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Stability conditions for thermodiffusion Timoshenko system with second sound

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Abstract. In this paper, we consider a new Timoshenko beam model with thermal and mass diffusion effects where heat and mass diffusion flux are governed by Cattaneo's law. Necessary and sufficient conditions for exponential stability are provided in terms of the physical parameters of the model. Firstly, by the C_0-semigroup theory, we prove the well-posedness of the considered problem. Then, we prove the lack of exponential stability of the system when one of these conditions is not valid. Finally, we prove in this case that the semigroup decays to zero polynomially as $1/\sqrt{t}$. Moreover, we show that the rate is optimal.

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1. Introduction

Beams represent the most common structural element in engineering and mechanics structures. Because of their ubiquity, they are widely studied, from mechanical and mathematical points of view. In materials mechanics, vibration has been known for a long time for a source of disturbance, discomfort, damage and destruction. However, the requirement for a more precise control of its vibrations has led to consider all the possible effects that a beam can undergo in a structure. One might perhaps think that the classical theory of thermoelasticity is a good model for explaining thermal conduction in this kind of problem. For a long time, the effects of diffusion have been ignored in the frame of the classical linear theory of thermoelasticity. Maybe we can think that the classical theory of thermoelasticity is a good model to explain the thermal conduction in contact problems. However, the research conducted in the development of high technologies after the second world war, confirmed that the fields of temperature and diffusion in solids cannot be ignored. So, the obvious question is what happens when the diffusion effect is considered with the thermal effect in contact problems. Diffusion can be defined as the random walk of a set of particles from regions of high concentration to regions of lower concentration. Thermodiffusion in an elastic solid is due to coupling of the fields of strain, temperature and mass diffusion. The processes of heat and mass diffusion play an important role in many contact engineering applications, such as satellites problems, returning space vehicles and aircraft landing on water or land.

Recently, Aouadi et al. [3] have considered the effect of mass diffusion effect in a thermo-Timoshenko beam. If the mass diffusion is taken into account in the Timoshenko equations, the evolution equations are given by

$$\rho_1\varphi_{tt} = S_x, \quad \rho_2\psi_{tt} = M_x - S, \quad \Psi_t = -q_x, \quad C_t = -\eta_x, \tag{1.1}$$

where φ is the transverse displacement and ψ is the rotation of the neutral axis due to bending. Here, $\rho_1 = \rho A$ and $\rho_2 = \rho I$, where $\rho > 0$ is the density, A is the cross-sectional area and I is the second moment of the cross-sectional area. By S, we denote the shear force and M is the bending moment, Ψ is the entropy, q is the heat flux, C is the concentration of the diffusive material in the elastic body, and

η is the mass diffusion flux. In this case, the constitutive equations with temperature and mass diffusion are given by [3]

$$\begin{aligned} &S=\kappa(\varphi_x+\psi),\quad M=b\psi_x+\gamma\theta+\beta C,\quad \Psi=-\gamma\psi_x+\rho_3\theta+\varpi C,\\ &P=\beta\psi_x+\varrho C-\varpi\theta, \end{aligned} \tag{1.2}$$

where b and κ stand for $b=EI$ and $\kappa=k_1GA$ where E, G and k_1 represent the Young's modulus, the modulus of rigidity and the transverse shear factor, respectively. Here, P is the chemical potential, ϖ is a measure of the thermodiffusion effect, ϱ is a measure of the diffusive effect, and γ and β are the coefficients of thermal and mass diffusion expansions, respectively. Substituting (1.2) into (1.1), we get the Timoshenko equations with thermodiffusion effects

$$\begin{aligned} \rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x&=0,\\ \rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)-\gamma\theta_x-\beta C_x&=0,\\ \rho_3\theta_t+\varpi C_t+q_x-\gamma\psi_{tx}&=0,\\ C_t+\eta_x&=0, \end{aligned} \tag{1.3}$$

where $(x,t)\in(0,l)\times\mathbb{R}^+$. We shall now formulate a different alternative form where the chemical potential P is considered as a state variable instead of the concentration C. This alternative form is obtained by substituting the last equation of (1.2) into $(1.3)_{2-4}$

$$\begin{aligned} \rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x&=0,\\ \rho_2\psi_{tt}-\alpha\psi_{xx}+\kappa(\varphi_x+\psi)-\gamma_1\theta_x-\gamma_2 P_x&=0,\\ c\theta_t+dP_t+q_x-\gamma_1\psi_{tx}&=0,\\ d\theta_t+rP_t+\eta_x-\gamma_2\psi_{tx}&=0, \end{aligned} \tag{1.4}$$

where

$$\alpha=b-\frac{\beta^2}{\varrho},\quad \gamma_1=\gamma+\frac{\beta\varpi}{\varrho},\quad \gamma_2=\frac{\beta}{\varrho},\quad c=\rho_3+\frac{\varpi^2}{\varrho},\quad d=\frac{\varpi}{\varrho},\quad r=\frac{1}{\varrho}$$

are physical positive constants.

Aouadi et al. [3] proved the well posedness of (1.4) with Dirichlet or Neumann boundary conditions when the temperature and the mass diffusion follow the Fourier's law and the Fick's law, respectively,

$$q=-K\theta_x,\qquad \eta=-\hbar P_x. \tag{1.5}$$

Then, they showed, without assuming the well-known equal wave speeds condition, the lack of exponential stability for the Neumann problem; meanwhile, one linear frictional damping is strong enough to guarantee the exponential stability for the Dirichlet problem.

The drawback of the Fourier law lies in the physical paradox of infinite propagation speed of (thermal) signals, a typical side-effect of parabolicity. A different model, removing this paradox, is the Cattaneo's law [8], namely the differential perturbation of (1.5)

$$\tau_0 q_t+q=-K\theta_x,\qquad \tau_1\eta_t+\eta=-\hbar P_x, \tag{1.6}$$

where the relaxation time τ_0 describes the time lag in the response of the heat flux to a gradient in the temperature, while τ_1 is the diffusion relaxation time, which will ensure that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to the other. In [2,6], Cattaneo's law is applied instead to Fick's law in order to remove the physical paradox that affects such a model.

Inserting (1.6) into (1.4), we get the thermodiffusion Timoshenko beam equations with second sound in $(x,t)\in(0,l)\times\mathbb{R}^+$

$$\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x=0, \tag{1.7}$$

$$\rho_2\psi_{tt}-\alpha\psi_{xx}+\kappa(\varphi_x+\psi)-\gamma_1\theta_x-\gamma_2 P_x=0, \tag{1.8}$$

$$
\begin{aligned}
c\theta_t + dP_t + q_x - \gamma_1\psi_{tx} &= 0, && (1.9)\\
\tau_0 q_t + q + K\theta_x &= 0, && (1.10)\\
rP_t + d\theta_t + \eta_x - \gamma_2\psi_{tx} &= 0, && (1.11)\\
\tau_1\eta_t + \eta + \hbar P_x &= 0, && (1.12)
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
&\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\\
&\theta(x,0)=\theta_0(x),\ P(x,0)=P_0(x),\ q(x,0)=q_0(x),\ \eta(x,0)=\eta_0(x) \qquad (1.13)
\end{aligned}
$$

and the boundary conditions

$$
\begin{aligned}
&\varphi(0,t)=\varphi(l,t)=\psi_x(0,t)=\psi_x(l,t)=0,\\
&\theta(0,t)=\theta(l,t)=P(0,t)=P(l,t)=0, \quad t\geq 0. \qquad (1.14)
\end{aligned}
$$

A natural question is under what necessary and sufficient conditions, the semigroup generated by Timoshenko systems is exponentially stable. Since there is not much literature on Timoshenko system with mass diffusion, the answer is known only in the case of Timoshenko systems with or without thermal effect. Indeed, in the case where the thermal effect is absent, the Timoshenko system is uniformly stable for weak solutions if

$$
\chi = \frac{\kappa}{\rho_1} - \frac{\alpha}{\rho_2} = 0. \qquad (1.15)
$$

Consequently, the number χ plays an important role in the asymptotic behavior of solutions to Timoshenko systems with or without thermal effect. Of course, since the Timoshenko system is a two by two system hyperbolic equations, many authors showed that the dissipation given by some damping terms is strong enough to stabilize the system exponentially regardless of whether the propagation velocities are equal or not (see, e.g., [10,11,14–16,21]).

When we consider the thermal effect in Timoshenko beam according to Fourier's law, $\tau_0 = 0$, Fernández Sare et al. [12] proved that the exponential stability can never occur when $\chi = 0$. Muñoz Rivera and Racke [17] proved several exponential decay results for the linearized system and a non-exponential stability result for the case of different wave speeds. Moreover, when $\chi \neq 0$, the authors showed the polynomial stability. Aouadi and Soufyane [5] showed that the dissipation product by the memory effect working at the boundary is sufficiently strong to produce a general decay obtained without imposing the condition (1.15). In [1], Almeida Júnior et al. considered a Timoshenko beam acting on shear force and proved that the resulting model is exponentially stable if and only if (1.15) holds.

Another option to remove the infinity speed of propagation is to consider the Cattaneo's law for the heat flux. In this case, the Timoshenko system is given by

$$
\begin{aligned}
\rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x &= 0,\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \gamma\theta_x &= 0,\\
\rho_3\theta_t + q_x + \gamma\psi_{xt} &= 0,\\
\tau_0 q_t + \beta q + \theta_x &= 0. && (1.16)
\end{aligned}
$$

The above model was studied by several authors, see, for example, [12,20,22] to quote a few. Fernández Sare and Racke [12] showed that, in the absence of the extra frictional damping, the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law. Precisely, it has been shown that (1.18) is no longer exponentially stable even if (1.15) holds. However, Santos et al. [22] proved that (1.18) is exponentially stable if and only if

$$
\chi_{\tau_0} = \left(\tau_0 - \frac{\rho_1}{\kappa\rho_3}\right)\left(\rho_2 - \frac{\rho_1 b}{\kappa}\right) - \frac{\rho_1\gamma^2\tau_0}{\kappa\rho_3} = 0. \qquad (1.17)
$$

Note that when $\tau_0 = 0$, the Cattaneo's law $(1.6)_1$ turns into the Fourier's law $(1.5)_1$ and the condition (1.17) is equivalent to (1.15), which tells at once that

$$\chi_{\tau_0} = 0 \Longleftrightarrow \chi = 0.$$

It is worth noting that Aouadi and Boulehmi [4] designed one feedback controller to make the solutions to non-uniform Timoshenko beam acting on shear force within Cattaneo's law decay exponentially regardless the value of χ_{τ_0}.

Later, Dell'Oro and Vittorino Pata [9] studied the same problem for the system

$$\begin{aligned} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x &= 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\theta_x &= 0, \\ \rho_3\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)ds + \gamma\psi_{xt} &= 0, \end{aligned} \tag{1.18}$$

describing a Timoshenko beam coupled with Gurtin–Pipkin heat conduction law for the heat flux. Then, they introduced the stability number

$$\chi_g = \left(\frac{\beta}{g(0)} - \frac{\rho_1}{\kappa\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{\kappa}\right) - \frac{\beta}{g(0)}\frac{\rho_1\gamma^2}{\kappa b\rho_3}. \tag{1.19}$$

They proved that the corresponding semigroup is exponentially stable if and only if $\chi_g = 0$. In particular, they showed that the corresponding semigroup remains stable (although not exponentially stable) also when $\chi_g \neq 0$. Moreover, they generalized previously known results on the Fourier–Timoshenko and the Cattaneo–Timoshenko beam models.

In this present work, we consider the Timoshenko system with mass diffusion and second sound effects, given by (1.7)–(1.14). We introduce some new numbers by the physical coefficients that characterize the exponential decay

$$\begin{aligned} \chi_0 &:= \big(d\gamma_2 - r\gamma_1\big)\frac{K}{\tau_0\gamma_1} - \big(d\gamma_1 - c\gamma_2\big)\frac{\hbar}{\tau_1\gamma_2}, \\ \chi_1 &:= \xi\big(\rho_2 - \frac{\alpha\rho_1}{\kappa}\big) - \left(\frac{\tau_0\gamma_1^2}{K} + \frac{\tau_1\gamma_2^2}{\hbar}\right)(1-\xi), \end{aligned}$$

where $\xi = 1 - \frac{\rho_1\Gamma}{\delta\kappa}$, $\delta = cr - d^2 > 0$ and $\Gamma = \big(d\gamma_2 - r\gamma_1\big)\frac{K}{\tau_0\gamma_1} = \big(d\gamma_1 - c\gamma_2\big)\frac{\hbar}{\tau_1\gamma_2}$ if $\chi_0 = 0$. If $\chi_0 = 0$, we prove that the semigroup associated with (1.7)–(1.14) is exponentially stable if and only if $\chi_1 = 0$. Otherwise, there is a lack of exponential stability. In this case, we prove that the semigroup decays as $1/\sqrt{t}$.

The method we use to show the lack of exponential stability is based on Gearhart–Herbst–Prüss–Huang theorem to dissipative systems. See also [13,19].

Theorem 1.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then, $S(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \varrho(\mathcal{A})$ and*

$$\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda\, I - \mathcal{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} < \infty, \tag{1.20}$$

where $\varrho(\mathcal{A})$ is the resolvent set of the linear operator $\mathcal{A}$.

On the other hand, to show the polynomial stability we use Theorem 2.4 in [7].

Theorem 1.2. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space with generator $\mathcal{A}$ such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. Then,*

$$\frac{1}{|\lambda|^\alpha}\|(i\lambda\, I - \mathcal{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \leq C, \quad \forall\lambda \in \mathbb{R} \Longleftrightarrow \|S(t)\mathcal{A}^{-1}\|_{\mathscr{L}(\mathscr{H})} \leq \frac{C}{t^{\frac{1}{\alpha}}}. \tag{1.21}$$

Throughout the paper, C will always stand for a generic positive constant.

