



Uniform stability for a semilinear non-homogeneous Timoshenko system with localized nonlinear damping

M. M. Cavalcanti, W. J. Corrêa , V. N. Domingos Cavalcanti, M. A. Jorge Silva and J. P. Zanchetta

Abstract. This work is concerned with a semilinear non-homogeneous Timoshenko system under the effect of two nonlinear localized frictional damping mechanisms. The main goal is to prove its uniform stability by imposing minimal amount of support for the damping and, as expected, without assuming any relation on the non-constant coefficients. This fact generalizes substantially the previous papers by Cavalcanti et al. (Z Angew Math Phys 65(6):1189–1206, 2014) and Santos et al. (Differ Integral Equ 27(1–2):1–26, 2014) at the levels of problem and method. It is worth mentioning that the methodologies of these latter cannot be applied to the semilinear case herein, namely when one considers the problem with nonlinear source terms. Thus, differently of Cavalcanti et al. (Z Angew Math Phys 65(6):1189–1206, 2014), Santos et al. (Differ Integral Equ 27(1–2):1–26, 2014), the proof of our main stability result relies on refined arguments of microlocal analysis due to Burq and Gérard (Contrôle Optimal des équations aux dérivées partielles, <http://www.math.u-psud.fr/~burq/articles/coursX.pdf>, 2001). As far as we know, it seems to be the first time that such a methodology has been employed to 1-D systems of Timoshenko type with nonlinear foundations.

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1. Introduction

In this work, we are going to address the following semilinear non-homogeneous Timoshenko system with localized damping

$$\rho_1(x)\varphi_{tt} - [k(x)(\varphi_x + \psi)]_x + f_1(\varphi) + \alpha_1(x)g_1(\varphi_t) = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.1)$$

$$\rho_2(x)\psi_{tt} - [b(x)\psi_x]_x + k(x)(\varphi_x + \psi) + f_2(\psi) + \alpha_2(x)g_2(\psi_t) = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.2)$$

subject to initial-boundary conditions

$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad t \geq 0, \quad (1.3)$$

$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad x \in \Omega, \quad (1.4)$$

where $\Omega := (0, L)$, with $L > 0$ denoting the beam length, $\varphi = \varphi(x, t)$ and $\psi = \psi(x, t)$ stand for transversal displacement and rotation angle of a filament of the beam, respectively, and

$$\rho_1(x) = \rho(x)A(x), \quad \rho_2(x) = \rho(x)I(x), \quad k(x) = k'GA(x), \quad b(x) = EI(x)$$

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are non-constant positive coefficients whose physical meanings are well known; namely, $\rho(x)$ is the mass density of the material, $A(x)$ is the area of a cross section, $I(x)$ is second moment of a cross section, k' is the shear factor coefficient, G is the shear modulus, and E is the modulus of elasticity.

The non-constant coefficients $\rho_1(x), \rho_2(x), k(x)$ and $b(x)$ are assumed to be smooth functions ($C^\infty(\Omega)$) and bounded from below and above by positive constants as well as their derivatives of first order; the localized damping coefficients $\alpha_1(x)$ and $\alpha_2(x)$ are supposed to be continuous and nonnegative functions in $[0, L]$, while the nonlinear feedback functions g_1 and g_2 are continuous, monotone increasing and zero at the origin. The nonlinear foundations f_1 and f_2 are assumed to growth polynomially under suitable conditions. All the precise assumptions, as well as the well-posed result, will be stated in Sect. 2.

In what concerns the stabilization of the Timoshenko system with two internal frictional dissipations, there are some results related to variations of the above model (1.1)–(1.4) in the literature. Indeed, Raposo et al. [29] proved the exponential stability for a linear problem related to equations (1.1) and (1.2) with constant coefficients $\rho_1, \rho_2, k, b, \alpha_1, \alpha_2 > 0$, null sources $f_1 = f_2 = 0$ and full linear damping $g_1(\varphi_t) = \varphi_t$, $g_2(\psi_t) = \psi_t$. In this occasion, since the problem is linear, it is used the semigroup theory to prove their stability result, see, e.g., [29, Theorem 3.1]. In 2014, both Cavalcanti et al. [8] and Santos et al. [30] considered equations (1.1) and (1.2) with constant coefficients $\rho_1, \rho_2, k, b > 0$, null sources $f_1 = f_2 = 0$ and two locally distributed nonlinear damping $\alpha_1(x)g_1(\varphi_t)$ and $\alpha_2(x)g_2(\psi_t)$. In both works [8, 30], it is proved that the stability of the corresponding energy is driven by a nonlinear ODE, see, for instance, [8, Theorem 3.1] and [30, Theorem 3.2]. To their proofs, keeping in mind that $f_1 = f_2 = 0$, it is employed the reduction principle [16] where the problem of decay rates with *nonlinear damping* is reduced to an appropriate stabilizability inequality for the *linear system*. In addition, stabilizability inequality for the linearly damped problem is achieved through an observability inequality obtained for the conservative system. Equivalence of these two observability inequalities owns its validity to the fact that the control action is bounded via locally distributed internal feedbacks. This allows the authors in [8] to engage the so called B^*L methodology presented in [25], whereas in [30] it is used contradiction arguments along with proper cut-off functions, once the observability inequality is obtained for the conservative system in both papers. More recently, Fatori et al. [19] and Ma et al. [27] have been considered problem (1.1)–(1.2) with constant coefficients $\rho_1, \rho_2, k, b, \alpha_1, \alpha_2 > 0$, linear dissipations $g_1(\varphi_t) = \varphi_t$ and $g_2(\psi_t) = \psi_t$, and also nonlinear source and external terms $f_1(\varphi, \psi) - h_1$, $f_2(\varphi, \psi) - h_2$ in the place of $f_1(\varphi)$, $f_2(\psi)$, respectively. Since the damping in the latter is full and linear, the long-time behavior of their problems is mainly achieved by means of standard multipliers and perturbed energy, see, e.g., [19, Lemma 4.6] and [27, Lemma 3.4].

The above chronologically referred papers [8, 19, 27, 29, 30] lead us to the main goal of the present article; namely, our purpose is to obtain general and uniform decay rate estimates for the energy corresponding to the semilinear Timoshenko system (1.1)–(1.4) with minimal support for the damping. As expected, when dealing with damping mechanisms for both displacements of system (1.1)–(1.2), we do not need to assume any relation on the coefficients. This yields a substantial generalization of the works [8, 19, 27, 29, 30] at the levels of result and method. In fact, since $f_1 \neq 0$ and $f_2 \neq 0$, the aforementioned methodologies employed by [8, 29, 30] are no longer valid here. Also, once we are dealing with locally distributed nonlinear damping $\alpha_1(x)g_1(\varphi_t)$ and $\alpha_2(x)g_2(\psi_t)$, the multiplier technique explored in [19, 27] fails as well because of the terms in the energy estimates that cannot be directly absorbed (this is not the case when the damping functions are supported on the whole domain as, for example, in [19, 27, 29]). When $f_1 = f_2 = 0$, in order to overcome this latter situation, special cut-off functions can be introduced in order to eliminate undesirable terms of higher order as considered previously in the works [8, 30]. In addition, since the coefficients $\rho_1(x), \rho_2(x), k(x)$ and $b(x)$ may depend on the x -variable, a new method is required and, for this purpose, the microlocal analysis (see, for instance, [5, 6, 20, 21, 26] and references therein) seems to be a very useful tool to prove a desired nonlinear observability inequality to problem (1.1)–(1.4) and, consequently, uniform decay rates estimates for the corresponding energy.

It seems to be the first time that microlocal analysis is invoked to stabilize a semilinear Timoshenko system where other methods can not be applied at a first glance. The next table summarizes the new contribution of the present paper at the levels of generality of problem (1.1)-(1.4) and the methodology for stability, when compared with [8, 19, 27, 29, 30].

Summary of Timoshenko systems with fully internal frictional damping					
Paper	Nonlinear sources	Variable coefficients	Localized damping coefficients	Nonlinear damping functions	Methodology
Raposo et al. [29]	NO	NO	NO	NO	Linear Semigroup
Cavalcanti et al. [8]	NO	NO	YES	YES	Observability, B^*L method
Santos et al. [30]	NO	NO	YES	YES	Observability, cut-off mult.
Fatori et al. [19]	YES	NO	NO	NO	Perturbed Energy
Ma et al. [27]	YES	NO	NO	NO	Perturbed Energy
Present Article	YES	YES	YES	YES	Observability, microlocal analysis

In conclusion, our main goal in the present paper is to work in a more general situation than [8, 29, 30] and still considering the minimum amount of supported damping to approach the localized semilinear Timoshenko problem (1.1)-(1.4). To do so, we follow the same spirit of [8, 30] with the difference that here, for the first time, we are going to exploit the microlocal analysis for 1-D systems of Timoshenko type.

