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Global multiplicity for very-singular elliptic problems with vanishing non-local terms

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Abstract. In this paper, we deal with issues related to global multiplicity of $W^{1,p}_{\text{loc}}(\Omega)$-solutions for the very-singular and non-local μ-problem

$$-g\left(\int_\Omega u^q\right)\Delta_p u = \mu u^{-\delta} + u^\beta \ \text{ in } \ \Omega, \quad u > 0 \ \text{ in } \Omega \quad \text{and} \ \ u = 0 \ \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\delta > 0$, $q > 0$, $0 < \beta \leq p-1$ and $g\colon [0,\infty) \to [0,\infty)$ is a continuous function that achieves critical values for the class of non-local problems (i.e., the level zero if $\beta < p-1$ and $1/\lambda_1$ if $\beta = p-1$, where λ_1 stands for the principal eigenvalue of the p-Laplacian in Ω under homogeneous Dirichlet boundary conditions). To overcome the difficulties arising from the geometry of g and the presence of very-singular term combined with a $(p-1)$-sublinear/asymptotically linear ones, we take advantage of a comparison principle for sub-supersolutions in $W^{1,p}_{\text{loc}}(\Omega)$-sense proved in Santos and Santos (Z Angew Math Phys 69:Art. 145, 2018), together with sub-supersolutions techniques and bifurcation theory.

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1. Introduction

In this paper, we deal with issues about global multiplicity of $W^{1,p}_{\text{loc}}(\Omega)$-solutions for the problem

$$(P_\mu)\ \begin{cases} -g\left(\displaystyle\int_\Omega u^q\right)\Delta_p u = \mu u^{-\delta} + u^\beta \text{ in } \Omega, \\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a smooth bounded domain, $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, $q > 0$, $\delta > 0$, $0 < \beta \leq p-1$, $\mu > 0$ is a real parameter and $g\colon [0,\infty) \to [0,\infty)$ is a continuous function that achieves the levels

$$\upsilon = \begin{cases} 0 & \text{if } 0 < \beta < p-1, \\ \lambda_1^{-1} & \text{if } \beta = p-1, \end{cases}$$

where $\lambda_1 > 0$ stands for the principal eigenvalue of the p-Laplacian in Ω under homogeneous Dirichlet boundary conditions. Throughout this work, we will call υ as a critical value for the non-local problem (P_μ) due to the statements in Theorems 1.2 and 1.3. Besides this, we set the meaning of global multiplicity when there exists a threshold parameter in such way the problem admits at least two solutions before

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it, at least one solution on it and no solution beyond these threshold whose solutions have $L^q(\Omega)$-norms controlled by two consecutive solutions of the equation $g(s) = \upsilon$, $s \geq 0$.

Non-local problems have caught a lot attention of a number of researchers in last years principally for non-local terms that do not achieve the critical values. Recently, some problems involving degenerate non-local terms (that is, $\upsilon = 0$) were dealt in [13,16,22] under smooth nonlinearities in the context of semilinear problems.

The kind of problem (P_μ) arises in various situations of practical interest such as systems of particles in thermodynamical equilibrium via gravitational (Coulomb) potential [1], $2-D$ fully turbulent behavior of real flow [4], plasma physics, among others. In population dynamics, for instance, systems of equations with non-local and singular terms arise in problems that model prey–predator interactions in a fragile environment, where the crowding at each point is associated with the distribution of the whole population in the habitat (see [12,14,15] and references therein).

Although the literature on non-local problems is vast, few authors have considered this class of problems in the presence of very(or strong)-singular nonlinearities. In [18,21] were considered such nonlinearities combined with ones that include the case $0 < \beta < p - 1$. Furthermore, in both papers, the case in which g assumes the critical value υ was not possible to be considered because of the approach and tools used there.