2. Well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (1.7)–(1.14) using semigroup theory. Introducing the vector function $U = (\varphi, v, \psi, \phi, \theta, q, P, \eta)^T$, where $v = \varphi_t$ and $\phi = \psi_t$, we consider the following Hilbert space

$$\mathcal{H} = H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L^2(0,l) \times L_*^2(0,l) \times L^2(0,l) \times L_*^2(0,l),$$

where

$$L_*^2(0,l) := \left\{ f \in L^2(0,l);\ \int_0^l f(x)\mathrm{d}x = 0 \right\} \quad \text{and} \quad H_*^1(0,l) := H^1(0,l) \cap L_*^2(0,l),$$

provided with the following inner product

$$\begin{aligned}\left\langle U_1, U_2 \right\rangle_{\mathcal{H}} &= \rho_1 \int_0^l v_1 \overline{v}_2 \mathrm{d}x + \rho_2 \int_0^l \phi_1 \overline{\phi}_2 \mathrm{d}x + \alpha \int_0^l \psi_{1,x} \overline{\psi}_{2,x} \mathrm{d}x \\ &+ \kappa \int_0^l (\varphi_{1,x} + \psi_1)\overline{(\varphi_{2,x} + \psi_2)} \mathrm{d}x + (c - d^2/r) \int_0^l \theta_1 \overline{\theta}_2 \mathrm{d}x \\ &+ \int_0^l \left(\frac{d}{\sqrt{r}} \theta_1 + \sqrt{r} P_1 \right) \overline{\left(\frac{d}{\sqrt{r}} \theta_2 + \sqrt{r} P_2 \right)} \mathrm{d}x \\ &+ \frac{\tau_0}{K} \int_0^l q_1 \overline{q}_2 \mathrm{d}x + \frac{\tau_1}{\hbar} \int_0^l \eta_1 \overline{\eta}_2 \mathrm{d}x\end{aligned}$$

with $cr - d^2 > 0$ for all $U_1 = (\varphi_1, v_1, \psi_1, \phi_1, \theta_1, q_1, P_1, \eta_1)^T$ and $U_2 = (\varphi_2, v_2, \psi_2, \phi_2, \theta_2, q_2, P_2, \eta_2)^T$ in $\mathcal{H}$ and norm given by

$$||U_1||_{\mathcal{H}}^2 := \left\langle U_1, U_1 \right\rangle_{\mathcal{H}}. \tag{2.1}$$

In order to prove the existence and uniqueness of solutions, we will use the semigroup theory [18]. Then, the system (1.7)–(1.12) can be rewritten as follows:

$$\begin{aligned} &U_t = \mathcal{A}U, \quad t > 0, \\ &U(0) = U_0, \end{aligned} \tag{2.2}$$

where $U = \big(\varphi, \varphi_t, \psi, \psi_t, \theta, q, P, \eta\big)^T$, $U_0 = \big(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, P_0, \eta_0\big)^T$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the operator defined by

$$\mathcal{A} := \left(\begin{array}{cccc} 0 & I(\cdot) & 0 & 0 \\ \kappa\rho_1^{-1}(\cdot)_{xx} & 0 & \kappa\rho_1^{-1}(\cdot)_x & 0 \\ 0 & 0 & 0 & I(\cdot) \\ -\kappa\rho_2^{-1}(\cdot)_x & 0 & \alpha\rho_2^{-1}(\cdot)_{xx} - \kappa\rho_2^{-1} I(\cdot) & 0 \\ 0 & 0 & 0 & -\delta^{-1}(\gamma_2 d - \gamma_1 r)(\cdot)_x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^{-1}(\gamma_1 d - \gamma_2 c)(\cdot)_x \\ 0 & 0 & 0 & 0 \end{array} \right.$$

$$
\begin{matrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\gamma_1\rho_2^{-1}(\cdot)_x & 0 & \gamma_2\rho_2^{-1}(\cdot)_x & 0 \\
0 & -r\delta^{-1}(\cdot)_x & 0 & d\delta^{-1}(\cdot)_x \\
-K\tau_0^{-1}(\cdot)_x & -\tau_0^{-1}I(\cdot) & 0 & 0 \\
0 & d\delta^{-1}(\cdot)_x & 0 & -c\delta^{-1}(\cdot)_x \\
0 & 0 & -\hbar\tau_1^{-1}(\cdot)_x & -\tau_1^{-1}I(\cdot)
\end{matrix}
\right)
$$

where $\delta := cr - d^2 > 0$ and $I(\cdot)$ is the identity operator. The domain of $\mathcal{A}$ is given by

$$
\begin{aligned}
D(\mathcal{A}) = \Big\{ & U \in \mathcal{H};\ \varphi \in H_0^1(0,l) \cap H^2(0,l), \\
& \psi \in H_*^1(0,l) \cap H^2(0,l),\ v, \theta, P \in H_0^1(0,l),\ q, \eta, \phi \in H_*^1(0,l) \Big\}.
\end{aligned}
$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$. We have the following existence and uniqueness result.

Theorem 2.1. *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.2). Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. The result follows from Lumer–Phillips Theorem provided we prove that $\mathcal{A}$ is a maximal monotone operator. In what follows, we prove that $\mathcal{A}$ is monotone. For any $U \in D(\mathcal{A})$, and using the inner product, we obtain

$$
\langle \mathcal{A}U, U \rangle_{\mathscr{H}} = -\frac{1}{K}\int_0^l q^2 \mathrm{d}x - \frac{1}{\hbar}\int_0^l \eta^2 \mathrm{d}x. \tag{2.3}
$$

Since $K > 0$ and $\hbar > 0$, it follows that $\langle \mathcal{A}U, U \rangle_{\mathscr{H}} \le 0$, which implies that $\mathcal{A}$ is dissipative. Next, we prove that the operator $I - \mathcal{A}$ is surjective. Given $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)^T \in \mathcal{H}$, we prove that there exists $U \in D(\mathcal{A})$ satisfying

$$
U - \mathcal{A}U = G, \tag{2.4}
$$

that is,

$$
\begin{aligned}
-v + \varphi &= g_1 \in H_0^1(0,l), \\
-\kappa(\varphi_x + \psi)_x + \rho_1 v &= \rho_1 g_2 \in L^2(0,l), \\
-\phi + \psi &= g_3 \in H_*^1(0,l), \\
-\alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x + \rho_2\phi &= \rho_2 g_4 \in L_*^2(0,l), \\
(d\gamma_2 - r\gamma_1)\phi_x + r q_x - d\eta_x + \delta\theta &= \delta g_5 \in L^2(0,l), \\
(1 + \tau_0)q + K\theta_x &= \tau_0 g_6, \in L^2(0,l), \\
(d\gamma_1 - c\gamma_2)\phi_x - d q_x + c\eta_x + \delta P &= \delta g_7 \in L^2(0,l), \\
(1 + \tau_1)\eta + \hbar P_x &= \tau_1 g_8 \in L^2(0,l).
\end{aligned} \tag{2.5}
$$

Suppose φ, ψ, q and η are found with the appropriate regularity. Then, $(2.5)_1$, $(2.5)_3$, $(2.5)_6$ and $(2.5)_8$ yield

$$\begin{aligned} v &= \varphi - g_1 \in H_0^1(0,l), \\ \phi &= \psi - g_3 \in H_*^1(0,l), \\ \theta_x &= \frac{\tau_0}{K} g_6 - \frac{1+\tau_0}{K} q \in L^2(0,l), \\ P_x &= \frac{\tau_1}{\hbar} g_8 - \frac{1+\tau_1}{\hbar} \eta \in L^2(0,l). \end{aligned} \tag{2.6}$$

From $(2.6)_{3,4}$, we have

$$\theta(x) = \frac{\tau_0}{K} \int_0^x g_6 \, \mathrm{d}x - \frac{1+\tau_0}{K} \int_0^x q \, \mathrm{d}x, \qquad P(x) = \frac{\tau_1}{\hbar} \int_0^x g_8 \, \mathrm{d}x - \frac{1+\tau_1}{\hbar} \int_0^x \eta \, \mathrm{d}x, \tag{2.7}$$

then $\theta(0) = \theta(1) = 0$ and $P(0) = P(1) = 0$. By using (2.6) and (2.7), it can easily be shown that φ, ψ, q and η satisfy

$$\begin{aligned} -\kappa(\varphi_x + \psi)_x + \rho_1 \varphi &= \rho_1(g_1 + g_2) \in L^2(0,l) \\ -\alpha\psi_{xx} + \kappa(\varphi_x + \psi) + \rho_2\psi + \frac{1+\tau_0}{K}\gamma_1 q + \frac{1+\tau_1}{\hbar}\gamma_2 \eta &= h_1 \in L_*^2(0,l) \\ -(d\gamma_2 - r\gamma_1)\psi_x - r q_x + d\eta_x + \delta \frac{1+\tau_0}{K} \int_0^x q(y)\mathrm{d}y &= h_2 \in L^2(0,l), \\ -(d\gamma_1 - c\gamma_2)\psi_x + d q_x - c\eta_x + \delta \frac{1+\tau_1}{\hbar} \int_0^x \eta(y)\mathrm{d}y &= h_3 \in L^2(0,l), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} h_1 &= \rho_2(g_3 + g_4) + \frac{\tau_0}{K}\gamma_1 g_6 + \frac{\tau_1}{\hbar}\gamma_2 g_8 \\ h_2 &= -(d\gamma_2 - r\gamma_1) g_{3,x} - \delta g_5 + \delta \frac{\tau_0}{K} \int_0^x g_6(y)\mathrm{d}y, \\ h_3 &= -(d\gamma_1 - c\gamma_2) g_{3,x} - \delta g_7 + \delta \frac{\tau_1}{\hbar} \int_0^x g_8(y)\mathrm{d}y. \end{aligned} \tag{2.9}$$

To solve (2.8) we consider the variational formulation

$$B\big((\varphi,\psi,q,\eta),(\tilde{\varphi},\tilde{\psi},\tilde{q},\tilde{\eta})\big) = F\big(\tilde{\varphi},\tilde{\psi},\tilde{q},\tilde{\eta}\big), \tag{2.10}$$

where $B : \left[H_0^1(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L_*^2(0,l)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B\big((\varphi,\psi,q,\eta),(\tilde{\varphi},\tilde{\psi},\tilde{q},\tilde{\eta})\big) &= \kappa \int_0^l (\varphi_x + \psi)(\tilde{\varphi}_x + \tilde{\psi}) \, \mathrm{d}x \\ &+ \frac{r\gamma_1(1+\tau_0)}{K(d\gamma_2 - r\gamma_1)} \int_0^l q\tilde{q} \, \mathrm{d}x + \frac{c\gamma_2(1+\tau_1)}{\hbar(d\gamma_1 - c\gamma_2)} \int_0^l \eta\tilde{\eta}\mathrm{d}x + \rho_1 \int_0^l \varphi\tilde{\varphi} \, \mathrm{d}x \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\alpha\int_0^l \psi_x\tilde{\psi}_x\,\mathrm{d}x+\rho_2\int_0^l \psi\tilde{\psi}\,\mathrm{d}x-\frac{1+\tau_0}{K}\gamma_1\int_0^l q\tilde{\psi}\mathrm{d}x\\
&-\frac{\gamma_2(1+\tau_1)}{\hbar}\int_0^l \eta\tilde{\psi}\,\mathrm{d}x+\frac{\gamma_1(1+\tau_0)}{K}\int_0^l \psi\tilde{q}\,\mathrm{d}x-\frac{d\gamma_1(1+\tau_0)}{K(d\gamma_2-r\gamma_1)}\int_0^l \eta\tilde{q}\,\mathrm{d}x\\
&+\frac{\delta\gamma_1}{d\gamma_2-r\gamma_1}\Big(\frac{1+\tau_0}{K}\Big)^2\int_0^l\Big(\int_0^x q(y)\mathrm{d}y\int_0^x\tilde{q}(y)\mathrm{d}y\Big)\,\mathrm{d}x\\
&+\frac{\gamma_2(1+\tau_1)}{\hbar}\int_0^l \psi\tilde{\eta}\,\mathrm{d}x-\frac{d\gamma_2(1+\tau_1)}{\hbar(d\gamma_1-c\gamma_2)}\int_0^l q\tilde{\eta}\,\mathrm{d}x\\
&+\frac{\delta\gamma_2}{d\gamma_1-c\gamma_2}\Big(\frac{1+\tau_1}{\hbar}\Big)^2\int_0^l\Big(\int_0^x \eta(y)\mathrm{d}y\int_0^x\tilde{\eta}(y)\mathrm{d}y\Big)\,\mathrm{d}x
\end{aligned}
$$

and $F:\big[H_0^1(0,l)\times H_*^1(0,l)\times L_*^2(0,l)\times L_*^2(0,l)\big]\to\mathbb{R}$ is the linear form

$$
\begin{aligned}
F(\tilde{\varphi},\tilde{\psi},\tilde{q},\tilde{\eta})=&\int_0^l \rho_1(g_1+g_2)\tilde{\varphi}\,\mathrm{d}x+\int_0^l h_1\tilde{\psi}\,\mathrm{d}x\\
&+\frac{\gamma_1(1+\tau_0)}{K(d\gamma_2-r\gamma_1)}\int_0^l h_2\int_0^x\tilde{q}(y)\,\mathrm{d}y\,\mathrm{d}x+\frac{\gamma_2(1+\tau_1)}{\hbar(d\gamma_1-c\gamma_2)}\int_0^l h_3\int_0^x\tilde{\eta}(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Now, for $V=H_0^1(0,l)\times H_*^1(0,l)\times L_*^2(0,l)\times L_*^2(0,l)$, one can easily see that B and F are bounded and B is coercive. Consequently, by Lax–Milgram Lemma, system (2.8) has a unique solution

$$\varphi\in H_0^1(0,l),\quad \psi\in H_*^1(0,l),\quad q,\ \eta\in L_*^2(0,l).$$

Substituting φ, ψ, q and η in $(2.5)_1$, $(2.5)_3$, $(2.5)_6$, and $(2.5)_8$, respectively, we obtain

$$v\in H_0^1(0,l),\quad \phi\in H_*^1(0,l),\quad \theta,\ P\in H_0^1(0,l).$$

Now, if $(\tilde{\varphi},\tilde{q},\tilde{\eta})\equiv(0,0,0)\in H_0^1(0,l)\times L_*^2(0,l)\times L_*^2(0,l)$, then (2.10) reduces to

$$
\begin{aligned}
&\kappa\int_0^l(\varphi_x+\psi)\tilde{\psi}\,\mathrm{d}x+\alpha\int_0^l\psi_x\tilde{\psi}_x\,\mathrm{d}x+\rho_2\int_0^l\psi\tilde{\psi}\,\mathrm{d}x-\frac{1+\tau_0}{K}\gamma_1\int_0^l q\tilde{\psi}\\
&-\frac{1+\tau_1}{\hbar}\gamma_2\int_0^l\eta\tilde{\psi}\,\mathrm{d}x=\int_0^l h_1\tilde{\psi}\,\mathrm{d}x,\quad\forall\tilde{\psi}\in H_*^1(0,l),
\end{aligned}
\tag{2.11}
$$

which implies

$$-\alpha\psi_{xx}=-\kappa(\varphi_x+\psi)-\rho_2\psi+\frac{1+\tau_0}{K}\gamma_1 q+\frac{1+\tau_1}{\hbar}\gamma_2\eta+h_1\in L^2(0,l).\tag{2.12}$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$\psi\in H^2(0,l)\cap H_*^1(0,l).$$