There is another scenario in Timoshenko systems involving *partially damped* problems. In this case, it is necessary to transfer dissipation from one displacement (equation) to another and, therefore, the uniform stability of the whole system will require an assumption on the coefficients usually called *equal speeds of wave propagation*. However, such types of partially damped systems represent a different situation and comparisons with them will not be given herein.

Our main results (Theorem 3.2 and Proposition 3.3) will be stated in Sect. 3, and their proofs will be given in Sect. 4 right after. In Sect. 5, we finish this paper with some remarks on the geodesics that can be built for the Timoshenko system.

2. Assumptions and well-posedness

Let us start by considering the following phase spaces

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

and

$$\mathcal{V} = [H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega) \times [H_0^1(\Omega) \cap H^2(\Omega)] \times H_0^1(\Omega).$$

Under the assumptions on the functions ρ_1, ρ_2, b, k , then \mathcal{H} is a Hilbert space with norm given by

$$\|(u, v, w, z)\|_{\mathcal{H}}^2 = \|\sqrt{\rho_1}v\|^2 + \|\sqrt{\rho_2}z\|^2 + \|\sqrt{b}w_x\|^2 + \|\sqrt{k}(u_x + w)\|^2,$$

for all $(u, v, w, z) \in \mathcal{H}$, where $\|\cdot\|$ stands for the usual $L^2(\Omega)$ -norm.

Denoting by $v = \varphi_t, z = \psi_t, U = (\varphi, v, \psi, z)$ and $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1)$, then system (1.1)-(1.4) can be rewritten as follows

$$\begin{cases} \frac{dU}{dt} = AU + BU + FU, & t > 0, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where the operators A , B and F are defined by

$$AU = \left(v, \frac{1}{\rho_1} [k(u_x + w)]_x, z, \frac{1}{\rho_2} [bw_x]_x - \frac{k}{\rho_2} (u_x + w) \right), \quad U \in D(A) = \mathcal{V}, \tag{2.2}$$

$$BU = \left(0, -\frac{\alpha_1}{\rho_1} g_1(v), 0, -\frac{\alpha_2}{\rho_2} g_2(z) \right), \quad U \in D(B) = \mathcal{H}, \tag{2.3}$$

$$FU = \left(0, -\frac{1}{\rho_1} f_1(\varphi), 0, -\frac{1}{\rho_2} f_2(\psi) \right), \quad U \in D(F) = \mathcal{H}. \tag{2.4}$$

For the sake of completeness, we provide the notion of solutions for the Cauchy problem (2.1) according to Barbu [2]. This is already recalled for Timoshenko systems in [8, 30].

Definition 2.1. One says that $U : [0, \infty[\rightarrow \mathcal{H}$ is a **strong solution** for (2.1) if U is continuous on $[0, \infty[$ and Lipschitz on every compact subset of $]0, \infty[$, $U(t)$ is differentiable a.e. on $]0, \infty[$ and

$$\frac{dU}{dt}(t) = AU(t) + BU(t) + FU(t) \quad \text{for a.e. } t \in]0, \infty[.$$

Definition 2.2. One says that $U : [0, \infty[\rightarrow \mathcal{H}$ is an **integral solution** for (2.1) if U is continuous on $[0, \infty[$, $U(0) = U_0$ and the following inequality holds

$$\frac{1}{2} \|U(t) - \Psi\|_{\mathcal{H}}^2 \leq \frac{1}{2} \|U(s) - \Psi\|_{\mathcal{H}}^2 + \int_s^t (A\Psi + BU(r) + FU(r), U(r) - \Psi)_{\mathcal{H}} dr,$$

for every $\Psi \in D(A)$ and $0 \leq s \leq t < \infty$, where $(\cdot, \cdot)_{\mathcal{H}}$ stands for the inner product of \mathcal{H} .

Now, we provide the assumptions on the localized damping coefficients, the dissipative feedback functions and the nonlinear source terms as follows.

Assumption 2.1. Let us consider nonnegative functions $\alpha_1, \alpha_2 \in C[0, L]$ such that

$$\alpha_i(x) \geq \alpha_0 > 0, \quad x \in I_i, i = 1, 2, \tag{2.5}$$

where I_1, I_2 are open intervals contained in $[0, L]$ satisfying $\omega := I_1 \cap I_2 \neq \emptyset$.

Assumption 2.2. In addition to Assumption 2.1, we suppose that ω geometrically controls $\Omega = (0, L)$, that is, there exists $T_0 > 0$ such that every geodesic of the metric $G_i(x)$, $i = 1, 2$, where $G_1 = \left(\frac{k}{\rho_1}\right)^{-1}$, $G_2 = \left(\frac{b}{\rho_2}\right)^{-1}$, traveling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.

Assumption 2.3. The functions g_1, g_2 are continuous, monotone increasing and satisfy

$$g_i(s)s > 0 \quad \text{for } s \neq 0, \tag{2.6}$$

$$k_i s^2 \leq g_i(s) \leq K_i s^2 \quad \text{for } |s| > 1, \tag{2.7}$$

for some positive constants k_i and K_i , $i = 1, 2$.

Assumption 2.4. The nonlinear terms $f_1, f_2 \in C^2(\mathbb{R})$ satisfy

$$f_i(0) = 0, \quad |f_i^{(j)}(s)| \leq k_0(1 + |s|)^{p-j}, \quad i = 1, 2, \quad j = 1, 2, \quad p > 1, \quad \forall s \in \mathbb{R}, \tag{2.8}$$

and their primitives $F_i(s) = \int_0^s f_i(\tau) d\tau$, $i = 1, 2$, verify

$$0 \leq F_i(s) \leq f_i(s)s, \quad \forall s \in \mathbb{R}, i = 1, 2. \tag{2.9}$$

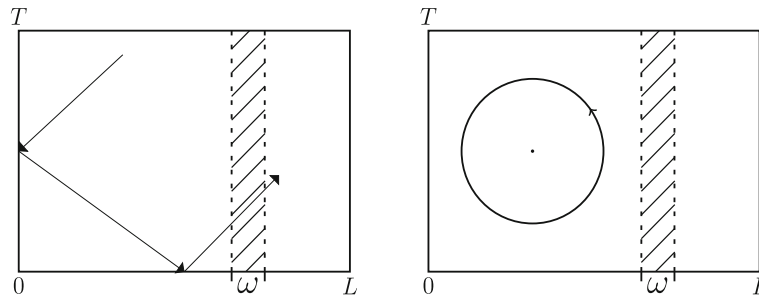


FIG. 1. At the left side the bicharacteristic (or its projection on $\Omega \times (0, T)$) is a straight line which enters the region $\omega \times (0, T)$. At the right side it is illustrated a trapped bicharacteristic that does not reach the damped area $\omega \times (0, T)$

Assumption 2.5. For every $T > 0$, the only solutions (η, ζ) lying in the space $C([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{V}')$, to system

$$\begin{cases} \rho_1(x)\eta_{tt} - (k(x)(\eta_x + \zeta))_x + V_1(x, t) = 0 & \text{in } (0, L) \times (0, T), \\ \rho_2(x)\zeta_{tt} - (b(x)\zeta_x)_x + k(x)(\eta_x + \zeta) + V_2(x, t) = 0 & \text{in } (0, L) \times (0, T), \\ \eta = \zeta = 0 & \text{in } \omega \times (0, T) \end{cases} \tag{2.10}$$

where $V_i(x, t) \in L^\infty([0, T]; L^1((0, L)))$, $i = 1, 2$ are the trivial solutions $(\eta, \zeta) = (0, 0)$.

Under the above assumptions, we make some comments as follows.

Remark 2.1. (a) Assumptions 2.1 and 2.3 are motivated by [8, 30].

(b) Assumption 2.4 is relatively standard for perturbation of wave-like systems. In particular, from (2.8) and the mean value theorem, there exist constants $C_i > 0$ such that

$$|f_i(r) - f_i(s)| \leq C_i (1 + |r|^{p-1} + |s|^{p-1}) |r - s|, \quad \forall r, s \in \mathbb{R}, i = 1, 2. \tag{2.11}$$

(c) For $V_i(x, t) \in L^\infty([0, T]; L^n((0, L)))$ Assumption 2.5 follows from the pioneer work of Ruiz [28]. According to Koch and Tataru [22] (see Theorem 8.15), in the more general case where $V_i(x, t) \in L^{\frac{n+1}{2}}([0, T]; L^{\frac{n+1}{2}}(0, L))$, the unique continuation result follows locally. Hence, under the conditions specified in [22], Assumption 2.5 is fulfilled.