The principal aim of this paper is to prove global multiplicity of positive solutions in the *loc*-sense for the problem (P_μ) under the assumptions:

(g_1) there exist $0 \leq s_1 < s_2$ such that $g(s_i) = 0$, $i = 1, 2$ and $g(s) \neq 0$ for $s \in (s_1, s_2)$,
(g_2) there exist $0 \leq s_1 < s_2$ such that $g(s_i) = 1/\lambda_1$, $i = 1, 2$ and $0 < g(s) \neq 1/\lambda_1$ for $s \in (s_1, s_2)$

if $\beta < p - 1$ and $\beta = p - 1$, respectively.

The main difficulty in treating the problem (P_μ) is due to the lack of variational structure that results from the presence of non-local terms and very-singular nonlinearities; however, we highlight that some variational approaches for some classes of non-local singular problems have been recently developed. See, for instance, [5–7].

Another difficulty comes from the fact that a priori estimates may become impracticable because of the geometry of g that is permitted by the assumptions (g_1) or (g_2). To overcome these difficulties, we take advantage of a comparison principle together with sub-supersolutions technique in $W^{1,p}_{\text{loc}}(\Omega)$ to establish a relationship between the branch of solutions of (P_μ) and the unbounded connected $\Sigma \subset \mathbb{R} \times C(\overline{\Omega})$ (given by Theorem 1.1) of positive $W^{1,p}_{\text{loc}}(\Omega)$-solutions for the local problem

$$(Q_\lambda) \begin{cases} -\Delta_p u = \lambda \left(u^{-\delta} + u^{\beta}\right) \text{ in } \Omega, \\ u > 0 \text{ in } \partial\Omega,\ u > 0 \text{ on } \Omega. \end{cases}$$

After these, by establishing estimates on solutions of (Q_λ) and exploring geometric properties of the unbounded connected Σ, we are able to prove our results of global multiplicity for the problem (P_μ). In this sense, the description of the structure of Σ is essential in our approach.

Before stating our first theorem, let us make clear our understanding on Dirichlet boundary condition and solution in this context.

Definition 1.1. We say that $u \leq 0$ on $\partial\Omega$ if $(u - \epsilon)^+ \in W^{1,p}_0(\Omega)$ for every $\epsilon > 0$ given. Furthermore, $u \geq 0$ if $-u \leq 0$ and $u = 0$ on $\partial\Omega$ if u is non-negative and non-positive on $\partial\Omega$.

About solutions.

Definition 1.2. We say that u is a $W^{1,p}_{\text{loc}}(\Omega)$-solution for the problem (P_μ) if $u \in L^q(\Omega)$, $\|u\|_q^q \neq s_i$ for all $i \in \{1, 2\}$, $u > 0$ in Ω (for each $K \subset\subset \Omega$ given there exists a positive constant c_K such that $u \geq c_K > 0$ in K) and

$$g\left(\int_\Omega u^q\right)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi \mathrm{d}x = \int_\Omega \left(\mu u^{-\delta}+u^\beta\right)\varphi \mathrm{d}x \text{ for all } \varphi\in C_c^\infty(\Omega).$$

We note that the condition $\|u\|_q^q \neq s_i$ for $i\in\{1,2\}$ is necessary due to the fact that any solution u of (P_μ) must satisfy

$$g(\|u\|_q^q) = 1/\lambda \text{ for some } \lambda\in \mathrm{Proj}_{\mathbb{R}}\Sigma,$$

see discussions above Lemma 4.1.

Besides these, we need to set that $\varphi_1\in W_0^{1,p}(\Omega)$ stands for the positive normalized ($\|\varphi_1\|_\infty = 1$) eigenfunction associated with the first eigenvalue $\lambda_1>0$, that is,

$$-\Delta_p\varphi_1 = \lambda_1\varphi_1^{p-1} \text{ in } \Omega,\ \varphi_1|_{\partial\Omega}=0,$$

and by $\phi\in W_0^{1,p}(\Omega)$ the unique positive solution of

$$-\Delta_p u = 1 \text{ in } \Omega, \quad u|_{\partial\Omega}=0.$$

Now, we are ready to state our first result for the local problem (Q_λ).

