the fibering map, it is natural to divide $\mathcal{N}^N$ into three subsets $\mathcal{N}^{N,-}$, $\mathcal{N}^{N,+}$ and $\mathcal{N}^{N,0}$ corresponding to the local maxima, local minima and points of inflexion of fibering map, respectively,

$$\mathcal{N}^{N,-} := \left\{u \in \mathcal{N}^N : Q_u''(1) < 0\right\}, \quad \mathcal{N}^{N,+} := \left\{u \in \mathcal{N}^N : Q_u''(1) > 0\right\},$$
$$\mathcal{N}^{N,0} := \left\{u \in \mathcal{N}^N : Q_u''(1) = 0\right\}.$$

We denote the unique positive solution of (S^*) by U, which achieves the best Sobolev constant S and $S^{N/2} = |U|_{2^*}^{2^*} = |\nabla U|_2^2$, where $|\cdot|_q$ is the standard norm in L^q for $q \geq 1$.

The results for the three- or four-dimensional cases can be stated as follows:

Theorem 1.1. *Assume $a > 0$ and $\lambda > 0$. Then, the following results hold:*

(i) For $N = 3$, problem (K_λ^*) has infinitely many distinct solutions $\{\varphi_i\}_{i=1}^\infty$. Moreover, φ_1 is a positive solution,

$$\inf_{u\in\mathcal{N}^3} I(u) = c_3 = I(\varphi_1) > 0, \quad I(\varphi_i) \to +\infty \ \text{ as } i \to +\infty.$$

(ii) For $N = 4$, problem (K_λ^*) has a positive solution φ_1 if and only if $\lambda < S^{-2}$. Moreover, for any $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that problem (K_λ^*) with $\lambda \in (0, \lambda_n)$ has n solutions $\{\varphi_i\}_{i=1}^n$ satisfying

$$\inf_{u\in\mathcal{N}^4} I(u) = c_4 = I(\varphi_1) > 0,$$
$$I(\varphi_1) < I(\varphi_2) < \cdots < I(\varphi_n) = \frac{a^2}{4(\lambda_n - \lambda)}.$$

If $N \geq 5$, the solutions of problem (K_λ^*) exist in pairs and the main results can be stated as follows:

Theorem 1.2. *Assume $N \geq 5$, $a > 0$ and $\lambda > 0$. Then, the following results hold:*

(i) Problem (K_λ^*) has two positive solutions $\varphi_{1,\pm}$ if and only if $\lambda \in (0, \Lambda_0)$. Both of them are two local minimum points on the Nehari manifolds

$$\inf_{u\in\mathcal{N}^{N,-}} I(u) = I(\varphi_{1,-}) = c_{N,-} > 0, \quad \inf_{u\in\mathcal{N}^{N,+}} I(u) = I(\varphi_{1,+}) = c_{N,+},$$

and $c_{N,-} > c_{N,+}$. Moreover, $c_{N,+} < 0$ for $\lambda \in (0, \Lambda_1)$ and $c_{N,+} > 0$ for $\lambda \in (\Lambda_1, \Lambda_0)$, where Λ_0 and Λ_1 are defined as follows, respectively,

$$\Lambda_0 := \frac{2}{N-2}\left(\frac{N-4}{a(N-2)}\right)^{\frac{N-4}{2}} S^{-\frac{N}{2}} \ \textit{and} \ \ \Lambda_1 := \frac{4}{N}\left(\frac{N-4}{aN}\right)^{\frac{N-4}{2}} S^{-\frac{N}{2}}. \tag{1.12}$$

(ii) For any $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that problem (K_λ^*) with $\lambda \in (0, \lambda_n)$ has $2n$ solutions $\varphi_{i,\pm} \in \mathcal{N}^{N,\pm}$, $i \in \{1, 2, \ldots, n\}$ and

$$I(\varphi_{i,-}) > I(\varphi_{i,+}),$$
$$I(\varphi_{1,\pm}) < I(\varphi_{2,\pm}) < \cdots < I(\varphi_{n,\pm}) < \frac{a^2}{N(N-4)\lambda}.$$

Remark 1.3. More details can be stated as follows for $N \geq 5$.

(i). It should be mentioned that 0 and $\varphi_{1,+}$ are two local minimum points of the energy functional I and one of them is the global minimum point in $D^{1,2}(\mathbb{R}^N)$. In other words, there hold,

$$I(0) = \inf_{u \in B_{r_\lambda}} I(u) \quad \text{and} \quad I(\varphi_{1,+}) = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus B_{r_\lambda}} I(u), \tag{1.13}$$

where $B_{r_\lambda} = \{u \in D^{1,2}(\mathbb{R}^N) : |\nabla u|_2 \leq r_\lambda\}$ and $r_\lambda := \xi_{1,-}$ is a constant defined by (1.21). Moreover,

$$\inf_{u \in D^{1,2}(\mathbb{R}^N)} I(u) = \begin{cases} I(\varphi_{1,+}), & \text{if } \lambda \in (0, \Lambda_1), \\ I(0), & \text{if } \lambda \in (\Lambda_1, \Lambda_0). \end{cases}$$

Actually, by the Sobolev inequalities and some basic computations, one obtains

$$\begin{aligned} I(0) \geq \inf_{u \in B_{r_\lambda}} I(u) &\geq \inf_{u \in B_{r_\lambda}} \left(\frac{a}{2}|\nabla u|_2^2 + \frac{\lambda}{4}|\nabla u|_2^4 - \frac{1}{2^*} S^{-\frac{2^*}{2}} |\nabla u|_2^{2^*} \right) \\ &= \inf_{t \in [0, r_\lambda]} \theta(t) = 0, \end{aligned}$$

where $\theta(t) = \frac{a}{2}t^2 + \frac{\lambda}{4}t^4 - \frac{1}{2^*} S^{-\frac{2^*}{2}} t^{2^*}$. Here $I(0) = \inf_{B_{r_\lambda}} I(u)$ holds.

On the one hand, for $u \in \mathcal{N}^{N,+}$, we have that $|\nabla u|_2 > \eta_\lambda > r_\lambda$ (see (1.23)), which implies $\mathcal{N}^{N,+} \subset B_{r_\lambda}^c$. Thus, $\inf_{u \in \mathcal{N}^{N,+}} I(u) \geq \inf_{u \in B_{r_\lambda}^c} I(u)$. On the other hand, it follows that

$$\inf_{u \in B_{r_\lambda}^c} I(u) \geq \inf_{t \geq r_\lambda} \theta(t) = \theta(\xi_{1,+}) = c_{N,+} = \inf_{u \in \mathcal{N}^{N,+}} I(u),$$

where $\xi_{1,+}$ is given in (1.21). Thus, $I(\varphi_{1,+}) = \inf_{u \in B_{r_\lambda}^c} I(u)$ holds. Moreover, by Theorem 1.2 (i), we have

$$\inf_{u \in D^{1,2}(\mathbb{R}^N)} I(u) = \begin{cases} I(\varphi_{1,+}), & \text{if } \lambda \in (0, \Lambda_1), \\ I(0), & \text{if } \lambda \in (\Lambda_1, \Lambda_0). \end{cases}$$

(ii) Moreover, $\varphi_{1,-}$ is a mountain pass solution. Actually, by (1.13), Theorem 1.2 (i) and $|\nabla \varphi_{1,+}|_2 > r_\lambda = |\nabla \varphi_{1,-}|_2$, we can check that

$$\max\{I(0), I(\varphi_{1,+})\} < I(\varphi_{1,-}) \leq \inf_{u \in \partial B_{r_\lambda}} I(u).$$

Thus, we can define a mountain pass level as following:

$$m_- := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)), \tag{1.14}$$

where $\Gamma = \{\gamma(t) \in C([0,1]) : \gamma(0) = 0 \text{ and } \gamma(1) = \varphi_{1,+}\}$. We claim that

$$c_{N,-} = m_-. \tag{1.15}$$

In fact, since $|\nabla \gamma(0)|_2 < r_\lambda < |\nabla \gamma(1)|_2$ for any $\gamma \in \Gamma$, there exists $t_\gamma \in (0,1)$ such that $\gamma(t_\gamma) \in \partial B_{r_\lambda}$. Thus, it follows that

$$\sup_{t \in [0,1]} I(\gamma(t)) \geq I(\gamma(t_\gamma)) \geq \inf_{u \in \partial B_{r_\lambda}} I(u) \geq I(\varphi_{1,-}),$$

which implies $c_{N,-} \leq m_-$. Let $\gamma_1(t) := t\varphi_{1,+}$ and $\gamma_1(t) \in \Gamma$. Then, we have

$$m_- \leq \sup_{t\in[0,1]} I(\gamma_1(t)) = \max_{t\in[0,1]} I(t\varphi_{1,+}) = I(\varphi_{1,-}),$$

which implies $c_{N,-} \geq m_-$. Then, $c_{N,-} = m_-$ holds.

Theorem 1.4. *Assume that $N \geq 3$, $a > 0$ and $\lambda < 0$, problem (K_λ^*) has infinitely many distinct solutions $\{\varphi_i\}_{i=1}^\infty$. Moreover, φ_1 is the positive ground state solution with $I(\varphi_1) = c_N > 0$ and $I(\varphi_i) < -\frac{a^2}{4\lambda}$.*

Remark 1.5. (i). Similar to Remark 1.3, the critical level c_N obtained in Theorem 1.1 (i)-(ii) and Theorem 1.4 are the mountain pass levels. In fact, for $a > 0$, if $N = 3$ or $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda < 0$, we take $T_0 > 0$ large such that $\|T_0U\| > \xi_1$ and $I(T_0U) < 0$, where ξ_1 comes from (1.19), (1.20) and (1.28),

$$\max\{I(0), I(T_0U)\} < I(\varphi_1) \leq \inf_{u\in\partial B_{\xi_1}} I(u).$$

Thus, we can define a mountain pass level as following:

$$m_- := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma(t) \in C([0,1]) : \gamma(0) = 0 \text{ and } \gamma(1) = T_0U\}$. It is evident to prove that

$$c_N = m_-.$$

(ii). We have a clear description of the above critical levels c_N and $c_{N,\pm}$ as follows. Set

$$\theta(t) = \frac{a}{2}t^2 + \frac{\lambda}{4}t^4 - \frac{1}{2^*}S^{-\frac{2^*}{2}}t^{2^*} \quad \text{for } t \geq 0. \tag{1.16}$$

If $N = 3$ or $N = 4$ and $\lambda < S^{-2}$ or $N \geq 5$ and $\lambda < 0$, the critical levels c_N defined in Theorem 1.1 (i)-(ii) and Theorem 1.4 can be rewritten as

$$c_N = \max_{t\geq 0} \theta(t). \tag{1.17}$$

If $N \geq 5$ and $0 < \lambda < \Lambda_0$, the critical levels $c_{N,\pm}$ defined in Theorem 1.2 (i) can be rewritten as

$$c_{N,-} = \max_{0\leq t\leq \eta_\lambda} \theta(t) \quad \text{and } c_{N,+} = \min_{t\geq \eta_\lambda} \theta(t), \tag{1.18}$$

where $\eta_\lambda = \left(\frac{2a}{(N-4)\lambda}\right)^{1/2}$. It is easy to check that (1.17) coincide with (1.5) for $N = 3$ and (1.8) for $N = 4$ and $0 < \lambda < S^{-2}$, (replace b by λ).

Here, we begin to give the proof of the above Theorem 1.1–1.4 by the multiplying argument.