Moreover, (2.11) is also true for any $u \in C^1([0,l]) \subset H^1_*(0,l)$. Hence, we have

$$\alpha \int_0^l \psi_x u_x \mathrm{d}x + \int_0^l \Big(\kappa(\varphi_x + \psi) + \rho_2 \psi - \frac{1+\tau_0}{K}\gamma_1 q - \frac{1+\tau_1}{\hbar}\gamma_2 \eta - h_1\Big) u \mathrm{d}x = 0$$

for all $u \in C^1([0,l])$. Thus, using integration by parts and bearing in mind (2.12), we obtain

$$\psi_x(l)u(l) - \psi_x(0)u(0) = 0, \quad \forall u \in C^1([0,l]).$$

Therefore, $\psi_x(0) = \psi_x(l) = 0$. In the same way, if $(\tilde{\psi}, \tilde{q}, \tilde{\eta}) \equiv (0,0,0) \in H^1_*(0,l) \times L^2_*(0,l) \times L^2_*(0,l)$, then we obtain

$$\varphi \in H^2(0,l) \cap H^1_0(0,l).$$

Recalling $\delta = cr - d^2 > 0$, the resolution of the system

$$-rq_x + d\eta_x = (d\gamma_2 - r\gamma_1)\psi_x - \delta \frac{1+\tau_0}{K} \int_0^x q(y)\mathrm{d}y + h_2 \in L^2(0,l),$$

$$dq_x - c\eta_x = (d\gamma_1 - c\gamma_2)\psi_x - \delta \frac{1+\tau_1}{\hbar} \int_0^x \eta(y)\mathrm{d}y + h_3 \in L^2(0,l),$$

gives

$$q, \ \eta \in H^1_*(0,l).$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of a unique $U \in D(\mathcal{A})$ such that (2.4) is satisfied. Consequently, $\mathcal{A}$ is a maximal operator. Hence, the result of Theorem 2.1 follows from Lumer–Phillips Theorem (see [18]). □

The previous lemmas lead to the next theorem.

Theorem 2.2. *The operator $\mathcal{A}$ generates a C_0-semigroup of contractions on the phase-space $\mathcal{H}$.*

Proof. Since the operator $\mathcal{A}$ is maximal dissipative in $\mathcal{H}$ and $D(\mathcal{A})$ is densely defined in $\mathcal{H}$, the proof follows from the Lumer–Phillips Corollary to the Hille–Yosida Theorem [18]. □

3. Exponential stability

We introduced the total energy of the system (1.7)–(1.14) given by

$$E(t) = \frac{\rho_1}{2}\int_0^l |\varphi_t|^2 \mathrm{d}x + \frac{\rho_2}{2}\int_0^l |\psi_t|^2 \mathrm{d}x + \frac{\alpha}{2}\int_0^l |\psi_x|^2 \mathrm{d}x + \frac{\kappa}{2}\int_0^l |\varphi_x + \psi|^2 \mathrm{d}x$$
$$+ \frac{1}{2}(c - d^2/r)\int_0^l |\theta|^2 \mathrm{d}x + \frac{1}{2}\int_0^l \left|\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right|^2 \mathrm{d}x$$
$$+ \frac{\tau_0}{2K}\int_0^l |q|^2 \mathrm{d}x + \frac{\tau_1}{2\hbar}\int_0^l |\eta|^2 \mathrm{d}x, \tag{3.1}$$

which is positive definite since $d^2 < cr$ and satisfies the dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\frac{1}{K}\int_0^l |q|^2\mathrm{d}x - \frac{1}{\hbar}\int_0^l |\eta|^2\mathrm{d}x, \quad t \geq 0. \tag{3.2}$$

Here, we will provide the necessary conditions on the coefficients ensuring the exponential stability of our model. From [19], we have to show that the resolvent is uniformly bounded over the imaginary axes. Note that, for any $U = (\varphi, v, \psi, \phi, \theta, q, P, \eta)^T \in D(\mathcal{A})$, the resolvent of our model is given by the equation $(i\lambda I - \mathcal{A})U = F$, i.e.,

$$i\lambda\varphi - v = f_1, \tag{3.3}$$
$$i\lambda\rho_1 v - \kappa(\varphi_x + \psi)_x = \rho_1 f_2, \tag{3.4}$$
$$i\lambda\psi - \phi = f_3, \tag{3.5}$$
$$i\lambda\rho_2\phi - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = \rho_2 f_4, \tag{3.6}$$
$$i\lambda\delta\theta + rq_x - d\eta_x + (d\gamma_2 - r\gamma_1)\phi_x = \delta f_5, \tag{3.7}$$
$$i\lambda\tau_0 q + q + K\theta_x = \tau_0 f_6, \tag{3.8}$$
$$i\lambda\delta P + c\eta_x - dq_x + (d\gamma_1 - c\gamma_2)\phi_x = \delta f_7, \tag{3.9}$$
$$i\lambda\tau_1\eta + \eta + \hbar P_x = \tau_1 f_8, \tag{3.10}$$

where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. From (2.3), it is easy to see that we have

$$\mathrm{Re}\Big\langle (i\lambda I - \mathcal{A})U, U\Big\rangle_{\mathcal{H}} = \frac{1}{K}\int_0^l |q|^2\mathrm{d}x + \frac{1}{\hbar}\int_0^l |\eta|^2\mathrm{d}x. \tag{3.11}$$

Furthermore,

$$\int_0^l |q|^2\mathrm{d}x + \int_0^l |\eta|^2\mathrm{d}x \leq C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.12}$$

Our starting point is to show that $i\mathbb{R} \subset \varrho(\mathcal{A})$. Note that from Theorem 2.1, one can deduce that $0 \in \varrho(\mathcal{A})$, therefore $\mathcal{A}^{-1}$ is bounded and it is a bijection between $\mathscr{H}$ and the domain $D(\mathcal{A})$. Since $D(\mathcal{A})$ has compact embedding into $\mathcal{H}$, it follows that $\mathcal{A}^{-1}$ is a compact operator, which implies that the spectrum of $\mathcal{A}$ is discrete.

Lemma 3.1. *Under the above notations, we have that* $i\mathbb{R} \subset \varrho(\mathcal{A})$.

Proof. Let us suppose that $\mathcal{A}$ has an imaginary eigenvalue. Then, we have that $\mathcal{A}U = i\lambda U$, $\lambda \in \mathbb{R}$. From (2.3), we get $q = \eta = 0$, which implies that $\theta = P = 0$. From Eqs. (3.5) and (3.7), we conclude that $\psi = \phi = 0$. Therefore, from (3.7) we get $\varphi = 0$. This implies that $U = 0$. But this is a contradiction, therefore there are not imaginary eigenvalues. □

Before we prove the technical lemmas that will support our propositions, we present the following definition.

Definition 3.1. Let $U = (u_1, u_2, u_3, u_4, u_5, u_6, , u_7, u_8)^T \in D(\mathcal{A})$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$. Then, we define the functional class $\mathfrak{R}$ given by

$$\mathfrak{R} := \left\{ \mathcal{R} = \int_0^l f_i u_j \,\mathrm{d}x;\ \big|\mathcal{R}\big| \leq C||U||_{\mathcal{H}}||F||_{\mathcal{H}},\ i, j \in \{1, \dots, 8\} \right\}. \tag{3.13}$$

Lemma 3.2. *Let $(\varphi, v, \psi, \phi, \theta, q, P, \eta)$ be a solution of system (1.7)–(1.14). There is a positive constant C independent of λ such that*

$$\int_0^l |\theta|^2 \mathrm{d}x + \int_0^l |P|^2 \mathrm{d}x \le C||\phi||_{L^2}\Big(||q||_{L^2} + ||\eta||_{L^2}\Big) + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.14}$$

Proof. Integrating Eq. (3.7) over $[x, l] \subset [0, l]$ and multiplying by $\int_0^l \overline{q}\mathrm{d}x$, we get

$$\begin{aligned}\mathcal{R} = i\lambda\delta \int_x^l \theta ds \int_0^l \overline{q}\mathrm{d}x + \big[rq(l) - d\eta(l) + (d\gamma_2 - r\gamma_1)\phi(l)\big] \int_0^l \overline{q}\mathrm{d}x \\ -rq \int_0^l \overline{q}\mathrm{d}x + d\eta \int_0^l \overline{q}\mathrm{d}x - (d\gamma_2 - r\gamma_1)\phi \int_0^l \overline{q}\mathrm{d}x,\end{aligned} \tag{3.15}$$

where $\mathcal{R} \in \mathfrak{R}$. Now integrating Eq. (3.8) and multiplying by $\frac{\delta}{\tau_0}\int_x^l \theta ds$

$$i\lambda\delta \int_x^l \theta ds \int_0^l \overline{q}\mathrm{d}x = \frac{\delta}{\tau_0}\int_x^l \theta ds \int_0^l \overline{q}\mathrm{d}x + \mathcal{R}. \tag{3.16}$$

From there, it follows that

$$\begin{aligned}\big[rq(l) - d\eta(l) + (d\gamma_2 - r\gamma_1)\phi(l)\big] \int_0^l \overline{q}\mathrm{d}x = -\frac{\delta}{\tau_0}\int_x^l \theta ds \int_0^l \overline{q}\mathrm{d}x + rq\int_0^l \overline{q}\mathrm{d}x \\ -d\eta \int_0^l \overline{q}\mathrm{d}x + (d\gamma_2 - r\gamma_1)\phi \int_0^l \overline{q}\mathrm{d}x + \mathcal{R}.\end{aligned}$$

Integrating over $[0, l]$ and using (3.12), we can conclude that

$$\begin{aligned}\left|\big[rq(l) - d\eta(l) + (d\gamma_2 - r\gamma_1)\phi(l)\big] \int_0^l \overline{q}\mathrm{d}x\right| \le C||\theta||_{L^2}||q||_{L^2} + C||\phi||_{L^2}||q||_{L^2} \\ + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.\end{aligned} \tag{3.17}$$

Similarly, integrating over $[0, x] \subset [0, l]$ Eq. (3.8), after multiplying by θ and integrating over $[0, l]$ we obtain

$$\underbrace{- i\lambda\tau_0 \int_0^l \int_0^x \overline{q} ds\theta \mathrm{d}x}_{I:=} + \int_0^l \int_0^x \overline{q} ds\theta \mathrm{d}x + K\int_0^l |\theta|^2 \mathrm{d}x = \mathcal{R}. \tag{3.18}$$

Using Eq. (3.7), we get

$$I = i\lambda\tau_0 \int_0^l \int_0^x \overline{q} ds\theta \mathrm{d}x$$

$$
\begin{aligned}
&= -\frac{\tau_0}{\delta}\int_0^l\int_0^x \overline{q}ds\big[rq_x - d\eta_x + (d\gamma_2 - r\gamma_1)\phi_x - \delta f_5\big]\mathrm{d}x \\
&= -\frac{\tau_0}{\delta}\big[rq(l) - d\eta(l) + (d\gamma_2 - r\gamma_1)\phi(l)\big]\int_0^l \overline{q}\mathrm{d}x + \frac{\tau_0 r}{\delta}\int_0^l |q|^2\mathrm{d}x \\
&\quad -\frac{\tau_0 d}{\delta}\int_0^l \overline{q}\eta\mathrm{d}x + \frac{\tau_0}{\delta}(d\gamma_2 - r\gamma_1)\int_0^l \overline{q}\phi\mathrm{d}x + \mathcal{R}.
\end{aligned}
$$

Substituting I in (3.18), we have

$$
\begin{aligned}
K\int_0^l |\theta|^2\mathrm{d}x &= -\frac{\tau_0}{\delta}\big[rq(l) - d\eta(l) + (d\gamma_2 - r\gamma_1)\phi(l)\big]\int_0^l \overline{q}\mathrm{d}x \\
&+ \frac{\tau_0 r}{\delta}\int_0^l |q|^2\mathrm{d}x - \frac{\tau_0 d}{\delta}\int_0^l \overline{q}\eta\mathrm{d}x + \frac{\tau_0}{\delta}(d\gamma_2 - r\gamma_1)\int_0^l \overline{q}\phi\mathrm{d}x \\
&- \int_0^l\int_0^x \overline{q}ds\theta\mathrm{d}x + \mathcal{R}.
\end{aligned}
$$

From (3.17) and (3.12), we have

$$
\int_0^l |\theta|^2\mathrm{d}x \le C||\theta||_{L^2}||q||_{L^2} + C||\phi||_{L^2}||q||_{L^2} + C||\eta||_{L^2}||q||_{L^2} + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}
$$

which reads

$$
\int_0^l |\theta|^2\mathrm{d}x \le C||\phi||_{L^2}||q||_{L^2} + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.19}
$$