(d) Assumption 2.2 is the so-called geometric control condition (G.C.C.) and it will be only used for the stability result. It is well-known that it is a necessary and sufficient for stabilization and control of the linear wave equation (see [1, 7, 10, 11, 18] and references therein). Since in the present paper we do not have any control of the geodesics because of the inhomogeneous medium, we assume that such an assumption must be considered, namely:

For all geodesic $t \in I \mapsto x(t) \in \Omega$ of the metric $G_1 = \left(\frac{k}{\rho_1}\right)^{-1}$ (or $G_2 = \left(\frac{b}{\rho_2}\right)^{-1}$), with $0 \in I$, there exists $t \geq 0$ such that $\alpha_1(x(t)) > 0$ (or $\alpha_2(x(t)) > 0$).

We also observe that if $G_1(x) = G_2(x) = \text{constant}$, the above condition holds true, once the bicharacteristics $t \mapsto (t, x(t), \tau, -\tau G(x(t))\dot{x}(t))$ (or its projection on $\Omega \times (0, T)$) are straight lines. On the other hand, for a general metric the above condition cannot hold if, for instance, G admits trapped bicharacteristics. See Fig. 1.

Under the above notations, assumptions and remarks, we are going to consider the existence and uniqueness result for the Cauchy problem (2.1), under the light of nonlinear semigroup method.

Theorem 2.3. *Let us suppose that Assumptions 2.1, 2.3 and 2.4 hold. Then, if $U_0 \in \mathcal{H}$, problem (2.1) has a unique integral solution. Moreover, if $U_0 \in \mathcal{V}$, the solution is strong one.*

Proposition 2.4. *Under the assumptions of Theorem 2.3, let us take $U_0 \in \mathcal{H}$ and consider $U \in C([0, \infty[; \mathcal{H})$ the respective unique integral solution of (2.1). Then, there exists a sequence of strong solutions U_n of (2.1) such that*

$$\lim_{n \rightarrow \infty} U_n = U \text{ in } C([0, T]; \mathcal{H}), \quad \forall T > 0.$$

Remark 2.2. To prove Theorem 2.3 and Proposition 2.4, we follow the same idea as presented in [8,30], by including the nonlinear source terms whose Assumption 2.4 leads them to locally Lipschitz perturbations. Indeed, from the definitions of operators A and B in (2.2)–(2.3) and using Assumptions 2.1 and 2.3, it is not so difficult to prove that A and B satisfy the hypotheses of [4, Theorems 3.1]. See also [2,3] for integral solutions. In addition, under Assumption 2.4 (see also (2.11)), one proves that operator F set in (2.4) is a locally Lipschitz perturbation (see Theorems 7.1 and 7.2 in [15]). Therefore, the existence and uniqueness of integral and strong solutions follow from the general theory given by [2–4] as well as the statement that every integral solution can be obtained as limit of strong solutions. Summarizing, problem (1.1)–(1.4) is well posed.

3. Asymptotic behavior: main results

3.1. Energy relation

We first introduce the energy functional associated to solutions of system (1.1)–(1.4) for all $t \geq 0$, namely,

$$E_U(t) = \int_0^L \left\{ \frac{\rho_1}{2} \varphi_t^2(t) + \frac{\rho_2}{2} \psi_t^2(t) + \frac{b}{2} \psi_x^2(t) + \frac{k}{2} (\varphi_x + \psi)^2(t) + F_1(\varphi(t)) + F_2(\psi(t)) \right\} dx. \quad (3.1)$$

As a consequence of Assumptions 2.1, 2.3 and 2.4, one can conclude without difficulties that the energy (3.1) is a monotone non-increasing (and nonnegative) function, which is stated in the next result.

Lemma 3.1. *Let $U = (\varphi, \varphi_t, \psi, \psi_t)$ be an integral (or strong) solution of (1.1)–(1.4). Then, the energy satisfies*

$$\frac{dE_U}{dt}(t) = - \int_0^L \left\{ \alpha_1(x) g_1(\varphi_t(t)) \varphi_t(t) + \alpha_2(x) g_2(\psi_t(t)) \psi_t(t) \right\} dx \leq 0, \quad \forall t > 0. \quad (3.2)$$

As a consequence, it follows that

$$E_U(t) = E_U(0) - \int_0^t \int_0^L \left\{ \alpha_1(x) g_1(\varphi_t(s)) \varphi_t(s) + \alpha_2(x) g_2(\psi_t(s)) \psi_t(s) \right\} dx ds, \quad t \geq 0. \quad (3.3)$$

Proof. Let us first consider a strong solution $U = (\varphi, \varphi_t, \psi, \psi_t)$ of (1.1)–(1.4). Thus, multiplying (1.1) by ψ_t , (1.2) by φ_t , integrating by parts on Ω and adding the resulting expressions, we easily get (3.2) and, consequently, (3.3) after integrating on $(0, t)$. From this and Proposition 2.4, the conclusion also holds true for the integral solution. \square

3.2. Nonlinear ODE relation

Before stating our main result, let us first introduce a nonlinear ODE that shall drive the energy stability. Such a constructive methodology was firstly introduced by Lasiecka and Tataru [23] and, subsequently, by some authors. See, for instance, [9,13,14] for wave models and [8,30] for the technique adapted to Timoshenko systems, from where we follow the same reasoning lines.

Assumption 2.3 allows us, according to [23] (see also [8, 9, 13, 14, 30]), to consider a function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$h(s) = h_1(s) + h_2(s), \quad s \in \mathbb{R},$$

where h_1, h_2 are continuous, concave, strictly increasing real functions satisfying

$$h_i(g_i(s)s) \geq s^2 + g_i^2(s), \quad |s| \leq 1, \quad i = 1, 2. \quad (3.4)$$

In this way, h has the same properties as its composing functions h_1 and h_2 . In addition, we define the auxiliary function r by

$$r(s) = h\left(\frac{s}{LT}\right), \quad s \in \mathbb{R}, \quad T > 0, \quad (3.5)$$

and so it is not difficult to see that $cI + r$ is invertible for any constant $c \geq 0$, once r is a monotone increasing function. Thus, for nonnegative constants c and M we define

$$p(s) = (cI + r)^{-1}(Ms), \quad s \in \mathbb{R}, \quad (3.6)$$

and observe that it is a continuous, positive and strictly increasing function with $p(0) = 0$. Keeping the above construction in mind, we finally define the nonlinear function

$$q(s) = s - (I + p)^{-1}(s), \quad s \in \mathbb{R},$$

and introduce the ODE driven by q as follows

$$\begin{cases} \frac{dS}{dt}(t) + q(S(t)) = 0, & t > 0 \\ S(0) = s_0. \end{cases} \quad (3.7)$$

Under the above construction, and relying on the results of [23], if p satisfies $p(s) > 0$ for $s > 0$, then it is possible to prove the following stabilization limit for S :

$$\lim_{t \rightarrow \infty} S(t) = 0.$$

3.3. Main results

Under the above notations and previous hypotheses, we are now in position to state our main result on decay rates of the energy associated with problem (1.1)–(1.4). More precisely, we have:

Theorem 3.2. (Asymptotic Behavior) *Let us suppose that Assumptions 2.1, 2.2, 2.3 and 2.4 hold and let $K > 0$ be any positive constant such that the initial energy $E_U(0) \leq K$. Then, there exists a time $T_0 > 0$ such that*

$$E_U(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \quad (3.8)$$

where $S(t)$ is the solution of problem (3.7) with $s_0 = E_U(0)$ and $\lim_{t \rightarrow \infty} S(t) = 0$.

Remark 3.1. We observe that although the decay rate (3.8) depends on the size of the initial energy, there are several examples of functions $g_i, i = 1, 2$, satisfying Assumption 2.3. Consequently, there are several explicit decay rates to solutions S of (3.7) as well as to the energy E_U by means of the inequality (3.8), see, for instance, [13, Section 8] from where we can similarly achieve such concrete decay rates. See also [8, 16, 24, 30].

We mention that the approach to the proof of Theorem 3.2 is different from that presented in [8, 30]. Here, the guiding idea is the one presented in [12] adapted to the context of Timoshenko systems where to prove the stabilization (3.8) reduces itself to prove an observability inequality, whose proof is now based on contradiction arguments in combination with microlocal analysis, see, e.g., Burq and Gérard [6]. Such an idea has also been considered previously in [17, 18] for different models. The result on observability inequality associated with problem (1.1)-(1.4) reads as follows:

Proposition 3.3. (Observability Inequality) *Under the assumptions of Theorem 3.2, there exist constants $T_0 > 0$ and $C = C(K, T) > 0$ such that*

$$E_U(0) \leq C \int_0^T \int_0^L \left\{ \alpha_1(x) [\varphi_t^2(t) + g_1^2(\varphi_t(t))] + \alpha_2(x) [\psi_t^2(t) + g_2^2(\psi_t(t))] \right\} dx dt, \tag{3.9}$$

for all $T > T_0$.