The proof of Theorem 1.1 (i).. Equation (S^*) admits infinitely many distinct solutions $\{u_i\}_{i=1}^\infty$ satisfying $|\nabla u_i|_2 \to +\infty$ as $i \to \infty$ (see Ding [11]). Without loss of generality, we assume $|\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_i|_2 \to +\infty$, where $u_1 = U$ is the positive solution. It is evident to check that the existence of a positive root K_i for equation $g_i(t) = 0$ defined by

$$g_i(t) = -t^4 + \lambda|\nabla u_i|_2^2 t^2 + a, \quad t \geq 0,$$

where $i \in \mathbb{N}$, $a > 0$ and $\lambda > 0$. Let $\varphi_i := K_i u_i$, then

$$\begin{aligned}-\Delta\varphi_i = -K_i\Delta u_i &= K_i\left(|u_i|^{2^*-2}u_i\right) \\ &= \frac{K_i^5}{a + \lambda K^2|\nabla u_i|_2^2}\left(|u_i|^{2^*-2}u_i\right) = \frac{1}{a + \lambda|\nabla\varphi_i|_2^2}\left(|\varphi_i|^{2^*-2}\varphi_i\right).\end{aligned}$$

Thus, $\{\varphi_i\}_{i=1}^{\infty}$ is a sequence of solutions for problem (K_λ^*) and $\varphi_i \in \mathcal{N}^{3,-}$ because of

$$Q''_{\varphi_i}(1) = -2\lambda|\nabla\varphi_i|_2^2\left(|\nabla\varphi_i|_2^2 + \frac{2a}{\lambda}\right) < 0.$$

Set $\xi_i := |\nabla\varphi_i|_2$, then ξ_i is the positive root of equation $f_i(t) = 0$ defined by

$$f_i(t) = -\left(|\nabla u_i|_2^{-1}t\right)^4 + \lambda t^2 + a, \quad t \geq 0,$$

where $i \in \mathbb{N}$, $a > 0$ and $\lambda > 0$. By the fact that $|\nabla u_i|_2 < |\nabla u_{i+1}|_2$ and $|\nabla u_i|_2 \to +\infty$ as $i \to \infty$ and some calculations, one has $f_i(t) < f_{i+1}(t)$ and

$$0 < \xi_1 < \xi_2 < \cdots < \xi_i \to +\infty \quad \text{as } i \to \infty. \tag{1.19}$$

Therefore, $I(\varphi_i) = \frac{1}{3}a|\nabla\varphi_i|_2^2 + \frac{1}{12}\lambda|\nabla\varphi_i|_2^4$. By (1.19), one obtains

$$0 < I(\varphi_1) < I(\varphi_2) < \cdots < I(\varphi_i) \to +\infty \quad \text{as } i \to \infty.$$

Moreover, it follows from $u \in \mathcal{N}^3$ and the Sobolev inequalities that

$$a|\nabla u|_2^2 + \lambda|\nabla u|_2^4 = |u|_6^6 \leq S^{-3}|\nabla u|_2^6,$$

which implies $|\nabla u|_2 \geq \xi_1$. Combing with $\xi_1 = |\nabla\varphi_1|_2$, it follows that

$$\inf_{u\in\mathcal{N}^3} I(u) = c_3 = I(\varphi_1) > 0.$$

The proof is completed. □

The Proof of Theorem 1.1 (ii). On the one hand, if $0 < \lambda < S^{-2}$, let $K_1 = \sqrt{\frac{a}{1-\lambda S^2}}$ and U be the positive solution of problem (S^*). Then, it is evident to check that $\varphi_1 := K_1U$ is a positive solution of problem (K_λ^*). On the other hand, if φ_1 is a positive solution of problem (K_λ^*), then $U = \left(a + \lambda|\nabla\varphi_1|_2^2\right)^{-1/2}\varphi_1$ is a positive solution of problem (S^*). Remind that $|\nabla U|_2^2 = S^{N/2}$, then $\xi_1 := |\nabla\varphi_1|_2$ is the root of equation $f(t) = 0$ defined by

$$f(t) = (\lambda - S^{-2})t^2 + a, \quad t \geq 0,$$

where $a > 0$ and $\lambda > 0$. The existence of a positive root for $f(t) = 0$ implies that $\lambda < S^{-2}$.

For any $n \in \mathbb{N}$, let $\{u_i\}_{i=1}^n$ be n solutions of equation (S^*) satisfying $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_n|_2$. Setting $\lambda_n = |\nabla u_n|_2^{-2}$, it is evident to check that the existence of a positive root K_i for equation $g_i(t) = 0$ defined by

$$g_i(t) = -(1 - \lambda|\nabla u_i|_2^2)t^2 + a, \quad t \geq 0,$$

where $i \in \{1, 2, \ldots, n\}$, $a > 0$ and $\lambda \in (0, \lambda_n)$. Let $\varphi_i := K_iu_i$, then $\{\varphi_i\}_{i=1}^n$ is a sequence of solutions for problem (K_λ^*) and $\varphi_i \in \mathcal{N}^{4,-}$ because of $Q''_{\varphi_i}(1) = -2a|\nabla\varphi_i|_2^2 < 0$. Set $\xi_i := |\nabla\varphi_i|_2$ for any $i \in \{1, 2, \ldots, n\}$, then ξ_i is the positive root of equation $f_i(t) = 0$ defined by

$$f_i(t) = -(|\nabla u_i|_2^{-2} - \lambda)t^2 + a, \quad t \geq 0,$$

where $i \in \{1, 2, \ldots, n\}$, $a > 0$ and $\lambda \in (0, \lambda_n)$. By some calculations and $|\nabla u_i|_2 < |\nabla u_{i+1}|_2$ for $i \in \{1, 2 \ldots, n-1\}$, one has $f_i(t) < f_{i+1}(t)$ and

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n = \sqrt{\frac{a}{|\nabla u_n|_2^{-2} - \lambda}}. \tag{1.20}$$

It follows from $I(\varphi_i) = \frac{1}{4}a|\nabla\varphi_i|_2^2$ and (1.20) that

$$0 < I(\varphi_1) < I(\varphi_2) < \cdots < I(\varphi_n) = \frac{a^2}{4(\lambda_n - \lambda)}.$$

By the Sobolev inequalities, for $u \in \mathcal{N}^4$, we obtain

$$a|\nabla u|_2^2 + \lambda|\nabla u|_2^4 = |u|_4^4 \leq S^{-2}|\nabla u|_2^4,$$

which implies that $|\nabla u|_2 \geq \xi_1$. Combing with $|\nabla \varphi_1|_2 = \xi_1$, it follows that

$$\inf_{u\in\mathcal{N}^4} I(u) = c_4 = I(\varphi_1) > 0.$$

The proof is completed. □

The proof of Theorem 1.2 (i).. On the one hand, by the assumption $\lambda \in (0, \Lambda_0)$, there exist two positive roots for equation $g(t) = 0$ defined by

$$g(t) = \lambda S^{\frac{N}{2}} t^2 - t^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

where $a > 0$. Without loss of generality, we denote the roots by $K_{1,\pm}$ and $K_{1,-} < K_{1,+}$. Let U be the positive solution of problem (S^*), then it is evident to check that $\varphi_{1,\pm} := K_{1,\pm}U$ are two positive solutions of problem (K^*_λ). On the other hand, if $\varphi_{1,\pm}$ are two positive solutions of problem (K^*_λ), then $U = \left(a + \lambda|\nabla\varphi_{1,\pm}|_2^2\right)^{-(N-2)/4} \varphi_{1,\pm}$ are the positive solutions of problem (S^*). Remind that $|\nabla U|_2^2 = S^{N/2}$. Set

$$\xi_{1,\pm} := |\nabla\varphi_{1,\pm}|_2, \tag{1.21}$$

then $\xi_{1,\pm}$ are the roots of equation $f(t) = 0$ defined by

$$f(t) = \lambda t^2 - \left(S^{-\frac{N}{4}} t\right)^{\frac{4}{N-2}} + a, \quad t \geq 0, \tag{1.22}$$

where $a > 0$ and $\lambda > 0$. The existence of two positive roots for $f(t) = 0$ implies that $\lambda \in (0, \Lambda_0)$.

Secondly, it follows from $u \in \mathcal{N}^N$ and the Sobolev inequalities that

$$a|\nabla u|_2^2 + \lambda|\nabla u|_2^4 = |u|_{2^*}^{2^*} \leq S^{-\frac{N}{N-2}} |\nabla u|_2^{\frac{2N}{N-2}},$$

which implies that $\xi_{1,-} \leq |\nabla u|_2 \leq \xi_{1,+}$. Therefore, for any $\mathcal{N}^N$, one obtains

$$Q''_u(1) = \frac{2(N-4)}{N-2}\lambda|\nabla u|_2^2 \left(|\nabla u|_2^2 - \frac{2a}{(N-4)\lambda}\right) = \frac{2(N-4)}{N-2}\lambda|\nabla u|_2^2 \left(|\nabla u|_2^2 - \eta_\lambda^2\right), \tag{1.23}$$

where $\eta_\lambda := (\frac{2a}{(N-4)\lambda})^{1/2}$. Combining with $\xi_{1,-} \leq |\nabla u|_2 \leq \xi_{1,+}$ for any $u \in \mathcal{N}^N$ and $\xi_{1,-} < \eta_\lambda < \xi_{1,+}$ (see Lemma 3.3), we can rewrite the Nehari manifolds as follows:

$$\mathcal{N}^{N,-} = \left\{u \in \mathcal{N}^N : \xi_{1,-} \leq |\nabla u|_2 < \eta_\lambda\right\}, \quad \mathcal{N}^{N,0} = \left\{u \in \mathcal{N}^N : |\nabla u|_2 = \eta_\lambda\right\},$$
$$\mathcal{N}^{N,+} = \left\{u \in \mathcal{N}^N : \eta_\lambda < |\nabla u|_2 \leq \xi_{1,+}\right\}.$$

Directly, $\varphi_{1,\pm} \in \mathcal{N}^{N,\pm}$. For any $u \in \mathcal{N}^N$, there holds,

$$I(u) = I(u) - \frac{1}{2^*}\langle I'(u), u\rangle = \frac{1}{N}a|\nabla u|_2^2 - \frac{N-4}{4N}\lambda|\nabla u|_2^4 := \mathcal{I}(|\nabla u|_2^2), \tag{1.24}$$

where

$$\mathcal{I}(t) = \frac{1}{N}at - \frac{N-4}{4N}\lambda t^2, \quad t \geq 0. \tag{1.25}$$

It is easy to check that $\mathcal{I}(t)$ is increasing in $(0, \eta_\lambda^2)$ and decreasing in $(\eta_\lambda^2, +\infty)$. $\mathcal{I}$ achieves its maximum point at η_λ^2. It follows from $\varphi_{1,\pm} \in \mathcal{N}^{N,\pm}$ and $0 < \xi_{1,-} < \eta_\lambda < \xi_{1,+}$ that

$$I(\varphi_{1,-}) \geq \inf_{u\in\mathcal{N}^{N,-}} I(u) \geq \inf_{\xi_{1,-}^2 \leq t < \eta_\lambda^2} \mathcal{I}(t) = \mathcal{I}(\xi_{1,-}^2) = I(\varphi_{1,-}) > 0,$$
$$I(\varphi_{1,+}) \geq \inf_{u\in\mathcal{N}^{N,+}} I(u) \geq \inf_{\xi_{1,+}^2 \geq t > \eta_\lambda^2} \mathcal{I}(t) = \mathcal{I}(\xi_{1,+}^2) = I(\varphi_{1,+}),$$

which imply that $\inf_{u\in\mathcal{N}^{N,\pm}} I(u) = I(\varphi_{1,\pm})$ and $I(\varphi_{1,-}) > 0$. Moreover, by the fact that

$$\xi_{1,+}^2 - \eta_\lambda^2 > \eta_\lambda^2 - \xi_{1,-}^2,$$

in Lemma 3.3, one obtains $I(\varphi_{1,+}) < I(\varphi_{1,-})$.

Lastly, it remains to prove that $I(\varphi_{1,+}) < 0$ for $\lambda \in (0, \Lambda_1)$ and $I(\varphi_{1,+}) > 0$ for $\lambda \in (\Lambda_1, \Lambda_0)$. Actually, it is evident to check that $\mathcal{I}(t) > 0$ for $t \in (0, 2\eta_\lambda^2)$, $\mathcal{I}(2\eta_\lambda^2) = 0$ and $\mathcal{I}(t) < 0$ for $t \in (2\eta_\lambda^2, +\infty)$. Then, it is sufficient to prove that $\xi_{1,+} > \sqrt{2}\eta_\lambda$ for $\lambda \in (0, \Lambda_1)$ and $\xi_{1,+} < \sqrt{2}\eta_\lambda$ for $\lambda \in (\Lambda_1, \Lambda_0)$. It follows that

$$f(\sqrt{2}\eta_\lambda) = 2\lambda\eta_\lambda^2 - (2S^{-\frac{N}{2}}\eta_\lambda^2)^{\frac{2}{N-2}} + a = \frac{aN}{N-4} - \left(\frac{4a}{(N-4)\lambda S^{N/2}}\right)^{\frac{2}{N-2}}.$$

Here, $f(\sqrt{2}\eta_\lambda) < 0$ for $\lambda \in (0, \Lambda_1)$ and $f(\sqrt{2}\eta_\lambda) > 0$ for $\lambda \in (\Lambda_1, \Lambda_0)$. Combining this and $f(\xi_{1,-}) = f(\xi_{1,+}) = 0$, one has that $\xi_{1,+} > \sqrt{2}\eta_\lambda$ for $\lambda \in (0, \Lambda_1)$ and $\xi_{1,+} < \sqrt{2}\eta_\lambda$ for $\lambda \in (\Lambda_1, \Lambda_0)$. The proof is completed. □