Theorem 1.1. (Local case, existence) *Suppose $\delta>0$ and $0<\beta\le p-1$. Then there exists an unbounded connected $\Sigma\subset\mathbb{R}\times C(\overline{\Omega})$) of $W^{1,p}_{\mathrm{loc}}(\Omega)$-solutions for the problem (Q_λ) such that $(0,0)\in\overline{\Sigma}$ and Σ is an increasing curve (i. e., if $(\lambda',u'),(\lambda,u)\in\Sigma$ with $\lambda'<\lambda$, then $u'<u$). Moreover, if:*

(i) $0<\beta<p-1$, *then* $\mathrm{Proj}_{\mathbb{R}}\Sigma=(0,\infty)$, *$\Sigma$ bifurcates from infinity at infinity and*

$$\max\left\{\left(\frac{\lambda}{\lambda_1}\right)^{\frac{1}{p-1+\delta}},\left(\frac{\lambda}{\lambda_1}\right)^{\frac{1}{p-1-\beta}}\right\}\varphi_1\le u\le K\max\left\{\lambda^{\frac{1}{p-1+\delta}},\lambda^{\frac{1}{p-1-\beta}}\right\}\phi^t, \tag{1.1}$$

for any $(\lambda,u)\in\Sigma$, where $t=(p-1)/(p-1+\delta)$ and $K>0$ is the unique solution of the equation

$$t^{p-1}K^{p-1} = K^{-\delta}+\|\phi\|_\infty^{t(\beta+\delta)}K^\beta,$$

(ii) $\beta=p-1$, *then* $\mathrm{Proj}_{\mathbb{R}}\Sigma=(0,\lambda_1)$, *$\Sigma$ bifurcates from infinity at $\lambda=\lambda_1$ and*

$$u\ge(\lambda_1-\lambda)^{\frac{-1}{p-1+\delta}}\lambda^{\frac{1}{p-1+\delta}}\varphi_1, \tag{1.2}$$

for any $(\lambda,u)\in\Sigma$.

Our second result can be stated as follows.

Theorem 1.2. (Non-local, non-existence) *Assume $\delta>0$, $0<\beta\le p-1$ and*

$$g(s)s^{(p-1-\beta)/q} < \frac{\|\varphi_1\|_q^{(p-1-\beta)}}{\lambda_1} \text{ for all } s>0. \tag{1.3}$$

Then there is no $W^{1,p}_{\mathrm{loc}}(\Omega)$-solution for the problem (P_μ), for any $\mu>0$.

The main result of this paper concerning existence of positive solution is the next one.

Theorem 1.3. (Non-local, existence) *Assume $\delta>0$ and one of the below assumptions:*

(a) $0<\beta<p-1$, (g_1) *and*

$$g(s_*)s_*^{(p-1-\beta)/q} > (K\|\phi^t\|_q)^{(p-1-\beta)} \text{ for some } s_*\in(s_1,s_2), \tag{1.4}$$

where $t,K>0$ are as in Theorem 1.1,

(b) $\beta=p-1$, (g_2) *and*

$$g(s_*) > 1/\lambda_1 \text{ for some } s_*\in(s_1,s_2). \tag{1.5}$$

Then there exists $0<\mu_1^*<\infty$ such that the problem (P_μ):

(i) *admits at least two solutions satisfying $\|u\|_q^q\in(s_1,s_2)$, for any $0<\mu<\mu_1^*$;*

(ii) *at least one solution satisfying* $\|u\|_q^q \in (s_1, s_2)$, *for* $\mu = \mu_1^*$;
(iii) *no solution satisfying* $\|u\|_q^q \in (s_1, s_2)$, *for any* $\mu > \mu_1^*$.

As an immediate consequence of the above theorem, we have the next result.