The proofs of Proposition 3.3 and Theorem 3.2 are given in the next section.

4. Proofs

4.1. Proof of Proposition 3.3

Let us assume that (3.9) does not hold. Then, there exists $T > 0$ (large enough) and a sequence $U^n = (\varphi^n, \varphi_t^n, \psi^n, \psi_t^n)$ of weak solutions of problem (1.1)–(1.4) in $(0, L) \times (0, T)$, that is,

$$\begin{cases} \rho_1(x) \varphi_{tt}^n - (k(x)(\varphi_x^n + \psi))_x + f_1(\varphi^n) + \alpha_1(x) g_1(\varphi_t^n) = 0, \\ \rho_2(x) \psi_{tt}^n - (b(x)\psi_x^n)_x + k(x)(\varphi_x^n + \psi) + f_2(\psi^n) + \alpha_2(x) g_2(\psi_t^n) = 0, \\ \varphi^n(0, \cdot) = \varphi^n(L, \cdot) = \psi^n(0, \cdot) = \psi^n(L, \cdot) = 0, \\ \varphi^n(\cdot, 0) = \varphi_0^n, \varphi_t^n(\cdot, 0) = \varphi_1^n, \psi^n(\cdot, 0) = \psi_0^n, \psi_t^n(\cdot, 0) = \psi_1^n, \end{cases} \tag{4.1}$$

such that $E_{U^n}(0) \leq K$ and

$$\lim_{n \rightarrow +\infty} \frac{E_{U^n}(0)}{\int_0^T \int_0^L (\alpha_1(x)((\varphi_t^n)^2 + (g_1(\varphi_t^n))^2) + \alpha_2(x)((\psi_t^n)^2 + (g_2(\psi_t^n))^2)) dx dt} = +\infty. \tag{4.2}$$

Thus, (4.2) yields

$$\lim_{n \rightarrow +\infty} \frac{\int_0^T \int_0^L (\alpha_1(x)((\varphi_t^n)^2 + (g_1(\varphi_t^n))^2) + \alpha_2(x)((\psi_t^n)^2 + (g_2(\psi_t^n))^2)) dx dt}{E_{U^n}(0)} = 0. \tag{4.3}$$

Now, we define:

$$\alpha_n := [E_{U^n}(0)]^{1/2}, \quad \tilde{\varphi}^n := \frac{\varphi^n}{\alpha_n}, \quad \tilde{\psi}^n = \frac{\psi^n}{\alpha_n}. \tag{4.4}$$

From (4.1) and (4.4), we can consider the following sequence of normalized problems in $(0, L) \times (0, T)$:

$$\begin{cases} \rho_1(x) \tilde{\varphi}_{tt}^n - (k(x)(\tilde{\varphi}_x^n + \tilde{\psi}^n))_x + \frac{1}{\alpha_n} f_1(\alpha_n \tilde{\varphi}^n) + \frac{1}{\alpha_n} \alpha_1(x) g_1(\alpha_n \tilde{\varphi}_t^n) = 0, \\ \rho_2(x) \tilde{\psi}_{tt}^n - (b(x)\tilde{\psi}_x^n)_x + k(x)(\tilde{\varphi}_x^n + \tilde{\psi}^n) + \frac{1}{\alpha_n} f_2(\alpha_n \tilde{\psi}^n) + \frac{1}{\alpha_n} \alpha_2(x) g_2(\alpha_n \tilde{\psi}_t^n) = 0, \\ \tilde{\varphi}^n(0) = \frac{\varphi_0^n}{\alpha_n}, \tilde{\varphi}_t^n(0) = \frac{\varphi_1^n}{\alpha_n}, \tilde{\psi}^n(0) = \frac{\psi_0^n}{\alpha_n}, \tilde{\psi}_t^n(0) = \frac{\psi_1^n}{\alpha_n}, \end{cases} \tag{4.5}$$

with the same boundary conditions for $\tilde{\varphi}^n$ and $\tilde{\psi}^n$. The functional energy associated with problem (4.5) is given by

$$E_{\tilde{U}^n}(t) = \int_0^L \left\{ \frac{\rho_1}{2} (\tilde{\varphi}_t^n)^2 + \frac{\rho_2}{2} (\tilde{\psi}_t^n)^2 + \frac{b}{2} (\tilde{\psi}_x^n)^2 + \frac{k}{2} (\tilde{\varphi}_x^n + \tilde{\psi}^n)^2 + \frac{1}{\alpha_n^2} F_1(\alpha_n \tilde{\varphi}^n) + \frac{1}{\alpha_n^2} F_2(\alpha_n \tilde{\psi}^n) \right\} dx, \tag{4.6}$$

for $t \geq 0$. From (4.4), a straightforward computation shows that

$$E_{\tilde{U}^n}(t) = \frac{1}{\alpha_n^2} E_{U^n}(t) \quad \text{for all } t \geq 0, n \in \mathbb{N},$$

which implies, in particular, that $E_{\tilde{U}^n}(0) = 1$ for all $n \in \mathbb{N}$. In order to achieve a contradiction, we are going to prove in what follows that $E_{\tilde{U}^n}(0)$ converges to zero as n goes to infinity.

Firstly, taking (4.3) and (4.4) into account we deduce

$$\lim_{n \rightarrow +\infty} \int_0^T \int_0^L \left\{ \alpha_1(x) \left[(\tilde{\varphi}_t^n)^2 + \frac{g_1^2(\alpha_n \tilde{\varphi}_t^n)}{\alpha_n^2} \right] + \alpha_2(x) \left[(\tilde{\psi}_t^n)^2 + \frac{g_2^2(\alpha_n \tilde{\psi}_t^n)}{\alpha_n^2} \right] \right\} dx dt = 0. \tag{4.7}$$

Further, from the boundedness $E_{\tilde{U}^n}(0) \leq 1$ we also deduce, for an eventual subsequence of $\{\tilde{U}^n\}$, that

$$(\tilde{\varphi}_t^n, \tilde{\psi}_t^n) \rightharpoonup (\tilde{\varphi}_t, \tilde{\psi}_t) \text{ weakly star in } L^\infty(0, T; (L^2(0, L))^2), \tag{4.8}$$

$$(\tilde{\varphi}^n, \tilde{\psi}^n) \rightharpoonup (\tilde{\varphi}, \tilde{\psi}) \text{ weakly star in } L^\infty(0, T; (H_0^1(0, L))^2), \tag{4.9}$$

$$(\tilde{\varphi}^n, \tilde{\psi}^n) \rightarrow (\tilde{\varphi}, \tilde{\psi}) \text{ strongly in } L^\infty(0, T; (C[0, L])^2). \tag{4.10}$$

We observe that α_n is bounded and, in addition, that $\alpha_n \rightarrow \alpha \in [0, K^{1/2}]$. We shall divide our proof in two cases: $\alpha > 0$ or $\alpha = 0$.

Case (i): $\alpha > 0$. Passing to the limit in (4.5) and using (4.7) along with assumptions on $g_i, i = 1, 2$, we arrive at

$$\begin{cases} \rho_1(x) \tilde{\varphi}_{tt} - (k(x)(\tilde{\varphi}_x + \tilde{\psi}))_x + \frac{1}{\alpha} f_1(\alpha \tilde{\varphi}) = 0 & \text{in } (0, L) \times (0, T), \\ \rho_2(x) \tilde{\psi}_{tt} - (b(x)\tilde{\psi}_x)_x + k(x)(\tilde{\varphi}_x + \tilde{\psi}) + \frac{1}{\alpha} f_2(\alpha \tilde{\psi}) = 0 & \text{in } (0, L) \times (0, T), \\ \tilde{\varphi}_t = \tilde{\psi}_t = 0 & \text{in } \omega \times (0, T), \end{cases} \tag{4.11}$$

and for $\tilde{\varphi}_t = \eta$ and $\tilde{\psi}_t = \zeta$, it holds in the distributional sense

$$\begin{cases} \rho_1(x) \eta_{tt} - (k(x)(\eta_x + \zeta))_x + f_1'(\alpha \tilde{\varphi}) \eta = 0 & \text{in } (0, L) \times (0, T), \\ \rho_2(x) \zeta_{tt} - (b(x)\zeta_x)_x + k(x)(\eta_x + \zeta) + f_2'(\alpha \tilde{\psi}) \zeta = 0 & \text{in } (0, L) \times (0, T), \\ \eta = \zeta = 0 & \text{in } \omega \times (0, T). \end{cases} \tag{4.12}$$

By noting that $f_1'(\alpha \tilde{\varphi}), f_2'(\alpha \tilde{\psi}) \in L^\infty(\Omega \times (0, T))$, then from Assumption 2.5 it results that $(\eta, \zeta) = (0, 0)$, which implies that $(\tilde{\varphi}_t, \tilde{\psi}_t) = (0, 0)$. Returning to (4.11), we infer

$$\begin{cases} -(k(x)(\tilde{\varphi}_x + \tilde{\psi}))_x + \frac{1}{\alpha} f_1(\alpha \tilde{\varphi}) = 0 & \text{in } (0, L) \times (0, T), \\ -(b(x)\tilde{\psi}_x)_x + k(x)(\tilde{\varphi}_x + \tilde{\psi}) + \frac{1}{\alpha} f_2(\alpha \tilde{\psi}) = 0 & \text{in } (0, L) \times (0, T), \end{cases}$$

and since $f_i(s)s \geq 0$ for $i = 1, 2$, we obtain

$$\int_0^L k(x)(\tilde{\varphi}_x + \tilde{\psi})^2 dx + \int_0^L b(x)(\tilde{\psi}_x)^2 dx \leq 0,$$

from which we conclude that $(\tilde{\varphi}, \tilde{\psi}) = (0, 0)$.