The Proof of Theorem 1.2 (ii). For any $n \in \mathbb{N}$, let $\{u_i\}_{i=1}^n$ be n solutions of equation (S^*) satisfying $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_n|_2$. Let

$$\lambda_n = \frac{2}{N-2}\left(\frac{N-4}{a(N-2)}\right)^{(N-4)/2} |\nabla u_n|_2^{-2},$$

it is evident to check that the existence of two positive roots $K_{i,\pm}$ ($K_{i,-} < K_{i,+}$) for equation $g_i(t) = 0$ defined by

$$g_i(t) = \lambda|\nabla u_i|_2^2 t^2 - t^{\frac{4}{N-2}} + a, \quad t \geq 0, \tag{1.26}$$

where $i \in \{1, 2, \ldots, n\}$, $a > 0$ and $\lambda \in (0, \lambda_n)$. Let $\varphi_{i,\pm} := K_{i,\pm}u_i$, then $\{\varphi_{i,\pm}\}_{i=1}^n$ are the solutions of equation (K_λ^*). Set $\xi_{i,\pm} := |\nabla\varphi_{i,\pm}|_2$ for $i \in \{1, 2, \ldots, n\}$, then $\xi_{i,\pm}$ are the roots of equation $f_i(t) = 0$ defined by

$$f_i(t) = \lambda t^2 - \left(|\nabla u_i|_2^{-1} t\right)^{\frac{4}{N-2}} + a, \quad t \geq 0, \tag{1.27}$$

where $i \in \{1, 2, \ldots, n\}$, $a > 0$ and $\lambda \in (0, \lambda_n)$. With a similar argument of Lemma 3.3, we obtain

$$\xi_{i,-} < \eta_\lambda < \xi_{i,+} \text{ and } \xi_{i,+}^2 - \eta_\lambda^2 > \eta_\lambda^2 - \xi_{i,-}^2,$$

which implies $\varphi_{i,\pm} \in \mathcal{N}^{N,\pm}$ and $I(\varphi_{i,-}) > I(\varphi_{i,+})$. From the assumption $|\nabla u_i|_2 < |\nabla u_{i+1}|_2$ for $i \in \{1, 2, \ldots, n-1\}$, it follows that $f_i(t) < f_{i+1}(t)$ and

$$0 < \xi_{1,-} < \xi_{2,-} < \cdots < \xi_{n,-} < \eta_\lambda < \xi_{n,+} < \cdots < \xi_{1,+}.$$

Combining with the monotonicity of $\mathcal{I}(t)$, the desired result holds,

$$I(\varphi_{1,\pm}) < I(\varphi_{2,\pm}) < \cdots < I(\varphi_{n,\pm}) < \frac{a^2}{N(N-4)\lambda}.$$

The proof is completed. □

The proof of Theorem 1.4. Let $\{u_i\}_{i=1}^\infty$ be a sequence of solutions for equation (S^*) and satisfy $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_i|_2 \to +\infty$. It is evident to check that the existence of a positive root K_i for equation $g_i(t) = 0$ defined by

$$g_i(t) = \lambda|\nabla u_i|_2^2 t^2 - t^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

where $i \in \mathbb{N}$, $a > 0$ and $\lambda < 0$. Let $\varphi_i := K_i u_i$, then $\{\varphi_i\}_{i=1}^\infty$ is a sequence of solutions of equation (K_λ^*). Set

$$\xi_i := |\nabla\varphi_i|_2, \tag{1.28}$$

then ξ_i is a positive root of $f_i(t) = 0$ defined by

$$f_i(t) = \lambda t^2 - \left(|\nabla u_i|_2^{-1} t\right)^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

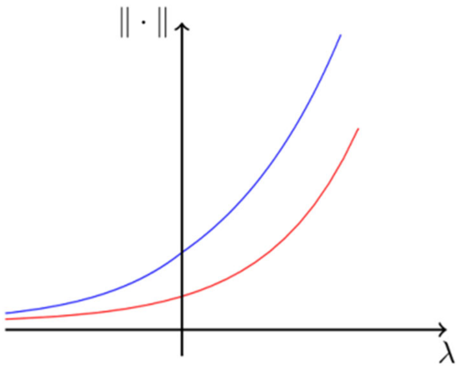

FIG. 1. $a > 0$ and $N = 3$

where $i \in \mathbb{N}$, $a > 0$ and $\lambda < 0$. From the assumption $|\nabla u_i|_2 < |\nabla u_{i+1}|_2$, it follows that $f_i(t) < f_{i+1}(t) < \lambda t^2 + a$ and

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n < \cdots < \sqrt{-\frac{a}{\lambda}}. \tag{1.29}$$

By (1.29), $a > 0$ and $\lambda < 0$, one has

$$Q''_{\varphi_i}(1) = \frac{2(N-4)}{N-2}\lambda|\nabla\varphi_i|_2^4 - \frac{4a}{(N-2)}|\nabla\varphi_i|_2^2 < 0,$$

which implies that $\varphi_i \in \mathcal{N}^{N,-}$. By (1.29) and the monotonicity of $\mathcal{I}(t)$, the desired result $I(\varphi_i) < -\frac{a^2}{4\lambda}$ holds. It follows from $u \in \mathcal{N}^N$ that

$$a|\nabla u|_2^2 + \lambda|\nabla u|_2^4 = |u|_{2^*}^{2^*} \le S^{-\frac{N}{N-2}}|\nabla u|_2^{\frac{2N}{N-2}},$$

which implies that $|\nabla u|_2 \ge \xi_1$. Combing with $|\nabla\varphi_1|_2 = \xi_1$, it follows that

$$\inf_{u\in\mathcal{N}^N} I(u) = c_N = I(\varphi_1) > 0.$$

The proof is completed. □

Remark 1.6. For $N \ge 5$, let $\{\lambda_n\}$ be a positive and decreasing sequence satisfying that $\lambda_n < \Lambda_0$ and $\lambda_n \to 0^+$ as $n \to \infty$. It follows from Theorem 1.2 (i) that $\varphi_{1,\pm}^n := K_{1,\pm}^n U$ being two positive solutions of problem (K_λ^*), where $K_{1,\pm}^n$ are the positive roots of the equation $\lambda_n S^{\frac{N}{2}} t^2 - t^{\frac{4}{N-2}} + a = 0$. It is evident to check that $a^{\frac{N-2}{4}} < K_{1,-}^n < \left(\frac{2}{(N-2)\lambda S^{N/2}}\right)^{\frac{N-2}{2(N-4)}} < K_{1,+}^n$ and

$$\lim_{n\to\infty} K_{1,-}^n = a^{\frac{N-2}{4}}, \quad \lim_{n\to\infty} K_{1,+}^n = \infty. \tag{1.30}$$

Moreover, let $\{\lambda_n\}$ be a negative and increasing sequence satisfying that $\lambda_n \to 0^-$. It follows from Theorem 1.4 that $\varphi_{1,\pm}^n := K_1^n U$ being a positive solutions of problem (K_λ^*), where $K_1^n < a^{\frac{N-2}{4}}$ and $\lim_{n\to\infty} K_1^n = a^{\frac{N-2}{4}}$. It is easy to check that $\varphi_1^0 := a^{\frac{N-2}{4}} U$ solves (K_λ^*) with $\lambda = 0$. Thus, from this aspect, the existence results are continuous in $\lambda = 0$.

From all the above analysis, we can give the following bifurcation and the red one is the positive solution.

In the last part of this section, we consider the case that $a \le 0$.

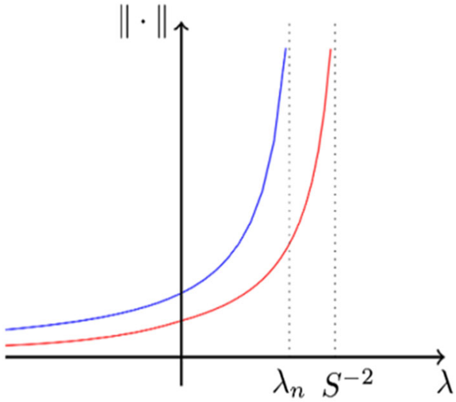

FIG. 2. $a > 0$ and $N = 4$

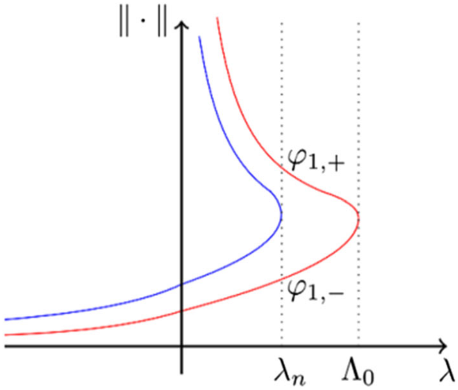

FIG. 3. $a > 0$ and $N \geq 5$

Theorem 1.7. *Assume $a = 0$ and $\lambda > 0$. Then, the following results hold:*

(*i*) *If $N = 3$, problem (K_λ^*) has infinitely many distinct solutions $\{\varphi_i\}_{i=1}^{\infty}$. Moreover, φ_1 is a positive ground state solution*

$$0 < \inf_{u \in \mathcal{N}^3} I(u) = \inf_{u \in \mathcal{N}^{3,-}} I(u) = I(\varphi_1), \quad I(\varphi_i) \to +\infty \ \ as \ i \to +\infty.$$

(*ii*) *If $N = 4$, problem (K_λ^*) admits a nontrivial solution φ if and only if $\lambda = \|w\|^{-2}$, where w is a nontrivial solution of problem (S^*). Moreover, $I(\varphi) = 0$.*

(*iii*) *If $N \geq 5$, problem (K_λ^*) has infinitely many distinct solutions $\{\varphi_i\}_{i=1}^{\infty}$. Moreover, φ_1 is a positive ground state solution*

$$\inf_{u \in \mathcal{N}^N} I(u) = \inf_{u \in \mathcal{N}^{N,+}} I(u) = I(\varphi_1), \quad I(\varphi_i) < 0, \ I(\varphi_i) \to 0 \ \ as \ i \to +\infty.$$

Proof. Since the proof is similar, we omit it here. □

Theorem 1.8. *Assume $a < 0$ and $\lambda > 0$. Then, the following results hold:*

(*i*) *For $N = 3$, problem (K^*_λ) has two sequences of distinct solutions $\{\varphi_{i,\pm}\}_{i=1}^\infty$.*

$$I(\varphi_{i,-}) < I(\varphi_{i,+}),$$
$$I(\varphi_{1,-}) < I(\varphi_{2,-}) < \cdots < I(\varphi_{i,-}) < \cdots < 0,$$
$$I(\varphi_{1,+}) < I(\varphi_{2,+}) < \cdots < I(\varphi_{i,+}) \to +\infty.$$

Moreover, if $\lambda > \lambda_1 = -4aS^{-3}$, then $\varphi_{1,\pm}$ are two positive solution. Both of them are local minimum points on the Nehari manifolds,

$$\inf_{u\in\mathcal{N}^{N,-}} I(u) = I(\varphi_{1,-}) < 0, \quad \inf_{u\in\mathcal{N}^{N,+}} I(u) = I(\varphi_{1,+}).$$

(*ii*) *For $N \geq 4$, problem (K^*_λ) has infinitely many distinct solutions $\{\varphi_i\}_{i=1}^\infty$ satisfying*

$$I(\varphi_1) < I(\varphi_2) < \cdots < I(\varphi_i) < \cdots < -\frac{a^2}{4\lambda}.$$

For $N = 4$ and $\lambda > \lambda_1 = S^{-2}$ or $N \geq 5$ and $\lambda > 0$, then φ_1 is a positive solution,

$$\inf_{u\in\mathcal{N}^N} I(u) = \inf_{u\in\mathcal{N}^{N,+}} I(u) = I(\varphi_1).$$

The proof Theorem 1.8 (i). Equation (S^*) admits infinitely many distinct solutions $\{u_i\}_{i=1}^\infty$ satisfying $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_i|_2 \to +\infty$. Set

$$i_0 := \min\{i \in \mathbb{N} : |\nabla u_i|_2 > \left(-4a/\lambda^2\right)^{1/4}\}.$$

Then, we can check that the existence of two positive roots $K_{i,\pm}$ ($K_{i,-} < K_{i,+}$) for equation $g_i(t) = 0$ defined by

$$g_i(t) = -t^4 + \lambda|\nabla u_{i+i_0-1}|_2^2 t^2 + a, \quad t \geq 0,$$

where $i \in \mathbb{N}$, $a < 0$ and $\lambda > 0$. Let $\varphi_{i,\pm} := K_{i,\pm}u_{i+i_0-1}$, $i \in \mathbb{N}$, then $\{\varphi_{i,\pm}\}_{i=1}^\infty$ are two sequences of solutions of equation (K^*_λ). Set $\xi_{i,\pm} := |\nabla\varphi_{i,\pm}|_2$ for any $i \in \mathbb{N}$, then $\xi_{i,\pm}$ admit the following forms:

$$\xi_{i,-} = \left(\frac{1}{2}(\lambda A_i - \sqrt{\lambda^2 A_i^2 + 4aA_i})\right)^{1/2}, \ \xi_{i,+} = \left(\frac{1}{2}(\lambda A_i + \sqrt{\lambda^2 A_i^2 + 4aA_i})\right)^{1/2},$$

where $A_i := |\nabla u_{i+i_0-1}|_2^4$. Then by some calculations and $A_i < A_{i+1}$, one has $f_i(t) < f_{i+1}(t) < \lambda t^2 + a$ and

$$\sqrt{-\frac{a}{\lambda}} < \xi_{i,-} < \cdots < \xi_{2,-} < \xi_{1,-} < \sqrt{-\frac{2a}{\lambda}} < \xi_{1,+} < \xi_{2,+} < \cdots < \xi_{i,+} \to +\infty, \tag{1.31}$$

$$\xi_{i,+}^2 + \frac{2a}{\lambda} > -\frac{2a}{\lambda} - \xi_{i,-}^2 > 0. \tag{1.32}$$

Then, $\varphi_{i,\mp} \in \mathcal{N}^{3,\pm}$ because of $\pm Q''_{\varphi_{i,\mp}}(1) = \pm 2\lambda|\nabla\varphi_{i,\mp}|_2^2\left(|\nabla\varphi_{i,\mp}|_2^2 + \frac{2a}{\lambda}\right) < 0$. Therefore, $I(\varphi_{i,\pm}) = \frac{1}{3}a|\nabla\varphi_{i,\pm}|_2^2 + \frac{1}{12}\lambda|\nabla\varphi_{i,\pm}|_2^4$. The desired results follow from (1.31) and (1.32).