On the other hand, combining Eqs. (3.7) and (3.9), we have

$$
i\lambda c\theta + i\lambda dP + q_x - \gamma_1\phi_x = cf_5 + df_7. \tag{3.20}
$$

Similarly, one can get

$$
\int_0^l |P|^2\mathrm{d}x \le C||\phi||_{L^2}\Big(||q||_{L^2} + ||\eta||_{L^2}\Big) + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.21}
$$

This concludes the proof. □

Lemma 3.3. *Let $(\varphi, v, \psi, \phi, \theta, q, P, \eta)$ be a solution of system (1.7)–(1.14). There is a positive constant C independent of λ such that for all $\varepsilon > 0$*

$$
\int_0^l |\phi|^2\mathrm{d}x \le \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2} + ||P||_{L^2}\Big) + C||U||_{\mathcal{H}}||F||_{\mathcal{H}} \tag{3.22}
$$

and

$$\int_0^l |\psi_x|^2 \mathrm{d}x \leq \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)+\frac{\varepsilon}{\lambda^2}\int_0^l |\varphi_x+\psi|^2\mathrm{d}x$$
$$+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.23}$$

Proof. Combining Eqs. (3.7) and (3.9), we have

$$i\lambda d\theta + i\lambda rP + \eta_x - \gamma_2\phi_x = df_5 + rf_7. \tag{3.24}$$

Multiplying by $\int_0^x \overline{\phi} ds$, we get

$$\gamma_2\int_0^l |\phi|^2\mathrm{d}x = \int_0^l \eta_x \int_0^x \overline{\phi}dsdx + i\lambda\int_0^l \Big(d\theta+rP\Big)\int_0^x \overline{\phi}dsdx + \mathcal{R}, \tag{3.25}$$

where $\mathcal{R} \in \mathfrak{R}$. From (3.6), we have

$$\gamma_2\int_0^l |\phi|^2\mathrm{d}x = -\frac{1}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\int_0^x \Big(\alpha\overline{\psi}_{xx} - \kappa\overline{(\varphi_x+\psi)} + \gamma_1\overline{\theta}_x$$
$$+\gamma_2\overline{P}_x + \rho_2\overline{f}_4\Big)dsdx + \int_0^l \eta_x\int_0^x \overline{\phi}dsdx + \mathcal{R}$$
$$= \int_0^l \eta_x\int_0^x \overline{\phi}dsdx - \frac{\alpha d}{\rho_2}\int_0^l \theta\overline{\psi}_x\mathrm{d}x - \frac{\alpha r}{\rho_2}\int_0^l P\overline{\psi}_x\mathrm{d}x$$
$$+\frac{\kappa}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\overline{\varphi}\mathrm{d}x + \frac{\kappa}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\int_0^x \overline{\psi}dsdx$$
$$+\frac{\gamma_1}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\overline{\theta}\mathrm{d}x + \frac{\gamma_2}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\overline{P}\mathrm{d}x + \mathcal{R}.$$

By using Cauchy–Schwarz inequality, we have

$$\gamma_2\int_0^l |\phi|^2\mathrm{d}x \leq C||\phi||_{L^2}||\eta||_{L^2} + C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||\psi_x||_{L^2}$$
$$+C\Big(||\theta||_{L^2}^2+||P||_{L^2}^2\Big) + \frac{\kappa}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\overline{\varphi}\mathrm{d}x$$
$$+\frac{\kappa}{\rho_2}\int_0^l \Big(d\theta+rP\Big)\int_0^x \overline{\psi}dsdx + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

From Eqs. (3.3) and (3.5), we infer that

$$\gamma_2\int_0^l |\phi|^2\mathrm{d}x \leq C||\phi||_{L^2}||\eta||_{L^2} + C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||\psi_x||_{L^2}$$

$$+C\Big(||\theta||_{L^2}^2+||P||_{L^2}^2\Big)-\frac{\kappa}{i\lambda\rho_2}\int_0^l\Big(d\theta+rP\Big)\overline{\nu}\mathrm{d}x$$
$$-\frac{\kappa}{i\lambda\rho_2}\int_0^l\Big(d\theta+rP\Big)\int_0^x\overline{\phi}dsdx+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$

and

$$\frac{\gamma_2}{2}\int_0^l|\phi|^2\mathrm{d}x\le\frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)+C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||\psi_x||_{L^2}$$
$$+C\Big(||\theta||_{L^2}^2+||P||_{L^2}^2\Big)+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

Now using Young's inequality, Lemma 3.2 and the estimate (3.12) we obtain

$$\int_0^l|\phi|^2\mathrm{d}x\le\frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)$$
$$+C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||\psi_x||_{L^2}+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.26}$$

On the other hand, multiplying Eq. (3.6) by $\overline{\psi}$

$$\alpha\int_0^l|\psi_x|^2\mathrm{d}x=-i\lambda\rho_2\int_0^l\phi\overline{\psi}\mathrm{d}x-\kappa\int_0^l(\varphi_x+\psi)\overline{\psi}\mathrm{d}x-\gamma_1\int_0^l\theta\overline{\psi}_x\mathrm{d}x$$
$$-\gamma_2\int_0^lP\overline{\psi}_x\mathrm{d}x+\mathcal{R}. \tag{3.27}$$

Substituting $\overline{\psi}$ by Eq. (3.5) and using the Cauchy–Schwarz and Young inequalities, we get

$$\alpha\int_0^l|\psi_x|^2\mathrm{d}x\le\rho_2\int_0^l|\phi|^2\mathrm{d}x+\frac{\kappa}{i\lambda}\int_0^l(\varphi_x+\psi)\overline{\phi}\mathrm{d}x-\gamma_1\int_0^l\theta\overline{\psi}_x\mathrm{d}x$$
$$-\gamma_2\int_0^lP\overline{\psi}_x\mathrm{d}x+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$
$$\le C||\phi||_{L^2}^2+\frac{\varepsilon}{\lambda^2}\int_0^l|\varphi_x+\psi|^2\mathrm{d}x+\frac{\alpha}{2}\int_0^l|\psi_x|^2\mathrm{d}x$$
$$+C\Big(||\theta||_{L^2}^2+||P||_{L^2}^2\Big)+C||U||_{\mathcal{H}}||F||_{\mathcal{H}},$$

for all $\varepsilon>0$. Finally, using Lemma 3.2 and Young's inequality we obtain

$$\int_0^l|\psi_x|^2\mathrm{d}x\le C||\phi||_{L^2}^2+\frac{\varepsilon}{\lambda^2}\int_0^l|\varphi_x+\psi|^2\mathrm{d}x+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.28}$$

Substituting (3.28) in (3.26) and using Lemma 3.2, we have

$$\int_0^l |\phi|^2 \mathrm{d}x \le \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)+C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||\phi||_{L^2}$$
$$+\frac{C}{|\lambda|}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)\Big(\int_0^l |\varphi_x+\psi|^2\mathrm{d}x\Big)^{1/2}$$
$$+C\Big(||\theta||_{L^2}+||P||_{L^2}\Big)\Big(||U||_{\mathcal{H}}||F||_{\mathcal{H}}\Big)^{1/2}+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

Note that

$$\frac{C}{|\lambda|}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)\Big(\int_0^l |\varphi_x+\psi|^2\mathrm{d}x\Big)^{1/2} \le \frac{C}{|\lambda|}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)||U||_{\mathcal{H}}.$$

Using the two estimates above with Young's inequality, we get (3.23). □

Now, we introduce two important stability numbers, associated with our model (1.7)–(1.14):

$$\chi_0 := \big(d\gamma_2 - r\gamma_1\big)\frac{K}{\tau_0\gamma_1} - \big(d\gamma_1 - c\gamma_2\big)\frac{\hbar}{\tau_1\gamma_2}, \tag{3.29}$$

$$\chi_1 := \xi\big(\rho_2 - \frac{\alpha\rho_1}{\kappa}\big) - \left(\frac{\tau_0\gamma_1^2}{K}+\frac{\tau_1\gamma_2^2}{\hbar}\right)(1-\xi), \tag{3.30}$$

where $\xi = 1-\frac{\rho_1\Gamma}{\delta\kappa}$, $\delta = cr-d^2>0$ and $\Gamma = \big(d\gamma_2 - r\gamma_1\big)\frac{K}{\tau_0\gamma_1} = \big(d\gamma_1 - c\gamma_2\big)\frac{\hbar}{\tau_1\gamma_2}$ if $\chi_0=0$. Otherwise, if $\chi_0\ne 0$, then χ_1 does not exist.

Lemma 3.4. *Let $(\varphi, v, \psi, \phi, \theta, q, P, \eta)$ be a solution of system (1.7)–(1.14). Assuming that $\chi_0=0$, there exists a positive constant C independent of λ such that*

$$\kappa|\xi|\left(1-\frac{C}{\lambda^2}\right)\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \le 2|\chi_1|\left|\int_0^l \phi\overline{v}_x\mathrm{d}x\right| + \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)$$
$$+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.31}$$

where $\xi\ne 0$ and for $|\lambda|$ large enough. Otherwise, if $\xi=0$, then we have

$$\kappa\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \le 2|c_1|\left|\int_0^l \phi\overline{v}_x\mathrm{d}x\right| + \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)$$
$$+|\lambda|^2C||U||_{\mathcal{H}}||F||_{\mathcal{H}}+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.32}$$

Proof. Multiplying Eq. (3.6) by $\overline{\varphi_x+\psi}$ and integrating by parts over $[0,l]$, we get

$$\kappa\int_0^l |\varphi_x+\psi|^2\mathrm{d}x = -i\lambda\rho_2\int_0^l \phi\overline{\varphi}_x\mathrm{d}x - i\lambda\rho_2\int_0^l \phi\overline{\psi}\mathrm{d}x - \alpha\int_0^l \psi_x\overline{(\varphi_x+\psi)}_x\mathrm{d}x$$
$$-\gamma_1\int_0^l \theta\overline{(\varphi_x+\psi)}_x\mathrm{d}x - \gamma_2\int_0^l P\overline{(\varphi_x+\psi)}_x\mathrm{d}x + \mathcal{R},$$

where $\mathcal{R} \in \mathfrak{R}$. Substituting $\overline{\varphi}_x$, $\overline{\psi}$ and $\overline{(\varphi_x+\psi)}_x$, respectively, by (3.3), (3.5) and (3.4), we get

$$\kappa\int_0^l |\varphi_x+\psi|^2\mathrm{d}x = \left(\rho_2-\frac{\alpha\rho_1}{\kappa}\right)\int_0^l \phi\overline{v}_x\mathrm{d}x + \rho_2\int_0^l |\phi|^2\mathrm{d}x + \underbrace{i\lambda\frac{\gamma_1\rho_1}{\kappa}\int_0^l \theta\overline{v}\mathrm{d}x}_{H_1:=}$$

$$+\underbrace{i\lambda\frac{\gamma_2\rho_1}{\kappa}\int_0^l P\overline{v}\mathrm{d}x}_{H_2:=} + \mathcal{R}. \tag{3.33}$$

Here, our aim is to obtain an expression for H_1 and H_2. For this, we multiply Eq. (3.7) by $\frac{\rho_1\gamma_1}{\kappa\delta}\overline{v}$ and integrating by parts we get

$$\begin{aligned} H_1 &= i\lambda\frac{\gamma_1\rho_1}{\kappa}\int_0^l \theta\overline{v}\mathrm{d}x \\ &= \frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{v}_x\mathrm{d}x - \frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{v}_x\mathrm{d}x + \left(r-d\gamma_2/\gamma_1\right)\gamma_1^2\frac{\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + \mathcal{R}. \end{aligned}$$

Substituting $\overline{v}_x$ by Eq. (3.3), we have

$$H_1 = \underbrace{-i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{\varphi}_x\mathrm{d}x}_{I_1:=} + \underbrace{i\lambda\frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{\varphi}_x\mathrm{d}x}_{I_2:=}$$

$$+\left(r-d\gamma_2/\gamma_1\right)\gamma_1^2\frac{\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + \mathcal{R}. \tag{3.34}$$

Analogously defining $H_2 := i\lambda\frac{\gamma_2\rho_1}{\kappa}\int_0^l P\overline{v}\mathrm{d}x$. Multiplying Eq. (3.9) by $\frac{\rho_1\gamma_2}{\delta\kappa}\overline{v}$ and using Eq. (3.3), we have

$$H_2 = -i\lambda\frac{c\gamma_2\rho_1}{\delta\kappa}\int_0^l \eta\overline{\varphi}_x\mathrm{d}x + i\lambda\frac{d\gamma_2\rho_1}{\delta\kappa}\int_0^l q\overline{\varphi}_x\mathrm{d}x$$

$$+\left(c-d\gamma_1/\gamma_2\right)\gamma_2^2\frac{\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + \mathcal{R}. \tag{3.35}$$

From here, the proof is divided into two steps to help the reader follow the arguments and the calculations.