Case (ii): $\alpha = 0$. In this case, we have $\alpha_n \rightarrow 0$. From the assumptions on f_1 , we note that we can write

$$f_1(s) = f'_1(0)s + R_1(s), \quad \text{where } |R_1(s)| \leq C(|s|^2 + |s|^p).$$

Thus,

$$\begin{aligned} \frac{1}{\alpha_n} f_1(\alpha_n \tilde{\varphi}^n) &= f'_1(0)\tilde{\varphi}^n + \frac{R(\alpha_n \tilde{\varphi}^n)}{\alpha_n}, \\ \frac{R_1(\alpha_n \tilde{\varphi}^n)}{\alpha_n} &\leq C(\alpha_n |\tilde{\varphi}^n|^2 + |\alpha_n|^{p-1} |\tilde{\varphi}^n|^p). \end{aligned}$$

We observe that

$$\alpha_n |\tilde{\varphi}^n|^2 + |\alpha_n|^{p-1} |\tilde{\varphi}^n|^p \rightarrow 0 \quad \text{in } L^2(0, T; L^2(0, L)),$$

since $\alpha_n \rightarrow 0$ and $\{\varphi^n\}$ is bounded in $L^\infty(0, T; H^1_0(0, L))$. As a consequence, from the compactness Aubin–Lions–Simon Lemma,

$$\frac{1}{\alpha_n} f_1(\alpha_n \tilde{\varphi}^n) \rightarrow f'_1(0)\tilde{\varphi} \quad \text{strongly in } L^2(0, T; L^2(0, L)). \tag{4.13}$$

Proceeding analogously for f_2 , we deduce

$$\frac{1}{\alpha_n} f_2(\alpha_n \tilde{\psi}^n) \rightarrow f'_2(0)\tilde{\psi} \quad \text{strongly in } L^2(0, T; L^2(0, L)). \tag{4.14}$$

Passing to the limit in (4.5) and using (4.13)–(4.14), we arrive at

$$\begin{cases} \rho_1(x)\tilde{\varphi}_{tt} - (k(x)(\tilde{\varphi}_x + \tilde{\psi}))_x + f'_1(0)\tilde{\varphi} = 0 & \text{in } (0, L) \times (0, T), \\ \rho_2(x)\tilde{\psi}_{tt} - (b(x)\tilde{\psi}_x)_x + k(x)(\tilde{\varphi}_x + \tilde{\psi}) + f'_2(0)\tilde{\psi} = 0 & \text{in } (0, L) \times (0, T), \\ \tilde{\varphi}_t = \tilde{\psi}_t = 0 & \text{in } \omega \times (0, T), \end{cases} \tag{4.15}$$

and for $\tilde{\varphi}_t = \eta$ and $\tilde{\psi}_t = \zeta$, it yields in the distributional sense

$$\begin{cases} \rho_1(x)\eta_{tt} - (k(x)(\eta_x + \zeta))_x + f'_1(0)\eta = 0 & \text{in } (0, L) \times (0, T), \\ \rho_2(x)\zeta_{tt} - (b(x)\zeta_x)_x + k(x)(\eta_x + \zeta) + f'_2(0)\zeta = 0 & \text{in } (0, L) \times (0, T) \\ \eta = \zeta = 0 & \text{in } \omega \times (0, T), \end{cases} \tag{4.16}$$

which implies from Assumption 2.5 that $(\eta, \zeta) = (0, 0)$, therefore $(\tilde{\varphi}_t, \tilde{\psi}_t) = (0, 0)$. Returning to (4.15) and proceeding verbatim what we have done before we deduce that $(\tilde{\varphi}, \tilde{\psi}) = (0, 0)$. As a consequence, one has

$$(\tilde{\varphi}^n_t, \tilde{\psi}^n_t) \rightharpoonup (0, 0) \quad \text{weakly star in } L^\infty(0, T; (L^2(0, L))^2), \tag{4.17}$$

$$(\tilde{\varphi}^n, \tilde{\psi}^n) \rightharpoonup (0, 0) \quad \text{weakly star in } L^\infty(0, T; (H^1_0(0, L))^2), \tag{4.18}$$

$$(\tilde{\varphi}^n, \tilde{\psi}^n) \rightarrow (0, 0) \quad \text{strongly in } L^\infty(0, T; (C([0, L]))^2). \tag{4.19}$$

Let us keep in mind that our objective is to prove that $E_{U^n}(0)$ converges to zero. For this purpose, let P_1 the wave operator defined by

$$P_1 := \rho_1(x)\partial_t^2 - \partial_x(k(x)\partial_x) = -\frac{1}{\alpha_n} f_1(\alpha_n \tilde{\varphi}^n) - \frac{1}{\alpha_n} \alpha_1(x) g_1(\alpha_n \tilde{\varphi}^n_t).$$

First, we will prove that

$$\tilde{\varphi}^n_t \rightarrow 0 \quad (\text{strongly}) \quad \text{in } L^2((0, L) \times (0, T)).$$

Indeed, from above convergence (4.17) we know that

$$\tilde{\varphi}^n_t \rightharpoonup 0 \quad (\text{weakly}) \quad \text{in } L^2((0, L) \times]0, T]). \tag{4.20}$$

So, let us consider $\mu_{\tilde{\varphi}}$ be the microlocal defect measure (m.d.m.) associated to $\{\tilde{\varphi}_t^n\}$ (which is assured by Theorem 5.5 in Burq–Gérard [6], see also [20]). Then, by (4.7), (4.13) and (4.19), we deduce

$$\partial_t P_1 \tilde{\varphi}^n = P_1 \tilde{\varphi}_t^n \rightarrow 0 \text{ (strongly) in } H_{loc}^{-1}((0, L) \times]0, T[). \tag{4.21}$$

Analogously, defining

$$P_2 := \rho_2(x) \partial_t^2 - \partial_x(b(x) \partial_x),$$

we also infer (similarly as above) that

$$\partial_t P_2 \tilde{\psi}^n = P_2 \tilde{\psi}_t^n \rightarrow 0 \text{ (strongly) in } H_{loc}^{-1}((0, L) \times]0, T[). \tag{4.22}$$

Taking into account that ω geometrically controls Ω , we deduce two facts:

- (i) The $\text{supp}(\mu_{\tilde{\varphi}})$ is contained in the characteristic set of the wave equation $\{\tau^2 = \frac{k(x)}{\rho_1(x)} |\xi|^2\}$, where $p_1(t, x, \tau, \xi) = \frac{1}{2} \left(-\tau^2 + \frac{k(x)}{\rho_1(x)} \xi^2 \right)$ denotes the principal symbol of P_1 .
- (ii) The m.d.m. $\mu_{\tilde{\varphi}}$ propagates along the bicharacteristic flow of this operator, which signifies, particularly, that if some point $\omega_0 = (t_0, x_0, \tau_0, \xi_0)$ does not belong to $\text{supp}(\mu_{\tilde{\varphi}})$, then the whole bicharacteristic issued from ω_0 is out of $\text{supp}(\mu_{\tilde{\varphi}})$.

Indeed, from (4.21) and Theorem 5.6 in Burq and Gérard [6] we deduce item (i). Furthermore, from Proposition 6.2 and Theorem 6.1 found in Burq and Gérard [6], we deduce that $\text{supp}(\mu_{\tilde{\varphi}})$ in $(\Omega \times (0, T)) \times S^1$, ($\Omega := (0, L)$) is a union of curves like

$$t \in I \cap (0, \infty) \mapsto m \pm(t) = \left(t, x(t), \frac{\pm 1}{\sqrt{1 + |G_1(x(t))x'(t)|}}, \frac{\pm G_1(x(t))x'(t)}{\sqrt{1 + |G_1(x(t))x'(t)|}} \right), \tag{4.23}$$

where $t \in I \mapsto x(t) \in \Omega$ is a geodesic associated to the metric $G_1 = (\frac{k}{\rho_1})^{-1}$.