Moreover, if $\lambda > -4aS^{-3}$, then $i_0 = 1$, that is, $u_1 = U$. It means that $\varphi_{1,\pm}$ are positive solutions. It follows from $u \in \mathcal{N}^4$ that

$$a|\nabla u|_2^2 + \lambda|\nabla u|_2^4 = |u|_6^6 \leq S^{-3}|\nabla u|_2^6,$$

which implies that $|\nabla u|_2 \leq \xi_{1,-}$ for $u \in \mathcal{N}^{3,+}$ and $|\nabla u|_2 \geq \xi_{1,+}$ for $u \in \mathcal{N}^{3,-}$. Combing with $\|\varphi_{1,\pm}\| = \xi_{1,\pm}$, it follows that

$$\inf_{u\in\mathcal{N}^{N,+}} I(u) = I(\varphi_{1,-}) < 0, \quad \inf_{u\in\mathcal{N}^{N,-}} I(u) = I(\varphi_{1,+}).$$

The proof is completed. □

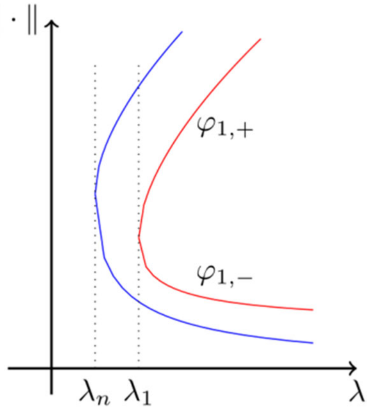

FIG. 4. $a < 0$ and $N = 3$

The Proof of Theorem 1.8 (ii). Equation (S^*) admits infinitely many distinct solutions $\{u_i\}_{i=1}^{+\infty}$ satisfying $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_i|_2 \to +\infty$.

$$i_0 = \begin{cases} \min\{i \in \mathbb{N} : |\nabla u_i|_2 > \lambda^{-1/2}\}, & \text{if } N = 4; \\ 1, & \text{if } N \geq 5. \end{cases}$$

Then, we can check that the existence of a positive root K_i for equation $g_i(t) = 0$ defined by

$$g_i(t) = \lambda|\nabla u_{i+i_0-1}|_2^2 t^2 - t^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

where $\lambda > 0$ and $i \in \mathbb{N}$. Let $\varphi_i := K_i u_{i+i_0-1}$, $i \in \mathbb{N}$, then $\{\varphi_i\}_{i=1}^{\infty}$ a sequence of solutions of equation (K_λ^*) . Set $\xi_i := |\nabla \varphi_i|_2$ for any $i \in \mathbb{N}$, then ξ_i is the positive solution of $f_i(t) = 0$, $i \in \mathbb{N}$,

$$f_i(t) = \lambda t^2 - \left(|\nabla u_{i+i_0-1}|_2^{-1} t\right)^{\frac{4}{N-2}} + a.$$

Then by some calculations and $|\nabla u_i|_2 < |\nabla u_{i+1}|_2$, one has $f_i(t) < f_{i+1}(t) < \lambda t^2 + a$ and

$$\sqrt{-\frac{a}{\lambda}} < \cdots < \xi_i < \cdots < \xi_2 < \xi_1. \tag{1.33}$$

Then, $\varphi_i \in \mathcal{N}^{4,+}$ for $Q''_{\varphi_i}(1) = -2a|\nabla \varphi_i|_2^2 > 0$ and $\varphi_i \in \mathcal{N}^{N,+}$ for

$$Q''_{\varphi_i}(1) = \frac{2(N-4)}{N-2}\lambda|\nabla u|_2^2 \left(|\nabla u|_2^2 - \frac{2a}{(N-4)\lambda}\right) > 0.$$

By (1.33), the following desired results hold,

$$I(\varphi_1) < I(\varphi_2) < \cdots < I(\varphi_i) < \cdots < -\frac{a^2}{4\lambda}.$$

Similarly, for $N = 4$ and $\lambda > S^{-2}$ or $N \geq 5$ and $\lambda > 0$, we can obtain that φ_1 is a positive solution satisfying

$$\inf_{u \in \mathcal{N}^N} I(u) = \inf_{u \in \mathcal{N}^{N,+}} I(u) = I(\varphi_1).$$

The proof is completed. □

We also have the following bifurcation and the red one is the positive solution.

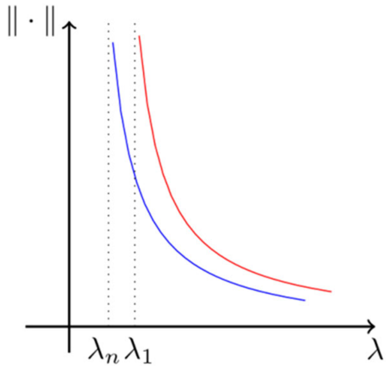

FIG. 5. $a < 0$ and $N = 4$

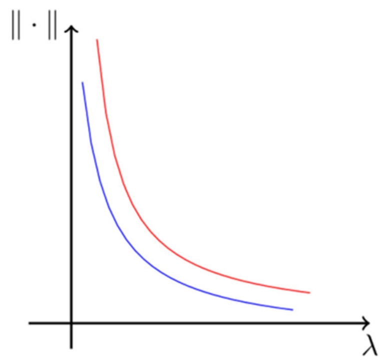

FIG. 6. $a < 0$ and $N \geq 5$

2. A perturbed problem

In this present paper, we consider the following critical Kirchhoff problem

$$-\Big(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)\Delta u+mu=\mu u^{p-1}+u^{2^*-1},\ u>0\ \text{ in }\mathbb{R}^N, \qquad (K^*_{\lambda,\mu})$$

where $a>0, m>0$ are two constants, $\lambda>0, \mu>0$ are two parameters, $2<p<2^*$ and $2^*=\frac{2N}{N-2}$ for $N\geq 3$. It can be easily checked that problem ($K^*_{\lambda,\mu}$) has no nontrivial solutions for any $m>0$ if $\mu=0$ by the Pohožaev identity.

Now, we state our existence results as follows.

Theorem 2.1. *Assume $a>0$, $m>0$ and $\lambda>0$. Then, the following results hold.*

(*i*) *If $N=3$, problem ($K^*_{\lambda,\mu}$) has a positive ground solution for one of the following cases:*

$$p\in(2,4)\ \textit{and}\ \mu>0\ \textit{large enough or}\ p\in(4,6)\ \textit{and for any}\ \mu>0.$$

(*ii*) *If $N=4$, there exists $0<\Lambda_2<S^{-2}$ such that problem ($K^*_{\lambda,\mu}$) has a positive ground solution for*

$$p\in(2,4),\quad 0<\lambda<\Lambda_2\ \textit{ and }\mu>0.$$

The critical Kirchhoff problem ($K^*_{\lambda,\mu}$) with $p \in (2,4)$ or $p \in (4,6)$ in three-dimensional case is considered in Proposition 3.8 of He and Li [16] and in Theorem 1.3 of Li and Ye [24], respectively. We list it in Theorem 2.1(i) for the sake of the completeness.

Theorem 2.2. *Assume $a > 0$, $m > 0$, $\mu > 0$, $p \in (2, 2^*)$ and $N \geq 5$.*

(*i*) *If $\lambda \in (0, \Lambda_1)$, problem ($K^*_{\lambda,\mu}$) has a positive ground state solution with negative energy, which is the global minima point of the corresponding functional, where*

$$\Lambda_1 = \frac{4}{N}\left(\frac{N-4}{aN}\right)^{\frac{N-4}{2}} S^{-\frac{N}{2}}.$$

(*ii*) *There exists $0 < \Lambda_2 < \Lambda_1$, problem ($K^*_{\lambda,\mu}$) with $\lambda \in (0, \Lambda_2)$ has two positive solutions.*

Remark 2.3. If $a = 1$ and $\lambda = 0$, the problem reduces to a semi-linear elliptic equation

$$-\Delta u + mu = \mu u^{p-1} + u^{2^*-1}, \ u > 0 \ \text{ in } \mathbb{R}^N. \tag{S^*_m}$$

The authors in Zhang and Zou [49] and Alves et al. [4] obtained a positive ground state solution for problem (S^*_m) either $N = 3$ and $p \in (4, 6)$ or $N \geq 4$ and $p \in (2, 2^*)$. A simple proof also is found in Proposition 1.1 of Akahori et al. [1], in which the authors proved that problem (S^*_m) admits a unique positive ground state solution in $H^1_{rad}(\mathbb{R}^N)$ if $N \geq 5$, $p \in (2, 2^*)$ and $m > 0$. However, from this point, we found a different result that problem ($K^*_{\lambda,\mu}$) admits two positive solutions if $\lambda > 0$ suitable small.

Since problem ($K^*_{\lambda,\mu}$) is an autonomous problem, we consider this problem on the space $H := H^1_{rad}(\mathbb{R}^3)$, the subspace formed by radially symmetric functions, with the norm

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \mathrm{d}x\right)^{1/2}.$$

To get the positive solution, the functional is defined by

$$J(u) := \frac{1}{2}\int\limits_{\mathbb{R}^N} (a|\nabla u|^2 + m|u|^2)\mathrm{d}x + \frac{\lambda}{4}|\nabla u|_2^4 - \frac{1}{p}\int\limits_{\mathbb{R}^N} u_+^p \mathrm{d}x - \frac{1}{2^*}\int\limits_{\mathbb{R}^N} u_+^{2^*}\mathrm{d}x, \tag{2.1}$$

for any $u \in H$, where $u_+ = \max\{u, 0\}$. Similar to the above section, we set that $Q_{u,m}(t) := J(tu)$ for any $u \in H$ and $t \geq 0$. Recall that U_ε is the unique positive solution of (S^*),

$$U_\varepsilon(x) = \left(\frac{\sqrt{N(N-2)\varepsilon}}{\varepsilon + |x|^2}\right)^{\frac{N-2}{2}}, \quad \varepsilon > 0, \ x \in \mathbb{R}^N \tag{2.2}$$

and $|\nabla U|_2^2 = |U|_{2^*}^{2^*} = S^{N/2}$. Similar to Brezis and Nirenberg [9], by some directly computations, it is evident to check that

$$|U_\varepsilon|_2^2 = \begin{cases} O(\varepsilon|\log\varepsilon|), & \text{if } N = 4,\\ O(\varepsilon), & \text{if } N \geq 5,\end{cases}$$

and

$$|U_\varepsilon|_p^p = O(\varepsilon^{\frac{2N-(N-2)p}{4}}) \ \text{ if } N \geq 4 \text{ and } \ 2 < p < 2^*.$$

Lemma 2.4. *Assume that $N = 4$, $a, m, \mu > 0$ and $0 < \lambda < S^{-2}$. Let $Q_{U_\varepsilon,m}(t) := J(tU_\varepsilon)$ for $t \geq 0$. There exists $t_\varepsilon > 0$ such that $Q'_{U_\varepsilon,m}(t_\varepsilon) = 0$ and $Q''_{U_\varepsilon,m}(t_\varepsilon) < 0$ for small $\varepsilon > 0$. Moreover, $I(t_\varepsilon U_\varepsilon) < c_4$, where c_4 is defined in Theorem 1.1(ii).*

Since the proof of Lemma 2.4 is similar, we omit it here and only prove the following lemma.