Corollary 1.1. *Let* $\delta > 0$ *and assume that for some* $k \in \mathbb{N}$, *there exist* $0 \le s_1 < s_2 < \cdots < s_{k+1}$ *such that*

(a) *either* $0 < \beta < p - 1$, (1.4) *holds on each* (s_i, s_{i+1}) *and*

$$(g_1)'\colon g(s_i) = 0,\ i = 1, 2, \dots, k \text{ and } g(s) \neq 0 \text{ for } s \in (s_1, s_2) \cup (s_2, s_3) \cup \cdots \cup (s_{k-1}, s_{k+1}),$$

(b) *or* $\beta = p - 1$, (1.5) *holds on each* (s_i, s_{i+1}) *and*

$$(g_2)'\colon g(s_i) = 1/\lambda_1,\ i = 1, 2, \dots, k \text{ and } 0 < g(s) \neq 1/\lambda_1 \text{ for } s \in (s_1, s_2) \cup (s_2, s_3) \cup \cdots \cup (s_k, s_{k+1}).$$

Then there exist $0 < \mu_* \le \mu^* < \infty$ *such that problem* (P_μ) *admits:*

(i) *at least* $2k$ *solutions satisfying*

$$0 \le s_1 < \|u_1\|_q^q < \|u_2\|_q^q < s_2 < \|u_3\|_q^q < \|u_4\|_q^q < s_3 < \cdots < s_k < \|u_{2k-1}\|_q^q < \|u_{2k}\|_q^q < s_{k+1},$$

for any $0 < \mu < \mu_*$,

(ii) *at least two solutions satisfying* $\|u\|_q^q \in (s_1, s_{k+1})$, *for any* $0 < \mu < \mu^*$,
(iii) *at least one solution satisfying* $\|u\|_q^q \in (s_1, s_{k+1})$, *for* $\mu = \mu^*$,
(iv) *no solution satisfying* $\|u\|_q^q \in (s_1, s_{k+1})$, *for any* $\mu > \mu^*$.

Below, we draw Figs. 2 and 4 to represent two possible diagrams of solutions for the problem (P_μ) depending on the geometry of the function g (Figs. 1, 3).

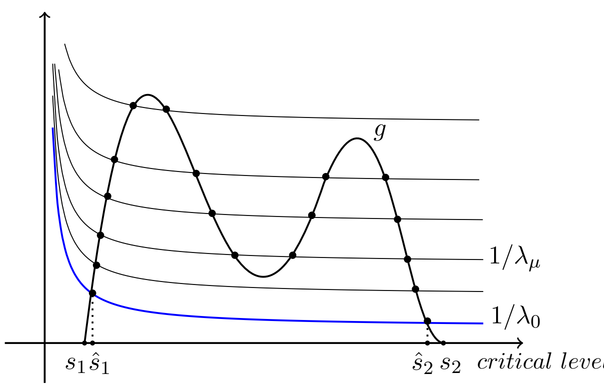


Fig. 1. Graphic of g versus the smallest level of $1/\lambda_\mu$

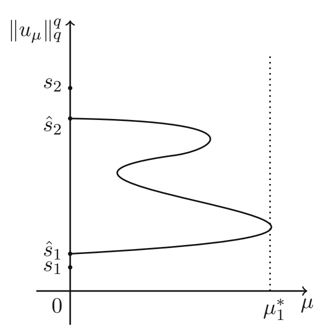


Fig. 2. The vertical projection is connected

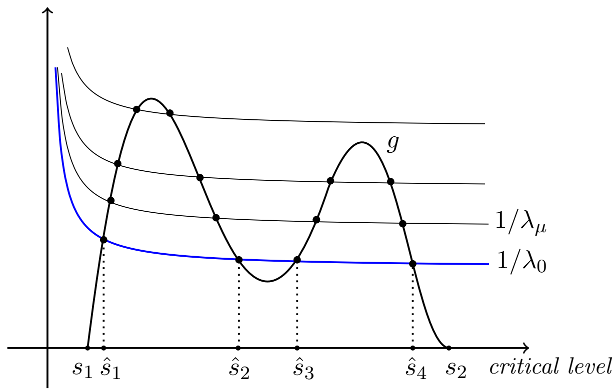


FIG. 3. Graphic of g versus the smallest level of $1/\lambda_\mu$

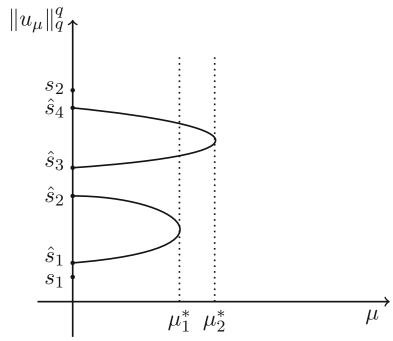


FIG. 4. The vertical projection is not connected

Let us highlight below some contributions of this paper to the literature.