Since by (4.7), we have $\tilde{\varphi}_t^n \rightarrow 0$ strongly in $L^2(\omega \times (0, T))$ then, from Remark 5.15 in Burq and Gérard [6], we have that $\mu_{\tilde{\varphi}} = 0$ in $\omega \times (0, T)$ and, consequently,

$$\text{supp}(\mu_{\tilde{\varphi}}) \subset (\Omega \setminus \omega) \times (0, T).$$

On the other hand, let $t_0 \in (0, +\infty)$ and let x be a geodesic of G_1 defined near t_0 . Once the geodesics inside $\Omega \setminus \omega$ enter necessarily in the region ω , then for any geodesic of the metric G_1 , with $0 \in I$ there exists $t > 0$ such that $m \pm(t)$ does not belong to $\text{supp}(\mu_{\tilde{\varphi}})$, so that $m \pm(t_0)$ does not belong as well and item (ii) follows.

Once the time t_0 and the geodesic x were taken arbitrarily, we conclude that $\text{supp}(\mu_{\tilde{\varphi}})$ is empty. Therefore, employing again [6, Remark 5.15], we have

$$\tilde{\varphi}_t^n \rightarrow 0 \text{ (strongly) in } L_{loc}^2((0, L) \times (0, T)). \tag{4.24}$$

Moreover, from (4.7) we claim that

$$\tilde{\varphi}_t^n \rightarrow 0 \text{ (strongly) in } L^2((0, L) \times (0, T)). \tag{4.25}$$

In fact, first of all, we observe that

$$\begin{aligned} \int_0^T \int_0^L |\tilde{\varphi}_t^n| dx dt &= \int_0^T \int_{\omega} |\tilde{\varphi}_t^n| dx dt + \int_0^T \int_{(0,L) \setminus \omega} |\tilde{\varphi}_t^n| dx dt \\ &:= L_1 + L_2. \end{aligned} \tag{4.26}$$

From (4.7), we have that $L_1 \rightarrow 0$ when $k \rightarrow \infty$. For L_2 consider the following decomposition:

$$\begin{aligned}
 L_2 &= \int_0^\varepsilon \int_{(0,L)\setminus\omega} |\tilde{\varphi}_t^n| \, dx \, dt + \int_\varepsilon^{T-\varepsilon} \int_{(0,L)\setminus\omega} |\tilde{\varphi}_t^n| \, dx \, dt + \int_{T-\varepsilon}^T \int_{(0,L)\setminus\omega} |\tilde{\varphi}_t^n| \, dx \, dt \\
 &:= J_1 + J_2 + J_3.
 \end{aligned}
 \tag{4.27}$$

Note that

$$J_1 = \int_0^\varepsilon \int_{(0,L)\setminus\omega} |\tilde{\varphi}_t^n| \, dx \, dt \leq 2\rho_1^{-1} \int_0^\varepsilon E_{\tilde{U}_n}(t) \, dt \leq 2\rho_1^{-1} \varepsilon E_{\tilde{U}_n}(0) \leq 2\rho_1^{-1} \varepsilon$$

since $E_{\tilde{U}_n}(0) \leq 1$. Therefore,

$$\lim_{n \rightarrow +\infty} J_1 \leq 2\rho_1^{-1} \varepsilon \quad \text{for all } T > \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\lim_{n \rightarrow +\infty} J_1 = 0$. Proceeding in the same way, we show that $\lim_{n \rightarrow +\infty} J_3 = 0$. Finally, from (4.24) we deduce that $\lim_{n \rightarrow +\infty} J_2 = 0$ as $n \rightarrow \infty$. Then, these facts guarantee us the statement given in (4.25), that is, the kinetic part of the energy with respect to the displacement goes to zero.

Proceeding analogously as above, we can also conclude that the kinetic part of the energy with respect to the rotation angle goes to zero, namely,

$$\tilde{\psi}_t^n \rightarrow 0 \text{ (strongly) in } L^2((0, L) \times (0, T)).
 \tag{4.28}$$

In what follows, we are going to recover the convergence of the potential part of the energy. To do so, let consider the equipartition of the energy; that is, let us first consider $\theta \in C_0^\infty(0, T)$; $0 \leq \theta \leq 1$ and $\theta = 1$ in $(\varepsilon, T - \varepsilon)$.

Now, let us take the multipliers $\tilde{\varphi}^n \theta(t)$ and $\tilde{\psi}^n \theta(t)$ in the first and second equations of (4.5), respectively. Then, adding the resulting expression, we get

$$\begin{aligned}
 & - \int_0^T \theta'(t) \int_0^L \rho_1 \tilde{\varphi}_t^n \tilde{\varphi}^n \, dx \, dt - \int_0^T \int_0^L \rho_1 |\tilde{\varphi}_t^n|^2 \, dx \, dt \\
 & - \int_0^T \theta'(t) \int_0^L \rho_2 \tilde{\psi}_t^n \tilde{\psi}^n \, dx \, dt - \int_0^T \int_0^L \rho_2 |\tilde{\psi}_t^n|^2 \, dx \, dt \\
 & + \int_0^T \theta(t) \int_0^L b(x) |\tilde{\psi}_x^n|^2 \, dx \, dt + \int_0^T \theta(t) \int_0^L k(x) (\tilde{\varphi}_x^n + \tilde{\psi}^n)^2 \, dx \, dt \\
 & + \frac{1}{\alpha_n} \int_0^T \theta(t) \int_0^L f_1(\alpha_n \tilde{\varphi}^n) \tilde{\varphi}^n \, dx \, dt + \frac{1}{\alpha_n} \int_0^T \theta(t) \int_0^L f_2(\alpha_n \tilde{\psi}^n) \tilde{\psi}^n \, dx \, dt \\
 & + \frac{1}{\alpha_n} \int_0^T \theta(t) \int_0^L \alpha_1 g_1(\alpha_n \tilde{\varphi}_t^n) \tilde{\varphi}^n \, dx \, dt + \frac{1}{\alpha_n} \int_0^T \theta(t) \int_0^L \alpha_2 g_2(\alpha_n \tilde{\psi}_t^n) \tilde{\psi}^n \, dx \, dt.
 \end{aligned}
 \tag{4.29}$$

Taking (4.7), (4.17), (4.18), (4.19), (4.25), (4.28) and (4.29) into account, yields

$$\lim_{n \rightarrow +\infty} \int_\varepsilon^{T-\varepsilon} \int_0^L |\tilde{\varphi}_x^n|^2 \, dx \, dt = \lim_{n \rightarrow +\infty} \int_\varepsilon^{T-\varepsilon} \int_0^L |\tilde{\psi}_x^n|^2 \, dx \, dt = 0,
 \tag{4.30}$$

and then

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{1}{\alpha_n} \int_{\epsilon}^{T-\epsilon} \int_0^L f_1(\alpha_n \tilde{\varphi}^n) \tilde{\varphi}^n dx dt &= \lim_{n \rightarrow +\infty} \frac{1}{\alpha_n^2} \int_{\epsilon}^{T-\epsilon} \int_0^L f_1(\varphi^n) \varphi^n dx dt = 0, \\ \lim_{n \rightarrow +\infty} \frac{1}{\alpha_n} \int_{\epsilon}^{T-\epsilon} \int_0^L f_2(\alpha_n \tilde{\psi}^n) \tilde{\psi}^n dx dt &= \lim_{n \rightarrow +\infty} \frac{1}{\alpha_n^2} \int_{\epsilon}^{T-\epsilon} \int_0^L f_2(\psi^n) \psi^n dx dt = 0. \end{aligned} \tag{4.31}$$