Lemma 2.5. *Assume that $N \geq 5$, $a, m, \mu > 0$ and $0 < \lambda < \Lambda_0$. Let $Q_{U_\varepsilon,m}(t) := J(tU_\varepsilon)$ for $t \geq 0$. There exist $0 < t_{\varepsilon,-} < t_{\varepsilon,+}$ such that $Q'_{U_\varepsilon,m}(t_{\varepsilon,\pm}) = 0$ and $\pm Q''_{U_\varepsilon,m}(t_{\varepsilon,\pm}) > 0$ for small $\varepsilon > 0$. Moreover, $I(t_{\varepsilon,\pm}U_\varepsilon) < c_{N,\pm}$, where $c_{N,\pm}$ is defined in Theorem 1.2 (i).*

Proof. By $\lambda \in (0, \Lambda_0)$, the following equation admits two positive roots $0 < K_- < K_+$,

$$\lambda S^{\frac{N}{2}} t^2 - t^{2^*-2} + a = 0.$$

It is evident to check that $\pm(2\lambda S^{N/2} K_\pm - (2^* - 2)K_\pm^{2^*-3}) > 0$. Moreover,

$$\frac{1}{2} a S^{\frac{N}{2}} K_\pm^2 + \frac{1}{4}\lambda S^N K_\pm^4 - \frac{1}{2^*} S^{\frac{N}{2}} K_\pm^{2^*} = c_{N,\pm}. \tag{2.3}$$

It is evident to check that there exists ε_0 such that $Q'_{U_\varepsilon,m}(t) = 0$ admits two positive roots $t_{\varepsilon,\pm}$ and $\pm Q''_{U_\varepsilon,m}(t_{\varepsilon,\pm}) > 0$ for $\varepsilon < \varepsilon_0$. Set $t_{\varepsilon,\pm} = K_\pm + \delta_{\varepsilon,\pm}$, then $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We claim that

$$\delta_{\varepsilon,\pm} = O(\varepsilon^{\frac{2N-(N-2)p}{4}}). \tag{2.4}$$

In fact, it follows from $Q'_{U_\varepsilon,m}(t_{\varepsilon,\pm}) = 0$ that

$$\begin{aligned} 0 &= a|\nabla U_\varepsilon|_2^2 + m|U_\varepsilon|_2^2 + \lambda|\nabla U_\varepsilon|_2^4 t_{\varepsilon,\pm}^2 - \mu|U_\varepsilon|_p^p t_{\varepsilon,\pm}^{p-2} - |U_\varepsilon|_{2^*}^{2^*} t_{\varepsilon,\pm}^{2^*-2} \\ &= aS^{N/2} + \lambda S^N t_{\varepsilon,\pm}^2 - S^{N/2} t_{\varepsilon,\pm}^{2^*-2} - O(\varepsilon^{\frac{2N-(N-2)p}{4}}) + o(\varepsilon^{\frac{2N-(N-2)p}{4}}) \\ &= C_{\lambda,\pm}\delta_{\varepsilon,\pm} - O(\varepsilon^{\frac{2N-(N-2)p}{4}}) + o(\delta_{\varepsilon,\pm}) + o(\varepsilon^{\frac{2N-(N-2)p}{4}}), \end{aligned}$$

where $C_{\lambda,\pm} = 2\lambda S^{N/2} K_\pm - (2^* - 2)K_\pm^{2^*-3}$ and $\pm C_{\lambda,\pm} > 0$. So our claim holds. It follows from (2.3) and (2.4) that

$$\begin{aligned} I(t_{\varepsilon,\pm}U_\varepsilon) &= \frac{1}{2} a|\nabla U_\varepsilon|_2^2 t_{\varepsilon,\pm}^2 + \frac{1}{4}\lambda|\nabla U_\varepsilon|_2^4 t_{\varepsilon,\pm}^4 - \frac{1}{2^*}|U_\varepsilon|_{2^*}^{2^*} t_{\varepsilon,\pm}^{2^*} \\ &\quad + \frac{1}{2} m|U_\varepsilon|_2^2 t_{\varepsilon,\pm}^2 - \frac{1}{p}\mu|U_\varepsilon|_p^p t_{\varepsilon,\pm}^p \\ &= \frac{1}{2} a S^{N/2} K_\pm^2 + \frac{1}{4}\lambda S^N K_\pm^4 - \frac{1}{2^*} S^{N/2} K_\pm^{2^*} \\ &\quad + S^{N/2}(a + \lambda S^{N/2} K_\pm^2 - K_\pm^{2^*-2}) t_\pm \delta_{\varepsilon,\pm} \\ &\quad + o(\delta_{\varepsilon,\pm}) - \mu C \varepsilon^{\frac{2N-(N-2)p}{4}} + o(\varepsilon^{\frac{2N-(N-2)p}{4}}) \\ &= c_{N,\pm} - \mu C \varepsilon^{\frac{2N-(N-2)p}{4}} + o(\varepsilon^{\frac{2N-(N-2)p}{4}}) + o(\delta_{\varepsilon,\pm}) < c_{N,\pm}. \end{aligned}$$

This completes the proof. □

Thanks to the radially symmetric functions space H and the compact embedding from H to $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*)$, we avoid to deal with the lack of the compactness caused by the translation. Thus, we have the following analysis on the Palais–Smale sequence by a standard argument (see Naimen [32] or Xie et al. [45]).

Proposition 2.6. *Let $\{u_n\}$ be a bounded Palais–Smale sequence for J in H. Then, $\{u_n\}$ has a subsequence which strongly converges in H. Otherwise, replacing $\{u_n\}$ if necessary by a subsequence, there exist a function $u_0 \in H$, a number $A \in \mathbb{R}$, a number $l \in \mathbb{N}$, l sequences of number $\{\sigma_n^i\} \subset \mathbb{R}^+$, points $\{y_n^i\} \subset \mathbb{R}^N$ and l functions $u^i \in D^{1,2}(\mathbb{R}^N)$, $i \in \{1, 2, \ldots, l\}$, which satisfy*

$$-(a + \lambda A)\Delta u_0 + m u_0 = \mu u_0^{p-1} + u_0^{2^*-1}, \quad u_0 > 0 \ \text{ in } \mathbb{R}^N \tag{2.5}$$

and

$$-(a + \lambda A)\Delta u^i = (u^i)^{2^*-1}, \quad u^i > 0 \ \text{ in } \ \mathbb{R}^N, \tag{2.6}$$

such that, up to subsequences, there hold

$$\begin{aligned} &\left|\nabla\left(u_n - u_0 - \sum_{i=1}^{l}(\sigma_n^i)^{-\frac{N-2}{2}}u^i(\frac{\cdot - y_n^i}{\sigma_n^i})\right)\right|_2 \to 0, \\ &|\nabla u_n|_2^2 \to A = |\nabla u_0|_2^2 + \sum_{i=1}^{l}|\nabla u^i|_2^2 \end{aligned} \tag{2.7}$$

and

$$J(u_n) \to J_1(u_0) + \sum_{i=1}^{l} J_2(u^i), \tag{2.8}$$

as $n \to \infty$, where $J_1(u)$ and $J_2(u)$ are defined by

$$\begin{aligned} J_1(u_0) &= \left(\frac{a}{2} + \frac{\lambda A}{4}\right)|\nabla u_0|_2^2 + \frac{1}{2}m|u_0|_2^2 - \frac{1}{p}\mu|u_0|_p^p - \frac{1}{2^*}|u_0|_{2^*}^{2^*}; \\ J_2(u^i) &= \left(\frac{a}{2} + \frac{\lambda A}{4}\right)|\nabla u^i|_2^2 - \frac{1}{2^*}|u^i|_{2^*}^{2^*}. \end{aligned} \tag{2.9}$$

Remark 2.7. (i). From the fact that equation (S^*) admits a unique positive solution and (2.6), it is evident to check that u^i can be rewritten as

$$u^i = (a + \lambda A)^{1/(2^*-2)}U_\varepsilon,$$

which implies that $|u^i|_{2^*}^{2^*} = S^{-2^*/2}|\nabla u^i|_2^{2^*}$ and $|\nabla u^i|_2^2 = |\nabla u^j|_2^2$ for any $i \neq j$. Moreover, the constant A can be rewritten as

$$A = |\nabla u_0|_2^2 + l|\nabla u^i|_2^2.$$

(ii). Set $j_1(t)$ and $j_2(t)$, respectively,

$$j_1(t) := \frac{1}{2}\left(a|\nabla u_0|_2^2 + m|u_0|_2^2\right)t^2 + \frac{1}{4}\lambda A|\nabla u_0|_2^2 t^4 - \frac{1}{p}\mu|u_0|_p^p t^p - \frac{1}{2^*}|u_0|_{2^*}^{2^*}t^{2^*}, \tag{2.10}$$

$$j_2(t) := \frac{1}{2}a|\nabla u^i|_2^2 t^2 + \frac{1}{4}\lambda A|\nabla u^i|_2^2 t^4 - \frac{1}{2^*}S^{-\frac{2^*}{2}}|\nabla u^i|_2^{2^*}t^{2^*}, \tag{2.11}$$

for $t \geq 0$. It follows from (2.5), (2.6) and (2.7) that

$$j_1(1) = J_1(u_0), \quad j_2(1) = J_2(u^i) \quad \text{and} \quad j_1'(1) = j_2'(1) = 0. \tag{2.12}$$

By a standard argument (see [7]), we get a Pohožaev identity from (2.5) as follows:

$$P(u_0) := \frac{1}{2^*}(a + \lambda A)|\nabla u_0|_2^2 + \frac{1}{2}m|u_0|_2^2 - \frac{1}{p}\mu|u_0|_p^p - \frac{1}{2^*}|u_0|_{2^*}^{2^*} = 0.$$

Thus, combining with $P(u_0) = j_1'(1) = 0$, one has

$$\frac{2^* - 2}{2}m|u_0|_2^2 = \frac{2^* - p}{p}\mu|u_0|_p^p. \tag{2.13}$$

Proposition 2.8. *For $N \geq 5$, then $J(u)$ is coercive, i.e., $J(u) \to +\infty$ as $\|u\| \to \infty$.*

Proof. By the Gagliardo–Nirenberg inequality in [14], we have that for all $u \in H$,

$$|u|_p \leq C|\nabla u|_2^\alpha |u|_2^{1-\alpha}, \tag{2.14}$$

where $\alpha = N(\frac{1}{2} - \frac{1}{p})$. Let $\delta_1 = (p-2)(N-4)/4$. Then, for $0 < \delta < \delta_1$, we set

$$q_1' = \frac{2}{2 - (p(1-\alpha) + \delta)} \quad \text{and} \quad q_2' = \frac{2}{p(1-\alpha) + \delta}.$$

It is evident to check that $q_1' > 1$, $q_2' > 1$ and $1/q_1' + 1/q_2' = 1$. Here by the Gagliardo–Nirenberg inequality and Young's inequality, one obtains

$$|u|_p^p \leq C|\nabla u|_2^{p\alpha}|u|_2^{p(1-\alpha)} \leq C\left(\frac{1}{q_1'}|\nabla u|_2^{q_1' p\alpha} + \frac{1}{q_2'}|u|_2^{q_2' p(1-\alpha)}\right) := C|\nabla u|_2^{q_1} + C|u|_2^{q_2}. \tag{2.15}$$

where $q_1 := q_1' p\alpha$ and $q_2 := q_2' p(1-\alpha)$. It follows from $\delta \in (0, \delta_1)$ that $q_1 < 4$ and $q_2 < 2$. By the above fact and the Sobolev inequalities, we obtain

$$\begin{aligned} J(u) &\geq \frac{1}{2}(a|\nabla u|_2^2 + m|u|_2^2) + \frac{\lambda}{4}|\nabla u|_2^4 - \frac{\mu}{p}|u|_p^p - \frac{1}{2^*}|u|_{2^*}^{2^*} \\ &\geq (\frac{1}{2}m|u|_2^2 - C|u|_2^{q_2}) + \frac{1}{2}a|\nabla u|_2^2 + \frac{\lambda}{4}|\nabla u|_2^4 - C|\nabla u|_2^{q_1} - \frac{1}{2^*}S^{-2^*/2}|\nabla u|_2^{2^*}, \end{aligned}$$

which implies that $J(u) \to +\infty$ as $\|u\| \to \infty$ because of $2^* < 4$, $q_1 < 4$ and $q_2 < 2$. □

The proof of Theorem 2.2 (i). Firstly, we define that

$$c_+ = \inf_{u \in H} J(u). \tag{2.16}$$

Combining with Proposition 2.8, Lemma 2.5 and Theorem 1.2(i), one has

$$-\infty < c_+ < c_{N,+} < 0. \tag{2.17}$$

It is evident to obtain a (PS) sequence $\{u_n\}$ of J at c_+ by Ekeland variational principle. By the coerciveness of J, $\{u_n\}$ is bounded in H. Assume that $u_n \rightharpoonup u_0$ weakly in H. We claim that $u_0 \neq 0$. Otherwise, if $u_0 = 0$, it follows from (2.11) and (2.12) that $A = l|\nabla u^i|_2^2$,

$$j_2(1) = J_2(u^i) = \frac{c_+}{l} < 0 \ \text{ and } \ j_2'(1) = 0,$$

which imply that $j_2''(1) > 0$. Thus, it obtains

$$0 < j_2''(1) = j_2''(1) - (2^* - 1)j_2'(1) = \frac{2(N-4)}{N-2}\lambda|\nabla u^i|_2^2\left(l|\nabla u^i|_2^2 - \frac{2a}{(N-4)\lambda}\right), \tag{2.18}$$

which implies $l|\nabla u^i|_2^2 > \eta_\lambda^2$. Moreover, by $j_2'(1) = 0$, one has

$$a(l^{1/2}|\nabla u|_2)^2 + \lambda(l^{1/2}|\nabla u|_2)^4 = S^{-\frac{2^*}{2}}l^{-\frac{2^*-2}{2}}(l^{1/2}|\nabla u|_2)^{2^*} \leq S^{-\frac{2^*}{2}}(l^{1/2}|\nabla u|_2)^{2^*},$$

which implies that $\xi_{1,-}^2 \leq l|\nabla u|_2^2 \leq \xi_{1,+}^2$. Thus, we obtain

$$\eta_\lambda^2 < l|\nabla u|_2^2 \leq \xi_{1,+}^2.$$

It follows that

$$\begin{aligned} c_+ = lj_2(1) - \frac{1}{2^*}j_2'(1) &= \frac{1}{N}(l|\nabla u^i|_2^2) - \frac{N-4}{4N}\lambda(l|\nabla u^i|_2^2)^2 \\ &\geq \frac{1}{N}(\xi_{1,+}^2) - \frac{N-4}{4N}\lambda(\xi_{1,+}^2)^2 = c_{N,+}, \end{aligned} \tag{2.19}$$

which is impossible for $c_+ < c_{N,+}$. Thus, $u_0 \neq 0$.