(1) Theorem 1.1 is new even for the Laplacian operator principally by describing the structure of the unbounded connected in $\mathbb{R} \times C(\overline{\Omega})$ of $W^{1,p}_{\text{loc}}(\Omega)$-solutions for the $(p-1)$-asymptotic linear case and presenting a λ-behavior of such solutions,
(2) Theorem 1.3 is new by considering non-local terms g that vanishes and showing global multiplicity results under assumptions on g constrained to the interval formed by two consecutive solutions of $g(s) = \upsilon$, $s \geq 0$,
(3) Corollary 1.1 establishes a local multiplicity result whose number of solutions is connected with the amount of solutions of the equation $g(s) = \upsilon$, $s \geq 0$,
(4) the arguments used by us complete some of those ones considered in [18,21], mainly by including the $(p-1)$-asymptotically linear case and vanishing non-local terms.

This work is organizing in the following way: In Sect. 2, we give a new proof of Theorem 2.1 in [21] by including $(p-1)$-asymptotic linear terms and correcting the test functions used there. In Sect. 3, we prove the existence of an unbounded connected of $W^{1,p}_{\text{loc}}(\Omega)$-solutions for a local problem together with properties of such solutions. Finally, in Sect. 4 we take advantage of the information obtained in the previous sections to prove Theorems 1.2 and 1.3, which concerns the non-local problem.

To end this section, we point out that throughout this paper, and we make use of the following notations:

- The norm in $L^p(\Omega)$ is denoted by $\| \cdot \|_p$.

- $W_0^{1,p}(\Omega)$ is the usual Sobolev space endowed with the norm $\|\nabla u\|_p^p = \int_\Omega |\nabla u|^p \mathrm{d}x$.
- $|U|$ stands for the Lebesgue measure of measurable set $U \subset \mathbb{R}^N$.
- $C_c^\infty(\Omega) = \Big\{u\colon \Omega \to \mathbb{R}\colon u \in C^\infty(\Omega) \text{ and supp } u \subset\subset \Omega\Big\}$.
- $c, c_1, c_2, \ldots$ denote positive constants.

2. Comparison principle for sub- and supersolutions in $W^{1,p}_{\mathrm{loc}}(\Omega)$

Below, let us define subsolution and supersolution to the problem

$$\begin{cases} -\Delta_p u = \lambda \left(u^{-\delta} + u^\beta\right) \text{ in } \Omega, \\ u > 0 \text{ in } \partial\Omega, \ u > 0 \text{ on } \Omega. \end{cases} \tag{2.1}$$

Definition 2.1. A function $\underline{v} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a subsolution of (2.1) if:

(i) there is a positive constant c_K such that $\underline{v} \geq c_K$ in K for each $K \subset\subset \Omega$ given;

(ii) the inequality

$$\int_\Omega |\nabla \underline{v}|^{p-2} \nabla \underline{v} \nabla \varphi \mathrm{d}x \leq \int_\Omega \left(\frac{a(x)}{\underline{v}^\delta} + b(x) \underline{v}^\beta \right) \varphi \mathrm{d}x \tag{2.2}$$

holds for all $0 \leq \varphi \in C_c^\infty(\Omega)$. A function $\overline{v} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying (i) and the reversed inequality in (2.2) is called a supersolution to problem (2.1).