Step 1. By adding and subtracting the terms $i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x$ and $i\lambda\frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x$ in (3.34) we get

$$\begin{aligned} H_1 = &-i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x + i\lambda\frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x \\ &-\frac{(d\gamma_2-r\gamma_1)\gamma_1\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x - i\lambda\frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x + \mathcal{R}. \end{aligned}$$

Then, substituting Eq. (3.5) in $\overline{\psi}$ we get

$$H_1 = -\Big(r - \frac{d\gamma_2}{\gamma_1}\Big)\frac{\gamma_1^2\rho_1}{\delta\kappa}\int_0^l \phi\overline{v}_x \mathrm{d}x - i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x$$

$$+i\lambda\frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{\phi}\mathrm{d}x + \frac{d\gamma_1\rho_1}{\delta\kappa}\int_0^l \eta\overline{\phi}\mathrm{d}x + \mathcal{R}. \tag{3.36}$$

Similarly, we have for Eq. (3.35)

$$H_2 = -\Big(c - d\gamma_1/\gamma_2\Big)\frac{\gamma_2^2\rho_1}{\delta\kappa}\int_0^l \phi\overline{v}_x \mathrm{d}x - i\lambda\frac{c\gamma_2\rho_1}{\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x$$

$$+i\lambda\frac{d\gamma_2\rho_1}{\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{c\gamma_2\rho_1}{\delta\kappa}\int_0^l \eta\overline{\phi}\mathrm{d}x + \frac{d\gamma_2\rho_1}{\delta\kappa}\int_0^l q\overline{\phi}\mathrm{d}x + \mathcal{R}.$$

Substituting H_1 and H_2 in (3.33), we have

$$\kappa\int_0^l |\varphi_x+\psi|^2\mathrm{d}x = c_1\int_0^l \phi\overline{v}_x\mathrm{d}x + \rho_2\int_0^l |\phi|^2\mathrm{d}x + \frac{\rho_1}{\delta\kappa}(d\gamma_2 - r\gamma_1)\int_0^l q\overline{\phi}\mathrm{d}x$$

$$+\frac{\rho_1}{\delta\kappa}(d\gamma_1 - c\gamma_2)\int_0^l \eta\overline{\phi}\mathrm{d}x + i\lambda\frac{\rho_1}{\delta\kappa}(d\gamma_2 - r\gamma_1)\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x$$

$$+i\lambda\frac{\rho_1}{\delta\kappa}(d\gamma_1 - c\gamma_2)\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x + \mathcal{R},$$

where $c_1 := \Big(\rho_2 - \frac{\alpha\rho_1}{\kappa}\Big) - \Big[\Big(r - \frac{d\gamma_2}{\gamma_1}\Big)\gamma_1^2 + \Big(c - \frac{d\gamma_1}{\gamma_2}\Big)\gamma_2^2\Big]\frac{\rho_1}{\delta\kappa}$. By using the Young's inequality and estimate (3.12), we get

$$\kappa\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \le |c_1|\Big|\int_0^l \phi\overline{v}_x\mathrm{d}x\Big| + 2\rho_2\int_0^l |\phi|^2\mathrm{d}x + \frac{\kappa}{2}\int_0^l |\varphi_x+\psi|^2\mathrm{d}x$$

$$+\lambda^2 C\int_0^l \Big(|q|^2 + |\eta|^2\Big)\mathrm{d}x + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$

and then

$$\frac{\kappa}{2}\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \le |c_1|\Big|\int_0^l \phi\overline{v}_x\mathrm{d}x\Big| + 2\rho_2\int_0^l |\phi|^2\mathrm{d}x$$

$$+|\lambda|^2 C||U||_{\mathcal{H}}||F||_{\mathcal{H}} + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

By using Lemma 3.3, (3.32) follows immediately.

Step 2. We aim to obtain the expressions of H_1 and H_2. Bearing in mind I_1 in Eq. (3.34), we multiply Eq. (3.8) by $\frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\overline{\varphi}_x$ to get

$$I_1 := i\lambda\frac{r\gamma_1\rho_1}{\delta\kappa}\int_0^l q\overline{\varphi}_x\mathrm{d}x = -\frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\varphi}_x\mathrm{d}x - \frac{Kr\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l \theta_x\overline{\varphi}_x\mathrm{d}x + \mathcal{R}.$$

Now, we add and subtract the term $\frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x$ to get

$$I_1 = -\frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x + \frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x - \frac{rK\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l \theta_x\overline{\varphi}_x\mathrm{d}x + \mathcal{R}.$$

Performing the same procedure in I_2 in Eq. (3.34), we have

$$I_2 = -\frac{d\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x + \frac{d\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x - \frac{d\hbar\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l P_x\overline{\varphi}_x\mathrm{d}x + \mathcal{R}.$$

Substituting I_1 and I_2 in (3.34), we obtain

$$\begin{aligned}H_1 =& \frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{d\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{r\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x\\ &+\frac{d\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x + \frac{rK\gamma_1\rho_1}{\tau_0\delta\kappa}\int_0^l \theta_x\overline{\varphi}_x\mathrm{d}x - \frac{d\hbar\gamma_1\rho_1}{\tau_1\delta\kappa}\int_0^l P_x\overline{\varphi}_x\mathrm{d}x\\ &-\frac{(d\gamma_2-r\gamma_1)\gamma_1\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + \mathcal{R}.\end{aligned}$$

Performing the same procedure in H_2 in Eq. (3.35), we have

$$\begin{aligned}H_2 =& \frac{c\gamma_2\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{d\gamma_2\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x - \frac{c\gamma_2\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x\\ &+\frac{d\gamma_2\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x + \frac{c\hbar\gamma_2\rho_1}{\tau_1\delta\kappa}\int_0^l P_x\overline{\varphi}_x\mathrm{d}x - \frac{dK\gamma_2\rho_1}{\tau_0\delta\kappa}\int_0^l \theta_x\overline{\varphi}_x\mathrm{d}x\\ &-\frac{(d\gamma_1-c\gamma_2)\gamma_2\rho_1}{\delta\kappa}\int_0^l \phi_x\overline{v}\mathrm{d}x + \mathcal{R}.\end{aligned}$$

Adding the simplifications H_1 and H_2, we get

$$\begin{aligned}H_1+H_2 =& \Big(r\gamma_1-d\gamma_2\Big)\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x\\ &+\Big(c\gamma_2-d\gamma_1\Big)\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{(\varphi_x+\psi)}\mathrm{d}x - \Big(r\gamma_1-d\gamma_2\Big)\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x\end{aligned}$$

$$-\Big(c\gamma_2-d\gamma_1\Big)\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x+\underbrace{\frac{\rho_1}{\delta\kappa}\Gamma\int_0^l(\gamma_1\theta_x+\gamma_2P_x)\overline{\varphi}_x\mathrm{d}x}_{J:=}$$

$$+\Big[\Big(c\gamma_2-d\gamma_1\Big)\gamma_2+\Big(r\gamma_1-d\gamma_2\Big)\gamma_1\Big]\frac{\rho_1}{\delta\kappa}\int_0^l\phi_x\overline{v}\mathrm{d}x+\mathcal{R}. \tag{3.37}$$

where $\Gamma:=\big(r\gamma_1-d\gamma_2\big)\frac{K}{\gamma_1\tau_0}=\big(c\gamma_2-d\gamma_1\big)\frac{\hbar}{\gamma_2\tau_1}$ since $\chi_0=0$. From J and (3.6), we obtain

$$J=i\lambda\frac{\rho_1\rho_2}{\delta\kappa}\Gamma\int_0^l\phi\overline{\varphi}_x\mathrm{d}x+\frac{\alpha\rho_1}{\delta\kappa}\Gamma\int_0^l\psi_x\overline{(\varphi_x+\psi)}_x\mathrm{d}x-\frac{\alpha\rho_1}{\delta\kappa}\Gamma\int_0^l|\psi_x|^2\mathrm{d}x$$
$$+\frac{\rho_1}{\delta}\Gamma\int_0^l|\varphi_x+\psi|^2\mathrm{d}x-\frac{\rho_1}{\delta}\Gamma\int_0^l(\varphi_x+\psi)\overline{\psi}\mathrm{d}x+\mathcal{R}.$$

Substituting $\overline{\varphi}_x$, $\overline{(\varphi_x+\psi)}_x$ and $\overline{\psi}_x$, respectively, by (3.3), (3.4) and (3.5), and making some algebraic manipulations, we get

$$J=-\left(\rho_2-\frac{\alpha\rho_1}{\kappa}\right)\frac{\rho_1}{\delta\kappa}\Gamma\int_0^l\phi\overline{v}_x\mathrm{d}x-\frac{\alpha\rho_1}{\delta\kappa}\Gamma\int_0^l|\psi_x|^2\mathrm{d}x+\frac{\rho_1}{\delta}\Gamma\int_0^l|\varphi_x+\psi|^2\mathrm{d}x$$
$$-\frac{\rho_1}{\delta}\Gamma\int_0^l(\varphi_x+\psi)\overline{\psi}\mathrm{d}x+\mathcal{R}.$$

Here, we replace J in (3.37) to get

$$H_1+H_2=-\left[\left(\rho_2-\frac{\alpha\rho_1}{\kappa}\right)\Gamma+\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\right]\frac{\rho_1}{\delta\kappa}\int_0^l\phi\overline{v}_x\mathrm{d}x$$
$$+\Delta_r\gamma_1\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x+\Delta_c\gamma_2\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l\eta\overline{(\varphi_x+\psi)}\mathrm{d}x$$
$$-\frac{\alpha\rho_1}{\delta\kappa}\Gamma\int_0^l|\psi_x|^2\mathrm{d}x+\frac{\rho_1}{\delta}\Gamma\int_0^l|\varphi_x+\psi|^2\mathrm{d}x-\frac{\rho_1}{\delta}\Gamma\int_0^l(\varphi_x+\psi)\overline{\psi}\mathrm{d}x$$
$$-\Delta_r\gamma_1\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x-\Delta_c\gamma_2\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l\eta\overline{\psi}\mathrm{d}x+\mathcal{R}, \tag{3.38}$$

where $\Delta_r:=r-d\gamma_2/\gamma_1$ and $\Delta_c:=c-d\gamma_1/\gamma_2$. Substituting (3.38) in (3.33), we obtain

$$\kappa\xi\int_0^l|\varphi_x+\psi|^2\mathrm{d}x=\chi_1\int_0^l\phi\overline{v}_x\mathrm{d}x+\rho_2\int_0^l|\phi|^2\mathrm{d}x+\Delta_r\gamma_1\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{(\varphi_x+\psi)}\mathrm{d}x$$
$$+\Delta_c\gamma_2\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l\eta\overline{(\varphi_x+\psi)}\mathrm{d}x-\frac{\alpha\rho_1}{\delta\kappa}\Gamma\int_0^l|\psi_x|^2\mathrm{d}x-\frac{\rho_1}{\delta}\Gamma\int_0^l(\varphi_x+\psi)\overline{\psi}\mathrm{d}x$$

$$-\Delta_r\gamma_1\frac{\rho_1}{\tau_0\delta\kappa}\int_0^l q\overline{\psi}\mathrm{d}x-\Delta_c\gamma_2\frac{\rho_1}{\tau_1\delta\kappa}\int_0^l \eta\overline{\psi}\mathrm{d}x+\mathcal{R},$$

where $\xi := 1-\frac{\rho_1}{\delta\kappa}\Gamma$ and χ_1 is given in (3.30). Taking the absolute value and using Young and Poincaré inequalities, we get

$$\kappa|\xi|\int_0^l|\varphi_x+\psi|^2\mathrm{d}x\le|\chi_1|\left|\int_0^l\phi\overline{v}_x\mathrm{d}x\right|+C\varepsilon\int_0^l|\varphi_x+\psi|^2\mathrm{d}x+\rho_2\int_0^l|\phi|^2\mathrm{d}x$$
$$+C\int_0^l|\psi_x|^2\mathrm{d}x+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

From $\varepsilon := \kappa|\xi|/2C$, we have

$$\kappa|\xi|\int_0^l|\varphi_x+\psi|^2\mathrm{d}x\le2|\chi_1|\left|\int_0^l\phi\overline{v}_x\mathrm{d}x\right|+C\Big(\int_0^l|\phi|^2\mathrm{d}x+\int_0^l|\psi_x|^2\mathrm{d}x\Big)$$
$$+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

By using Lemma 3.3, we obtain

$$\kappa|\xi|\int_0^l|\varphi_x+\psi|^2\mathrm{d}x\le2|\chi_1|\left|\int_0^l\phi\overline{v}_x\mathrm{d}x\right|+\frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)$$
$$+\frac{\varepsilon_1}{\lambda^2}C\int_0^l|\varphi_x+\psi|^2\mathrm{d}x+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.$$

Choosing, $\varepsilon_1 := \kappa|\xi|$, (3.31) follows immediately. This completes the proof of the lemma. □

Lemma 3.5. *Let $(\varphi, v, \psi, \phi, \theta, q, P, \eta)$ be a solution of system (1.7)–(1.14). Then, there exists a positive constant C such that*

$$\left(c-\frac{d^2}{r}\right)\int_0^l|\theta|^2\mathrm{d}x+\int_0^l\left|\frac{d}{\sqrt{r}}\theta+\sqrt{r}P\right|^2\mathrm{d}x\le\frac{C}{|\lambda|}||U||^2_{\mathcal{H}}+C\varepsilon||U||^2_{\mathcal{H}}$$
$$+C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.39}$$

Proof. Multiplying Eqs. (3.24) and (3.20) by $\overline{P}$ and $\overline{\theta}$, respectively, we get

$$i\lambda d\int_0^l\theta\overline{P}\mathrm{d}x+i\lambda r\int_0^l|P|^2\mathrm{d}x-\int_0^l\eta\overline{P}_x\mathrm{d}x+\gamma_2\int_0^l\phi\overline{P}_x\mathrm{d}x=\mathcal{R} \tag{3.40}$$

and

$$i\lambda\frac{d^2}{r}\int_0^l|\theta|^2\mathrm{d}x+i\lambda\left(c-\frac{d^2}{r}\right)\int_0^l|\theta|^2\mathrm{d}x+i\lambda d\int_0^lP\overline{\theta}\mathrm{d}x-\int_0^lq\overline{\theta}_x\mathrm{d}x$$
$$+\gamma_1\int_0^l\phi\overline{\theta}_x\mathrm{d}x=\mathcal{R}, \tag{3.41}$$

where $\mathcal{R} \in \mathfrak{R}$. Adding Eqs. (3.40) and (3.41), we have

$$i\lambda\left(c-\frac{d^2}{r}\right)\int_0^l |\theta|^2\mathrm{d}x + i\lambda\int_0^l \left|\frac{d}{\sqrt{r}}\theta+\sqrt{r}P\right|^2\mathrm{d}x - \int_0^l q\overline{\theta}_x\mathrm{d}x$$
$$-\int_0^l \eta\overline{P}_x\mathrm{d}x = -\gamma_1\int_0^l \phi\overline{\theta}_x\mathrm{d}x - \gamma_2\int_0^l \phi\overline{P}_x\mathrm{d}x + \mathcal{R}.$$