It remains to prove that $u_n \to u_0$ strongly in H. Arguing indirectly, let $\overline{u}_n = u_n - u_0 \in H$. Thus, $\overline{u}_n \rightharpoonup 0$ weakly in H and $\lim_{n\to\infty} |\nabla \overline{u}_n|_2^2 = l|\nabla u^i|_2^2$. Set that

$$u_{n,s} = (1+s)^{\frac{1}{2}}u_0 + (1-es)^{\frac{1}{2}}\overline{u}_n \in H,$$

where $e = |\nabla u_0|_2^2/(l|\nabla u^i|_2^2)$ and $s \in (-1, 1/e)$ is a parameter. Then,

$$\lim_{n\to\infty} |\nabla u_{n,s}|_2^2 = |\nabla(1+s)^{\frac{1}{2}}u_0|_2^2 + \lim_{n\to\infty} |\nabla(1-es)^{\frac{1}{2}}\overline{u}_n|_2^2 = A$$

for $s \in (-1, 1/e)$. Set $h(s) := \lim_{n\to\infty} J(u_{n,s})$. Thus, we know that $h(0) = c_+$ and

$$\begin{aligned} h(s) = & \frac{1}{2}\left(a|\nabla u_0|_2^2 + m|u_0|_2^2\right)(1+s) + \frac{1}{4}\lambda A|\nabla u_0|_2^2(1+s) \\ & - \frac{1}{p}\mu|u_0|_p^p(1+s)^{p/2} - \frac{1}{2^*}|u_0|_{2^*}^{2^*}(1+s)^{2^*/2} \\ & + l\left(\frac{1}{2}a|\nabla u^i|_2^2(1-es) + \frac{1}{4}\lambda A|\nabla u^i|_2^2(1-es) - \frac{1}{2^*}|u^i|_{2^*}^{2^*}(1-es)^{2^*/2}\right). \end{aligned}$$

By some direct calculations, it obtains

$$\begin{aligned} h'(0) &= \frac{1}{2}\left(a|\nabla u_0|_2^2 + m|u_0|_2^2\right) + \frac{1}{4}\lambda A|\nabla u_0|_2^2 - \frac{1}{2}\mu|u_0|_p^p - \frac{1}{2}|u_0|_{2^*}^{2^*} - \frac{1}{2}j_1'(1) \\ & \quad - el\left(\frac{1}{2}a|\nabla u^i|_2^2 + \frac{1}{4}\lambda A|\nabla u^i|_2^2 - \frac{1}{2}|u^i|_{2^*}^{2^*} - \frac{1}{2}j_2'(1)\right) \\ &= -\frac{1}{4}\lambda A|\nabla u_0|_2^2 + el\frac{1}{4}\lambda A|\nabla u^i|_2^2 = -\frac{1}{4}\lambda A\left(|\nabla u_0|_2^2 - el|\nabla u^i|_2^2\right) = 0 \end{aligned}$$

and

$$h''(0) = -\frac{1}{4}\mu(p-2)|u_0|_p^p - \frac{1}{4}(2^*-2)|u_0|_{2^*}^{2^*} - \frac{1}{4}(2^*-2)e^2 l|u^i|_{2^*}^{2^*} < 0.$$

That is $h(0) = c_+$, $h'(0) = 0$ and $h''(0) < 0$. Thus, there exists s_0 small, such that $h(s_0) < h(0)$. Combining with $h(s_0) = \lim_{n\to\infty} J(u_{n,s_0})$, we have $J(u_{n,s_0}) < c_+$ for n large enough, which is impossible because of the definition of c_+. The proof is completed. □

In what follows, we will prove the existence of the solution for $N = 4$ and the second solution for $N \geq 5$. With the help of the fibering map, it is useful to understand the structure of $\mathcal{N}_{\mu,m}^N$ for $N \geq 4$, where

$$\mathcal{N}_{\mu,m}^N = \left\{u \in H\backslash\{0\} : Q'_{u,m}(1) = 0\right\}. \tag{2.20}$$

We define that for $N \geq 4$,

$$\mathcal{N}_{\mu,m}^{N,-} = \left\{u \in \mathcal{N}_{\mu,m}^N : Q''_{u,m}(1) < 0\right\}, \quad \mathcal{N}_{\mu,m}^{N,0} = \left\{u \in \mathcal{N}_{\mu,m}^N : Q''_{u,m}(1) = 0\right\}.$$

It follows from the Sobolev inequalities that for any $u \in \mathcal{N}_{\mu,m}^N$,

$$\min\{a,\ m\}\|u\|^2 \leq a|\nabla u|_2^2 + m|u|_2^2 + \lambda|\nabla u|_2^4 = \mu|u|^p + |u|_{2^*}^{2^*} \leq \mu C_p\|u\|^p + S^{-\frac{2^*}{2}}\|u\|^{2^*}.$$

By $2 < p < 2^*$, there exists $\rho_0 > 0$ such that $\|u\| \geq \rho_0$ for any $u \in \mathcal{N}_{\mu,m}^N$.

For $N \geq 4$, set

$$c_- := \inf_{\mathcal{N}_{\mu,m}^{N,-}} J(u). \tag{2.21}$$

It follows from Lemma 2.4 and Lemma 2.5 that $\mathcal{N}_{\mu,m}^{N,-} \neq \emptyset$. By $u \in \mathcal{N}_{\mu,m}^{N,-}$, we obtain

$$Q''_{u,m}(1) = (2-2^*)(a|\nabla u|_2^2 + m|u|_2^2) + (4-2^*)\lambda|\nabla u|_2^4 - (p-2^*)\mu|u|_p^p.$$

It follows that

$$\begin{aligned} J(u) &= Q_{u,m}(1) - \frac{1}{2^*}Q'_{u,m}(1) \\ &= (\frac{1}{2} - \frac{1}{2^*})(a|\nabla u|_2^2 + m|u|_2^2) + (\frac{1}{4} - \frac{1}{2^*})\lambda|\nabla u|_2^4 + (\frac{1}{2^*} - \frac{1}{p})\mu|u|_p^p \\ &= \frac{p-2}{Np}(a|\nabla u|_2^2 + m|u|_2^2) + \frac{(N-4)(4-p)}{4Np}\lambda|\nabla u|_2^4 - \frac{1}{2^*p}Q''_{u,m}(1) \\ &> \frac{p-2}{Np}\min\{a,m\}\rho_0^2, \end{aligned} \tag{2.22}$$

which implies that $c_- > 0$. To get a PS sequence, we set

$$c_0 := \inf\left\{\liminf_{n\to\infty} J(u_n) : \{u_n\} \in \widetilde{\mathcal{N}}^{N,0}\right\},$$

where $\widetilde{\mathcal{N}}^{N,0} := \left\{\{u_n\} \subset \mathcal{N}^N_{\mu,m} : \lim_{n\to\infty} Q''_{u_n,m}(1) = 0\right\}$. If $\widetilde{\mathcal{N}}^{N,0} = \emptyset$, we set $c_0 = \infty$.

Lemma 2.9. *There exists $0 < \Lambda_2 < S^{-2}$ for $N = 4$ or $0 < \Lambda_2 < \Lambda_1$ for $N \geq 5$ such that*

$$c_- < \frac{(p-2)^2}{4p(4-p)\lambda \max\{a^{-2}, 1\}} \leq c_0, \tag{2.23}$$

for $0 < \lambda < \Lambda_2$.

Proof. Firstly, we prove the second inequality. Let $\{u_n\} \in \widetilde{\mathcal{N}}^{N,0}$, then

$$Q''_{u_n,m}(1) = (2-p)(a|\nabla u_n|_2^2 + m|u_n|_2^2) + (4-p)\lambda|\nabla u_n|_2^4 + (p-2^*)|u_n|_{2^*}^{2^*},$$

which implies that

$$(2-p)(a|\nabla u_n|_2^2 + m|u_n|_2^2) + (4-p)\lambda|\nabla u_n|_2^4 = (2^*-p)|u_n|_{2^*}^{2^*} + Q''_{u_n,m}(1) \geq Q''_{u_n,m}(1).$$

Then, we have

$$\lambda|\nabla u_n|_2^4 \geq \frac{p-2}{4-p}(a|\nabla u_n|_2^2 + m|u_n|_2^2) + \frac{1}{4-p}Q''_{u_n,m}(1) \tag{2.24}$$

and

$$\begin{aligned} a|\nabla u_n|_2^2 + m|u_n|_2^2 \geq & \frac{p-2}{\lambda(4-p)\max\{a^{-2},1\}} \\ & + \frac{1}{\lambda(4-p)\max\{a^{-2},1\}(a|\nabla u_n|_2^2 + m|u_n|_2^2)}Q''_{u_n,m}(1). \end{aligned} \tag{2.25}$$

By (2.24), we have

$$\begin{aligned} J(u_n) &= Q_{u_n,m}(1) - \frac{1}{2^*}Q'_{u_n,m}(1) \\ &= (\frac{1}{2} - \frac{1}{2^*})(a|\nabla u_n|_2^2 + m|u_n|_2^2) + (\frac{1}{4} - \frac{1}{2^*})\lambda|\nabla u_n|_2^4 + (\frac{1}{2^*} - \frac{1}{p})\mu|u_n|_p^p \\ &= \frac{p-2}{Np}(a|\nabla u_n|_2^2 + m|u_n|_2^2) + \frac{(N-4)(4-p)}{4Np}\lambda|\nabla u_n|_2^4 - \frac{1}{2^*p}Q''_{u_n,m}(1) \\ &\geq \frac{p-2}{4p}(a|\nabla u_n|_2^2 + m|u_n|_2^2) - \frac{1}{4p}Q''_{u_n,m}(1). \end{aligned}$$

With the help of (2.25), it obtains

$$\liminf_{n\to\infty} J(u_n) \geq \frac{(p-2)^2}{4p(4-p)\lambda\max\{a^{-2},1\}}.$$

By the arbitrariness of $\{u_n\} \in \widetilde{\mathcal{N}}^{N,0}$ and the definition of c_0, we have

$$\frac{(p-2)^2}{4p(4-p)\lambda\max\{a^{-2},1\}} \leq c_0.$$

For $N = 4$, we have $c_N^{\lambda_1} \leq c_N^{\lambda_2}$ for $0 < \lambda_1 \leq \lambda_2 < S^{-2}$ directly from

$$c_4^{\lambda_2} = \max_{t\geq 0} I_{\lambda_2}(tU) \geq \max_{t\geq 0} I_{\lambda_1}(tU) = c_4^{\lambda_1}.$$

Similarly, for $N \geq 5$, we have $c_{N,-}^{\lambda_1} \leq c_{N,-}^{\lambda_2}$ for $0 < \lambda_1 \leq \lambda_2 < \Lambda_0$ from

$$c_{N,-}^{\lambda_2} = \max_{0\leq t\leq K_+^{\lambda_2}} I_{\lambda_2}(tU) \geq \max_{0\leq t\leq K_+^{\lambda_2}} I_{\lambda_1}(tU) = c_{N,-}^{\lambda_1}.$$

Moreover, by Remark 1.6, we can easily obtain that

$$\lim_{\lambda\to 0^+} c_4^\lambda = \frac{1}{4}(aS)^2 \quad\text{and}\quad \lim_{\lambda\to 0^+} c_{N,-}^\lambda = \frac{1}{N}(aS)^{N/2}.$$

Thus, c_4^λ and $c_{N,-}^\lambda$ are nondecreasing with respect to $\lambda \in (0, S^{-2})$ and $\lambda \in (0, \Lambda_1)$, respectively. Combining with the above inequalities, there exists Λ_2 such that

$$c_- < c_4 \ (c_{N,-}^\lambda) < \frac{(p-2)^2}{4p(4-p)\lambda \max\{a^{-2}, 1\}},$$

for $0 < \lambda < \Lambda_2$. The proof is completed. □

The proof of Theorem 2.1 (i) and Theorem 2.2 (ii). We divide the proof into two steps.

Step 1. Construct a PS sequence for J at level c_- in $\mathcal{N}_{\mu,m}^{N,-}$.

From the boundedness of $\mathcal{N}_{\mu,m}^N$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_{\mu,m}^{N,-} \cup \mathcal{N}_{\mu,m}^{N,0}$ satisfying that

$$J(u_n) \le \inf_{\mathcal{N}_{\mu,m}^{N,-} \cup \mathcal{N}^{N,0}} J(u) + \frac{1}{n} \quad\text{and}\quad J(w) \ge J(u_n) - \frac{1}{n}\|u_n - w\|,$$

for any $w \in \mathcal{N}_{\mu,m}^{N,-} \cup \mathcal{N}_{\mu,m}^{N,0}$, by Ekeland variational principle. With the help of Lemma 2.9, we know that

$$\inf_{\mathcal{N}_{\mu,m}^{N,-}} J(u) = c_- \quad\text{and}\quad \{u_n\} \subset \mathcal{N}_{\mu,m}^{N,-}.$$

Then, it is evident to check that $J'(u_n) \to 0$ as $n \to \infty$ since $\lim_{n\to\infty} |Q''_{u_n,m}(1)| > 0$. Thus $\{u_n\}$ is a PS sequence for J at level c_- in $\mathcal{N}_{\mu,m}^{N,-}$.