To state the comparison principle, let set the following assumptions:

(B_1) $\beta \in (0, p-1),\ b \in L^{(\frac{p^*}{\beta+1})'}(\Omega)$ and $a + b > 0$ in Ω,
(B_2) $\beta = p-1,\ b \in L^r(\Omega)$ for some $r > N/p, a > 0$ in Ω and $1 < \lambda_1(b)$,
where $\lambda_1(b) > 0$ is the principal eigenvalue of the problem

$$(EP) \ \begin{cases} -\Delta_p u = \lambda b(x) |u|^{p-2} u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega. \end{cases}$$

Theorem 2.1. ($W^{1,p}_{\mathrm{loc}}(\Omega)$-Comparison Principle) *Assume that either* (B_1) *or* (B_2) *holds. Moreover, suppose one of the following assumptions:*

(A_1) $a \in L^{(\frac{p^*}{1-\delta})'}(\Omega)$, *if* $0 < \delta < 1$,
(A_2) $a \in L^1(\Omega)$, *if* $\delta > 1$,
(A_3) $a \in L^s(\Omega)$ *for some* $s > 1$, *if* $\delta = 1$.

If $\underline{v}, \overline{v} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *are subsolution and supersolution of* (2.1), *respectively, with* $\underline{v} \leq 0$ *in* $\partial\Omega$, *then* $\underline{v} \leq \overline{v}$ *a.e. in* Ω. *Besides this, if in addition* $\underline{v}, \overline{v} \in W_0^{1,p}(\Omega)$ *and* (2.2) *is satisfied for all* $0 \leq \varphi \in W_0^{1,p}(\Omega)$, *then the same conclusion holds even for* $a \in L^1(\Omega)$ *in* (A_3).

Let us emphasize the proof of Theorem 2.1 under the assumption (B_2), since the proof on the hypothesis (B_1) was already proved by the two last authors in [21]. However, there is a failure in the construction of the test functions used there. Here, we redo such proof highlighting the correct construction of the test functions.

Let us begin the proof of Theorem 2.1 defining for each $\epsilon > 0$, the functional $J_\epsilon\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_\epsilon(\omega) = \frac{1}{p} \int_\Omega |\nabla \omega|^p \mathrm{d}x - \int_\Omega F_\epsilon(x, \omega) \mathrm{d}x,$$

where $F_\epsilon(x,t)=\int\limits_0^t f_\epsilon(x,s)ds$, with

$$f_\epsilon(x,s)=\begin{cases} a(x)(s+\epsilon)^{-\delta}+b(x)(s+\epsilon)^{\beta} & \text{if } s\geq 0\\ a(x)\epsilon^{-\delta}+b(x)\epsilon^{\beta} & \text{if } s<0\end{cases}$$

and denote by $\mathscr{C}$ the convex and closed set

$$\mathscr{C}=\left\{\omega\in W_0^{1,p}(\Omega)\colon 0\leq\omega\leq\overline{v}\right\},$$

where $\overline{v}\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is the supersolution to problem (2.1).

Lemma 2.1. *Assume either (B_1) or (B_2) and that one of the hypotheses (A_1), (A_2) or (A_3) holds. Then, the functional J_ϵ is coercive and weakly lower semicontinuous on $\mathscr{C}$.*

Proof. The proof has been done in [21] when (B_1) holds. Let us assume (B_2). So, it follows from (EP) that

$$\frac{1}{p}\int\limits_\Omega b\left(\tau\frac{w}{\tau}+(1-\tau)\frac{\epsilon}{(1-\tau)}\right)^p\mathrm{d}x\leq\frac{1}{p}\frac{\tau^{1-p}}{\lambda_1(b)}\|\nabla w\|_p^p+\frac{1}{p}(1-\tau)^{1-p}\epsilon^p\int\limits_\Omega b(x)\mathrm{d}x,$$

for $\tau\in(\lambda_1(b)^{-1/(p-1)},1)$, whence

$$J_\epsilon(\omega)\geq\begin{cases}\frac{1}{p}\left(1-\frac{\tau^{1-p}}{\lambda_1(b)}\right)\|\nabla\omega\|_p^p-C\left[\|a\|_{(\frac{p^*}{1-\delta})'}\|\omega\|_{p^*}^{1-\delta}+1\right] & \text{if } \ 0<\delta<1,\\ \frac{1}{p}\left(1-\frac{\tau^{1-p}}{\lambda_1(b)}\right)\|\nabla\omega\|_p^p-C\left[\|a\|_s\|\omega\|_{p^*}^{t}+1\right] & \text{if } \ \delta=1,\\ \frac{1}{p}\left(1-\frac{\tau^{1-p}}{\lambda_1(b)}\right)\|\nabla\omega\|_p^p-C & \text{if } \ \delta>1,\end{cases}$$

which leads to the coerciveness of J_ϵ.