Substituting $\overline{\theta}_x$ and $\overline{P}_x$, respectively, by (3.8) and (3.9), we obtain

$$i\lambda\left(c-\frac{d^2}{r}\right)\int_0^l |\theta|^2\mathrm{d}x + i\lambda\int_0^l \left|\frac{d}{\sqrt{r}}\theta+\sqrt{r}P\right|^2\mathrm{d}x + \frac{(1-i\lambda\tau_0)}{1+\lambda^2\tau_0^2}K\int_0^l |\theta_x|^2\mathrm{d}x$$
$$+\frac{(1-i\lambda\tau_1)}{1+\lambda^2\tau_1^2}\hbar\int_0^l |P_x|^2\mathrm{d}x = -\gamma_1\int_0^l \phi\overline{\theta}_x\mathrm{d}x - \gamma_2\int_0^l \phi\overline{P}_x\mathrm{d}x + \mathcal{R}. \tag{3.42}$$

Taking the real part and using Eq. (3.6), we get

$$\begin{aligned}
\mathscr{I} &= \frac{K}{1+\lambda^2\tau_0^2}\int_0^l |\theta_x|^2\mathrm{d}x + \frac{\hbar}{1+\lambda^2\tau_1^2}\int_0^l |P_x|^2\mathrm{d}x \\
&= -\mathrm{Re}\left\{\int_0^l \phi(\gamma_1\overline{\theta}_x+\gamma_2\overline{P}_x)\mathrm{d}x\right\} + \mathcal{R} \text{ (use (3.6))} \\
&= -\mathrm{Re}\left\{\int_0^l \phi\Big(-i\lambda\rho_2\overline{\phi}-\alpha\overline{\psi}_{xx}+\kappa\overline{(\varphi_x+\psi)}\Big)\mathrm{d}x\right\} + \mathcal{R} \\
&= -\mathrm{Re}\left\{\alpha\int_0^l \phi_x\overline{\psi}_x\mathrm{d}x + \kappa\int_0^l \phi\overline{(\varphi_x+\psi)}\mathrm{d}x\right\} + \mathcal{R}.
\end{aligned}$$

Adding and subtracting v_x and using Poincaré inequality yields

$$\mathscr{I} = -\mathrm{Re}\left\{\alpha\int_0^l \phi_x\overline{\psi}_x\mathrm{d}x\right\} - \mathrm{Re}\left\{\kappa\int_0^l (v_x+\phi)\overline{(\varphi_x+\psi)}\mathrm{d}x\right.$$
$$\left.+\kappa\int_0^l v\overline{(\varphi_x+\psi)}_x\mathrm{d}x\right\} + \mathcal{R}.$$

Substituting $\overline{\phi}_x$ and $\overline{(\varphi_x+\psi)}_x$, respectively, by (3.5) and (3.4), we get

$$\mathscr{I} = -\mathrm{Re}\left\{\kappa\int_0^l (v_x+\phi)\overline{(\varphi_x+\psi)}\mathrm{d}x\right\} + \mathcal{R}. \tag{3.43}$$

Combining Eqs. (3.3) and (3.5) we obtain

$$v_x+\phi = i\lambda(\varphi_x+\psi) - (f_{1,x}+f_3). \tag{3.44}$$

Substituting (3.44) into (3.43), we have

$$\mathscr{I} = -\mathrm{Re}\left\{ i\lambda\kappa \int_0^l |\varphi_x + \psi|^2 \mathrm{d}x \right\} + \mathcal{R} \leq C||U||_{\mathcal{H}}||F||_{\mathcal{H}},$$

which leads to

$$\int_0^l |\theta_x|^2 \mathrm{d}x + \int_0^l |P_x|^2 \mathrm{d}x \leq \lambda^2 C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{3.45}$$

On the other hand, taking the imaginary part we have

$$\left(c - \frac{d^2}{r}\right) \int_0^l |\theta|^2 \mathrm{d}x + \int_0^l \left| \frac{d}{\sqrt{r}}\theta + \sqrt{r}P \right|^2 \mathrm{d}x = \frac{\tau_0 K}{1+\lambda^2\tau_0^2} \int_0^l |\theta_x|^2 \mathrm{d}x$$
$$+ \frac{\tau_1 \hbar}{1+\lambda^2\tau_1^2} \int_0^l |P_x|^2 \mathrm{d}x - \frac{\gamma_1}{\lambda} \mathrm{Im}\left\{ \int_0^l \phi \bar{\theta}_x \mathrm{d}x \right\} - \frac{\gamma_2}{\lambda} \mathrm{Im}\left\{ \int_0^l \phi \overline{P}_x \mathrm{d}x \right\} + \mathcal{R}.$$

By using the Young's inequality, we obtain

$$\left(c - \frac{d^2}{r}\right) \int_0^l |\theta|^2 \mathrm{d}x + \int_0^l \left| \frac{d}{\sqrt{r}}\theta + \sqrt{r}P \right|^2 \mathrm{d}x \leq \frac{\tau_0}{1+\lambda^2\tau_0^2} K \int_0^l |\theta_x|^2 \mathrm{d}x$$
$$+ \frac{\tau_1}{1+\lambda^2\tau_1^2} \hbar \int_0^l |P_x|^2 \mathrm{d}x + \frac{K}{\lambda^2\tau_0} \int_0^l |\theta_x|^2 \mathrm{d}x + \frac{\gamma_1^2\tau_0}{4K} \int_0^l |\phi|^2 \mathrm{d}x$$
$$+ \frac{\hbar}{\lambda^2\tau_1} \int_0^l |P_x|^2 \mathrm{d}x + \frac{\gamma_2^2\tau_1}{4\hbar} \int_0^l |\phi|^2 \mathrm{d}x,$$

Using $1 + \lambda^2\tau_0^2 > \lambda^2\tau_0^2$ and $1 + \lambda^2\tau_1^2 > \lambda^2\tau_1^2$ yields

$$\left(c - \frac{d^2}{r}\right) \int_0^l |\theta|^2 \mathrm{d}x + \int_0^l \left| \frac{d}{\sqrt{r}}\theta + \sqrt{r}P \right|^2 \mathrm{d}x \leq \frac{2K}{\lambda^2\tau_0} \int_0^l |\theta_x|^2 \mathrm{d}x + \frac{2\hbar}{\lambda^2\tau_1} \int_0^l |P_x|^2 \mathrm{d}x$$
$$+ \frac{1}{4}\left(\frac{\gamma_1^2\tau_0}{K} + \frac{\gamma_2^2\tau_1}{\hbar}\right) \int_0^l |\phi|^2 \mathrm{d}x \tag{3.46}$$

Substituting (3.45) in (3.46) and using Lemma 3.3, we obtain (3.39). □

Lemma 3.6. *Let $(\varphi, v, \psi, \phi, \theta, q, P, \eta)$ be a solution of system (1.7)–(1.14). There then exists a positive constant ε independent of λ such that*

$$\rho_1 \int_0^l |v|^2 \mathrm{d}x \leq C \int_0^l |\varphi_x + \psi|^2 \mathrm{d}x + \frac{C}{|\lambda|}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \tag{3.47}$$

for $|\lambda| > 1$ large enough.

Proof. Multiplying Eq. (3.4) by $-i\lambda^{-1}\overline{v}$, we have

$$\rho_1 \int_0^l |v|^2 \mathrm{d}x - \frac{i\kappa}{\lambda} \int_0^l (\varphi_x + \psi)\overline{v}_x \mathrm{d}x = \mathcal{R}, \tag{3.48}$$

where $\mathcal{R} \in \mathfrak{R}$. From (3.3) we have $\overline{v}_x = -(i\lambda\overline{\varphi}_x + \overline{f}_{1,x})$ and consequently,

$$\rho_1 \int_0^l |v|^2 \mathrm{d}x \leq \kappa \int_0^l |\varphi_x + \psi|^2 \mathrm{d}x + \kappa \int_0^l (\varphi_x + \psi)\overline{\psi} \mathrm{d}x + \mathcal{R}. \tag{3.49}$$

By using the Young's inequality and Lemmas 3.3 and 3.5, we obtain (3.47). □

Theorem 3.1. *If $\chi_0 = 0$, then system (1.7)–(1.14) is exponentially stable if and only if $\chi_1 = 0$.*

Proof. (i) Sufficiency: From Lemmas 3.2–3.6 and estimate (3.12), we get if $\chi_0 = 0$

$$||U||^2_{\mathcal{H}} \leq |\chi_1| C \left| \int_0^l \phi \overline{v}_x \mathrm{d}x \right| + \frac{C}{|\lambda|} ||U||^2_{\mathcal{H}} + C\varepsilon ||U||^2_{\mathcal{H}} + C||U||_{\mathcal{H}} ||F||_{\mathcal{H}}. \tag{3.50}$$

Since $\chi_1 = 0$ we have

$$\left(1 - \frac{C}{|\lambda|} - C\varepsilon\right) ||U||^2_{\mathcal{H}} \leq C||U||_{\mathcal{H}} ||F||_{\mathcal{H}}. \tag{3.51}$$

Choosing $|\lambda|$ large enough and ε sufficiently small, we concluded the sufficiency condition.

(ii) Necessity: we show that the semigroup $S(t)$ is not exponentially stable when the stability number χ_1 is different from zero. The proof is based on Theorem 1.1. The strategy consists of verifying that condition (1.20) fails to hold. To this end, let us assume that there exists $U = (\varphi, v, \psi, \phi, \theta, q, P, \eta) \in \mathcal{H}$ such that $||U||_{\mathcal{H}} \neq 0$. Without loss of generality we can take $f_1 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = 0$ and choose $f_2(x) = \rho_1^{-1} \sin(\beta_n x)$ in system (3.3)–(3.10) such that $F = \big(0, f_2, 0, 0, 0, 0, 0, 0\big)$ is limited in $\mathcal{H}$. Because of the boundary conditions (1.14), we can suppose that

$$\varphi(x) = \widehat{A}_1 \sin(\beta_n x), \quad \psi(x) = \widehat{A}_2 \cos(\beta_n x), \quad \theta(x) = \widehat{A}_3 \sin(\beta_n x),$$
$$q(x) = \widehat{A}_4 \cos(\beta_n x), \quad P(x) = \widehat{A}_5 \sin(\beta_n x), \quad \eta(x) = \widehat{A}_6 \cos(\beta_n x),$$

where $\widehat{A}_i$ $(i = 1, \ldots, 5)$ are constant and $\beta_n := n\pi/l$. Therefore, system (3.3)–(3.10) is equivalent to

$$\big(-\lambda^2\rho_1 + \kappa\beta_n^2\big)\widehat{A}_1 + \kappa\beta_n\widehat{A}_2 = 1, \tag{3.52}$$
$$\kappa\beta_n\widehat{A}_1 + \big(-\lambda^2\rho_2 + \alpha\beta_n^2 + \kappa\big)\widehat{A}_2 - \gamma_1\beta_n\widehat{A}_3 - \gamma_2\beta_n\widehat{A}_5 = 0, \tag{3.53}$$
$$-i\lambda(d\gamma_2 - r\gamma_1)\beta_n\widehat{A}_2 + i\lambda\delta\widehat{A}_3 - r\beta_n\widehat{A}_4 + d\beta_n\widehat{A}_6 = 0, \tag{3.54}$$
$$(1 + i\lambda\tau_0)\widehat{A}_4 + K\beta_n\widehat{A}_3 = 0, \tag{3.55}$$
$$-i\lambda(d\gamma_1 - c\gamma_2)\beta_n\widehat{A}_2 + d\beta_n\widehat{A}_4 + i\lambda\delta\widehat{A}_5 - c\beta_n\widehat{A}_6 = 0, \tag{3.56}$$
$$(1 + i\lambda\tau_1)\widehat{A}_6 + \hbar\beta_n\widehat{A}_5 = 0. \tag{3.57}$$

By using elementary row operation in (3.54) and (3.56), we obtain

$$\big(-\lambda^2\rho_1 + \kappa\beta_n^2\big)\widehat{A}_1 + \kappa\beta_n\widehat{A}_2 = 1, \tag{3.58}$$
$$\kappa\beta_n\widehat{A}_1 + \big(-\lambda^2\rho_2 + \alpha\beta_n^2 + \kappa\big)\widehat{A}_2 - \gamma_1\beta_n\widehat{A}_3 - \gamma_2\beta_n\widehat{A}_5 = 0, \tag{3.59}$$
$$i\lambda\big(\Delta_r\gamma_1^2 + \Delta_c\gamma_2^2\big)\beta_n\widehat{A}_2 + i\lambda\delta\gamma_1\widehat{A}_3 - \Delta_r\gamma_1\beta_n\widehat{A}_4 + i\lambda\gamma_2\delta\widehat{A}_5 - \Delta_c\gamma_2\beta_n\widehat{A}_6 = 0, \tag{3.60}$$
$$(1 + i\lambda\tau_0)\widehat{A}_4 + K\beta_n\widehat{A}_3 = 0, \tag{3.61}$$

$$
\begin{aligned}
&i\lambda\big(\Delta_r+\Delta_c\big)\gamma_1\gamma_2\beta_n\widehat{A}_2+i\lambda\delta\gamma_2\widehat{A}_3-\big(r-d\gamma_1/\gamma_2\big)\gamma_2\beta_n\widehat{A}_4\\
&\quad+i\lambda\gamma_1\delta\widehat{A}_5-\big(c-d\gamma_2/\gamma_1\big)\gamma_1\beta_n\widehat{A}_6=0, \qquad (3.62)\\
&(1+i\lambda\tau_1)\widehat{A}_6+\hbar\beta_n\widehat{A}_5=0, \qquad (3.63)
\end{aligned}
$$

where $\Delta_r := r - d\gamma_2/\gamma_1$ and $\Delta_c := c - d\gamma_1/\gamma_2$. So we have that

$$
\widehat{A}_4=-\frac{K\beta_n}{1+i\lambda\tau_0}\widehat{A}_3 \quad \text{and} \quad \widehat{A}_6=-\frac{\hbar\beta_n}{1+i\lambda\tau_1}\widehat{A}_5. \qquad (3.64)
$$