Step 2. Claim $u_n \to u_0$ strongly in H.

If $N = 4$, then we have

$$c_- + o(1) = J(u_n) = Q_{u_n,m}(1) - \frac{1}{4}Q'_{u_n}(1) > \frac{p-2}{4p}\min\{a, m\}\|u_n\|^2$$

implies the boundedness of $\{u_n\}$. Without loss of generality, we assume that $u_n \rightharpoonup u_0$ weakly in H as $n \to \infty$. It remains to prove that $u_n \to u_0$ strongly in H. Arguing indirectly, by Proposition 2.6, Lemma 2.4, and (2.12), we get

$$\lim_{n\to\infty} J(u_n) = c_- = j_1(1) + l j_2(1) < c_4.$$

Firstly, we have $j_1(1) = j_1(1) - \frac{1}{4}j_1'(1) = \frac{1}{4}a|\nabla u_0|_2^2 \ge 0$. Let

$$\theta(t) = \frac{a}{2}t^2 + \frac{\lambda}{4}t^4 - \frac{1}{4}S^{-2}t^4 \quad\text{for } t \ge 0.$$

It follows from $j_1'(1) = 0$ that

$$a|\nabla u^i|_2^2 t^2 + \lambda|\nabla u^i|_2^4 \le a|\nabla u^i|_2^2 t^2 + \lambda A|\nabla u^i|_2^2 = S^{-2}|\nabla u^i|_2^4,$$

which implies that $\theta'(|\nabla u^i|_2) \le 0$. It follows $j_1'(1) = 0$ and $j_2''(1) < 0$ that

$$\begin{aligned} j_2(1) = \max_{t\in[0,1]} j_2(t) &\ge \max_{t\in[0,1]} \left(\frac{1}{2}a|\nabla u^i|_2^2 t^2 + \frac{1}{4}\lambda|\nabla u^i|_2^4 t^4 - \frac{1}{4}S^{-2}|\nabla u^i|_2^4 t^4\right) \\ &= \max_{0\le t\le|\nabla u^i|_2} \theta(t) = c_4, \end{aligned}$$

which is impossible for $j_2(1) \le j_1(1) + l j_2(1) = c_- < c_4$. Thus, $u_n \to u_0$ strongly in H.

If $N \geq 5$, the sequence $\{u_n\}$ is bounded in H from the coerciveness of $J(u)$ in H. We also assume that $u_n \rightharpoonup u_0$ weakly in H as $n \to \infty$. It remains to prove that $u_n \to u_0$ strongly in H. Arguing indirectly, by Proposition 2.6 and (2.12), we get

$$\lim_{n\to\infty} J(u_n) = c_- = j_1(1) + l j_2(1) - \frac{1}{2^*} j_1'(1) - \frac{1}{2^*} l j_2'(1)$$
$$= \left(\frac{1}{2} - \frac{1}{2^*}\right) aA + \left(\frac{1}{4} - \frac{1}{2^*}\right)\lambda A^2 > 0,$$

which implies that $0 < A < 2\eta_\lambda^2 = \frac{4a}{(N-4)\lambda}$. Then, we have

$$j_1(1) = j_1(1) - \frac{1}{2^*} j_1'(1) = \left(\frac{1}{2} - \frac{1}{2^*}\right) a|\nabla u_0|_2^2 + \left(\frac{1}{4} - \frac{1}{2^*}\right)\lambda A|\nabla u_0|_2^2 \geq 0, \tag{2.26}$$

We claim that

$$j_2''(1) \leq 0.$$

In fact, it obviously holds if $u_0 = 0$ because of

$$0 \geq \lim_{n\to\infty} Q''_{u_n} = j_1''(1) + j_2''(1) = j_2''(1).$$

If $u_0 \neq 0$, arguing indirectly, we assume $j_2''(1) > 0$, that is,

$$j_2''(1) = j_2''(1) - (2^* - 1) j_2'(1) = \left((2 - 2^*)a + (4 - 2^*)\lambda A\right)|\nabla u^i|_2^2 > 0.$$

Then,

$$0 \geq \lim_{n\to\infty} Q''_{u_n} = j_1''(1) + j_2''(1) > j_1''(1) = j_1''(1) - (2^* - 1) j_1'(1)$$
$$= (2 - 2^*)a|\nabla u_0|_2^2 + (4 - 2^*)\lambda A|\nabla u_0|_2^2 + (2 - 2^*)m|u_0|_2^2 + (2^* - p)\mu|u_0|_p^p$$
$$= \left((2 - 2^*)a + (4 - 2^*)\lambda A\right)|\nabla u_0|_2^2 + \frac{(2^* - p)(p-2)}{p}\mu|u_0|_p^p$$
$$\geq \frac{(2^* - p)(p-2)}{p}\mu|u_0|_p^p > 0,$$

which is impossible. Thus, our claim $j_2''(1) \leq 0$ holds. Recall that

$$\theta(t) = \frac{a}{2}t^2 + \frac{\lambda}{4}t^4 - \frac{1}{2^*}S^{-\frac{2^*}{2}}t^{2^*} \quad \text{for } t \geq 0.$$

It follows from $j_1'(1) = 0$ that

$$a|\nabla u^i|_2^2 t^2 + \lambda|\nabla u^i|_2^4 \leq a|\nabla u^i|_2^2 t^2 + \lambda A|\nabla u^i|_2^2 = S^{-\frac{2^*}{2}}|\nabla u^i|_2^{2^*},$$

which implies that $\theta'(|\nabla u^i|_2) \leq 0$. It follows from $j_1'(1) = 0$ and $j_2''(1) \leq 0$ that

$$j_2(1) = \max_{t\in[0,1]} j_2(t) \geq \max_{t\in[0,1]} \left(\frac{1}{2}a|\nabla u^i|_2^2 t^2 + \frac{1}{4}\lambda|\nabla u^i|_2^4 t^4 - \frac{1}{2^*}S^{-\frac{2^*}{2}}|\nabla u^i|_2^{2^*} t^{2^*}\right)$$
$$= \max_{0\leq t\leq|\nabla u^i|_2} \theta(t) = c_{N,-},$$

which is impossible for $j_2(1) < j_1(1) + j_2(1) \leq c_- < c_{N,-}$. □

Remark 2.10. The existence and multiplicity results also can be obtained by the rescaling argument as follows. Problem $(K^*_{\lambda,\mu})$ has a positive solution for any $\mu > 0$ and $m > 0$ if one of the following cases holds,

- $N = 3$, $p \in (4, 6)$ and $\lambda > 0$;
- $N = 4$, $p \in (2, 4)$ and $0 < \lambda < S^{-2}$.

Problem $(K^*_{\lambda,\mu})$ has two positive solutions for any $\mu > 0$ and $m > 0$ if

- $N \geq 5$, $p \in (2, 2^*)$ and $0 < \lambda < \Lambda_0$, where

$$\Lambda_0 = \frac{2}{N-2}\left(\frac{N-4}{a(N-2)}\right)^{(N-4)/2} S^{-\frac{N}{2}}.$$

Actually, the authors in Akahori et al. [1] obtained a positive ground state solution u_1 of the following problem

$$-\Delta u + mu = \mu u^{p-1} + u^{2^*-1},\ u > 0 \text{ in } \mathbb{R}^N, \tag{S_m^*}$$

where $m > 0$, $\mu > 0$, $p \in (4, 6)$ for $N = 3$ and $p \in (2, 2^*)$ for $N \geq 4$. It follows from that Lemma A.2 in Akahori et al. [1] that the ground state solution u_1 satisfies

$$J_{0,m}(u_1) < \frac{1}{N} S^{\frac{N}{2}}, \tag{2.27}$$

where $J_{0,m}(u) = \frac{1}{2}(|\nabla u|_2 + m|u|_2^2) - \frac{1}{p}\mu|u|_p^p - \frac{1}{2^*}|u|_{2^*}^{2^*}$. Since u_1 is a solution, we have

$$P_1(u_1) = |\nabla u_1|_2 + m|u_1|_2^2 - \mu|u_1|_p^p - |u_1|_{2^*}^{2^*} = 0 \tag{2.28}$$

and $P_2(u_1) = \frac{1}{2^*}|\nabla u_1|_2 + \frac{1}{2}m|u_1|_2^2 - \frac{1}{p}\mu|u_1|_p^p - \frac{1}{2^*}|u_1|_{2^*}^{2^*} = 0$, which implies that

$$\frac{2^*-2}{2} m|u_1|_2^2 = \frac{2^*-p}{p}\mu|u_1|_p^p. \tag{2.29}$$

By (2.27), (2.28) and (2.29), we obtain that

$$\begin{aligned}\frac{1}{N}S^{\frac{N}{2}} >& J_{0,m}(u_1) = J_{0,m}(u_1) - \frac{1}{2^*}P_1(u_1)\\ &= (\frac{1}{2} - \frac{1}{2^*})(|\nabla u_1|_2^2 + m|u_1|_2^2) + (\frac{1}{2^*} - \frac{1}{p})|u_1|_{2^*}^{2^*} = \frac{1}{N}|\nabla u_1|_2^2.\end{aligned}$$

which implies that

$$|\nabla u_1|_2^2 < S^{\frac{N}{2}}. \tag{2.30}$$

Now, we try to prove the main results. Firstly, we consider the existence of positive roots of equation $G_1(t) = 0$ defined by

$$G_1(t) = at^{N+2} - t^N + \lambda|\nabla u_1|_2^2 t^4. \tag{2.31}$$

Combining with (2.30), if $N = 3$ and $\lambda > 0$ or $N = 4$ and $0 < \lambda < S^{-2}$, then $G_1(t) = 0$ has a positive root. If $N = 5$ and

$$0 < \lambda < \Lambda_0 = \frac{2}{N-2}\left(\frac{N-4}{a(N-2)}\right)^{(N-4)/2} S^{-\frac{N}{2}}, \tag{2.32}$$

then $G_1(t) = 0$ has two positive roots.

Let $\varphi_1(x) := u_1(\delta x)$, where $\delta > 0$ such that $G_1(\delta) = 0$ and u_1 is a solution of (S_m^*). Then, φ_1 is a solution of ($K_{\lambda,\mu}^*$). In fact, $G_1(\delta) = 0$ implies that $\left(a + \lambda\delta^{2-N}|\nabla u_1|_2^2\right)\delta^2 = 1$. Then, we obtain

$$\begin{aligned}-\Delta\varphi_1(x) &= -\delta^2\Delta u_1(\delta x) = \delta^2\left(\mu u_1(\delta x)^{p-1} + u_1(\delta x)^{2^*-1} - mu_1(\delta x)\right)\\ &= \frac{1}{a + \lambda\delta^{2-N}|\nabla u_1|_2^2}\left(\mu\varphi_1(x)^{p-1} + \varphi_1(x)^{2^*-1} - m\varphi_1(x)\right)\\ &= \frac{1}{a + \lambda|\nabla\varphi_1|_2^2}\left(\mu\varphi_1(x)^{p-1} + \varphi_1(x)^{2^*-1} - m\varphi_1(x)\right),\end{aligned} \tag{2.33}$$

which implies that φ_1 is a solution of ($K_{\lambda,\mu}^*$). By the above argument, if $N = 3$ and $\lambda > 0$ or $N = 4$ and $0 < \lambda < S^{-2}$, then problem ($K_{\lambda,\mu}^*$) has a positive solution. If $N = 5$ and $0 < \lambda < \Lambda_0$, then problem ($K_{\lambda,\mu}^*$) has two positive solutions.

Compared with these results, we have more information about the energies of the solutions obtained in Theorem 2.1 and 2.2. Though the exact intervals of λ for the existence of solutions seem different, we tend to believe that the solutions obtained in the rescaling argument and in Theorems 2.1 or 2.2 are the same.

Moreover, some existence results also can be obtained if $\lambda < 0$ by the same method and we do not state it here.

3. Some general results and an auxiliary lemma

In this section, we consider a general nonlocal equation by the multiplying argument

$$\begin{cases} -M_1\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u = M_2\left(\int\limits_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\right)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where M_i: $[0,+\infty) \to (0,\infty)$, are two continuous functions, $i = 1, 2$. Actually, the case that $M_1(t) = a+\lambda t$ and $M_2(t) = 1$ has been considered in Section 1. As to the general cases, we have the following results:

Theorem 3.1. *If there exists a positive root of the following equation,*

$$M_1\left(t^{(N-2)/2}A\right) - M_2\left(t^{N/2}A\right)t = 0, \tag{3.2}$$

where $A = S^{N/2}$, problem (3.1) has a positive solution. Moreover, if there exists a positive root of equation (3.2) for any positive parameter A, problem (3.1) has infinitely many distinct solutions.