Moreover, from (4.31) and regarding assumption (2.9), we conclude

$$\lim_{n \rightarrow +\infty} \frac{1}{\alpha_n^2} \int_{\epsilon}^{T-\epsilon} \int_0^L F_1(\varphi^n) dx dt = \lim_{n \rightarrow +\infty} \frac{1}{\alpha_n^2} \int_{\epsilon}^{T-\epsilon} \int_0^L F_2(\psi^n) dx dt = 0,$$

which implies, jointly with all the above convergences, that

$$\int_{\epsilon}^{T-\epsilon} E_{\tilde{U}^n}(t) dt \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Since the energy is non-increasing, we obtain

$$(T - 2\epsilon)E_{\tilde{U}^n}(T - \epsilon) \rightarrow 0.$$

From this and using the energy identify (see (3.3))

$$\begin{aligned} E_{\tilde{U}^n}(T - \epsilon) - E_{\tilde{U}^n}(\epsilon) &= -\frac{1}{\alpha_n} \left[\int_{\epsilon}^{T-\epsilon} \int_0^L \left(\alpha_1(x) (|\tilde{\varphi}_t^n|^2 + g_1(\alpha_n \tilde{\varphi}_t^n) \tilde{\varphi}_t^n) + \alpha_2(x) (|\tilde{\psi}_t^n|^2 + \tilde{g}_2(\alpha_n \tilde{\psi}_t^n) \tilde{\psi}_t^n) \right) dx dt \right] \end{aligned}$$

along with the convergence (4.7), we arrive at

$$E_{\tilde{U}^n}(\epsilon) \rightarrow 0. \tag{4.32}$$

Thus, from (4.32) and using again the energy identify and the limit in (4.7), we finally conclude that

$$\begin{aligned} E_{\tilde{U}^n}(0) &= - [E_{\tilde{U}^n}(\epsilon) - E_{\tilde{U}^n}(0)] + E_{\tilde{U}^n}(\epsilon) \\ &= \int_0^{\epsilon} \int_0^L \left(\alpha_1(x) (|\tilde{\varphi}_t^n|^2 + g_1(\alpha_n \tilde{\varphi}_t^n) \tilde{\varphi}_t^n) + \alpha_2(x) (|\tilde{\psi}_t^n|^2 + \tilde{g}_2(\alpha_n \tilde{\psi}_t^n) \tilde{\psi}_t^n) \right) dx dt \\ &\quad + E_{\tilde{U}^n}(\epsilon) \rightarrow 0. \end{aligned}$$

Hence, $E_{\tilde{U}^n}(0) \rightarrow 0$ when $n \rightarrow +\infty$, which concludes the desired contradiction.

This completes the proof of (3.9) as desired and finishes the proof of Proposition 3.3. □

The conclusion of the proof of Theorem 3.2 follows the verbatim the same arguments as presented in the references [8,30]. For the sake of completeness, we also provide it here as follows.

4.2. Proof of Theorem 3.2

Let $T > T_0$, where T_0 comes from the observability inequality. From (3.9) we have

$$E_U(0) \leq \bar{C}_T \int_0^T \int_0^L \left\{ \alpha_1(x) [\varphi_t^2 + g_1(\varphi_t)^2] + \alpha_2(x) [\psi_t^2 + g_2(\psi_t)^2] \right\} dx dt, \tag{4.33}$$

for some constant $\overline{C}_T > 0$.

Now, given a strong solution $U = (\varphi, \varphi_t, \psi, \psi_t)$ of (2.1), we define the following sets

$$\begin{aligned} \Sigma_\varphi &= \{(x, t) \in \Omega \times (0, T); |\varphi_t(x, t)| > 1\} \text{ and } \Gamma_\varphi = \Omega \times (0, T) \setminus \Sigma_\varphi, \\ \Sigma_\psi &= \{(x, t) \in \Omega \times (0, T); |\psi_t(x, t)| > 1\} \text{ and } \Gamma_\psi = \Omega \times (0, T) \setminus \Sigma_\psi. \end{aligned}$$

Our strategy here is to estimate the integrals at the right side of (4.33). First, note that

$$\begin{aligned} \int_0^T \int_0^L \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt &= \int_{\Sigma_\varphi} \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt \\ &\quad + \int_{\Gamma_\varphi} \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt. \end{aligned} \tag{4.34}$$

From Assumption 2.1, we have

$$\int_{\Sigma_\varphi} \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt \leq (k_1^{-1} + K_1) \int_{\Sigma_\varphi} \alpha_1(x)g_1(\varphi_t)\varphi_t dx dt. \tag{4.35}$$

Now, using (3.4) we obtain

$$\begin{aligned} \int_{\Gamma_\varphi} \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt &\leq \int_{\Gamma_\varphi} \alpha_1(x)h_1\left(g_1(\varphi_t)\varphi_t\right) dx dt \\ &\leq \int_{\Gamma_\varphi} (1 + \|\alpha_1\|_\infty)h_1\left(\frac{\alpha_1}{1 + \|\alpha_1\|_\infty}g_1(\varphi_t)\varphi_t\right) dx dt \\ &\leq (1 + \|\alpha_1\|_\infty) \int_{\Gamma_\varphi} h_1\left(\alpha_1(x)g_1(\varphi_t)\varphi_t\right) dx dt \\ &\leq (1 + \|\alpha_1\|_\infty)LT h_1\left(\frac{1}{LT} \int_0^T \int_0^L \alpha_1(x)g_1(\varphi_t)\varphi_t dx dt\right) \end{aligned} \tag{4.36}$$

where the last inequality is obtained using the Jensen’s inequality. Therefore, using (4.35) and (4.36),

$$\begin{aligned} \int_0^T \int_0^L \alpha_1(x)(\varphi_t^2 + g_1(\varphi_t)^2) dx dt &\leq (k_1^{-1} + K_1) \int_0^T \int_0^L \alpha_1(x)g_1(\varphi_t)\varphi_t dx dt \\ &\quad + (1 + \|\alpha_1\|_\infty)LT h_1\left(\frac{1}{LT} \int_0^T \int_0^L \alpha_1(x)g_1(\varphi_t)\varphi_t dx dt\right). \end{aligned} \tag{4.37}$$

Analogously, we can conclude that

$$\begin{aligned} \int_0^T \int_0^L \alpha_2(x)(\psi_t^2 + g_1(\psi_t)^2) dx dt &\leq (k_2^{-1} + K_2) \int_0^T \int_0^L \alpha_2(x)g_2(\psi_t)\psi_t dx dt \\ &\quad + (1 + \|\alpha_2\|_\infty)LT h_2\left(\frac{1}{LT} \int_0^T \int_0^L \alpha_2(x)g_2(\psi_t)\psi_t dx dt\right). \end{aligned} \tag{4.38}$$

Since each h_i is an increasing function, then combining the energy identity, (4.37) and (4.38), we have

$$E_U(T) \leq \bar{C} \sum_{i=1}^2 (k_i^{-1} + K_i) \int_0^T \int_0^L \left\{ \alpha_1(x) g_1(\varphi_t) \varphi_t + \alpha_2(x) g_2(\psi_t) \psi_t \right\} dx dt + \bar{C} L T \sum_{i=1}^2 (1 + \|\alpha_i\|_\infty) r \left(\int_0^T \int_0^L \alpha_1(x) g_1(\varphi_t) \varphi_t + \alpha_2(x) g_2(\psi_t) \psi_t dx dt \right),$$

where r was defined in (3.5). Setting the constants

$$M = \frac{1}{\bar{C} L T \sum_{i=1}^2 (1 + \|\alpha_i\|_\infty)} \quad \text{and} \quad c = \frac{\sum_{i=1}^2 (k_i^{-1} + K_i)}{L T \sum_{i=1}^2 (1 + \|\alpha_i\|_\infty)},$$

and using (3.2), we arrive at

$$M E_U(T) \leq c \int_0^T L T \int_0^L \left\{ \alpha_1(x) g_1(\varphi_t) \varphi_t + \alpha_2(x) g_2(\psi_t) \psi_t \right\} dx dt + r \left(\int_0^T \int_0^L \alpha_1(x) g_1(\varphi_t) \varphi_t + \alpha_2(x) g_2(\psi_t) \psi_t dx dt \right) = (cI + r) \left(E_U(0) - E_U(T) \right).$$

Using the notation introduced in (3.6), the previous inequality can be rewritten as

$$p(E_U(T)) \leq E_U(0) - E_U(T). \tag{4.39}$$

To finish the proof, we replace T by $(m + 1)T$ (respectively, 0 by mT) in (4.39), $m \in \mathbb{N}$, in order to obtain

$$E_U((m + 1)T) + p(E_U((m + 1)T)) \leq E_U(mT), \quad \text{for } m = 0, 1, \dots.$$

Thus, using [23, Lemma 3.3] with $s_m = E_U(mT)$, we can conclude that

$$E_U(mT) \leq S(m), \quad m = 0, 1, \dots.$$

Finally, observing that for every $t > T$ we can find $m \in \mathbb{N}$ and $\tau \in [0, T]$ such that $t = mT + \tau$, then

$$E_U(t) \leq E_U(mT) \leq S(m) \leq S\left(\frac{t - \tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right) \quad \text{for all } t > T,$$

where we have used the fact that the solution S of (3.7) is dissipative.