Therefore, we can rewrite the above system as

$$
\begin{pmatrix}
p_1(\lambda) & \kappa\beta_n & 0 & 0\\
\kappa\beta_n & p_2(\lambda) & -\gamma_1\beta_n & -\gamma_2\beta_n\\
0 & p_3(\lambda) & p_4(\lambda) & p_5(\lambda)\\
0 & p_6(\lambda) & p_7(\lambda) & p_8(\lambda)
\end{pmatrix}
\begin{pmatrix}\widehat{A}_1\\ \widehat{A}_2\\ \widehat{A}_3\\ \widehat{A}_5\end{pmatrix}
=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad (3.65)
$$

where

$$
\begin{aligned}
&p_1(\lambda):=-\lambda^2\rho_1+\kappa\beta_n^2, \quad p_2(\lambda):=-\lambda^2\rho_2+\alpha\beta_n^2+\kappa,\\
&p_3(\lambda):=i\lambda\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\beta_n, \quad p_4(\lambda):=i\lambda\delta\gamma_1+\frac{\Delta_r\gamma_1 K}{1+i\lambda\tau_0}\beta_n^2,\\
&p_5(\lambda):=i\lambda\delta\gamma_2+\frac{\Delta_c\gamma_2\hbar}{1+i\lambda\tau_1}\beta_n^2, \quad p_6(\lambda):=i\lambda\beta_n\Big(\Delta_r+\Delta_c\Big)\gamma_1\gamma_2,\\
&p_7(\lambda):=i\lambda\delta\gamma_2+\frac{\big(r-d\gamma_1/\gamma_2\big)\gamma_2 K}{1+i\lambda\tau_0}\beta_n^2,\\
&p_8(\lambda):=i\lambda\delta\gamma_1+\frac{\big(c-d\gamma_2/\gamma_1\big)\gamma_1\hbar}{1+i\lambda\tau_1}\beta_n^2.
\end{aligned}
$$

Now we choose $\lambda^2 \equiv \lambda_n^2 = \frac{\kappa}{\rho_1}\beta_n^2 - \frac{\sigma}{\rho_1}$ which gives $p_1(\lambda) = \sigma$, where $\sigma \in \mathbb{R}$ is going to be fixed later. The resolution of Eq. (3.65) gives us

$$
\widehat{A}_1=\frac{W}{\sigma W-\kappa^2\beta_n^2}, \qquad (3.66)
$$

where

$$
W=p_2(\lambda)+\beta_n\frac{\gamma_1\Big(p_3(\lambda)p_8(\lambda)-p_5(\lambda)p_6(\lambda)\Big)-\gamma_2\Big(p_3(\lambda)p_7(\lambda)-p_4(\lambda)p_6(\lambda)\Big)}{p_4(\lambda)p_8(\lambda)-p_5(\lambda)p_7(\lambda)}. \qquad (3.67)
$$

For this case, we consider the following asymptotic equivalences

$$
\begin{aligned}
p_3(\lambda)p_7(\lambda)\sim&-\beta_n^3\frac{\delta\kappa}{\rho_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_2\xi+\beta_n^3\frac{dK}{\tau_0\gamma_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\big(\gamma_2^2-\gamma_1^2\big)\\
&+i\lambda\beta_n\frac{\delta}{\tau_0}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_2+\beta_n\frac{\delta\sigma}{\rho_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_2\xi, \qquad (3.68)\\
p_3(\lambda)p_8(\lambda)\sim&-\beta_n^3\frac{\delta\kappa}{\rho_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_1\xi+\beta_n^3\frac{d\hbar}{\tau_1\gamma_2}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\big(\gamma_1^2-\gamma_2^2\big)\\
&+i\lambda\beta_n\frac{\delta}{\tau_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_1+\beta_n\frac{\delta\sigma}{\rho_1}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\gamma_1\xi, \qquad (3.69)\\
p_4(\lambda)p_6(\lambda)\sim&-\beta_n^3\frac{\delta\kappa}{\rho_1}\Big(\Delta_r+\Delta_c\Big)\gamma_1^2\gamma_2\xi+\beta_n\frac{\delta\sigma}{\rho_1}\Big(\Delta_r+\Delta_c\Big)\gamma_1^2\gamma_2\xi\\
&+i\lambda\beta_n\frac{\delta}{\tau_0}\Big(\Delta_r+\Delta_c\Big)\gamma_1^2\gamma_2, \qquad (3.70)
\end{aligned}
$$

$$\begin{aligned} p_4(\lambda)p_8(\lambda) \sim & -\beta_n^4 \frac{\delta^2\kappa^2\gamma_1^2}{\lambda^2\rho_1^2}\xi^2 + \beta_n^4 \frac{\delta\kappa\big(\gamma_1^2-\gamma_2^2\big)d\hbar\gamma_1}{\lambda^2\rho_1\tau_1\gamma_2}\xi \\ & +2\beta_n^2\frac{\sigma\delta^2\kappa\gamma_1^2}{\lambda^2\rho_1^2}\xi^2 - \beta_n^2\frac{\delta^2\kappa\big(\tau_0+\tau_1\big)\gamma_1^2}{i\lambda\rho_1\tau_0\tau_1}\xi + \frac{\delta^2\sigma\big(\tau_0+\tau_1\big)\gamma_1^2}{i\lambda\rho_1\tau_0\tau_1}\xi \\ & +\beta_n^2\frac{\delta\big(\gamma_1^2-\gamma_2^2\big)d\hbar\gamma_1}{i\lambda\tau_0\tau_1\gamma_2} - \beta_n^2\frac{\delta\sigma\big(\gamma_1^2-\gamma_2^2\big)d\hbar\gamma_1}{\lambda^2\rho_1\tau_1\gamma_2}\xi - \frac{\sigma^2\delta^2\gamma_1^2}{\lambda^2\rho_1^2}\xi^2 + \frac{\delta^2\gamma_1^2}{\tau_0\tau_1}, \end{aligned} \tag{3.71}$$

$$\begin{aligned} p_5(\lambda)p_6(\lambda) \sim & -\beta_n^3\frac{\delta\kappa}{\rho_1}\Big(\Delta_r+\Delta_c\Big)\gamma_1\gamma_2^2\xi + \beta_n\frac{\delta\sigma}{\rho_1}\Big(\Delta_r+\Delta_c\Big)\gamma_1\gamma_2^2\xi \\ & +i\lambda\beta_n\frac{\delta}{\tau_1}\Big(\Delta_r+\Delta_c\Big)\gamma_1\gamma_2^2, \end{aligned} \tag{3.72}$$

$$\begin{aligned} p_5(\lambda)p_7(\lambda) \sim & -\beta_n^4\frac{\delta^2\kappa^2\gamma_2^2}{\lambda^2\rho_1^2}\xi^2 + \beta_n^4\frac{\delta\kappa\big(\gamma_2^2-\gamma_1^2\big)dK\gamma_2}{\lambda^2\rho_1\tau_0\gamma_1}\xi + 2\beta_n^2\frac{\sigma\delta^2\kappa\gamma_2^2}{\lambda^2\rho_1^2}\xi^2 \\ & -\beta_n^2\frac{\delta^2\kappa\big(\tau_0+\tau_1\big)\gamma_2^2}{i\lambda\rho_1\tau_0\tau_1}\xi + \frac{\delta^2\sigma\big(\tau_0+\tau_1\big)\gamma_2^2}{i\lambda\rho_1\tau_0\tau_1}\xi + \frac{\delta^2\gamma_2^2}{\tau_0\tau_1} \\ & +\beta_n^2\frac{\delta\big(\gamma_2^2-\gamma_1^2\big)dK\gamma_2}{i\lambda\tau_0\tau_1\gamma_1} - \beta_n^2\frac{\delta\sigma\big(\gamma_2^2-\gamma_1^2\big)dK\gamma_2}{\lambda^2\rho_1\tau_0\gamma_1}\xi - \frac{\sigma^2\delta^2\gamma_2^2}{\lambda^2\rho_1^2}\xi^2. \end{aligned} \tag{3.73}$$

Because of this, note that the following asymptotic equivalences hold:

$$\begin{aligned} & \gamma_1\beta_n\Big(p_3(\lambda)p_8(\lambda) - p_5(\lambda)p_6(\lambda)\Big) - \gamma_2\beta_n\Big(p_3(\lambda)p_7(\lambda) - p_4(\lambda)p_6(\lambda)\Big) \\ & \sim -\frac{\delta\kappa}{\rho_1}\beta_n^4\big(\gamma_1^2-\gamma_2^2\big)\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big)\xi \\ & +\beta_n^4 d\bigg(\frac{\hbar\gamma_1}{\tau_1\gamma_2}+\frac{K\gamma_2}{\tau_0\gamma_1}\bigg)\big(\gamma_1^2-\gamma_2^2\big)\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big) + O(\lambda^3) := \Upsilon_1 \end{aligned} \tag{3.74}$$

and

$$\begin{aligned} & p_4(\lambda)p_8(\lambda) - p_5(\lambda)p_7(\lambda) \sim -\lambda^2\delta^2\big(\gamma_1^2-\gamma_2^2\big)\xi^2 \\ & +\lambda^2\frac{\rho_1}{\kappa}d\delta\bigg(\frac{\hbar\gamma_1}{\tau_1\gamma_2}+\frac{K\gamma_2}{\tau_0\gamma_1}\bigg)\big(\gamma_1^2-\gamma_2^2\big)\xi + O(\lambda) := \Upsilon_2. \end{aligned} \tag{3.75}$$

Suppose that $\chi_0 = 0$, $\gamma_1 \neq \gamma_2$, $\xi \neq 0$ and $\chi_1 \neq 0$. Then, from (3.67), (3.74) and (3.75) we get

$$\begin{aligned} W & \sim p_2(\lambda) + \frac{\Upsilon_1}{\Upsilon_2} \\ & = p_2(\lambda) + \delta\beta_n^2\frac{\frac{\kappa}{\rho_1}\beta_n^2\Upsilon_3\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big) + O(\lambda)}{\lambda^2\delta^2\xi\Upsilon_3 + O(\lambda)} \\ & = p_2(\lambda) + \frac{\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big)}{\delta\xi}\beta_n^2\delta\beta_n^2\frac{O(\lambda)}{\lambda^2\delta^2\xi\Psi_3 + O(\lambda)} \\ & \sim p_2(\lambda) + \frac{\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big)}{\delta\xi}\beta_n^2 + O(\lambda), \end{aligned}$$

where we have used $\gamma_1 \neq \gamma_2$ and $\Upsilon_3 = \bigg[\xi - \frac{\rho_1}{\kappa}\frac{d}{\delta}\bigg(\frac{\hbar\gamma_1}{\tau_1\gamma_2}+\frac{K\gamma_2}{\tau_0\gamma_1}\bigg)\bigg]\big(\gamma_1^2-\gamma_2^2\big)$. From $p_2(\lambda) := -\lambda^2\rho_2 + \alpha\beta_n^2 + \kappa$ we have

$$W \sim \bigg(-\frac{\kappa\rho_2}{\rho_1} + \alpha + \frac{\big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\big)}{\delta\xi}\bigg)\beta_n^2 + \kappa + O(\lambda). \tag{3.76}$$

Taking $-\frac{\kappa\rho_2}{\rho_1}+\alpha+\frac{\left(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\right)}{\delta\xi}=\frac{\kappa^2}{\sigma}$ we have

$$W\sim\frac{\kappa^2}{\sigma}\beta_n^2+\kappa+O(\lambda),$$

where $\sigma:=-\rho_1\kappa\xi/\chi_1$. From (3.66), we have

$$\widehat{A}_1\sim\frac{\frac{\kappa^2}{\sigma}\beta_n^2+\kappa+O(\lambda)}{\sigma\kappa+O(\lambda)}=\frac{\frac{\chi_1^2}{\rho_1^2\kappa\xi^2}\beta_n^2-\frac{\chi_1}{\rho_1\kappa\xi}+O(\lambda)}{1+O(\lambda)}\sim\lambda\frac{\chi_1^2}{\rho_1\kappa^2\xi^2},$$

since $\xi\neq 0$ and $\chi_1\neq 0$. Therefore,

$$\begin{aligned}||U||_{\mathcal{H}}^2\geq\rho_1\int_0^l|v|^2\mathrm{d}x&=|\lambda|^2|\widehat{A}_1|^2\rho_1\int_0^l|\sin(\beta_n x)|^2\mathrm{d}x\\&=|\lambda|^2|O(\lambda)|^2\frac{\chi_1^4}{\rho_1\kappa^4\xi^4}\frac{l}{2}=|O(\lambda^4)|\frac{\chi_1^4}{\rho_1\kappa^4\xi^4}\end{aligned}$$

which implies that $||U||_{\mathcal{H}}\to\infty$ as $\lambda\to\infty$. □

Remark 3.1. One can also show the lack of exponential stability of system (1.7)–(1.14) under the conditions $\chi_0=0$, $\xi=0$ and $\gamma_1\neq\gamma_2$. In fact, if one chooses $\lambda^2\equiv\lambda_n^2=\frac{\kappa}{\rho_1}\beta_n^2$ which gives $p_1(\lambda)=0$. The resolution of Eq. (3.65) gives us

$$\begin{aligned}\widehat{A}_1=&-\frac{p_2(\lambda)}{\kappa^2\beta_n^2}\\&+\frac{\gamma_1\Big(p_3(\lambda)p_8(\lambda)-p_5(\lambda)p_6(\lambda)\Big)-\gamma_2\Big(p_3(\lambda)p_7(\lambda)-p_4(\lambda)p_6(\lambda)\Big)}{-\kappa^2\beta_n\Big(p_4(\lambda)p_8(\lambda)-p_5(\lambda)p_7(\lambda)\Big)}.\end{aligned}$$