Proof. Firstly, we denote the positive root of the following equation by t_1,

$$M_1\left(t^{(N-2)/2}S^{N/2}\right) - M_2\left(t^{N/2}S^{N/2}\right)t = 0.$$

Thus, let $\varphi_1 = t_1^{(N-2)/4}U$, we can easily check that

$$\begin{aligned} -\Delta\varphi_1 &= -t_1^{(N-2)/4}\Delta U = t_1^{\frac{1}{2^*-2}}U^{2^*-1} = t_1^{-1}\left(t^{\frac{1}{2^*-2}}U\right)^{2^*-1} \\ &= \frac{M_2(S^{N/2}t^{\frac{N}{2}})}{M_1(S^{N/2}t^{\frac{N-2}{2}})}\varphi_1^{2^*-1} = \frac{M_2(|U|_{2^*}^{2^*}t^{\frac{N}{2}})}{M_1(|\nabla U|_2^2 t^{\frac{N-2}{2}})}\varphi_1^{2^*-1} = \frac{M_2(|\varphi_1|_{2^*}^{2^*})}{M_1(|\nabla\varphi_1|_2^2)}\varphi_1^{2^*-1} \end{aligned}$$

and φ_1 is a positive solution of problem (3.1).

Similarly, let $\{u_i\}_{i=1}^{\infty}$ be a sequence of solutions for equation (S^*) satisfying $|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \cdots < |\nabla u_i|_2 \to +\infty$. By the assumption, there exists a positive root t_i for the following equation:

$$M_1\left(t^{(N-2)/2}A_i\right) - M_2\left(t^{N/2}A_i\right)t = 0,$$

where $A_i = |\nabla u_i|_2^2 = |u_i|_{2^*}^{2^*}$. Thus, $\varphi_i = t_i u_i$ solves problem (3.1). Then, problem (3.1) admits a sequence of solutions $\{\varphi_i\}_{i=1}^{\infty}$, and the proof is completed. □

In the last part of this section, we investigate the existence of positive solutions for two linear growth terms case

$$\begin{cases} -\left(a + \lambda\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u = \left(1 + \mu\int\limits_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\right)u^{2^*-1}, \quad u > 0 \ \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.3}$$

where $a > 0$, $\lambda > 0$ and $\mu \in \mathbb{R}$.

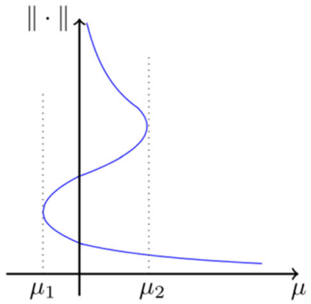

FIG. 7. $N \leq 5$ and $\lambda \epsilon (0 \Lambda_0)$

Theorem 3.2. *Assume that $a > 0$ and $\lambda > 0, \mu \in \mathbb{R}$. The following statements hold.*

(*i*) *If $N = 3$ or 4, problem (3.3) has a positive solution for $\mu \in (0, +\infty)$.*

(*ii*) *If $N \geq 5$ and $\lambda \in (0, \Lambda_0)$, there exist $\mu_1 < 0 < \mu_2$ such that problem (3.3) has two positive solutions for $\mu \in (\mu_1, 0)$, three positive solutions for $\mu \in (0, \mu_2)$ and a positive solution for $\mu \in (\mu_2, +\infty)$.*

(*iii*) *If $N \geq 5$ and $\lambda \in (\Lambda_0, \overline{\Lambda})$, there exist $\mu_4 > \mu_3 > 0$ such that problem (3.3) has three positive solutions for $\mu \in (\mu_3, \mu_4)$ and a positive solution for $\mu \in (0, \mu_3) \cup (\mu_4, +\infty)$, where $\overline{\Lambda}$ is defined by a exact form,*

$$\overline{\Lambda} = \frac{N}{4} \left(\frac{N}{N+2} \right)^{\frac{N-4}{2}} \Lambda_0. \tag{3.4}$$

(*iv*) *If $N \geq 5$ and $\lambda \in (\overline{\Lambda}, +\infty)$, problem (3.3) has a positive solution for $\mu \in (0, +\infty)$.*

The proof of Theorem 3.2. Setting that $G(t) := G_1(t) - \mu S^{\frac{N}{2}}$, where

$$G_1(t) := at^{N+2} - t^N + \lambda S^{\frac{N}{2}} t^4, \quad t > 0.$$

It is obvious that $G_1(0) = 0$ and $G_1(t) \to +\infty$ as $t \to +\infty$. If $N = 3$ or 4, then $G(t) = 0$ admits a positive root denoted by t_1. Let $K = t_1^{-(N-2)/2}$ and U be the positive solution of problem (S^*), then it is evident to check that $\varphi = KU$ solves problem (3.3). Thus, (*i*) holds.

It remains to prove that $N \geq 5$. By some calculations, for $\lambda \in (0, \overline{\Lambda})$, $G_1(t)$ admits a local maximum point $t_{\max}$ and a local minimum point $t_{\min}$ satisfying $0 < t_{\max} < t_{\min}$ and

$$G_1(t_{\max}) > 0 > G_1(t_{\min}) \text{ for } \lambda \in (0, \Lambda_0),$$
$$G_1(t_{\max}) > G_1(t_{\min}) > 0 \text{ for } \lambda \in (\Lambda_0, \overline{\Lambda}).$$

(I). For $\lambda \in (0, \Lambda_0)$, set that $\mu_1 = -G_1(t_{\min}) S^{-N/2}$ and $\mu_2 = G_1(t_{\max}) S^{-N/2}$. Then, $G(t)$ admits two roots for $\mu \in (\mu_1, 0)$, three roots for $\mu \in (0, \mu_2)$ and a root for $\mu \in (\mu_2, +\infty)$.

(II). For $\lambda \in (\Lambda_0, \overline{\Lambda})$, set that $\mu_3 = G_1(t_{\min}) S^{-N/2}$ and $\mu_4 = G_1(t_{\max}) S^{-N/2}$. Then, $G(t)$ admits three roots for $\mu \in (\mu_3, \mu_4)$ and a root for $\mu \in (0, \mu_3) \cup (\mu_4, +\infty)$.

(III). For $\lambda \in (\overline{\Lambda}, +\infty)$, then $G(t)$ admits a root for $\mu \in (0, +\infty)$.

Lastly, assume that t_i is a root of $G(t) = 0$. Let $K_i := t_i^{-(N-2)/2}$ and U be the positive solution of problem (S^*), then it is evident to check that $\varphi_i = K_i U$ solves problem (3.3). The desired results from (I)–(III). □

By some careful analysis on the roots of $G(t) = 0$, we can give the following bifurcation of the positive solutions.

We complete this paper by proving an auxiliary lemma.

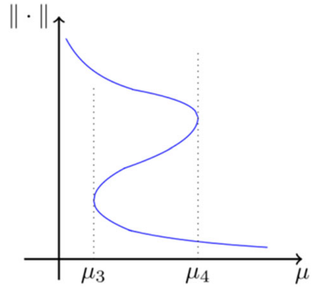

FIG. 8. $N \le 5$ and $\lambda\epsilon(\Lambda_0\underline{\Lambda})$

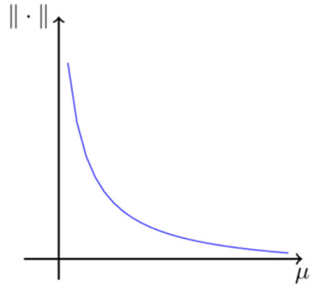

FIG. 9. $N \le 5$ and $\lambda\epsilon(\underline{\Lambda}+\infty)$

Lemma 3.3. *Let $N \ge 5$, $a > 0$ and*

$$f(t) = \lambda t^2 - \left(S^{-N/4}t\right)^{\frac{4}{N-2}} + a, \quad t \ge 0.$$

For $\lambda \in (0, \Lambda_0)$, we denote the roots of $f(t) = 0$ by $\xi_{1,-}$ and $\xi_{1,+}$ $(0 < \xi_{1,-} < \xi_{1,+})$. Let $\eta_\lambda = (\frac{2a}{(N-4)\lambda})^{\frac{1}{2}}$, then $\xi_{1,-} < \eta_\lambda < \xi_{1,+}$ and

$$\xi_{1,+}^2 - \eta_\lambda^2 > \eta_\lambda^2 - \xi_{1,-}^2. \tag{3.5}$$

Proof. For $\lambda \in (0, \Lambda_0)$, there exists ζ_λ such that

$$f'(\zeta_\lambda) = 0, \quad f(\zeta_\lambda) = \min_{t\ge 0} f(t), \quad \text{where} \ \ \zeta_\lambda = (\frac{2}{(N-2)\lambda})^{\frac{N-2}{2(N-4)}} S^{-\frac{N}{2(N-4)}}.$$

Therefore, $\xi_{1,-} < \zeta_\lambda < \xi_{1,+}$ and $\eta_\lambda^2 < \zeta_\lambda^2$ because of $0 < \lambda < \Lambda_0$. By $\lambda \in (0, \Lambda_0)$, then $f(\eta_\lambda) < 0$ implies $\xi_{1,-} < \eta_\lambda < \xi_{1,+}$.

If we set that $\widetilde{\zeta}_\lambda := \zeta_\lambda^2$, $\widetilde{\xi}_\pm := \xi_{1,\pm}^2$ and

$$k(t) := f(t^{1/2}) = \lambda t - S^{-\frac{N}{N-2}} t^{\frac{2}{N-2}} + a \ \text{ for } t > 0,$$

then $f'(\zeta_\lambda) = k'(\widetilde{\zeta}_\lambda) = f(\xi_{1,\pm}) = k(\widetilde{\xi}_\pm) = 0$. By the above setting, to prove (3.5), it is sufficient to prove that

$$\widetilde{\xi}_+ - \widetilde{\zeta}_\lambda > \widetilde{\zeta}_\lambda - \widetilde{\xi}_-. \tag{3.6}$$

We claim that

$$k(\widetilde{\zeta}_\lambda - t) > k(\widetilde{\zeta}_\lambda + t) \ \text{ for any } \ 0 < t < \widetilde{\zeta}_\lambda. \tag{3.7}$$

The desired result (3.6) follows from the claim (3.7). If (3.6) does not hold, that is, $\widetilde{\xi}_+ - \widetilde{\zeta}_\lambda \le \widetilde{\zeta}_\lambda - \widetilde{\xi}_-$. Setting $\delta := \widetilde{\xi}_+ - \widetilde{\zeta}_\lambda$, then $\delta \in (0, \widetilde{\zeta}_\lambda)$. It follows from $\widetilde{\xi}_- \le 2\widetilde{\zeta}_\lambda - \widetilde{\xi}_+ < \widetilde{\xi}_+$ and (3.7) that

$$0 = k(\widetilde{\xi}_-) \ge k(2\widetilde{\zeta}_\lambda - \widetilde{\xi}_+) = k(\widetilde{\zeta}_\lambda - \delta) > k(\widetilde{\zeta}_\lambda + \delta) = k(\widetilde{\xi}_+) = 0,$$

which is a contradiction.

Lastly, it remains to prove our claim (3.7). In fact, by some basic calculations, it follows that

$$k'(t) = \lambda - \frac{2}{N-2} S^{-\frac{N}{N-2}} t^{-\frac{N-4}{N-2}}, \quad k''(t) = \frac{2(N-4)}{(N-2)^2} S^{-\frac{N}{N-2}} t^{-\frac{2(N-3)}{N-2}},$$
$$k'''(t) = -\frac{4(N-3)(N-4)}{(N-2)^3} S^{-\frac{N}{N-2}} t^{-\frac{3N-8}{N-2}}.$$

Since $N \ge 5$, $k'''(t) < 0$ for any $t > 0$. Then, one obtains that for any $0 < t < \widetilde{\zeta}_\lambda$,

$$k''(\widetilde{\zeta}_\lambda - t) > k''(\widetilde{\zeta}_\lambda + t).$$

For $0 < s < \widetilde{\zeta}_\lambda$, integrating the above inequality from 0 to s gives us

$$\int_{\widetilde{\zeta}_\lambda - s}^{\widetilde{\zeta}_\lambda} k''(t)\mathrm{d}t > \int_{\widetilde{\zeta}_\lambda}^{\widetilde{\zeta}_\lambda + s} k''(t)\mathrm{d}t. \tag{3.8}$$

Moreover, by $k'(\widetilde{\zeta}_\lambda) = 0$ and (3.8), then one obtains

$$-k'(\widetilde{\zeta}_\lambda - s) = \int_{\widetilde{\zeta}_\lambda - s}^{\widetilde{\zeta}_\lambda} k''(t)\mathrm{d}t > \int_{\widetilde{\zeta}_\lambda}^{\widetilde{\zeta}_\lambda + s} k''(t)\mathrm{d}t = k'(\widetilde{\zeta}_\lambda + s), \tag{3.9}$$

Similarly, integrating the inequality (3.9) again from 0 to t, it follows that $k(\widetilde{\zeta}_\lambda - t) > k(\widetilde{\zeta}_\lambda + t)$ for $0 < t < \widetilde{\zeta}_\lambda$. The proof is completed. □

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