Therefore, the proof of Theorem 3.2 is complete. □

5. Further remarks

We finish this work by giving some examples of 1-D metrics

$$G_1 = (k/\rho_1)^{-1} \quad \text{or} \quad G_2 = (b/\rho_2)^{-1}$$

that satisfy the geometric control condition. It is enough to choose one them, for example G_1 . To this purpose, let us remember that

$$p_1(t, x, \tau, \xi) = \frac{1}{2} \left(\frac{k(x)}{\rho_1(x)} \xi^2 - \tau^2 \right)$$

is the principal symbol of the wave operator $P_1 = \rho_1 \partial_t^2 - [k(x)\partial_x]_x$. In order to obtain the Hamiltonian flow, keeping in mind that

$$H_{p_1}(t, x, \tau, \xi) = \left(\frac{\partial p_1}{\partial \tau}, \frac{\partial p_1}{\partial \xi}, -\frac{\partial p_1}{\partial t}, -\frac{\partial p_1}{\partial x} \right),$$

one has in our case

$$\frac{\partial p_1}{\partial \tau} = -\tau, \quad \frac{\partial p_1}{\partial \xi} = \frac{k(x)}{\rho_1(x)}\xi, \quad -\frac{\partial p_1}{\partial t} = 0, \quad -\frac{\partial p_1}{\partial x} = -\frac{1}{2} \left(\frac{k(x)}{\rho_1(x)} \right)' \xi^2,$$

where the subscript $'$ denotes the x -derivative of the function $\frac{k(x)}{\rho_1(x)}$. Thus,

$$\dot{t}(s) = -\tau(x(s)), \quad \dot{x}(s) = \frac{k(x)}{\rho_1(x)}\xi(s), \quad \dot{\tau}(s) = 0, \quad \dot{\xi}(s) = -\frac{1}{2} \left(\frac{k(x)}{\rho_1(x)} \right)' \xi^2(s),$$

where $\dot{}$ stands for the time derivative.

We observe that once $\dot{\tau}(s) = 0$, then $\tau(s) = \tau_0 = \text{constant}$. Now, being $p = 0$ on each null bicharacteristic, yields

$$\frac{k(x)}{\rho_1(x)}\xi^2(s) = \tau^2(s) = \tau_0^2, \quad \forall s \in I.$$

On the other hand from the above relationships, we deduce that

$$\left(\frac{\rho_1}{k} \right) (x(s))(\dot{x}(s))^2 = \left(\frac{k}{\rho_1} \right) (x(s))\xi^2(s) = \tau_0^2, \quad \forall s \in I,$$

and from the second identity on the right hand side of the last identity, we obtain

$$\xi(s) = \pm \sqrt{\left(\frac{\rho_1}{k} \right) (x(s))} \tau_0,$$

from which we deduce that

$$\dot{x}(s) = \pm \sqrt{\left(\frac{k}{\rho_1} (x(s)) \right)} \tau_0. \tag{5.1}$$

Analogously, one can conclude

$$\dot{x}(s) = \pm \sqrt{\left(\frac{b}{\rho_2} (x(s)) \right)} \tau_0. \tag{5.2}$$

The above formulas (5.1) and (5.2) allow us to deduce concrete examples where the geometric control condition (GCC) holds true; that is, Assumption 2.2 is not empty.

Example 5.1. Let us first consider three cases where the GCC is verified.

- (a) Assuming $\frac{k}{\rho_1}(x) = c_0 > 0$, then $x(t) = x(0) \pm c_0^{1/2} \tau_0 t$, for some $x(0) \in (0, L)$ and $t \in I \cap (0, \infty)$. Hence, the GCG is ensured.
- (b) Considering $\frac{k}{\rho_1}(x) = x$, then we deduce easily that $x(t) = ((x(0))^2 \pm \frac{\tau_0}{2} t)^2$ and, therefore, the CGC also holds true.
- (c) If $\left(\frac{k}{\rho_1}(x) \right) = x^2$ we infer that $x(t) = \pm |x(0)| e^{\pm \tau_0 t}$ and the GCG is also verified.

Finally, we consider an example where GCG fails.

Example 5.2. Let us consider a family of circles centering in the x -axis

$$f(s, x(s), \lambda) := (s - 2\lambda)^2 + x(s)^2 - \lambda^2 = 0. \quad (5.3)$$

Performing differentiation with respect to s , we have

$$\frac{s + x(s)\dot{x}(s)}{2} = \lambda. \quad (5.4)$$

Substituting (5.4) in (5.3), it follows that

$$3x(s)[\dot{x}(s)]^2 - 2sx(s)\dot{x}(s) + 4x^2(s) - s^2 = 0. \quad (5.5)$$

Looking (5.5) as a second-order equation in the variable $\dot{x}(s)$, it follows that its roots are

$$\dot{x}(s) = \frac{2sx(s) \pm \sqrt{-48x(s)^3 + 4s^2x(s)^2 + 12s^2x(s)}}{6x(s)}. \quad (5.6)$$

On the other hand, recalling (5.1), one gets

$$\dot{x}(s) = \pm \sqrt{\left(\frac{k}{\rho_1}(x(s))\right)\tau_0},$$

and combining it with (5.6), we infer

$$\frac{k}{\rho_1}(x(s)) = \left(\frac{2sx(s) \pm \sqrt{-48x(s)^3 + 4s^2x(s)^2 + 12s^2x(s)}}{6x(s)\tau_0}\right)^2. \quad (5.7)$$

Now, using computational analysis to solve the above nonlinear algebraic equation¹, we obtain that the solutions of the ODE are of the form:

$$x(s) = \pm \sqrt{\frac{\pm 4is \sinh(3c_1) - 4is \cosh(3c_1) + \sinh(6c_1) + \cosh(6c_1) - 3s^2}{3}}, \quad (5.8)$$

for some real constant c_1 . Since

$$x_0 = x(0) = \pm \sqrt{\frac{\sinh(6c_1) + \cosh(6c_1)}{3}} = \pm \sqrt{\frac{e^{6c_1}}{3}},$$

then

$$c_1 = \frac{\ln(3x_0^2)}{6}. \quad (5.9)$$

We observe that there are cases for λ where the circle (in the family (5.3)) intercepts the damped area $\omega \times (0, T)$. Our next goal is to choose suitable values of λ where this situation does not occur. Indeed, fixing λ such that

$$|x - 2\lambda| > \lambda, \forall x \in \omega, \quad (5.10)$$

that is, $\lambda > x$ or $\lambda < \frac{x}{3}$, for all $x \in \omega$, we conclude that the trapped bicharacteristics associated with the curves (5.3), which are solutions given in (5.8)–(5.9), do not reach the damped area $\omega \times (0, T)$, taking into account the restriction (5.10) on λ , see the illustrative case in Fig.f2. Therefore, the GCG fails.

¹One can use, for instance, the easy tool Wolfram. See on the website: [https://www.wolframalpha.com/input/?i=3*\(x\(s\)\)^5E2*\(x%27\(s\)\)^5E2-2*s*x\(s\)*x%27\(s\)%2B4*\(x\(s\)\)^5E2-s%5E2%3D0](https://www.wolframalpha.com/input/?i=3*(x(s))^5E2*(x%27(s))^5E2-2*s*x(s)*x%27(s)%2B4*(x(s))^5E2-s%5E2%3D0)

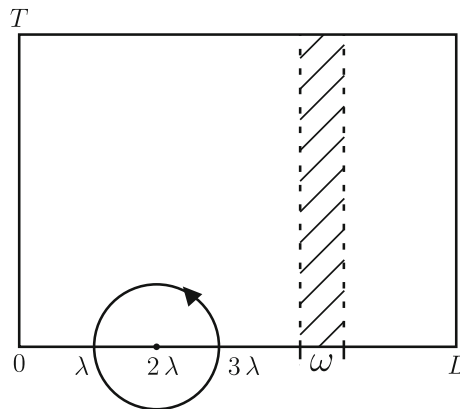


FIG. 2. The solutions of trapped bicharacteristics associated with the curves (5.3) do not reach the damped area $\omega \times (0, T)$, where λ satisfies (5.10)

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M. M. Cavalcanti
Department of Mathematics
State University of Maringá
Maringá PR87020-900
Brazil
e-mail: mmcavalcanti@uem.br

W. J. Corrêa
Federal Technological University of Paraná
Campus Campo Mourão
Campo Mourão PR87301-899
Brazil
e-mail: wcorrea@utfpr.edu.br

V. N. Domingos Cavalcanti
Departamento de Mathematics
State University of Maringá
Maringá PR87020-900
Brazil
e-mail: vndcavalcanti@uem.br

M. A. Jorge Silva
Department of Mathematics
State University of Londrina
Londrina PR86057-970
Brazil
e-mail: marcioajs@uel.br

J. P. Zanchetta
Department of Mathematics
State University of Maringá
Maringá PR87020-900
Brazil
e-mail: janainazanchetta@yahoo.com.br

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