Since $\lambda^2=\frac{\kappa}{\rho_1}\beta_n^2$, we consider the following asymptotic equivalences

$$\begin{aligned}p_3(\lambda)p_7(\lambda)&\sim-\lambda^3\frac{\rho_1^{1/2}}{\kappa^{1/2}}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\Big(\delta\gamma_2\xi-\frac{\rho_1}{\kappa}\frac{dK}{\gamma_1\tau_0}(\gamma_2^2-\gamma_1^2)\Big),\\p_3(\lambda)p_8(\lambda)&\sim-\lambda^3\frac{\rho_1^{1/2}}{\kappa^{1/2}}\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\Big(\delta\gamma_1\xi+\frac{\rho_1}{\kappa}\frac{d\hbar}{\gamma_2\tau_1}(\gamma_2^2-\gamma_1^2)\Big),\\p_4(\lambda)p_6(\lambda)&\sim-\lambda^2(\Delta_r+\Delta_c)\delta\gamma_1^2\gamma_2\frac{\rho_1^{1/2}}{\kappa^{1/2}}\Big(\lambda\xi-\frac{i}{\tau_0}\Big),\\p_4(\lambda)p_8(\lambda)&\sim-\lambda^2\delta^2\gamma_1^2\xi^2+\lambda(\gamma_1^2-\gamma_2^2)\frac{\rho_1\delta d\hbar\gamma_1}{\kappa\gamma_2\tau_1}\Big(\lambda\xi-\frac{i}{\tau_0}\Big),\\p_5(\lambda)p_6(\lambda)&\sim-\lambda^2(\Delta_r+\Delta_c)\delta\gamma_1\gamma_2^2\frac{\rho_1^{1/2}}{\kappa^{1/2}}\Big(\lambda\xi-\frac{i}{\tau_1}\Big),\\p_5(\lambda)p_7(\lambda)&\sim-\lambda^2\delta^2\gamma_1^2\xi^2-\lambda(\gamma_1^2-\gamma_2^2)\frac{\rho_1\delta dK\gamma_2}{\kappa\gamma_1\tau_0}\Big(\lambda\xi-\frac{i}{\tau_1}\Big),\end{aligned}\tag{3.77}$$

Consequently, by setting

$$\Upsilon_4=-\Big[\lambda\delta^2\xi^2-\lambda\frac{\rho_1\delta d}{\kappa}\Big(\frac{\gamma_1\hbar}{\gamma_2\tau_1}+\frac{\gamma_2K}{\gamma_1\tau_0}\Big)\xi-i\frac{\rho_1\delta d}{\kappa\tau_0\tau_1}\Big(\frac{\gamma_1\hbar}{\gamma_2}+\frac{\gamma_2K}{\gamma_1}\Big)\Big](\gamma_1^2-\gamma_2^2),$$

we get

$$\begin{aligned}\widehat{A}_1 \sim & \frac{\lambda\Big(\Delta_r\gamma_1^2+\Delta_c\gamma_2^2\Big)\Big(\xi\delta-d\frac{\rho_1}{\kappa}\big(\frac{\gamma_1\hbar}{\gamma_2\tau_1}+\frac{\gamma_2 K}{\gamma_1\tau_0}\big)\Big)\big(\gamma_1^2-\gamma_2^2\big)}{\kappa^2\Upsilon_4}\\ & -\frac{i\frac{\delta\gamma_1^2\gamma_2^2}{\tau_0\tau_1}\Big(\Delta_r+\Delta_c\Big)\big(\tau_1-\tau_0\big)}{\kappa^2\Upsilon_4}+\frac{1}{\kappa\rho_1}\Big(\rho_2-\frac{\alpha\rho_1}{\kappa}\Big)-\frac{1}{\lambda^2\rho_1},\\ \sim & \frac{\rho_2}{\kappa\rho_1}-\frac{\alpha\rho_1}{\kappa^2}-\frac{1}{\lambda^2\rho_1}+\frac{O(\lambda^3)\big(\gamma_1^2-\gamma_2^2\big)(\xi+1)}{O(\lambda^3)\big(\gamma_1^2-\gamma_2^2\big)(\xi^2+\xi)+O(\lambda^2)\big(\gamma_1^2-\gamma_2^2\big)}\\ & +\frac{O(\lambda^3)\big(\gamma_1^2-\gamma_2^2\big)(\xi+1)}{O(\lambda^3)\big(\gamma_1^2-\gamma_2^2\big)(\xi^2+\xi)+O(\lambda^2)\big(\gamma_1^2-\gamma_2^2\big)}.\end{aligned} \tag{3.78}$$

Taking $\chi_0=0$ and $\xi=0$, we have

$$\widehat{A}_1 \sim \frac{1}{\kappa\rho_1}\Big(\rho_2-\frac{\alpha\rho_1}{\kappa}\Big)-\frac{1}{\lambda^2\rho_1}+\frac{O(\lambda^3)\big(\gamma_1^2-\gamma_2^2\big)}{O(\lambda^2)\big(\gamma_1^2-\gamma_2^2\big)}+\frac{O(\lambda^2)(\tau_1-\tau_0)}{O(\lambda^2)\big(\gamma_1^2-\gamma_2^2\big)},$$

since $\gamma_1\neq\gamma_2$. Consequently,

$$\widehat{A}_1 \sim O(\lambda)+\frac{1}{\kappa\rho_1}\Big(\rho_2-\frac{\alpha\rho_1}{\kappa}\Big)-\frac{1}{\lambda^2\rho_1}+O(1).$$

Therefore,

$$\begin{aligned}||U||_{\mathcal{H}}^2 \geq \rho_1\int_0^l |v|^2\mathrm{d}x = |\lambda|^2|\widehat{A}_1|^2\rho_1\int_0^l |\sin(\beta_n x)|^2\mathrm{d}x &= |\lambda|^2|O(\lambda)|^2\frac{l}{2}\rho_1\\ &= |O(\lambda^4)|,\end{aligned}$$

which implies that $||U||_{\mathcal{H}}\to\infty$ as $\lambda\to\infty$.

4. Polynomial decay

In this section, we will show the solutions to system (1.7)–(1.14) decay to zero polynomially as $1/\sqrt{t}$ by using the Borichev and Tomilov's result [7]:

Theorem 4.1. *Let us suppose that $\chi_0=0$, $\xi\neq0$ and $\chi_1\neq0$. Then, the semigroup associated with system (1.7)–(1.14) is polynomially stable, i.e.,*

$$||S(t)U_0||_{\mathcal{H}} \leq \frac{C}{t^{1/2}}||U_0||_{D(\mathcal{A})}, \quad \forall\, t>0, \quad U_0\in D(\mathcal{A}). \tag{4.1}$$

In the particular case $\xi=0$, (4.1) holds as well. Moreover, this rate of decay is optimal.

Proof. Let us suppose that $\xi\neq0$ and $\chi_1\neq0$. It follows from Lemma 3.4 and Eq. (3.3) that

$$\begin{aligned}\kappa|\xi|(1-\frac{C}{\lambda^2})\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \leq 2|\lambda||\chi_1|\Big|\int_0^l \phi\overline{\varphi}_x\mathrm{d}x\Big|+\frac{C}{|\lambda|}||U||_{\mathcal{H}}(||\theta||_{L^2}+||P||_{L^2})\\ +C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.\end{aligned}$$

By using the Young's inequality, Lemma 3.2 and the estimate (3.12), we get

$$\kappa|\xi|(1-\frac{C}{\lambda^2})\int_0^l |\varphi_x+\psi|^2\mathrm{d}x \le 2|\lambda||\chi_1|\left|\int_0^l \phi\overline{\varphi}_x\mathrm{d}x\right| + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + \varepsilon||\phi||_{L^2}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{4.2}$$

On the other hand, combining Lemmas 3.2, 3.3 and the estimate (3.12) yields

$$\int_0^l |\phi|^2\mathrm{d}x \le \frac{C\varepsilon}{\lambda^2}||U||_{\mathcal{H}}^2 + C_\varepsilon||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \quad \forall\,\varepsilon>0. \tag{4.3}$$

From Lemmas 3.2, 3.3, 3.5 and 3.6 and by the inequalities (4.2), we have

$$\begin{aligned}||U||_{\mathcal{H}}^2 &\le |\lambda|C\left|\int_0^l \phi\overline{\varphi}_x\mathrm{d}x\right| + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + \frac{C}{|\lambda|}||U||_{\mathcal{H}}\Big(||\theta||_{L^2}+||P||_{L^2}\Big)\\ &\quad + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}\\ &\le |\lambda|C\left|\int_0^l \phi\overline{\varphi}_x\mathrm{d}x\right| + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}\\ &\le \lambda^2C\int_0^l|\phi|^2\mathrm{d}x + \varepsilon\int_0^l|\varphi_x|^2\mathrm{d}x + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2\\ &\quad + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.\end{aligned} \tag{4.4}$$

Since $\int_0^l|\varphi_x|^2\mathrm{d}x \le 2\int_0^l|\varphi_x+\psi|^2\mathrm{d}x + 2c_p\int_0^l|\psi_x|^2\mathrm{d}x$ yields

$$\begin{aligned}||U||_{\mathcal{H}}^2 &\le \lambda^2C\int_0^l|\phi|^2\mathrm{d}x + C\varepsilon\int_0^l|\varphi_x+\psi|^2\mathrm{d}x + C\varepsilon\int_0^l|\psi_x|^2\mathrm{d}x\\ &\quad + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}\\ &\le \lambda^2C\int_0^l|\phi|^2\mathrm{d}x + C\varepsilon||U||_{\mathcal{H}}^2 + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}\\ &\le \lambda^2C\int_0^l|\phi|^2\mathrm{d}x + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.\end{aligned} \tag{4.5}$$

From (4.3), we obtain

$$\begin{aligned}||U||_{\mathcal{H}}^2 &\le C\varepsilon||U||_{\mathcal{H}}^2 + \lambda^2C_\varepsilon||U||_{\mathcal{H}}||F||_{\mathcal{H}} + \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}\\ &\le \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + \lambda^2C_\varepsilon||U||_{\mathcal{H}}||F||_{\mathcal{H}} + C||U||_{\mathcal{H}}||F||_{\mathcal{H}}.\end{aligned} \tag{4.6}$$

Using the fact that $C||U||_{\mathcal{H}}||F||_{\mathcal{H}} \le \lambda^2C_\varepsilon||U||_{\mathcal{H}}||F||_{\mathcal{H}})$, the above estimation becomes

$$||U||_{\mathcal{H}}^2 \le \frac{C}{\lambda^2}||U||_{\mathcal{H}}^2 + C\varepsilon||U||_{\mathcal{H}}^2 + \lambda^2C_\varepsilon||U||_{\mathcal{H}}||F||_{\mathcal{H}}$$

TABLE 1. Abbreviations: Yes (Y), Not (N) and not applicable (N/A)

Stability conditions			Lack of decay	Decay types	Optimal rate
$\chi_0 = 0$	with	$\xi = 0$	Y	Polynomial	Y
$\chi_0 = 0$	with	$\xi \neq 0$ $\chi_1 \neq 0$	Y	Polynomial	Y
$\chi_0 = 0$	with	$\xi \neq 0$ $\chi_1 = 0$	N	Exponential	N/A

$$\leq \frac{C}{\lambda^2}||U||^2_{\mathcal{H}} + C\varepsilon||U||^2_{\mathcal{H}} + \lambda^4 C||F||^2_{\mathcal{H}}.$$

Therefore,

$$\left(1 - \frac{C}{\lambda^2} - C\varepsilon\right)||U||^2_{\mathcal{H}} \leq \lambda^4 C||F||^2_{\mathcal{H}}.$$

Choosing $|\lambda|$ large enough and ε sufficiently small, we get

$$\frac{1}{\lambda^2}||U||_{\mathcal{H}} \leq C||F||_{\mathcal{H}},$$

which is equivalent to

$$||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C\lambda^2.$$

Then, using Theorem 1.2 (see (1.21)), we obtain

$$||S(t)\mathcal{A}^{-1}||_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}} \Longrightarrow ||S(t)\mathcal{A}^{-1}F||_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}}||F||_{\mathcal{H}}.$$

From Theorem 2.1, we conclude that $0 \in \rho(\mathcal{A})$, it follows that $\mathcal{A}$ is onto over $\mathcal{H}$, then taking $\mathcal{A}U_0 = F$, we get

$$||S(t)U_0||_{\mathcal{H}} \leq \frac{C}{t^{1/2}}||U_0||_{D(\mathcal{A})}, \quad \forall\, t > 0, \quad U_0 \in D(\mathcal{A}),$$

Therefore, the solution decays polynomially.

In case $\xi = 0$, we use the same ideas as above. So the polynomial decay holds.

Finally, to show the optimality we follow the same ideas of the proof of Theorem 3.1 (ii) or Remark 3.1. Note that in case $\chi_0 = 0$ with $\xi = 0$ or in case $\chi_0 = 0$ with $\xi \neq 0$ and $\chi_1 \neq 0$, we have the inequality

$$||U||^2_{\mathcal{H}} \geq \lambda^4 C_0, \tag{4.7}$$

for $|\lambda|$ large enough. If we assume that the rate of decay can be improved from $1/t^{1/2}$ to $1/t^{1/(2-\varsigma)}$ for some $\varsigma > 0$, then we will have that

$$\frac{1}{|\lambda|^{2-\varsigma}}||U||_{\mathcal{H}}$$

must be bounded. But this is not possible because of the inequality (4.7). The proof is now complete. □

5. Conclusion

(a) Note that when $\tau_0 = \tau_1 = 0$, Cattaneo's law turns into the Fourier's law for heat transmission and Fick's law for diffusion transmission. In that case the number χ_0 does not exist and consequently $\Gamma \equiv 0$ and the condition of exponential stability over the new number $\chi_1 = 0$ is equivalent to the one over the old stability number $\chi = 0$. That is, we get the same result as proved by Aouadi et al. [3] for a Timoshenko system with thermodiffusion effects in the case $\tau_0 = \tau_1 = 0$.
(b) Table 1 summarizes the different types of decay obtained for the system (1.7)–(1.12) for different numbers of stability.

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