The proof that J_ϵ is weakly lower semicontinuous on $\mathscr{C}$ under (B_2) is the same as done in [21] for the case (B_1). This finishes the proof of Lemma. □

As a consequence of $\mathscr{C}$ being convex and closed in the $W_0^{1,p}(\Omega)$-topology, we conclude by Lemma 2.1 that there exists a $\omega_0\in\mathscr{C}$ such that

$$J_\epsilon(\omega_0)=\inf_{\omega\in\mathscr{C}}J_\epsilon(\omega)$$

and from this, by redoing the same steps as done in [21], we obtain the inequality

$$\int\limits_\Omega|\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\varphi\mathrm{d}x\geq\int\limits_\Omega\left[a(\omega_0+\epsilon)^{-\delta}+b(\omega_0+\epsilon)^{\beta}\right]\varphi\mathrm{d}x \tag{2.3}$$

for all $\varphi\geq 0$ in $C_c^\infty(\Omega)$.

Proof of Theorem 2.1-Conclusion. For each $\epsilon>0$ and $n\in\mathbb{N}$, let us set $v_n:=\min\{\underline{v},n\}$,

$$\Omega_\epsilon:=\{x\in\Omega\colon \underline{v}(x)>\omega_0(x)+\epsilon\}\,,\ \Omega_\epsilon^n:=\{x\in\Omega\colon v_n(x)>\omega_0(x)+\epsilon\}$$

and the functions

$$\xi_1:=\left[v_n^p-(\omega_0+\epsilon)^p\right]^+\underline{v}^{1-p}\ \text{ and }\ \xi_2:=\left[v_n^p-(\omega_0+\epsilon)^p\right]^+(\omega_0+\epsilon)^{1-p}.$$

We claim that $|\Omega_\epsilon|=0$. On the contrary, there would exist some $n_0\in\mathbb{N}$ such that $|\Omega_\epsilon^n|>0$ for all $n>n_0$. In this case, $\xi_1\not\equiv 0$ and $\xi_2\not\equiv 0$. Moreover,

$$\nabla\xi_1=\left[p\frac{v_n^{p-1}}{\underline{v}^{p-1}}\nabla v_n-p\frac{(\omega_0+\epsilon)^{p-1}}{\underline{v}^{p-1}}\nabla\omega_0+(p-1)\frac{(\omega_0+\epsilon)^p}{\underline{v}^p}\nabla\underline{v}-(p-1)\frac{v_n^p}{\underline{v}^p}\nabla\underline{v}\right]\chi_{\overline{\Omega_\epsilon^n}}$$

and

$$\nabla\xi_2 = \left[\frac{pv_n^{p-1}}{(\omega_0+\epsilon)^{p-1}}\nabla v_n - \nabla(\omega_0+\epsilon) - (p-1)\frac{v_n^p}{(\omega_0+\epsilon)^p}\nabla(\omega_0+\epsilon)\right]\chi_{\overline{\Omega_\epsilon^n}}.$$

Since $\underline{v} \le 0$ on $\partial\Omega$, we have $(\underline{v}-\epsilon)^+ \in W_0^{1,p}(\Omega)$, that is, $|\nabla\underline{v}| \in L^p(\mathscr{O}_\epsilon)$, where

$$\mathscr{O}_\epsilon := \{x \in \Omega\colon \underline{v} \ge \epsilon\}.$$

By combining this information with the facts that $\overline{\Omega_\epsilon^n} \subset \mathscr{O}_\epsilon$ and $0 \le \omega_0 \in W_0^{1,p}(\Omega)$, we conclude that

$$|\nabla\xi_1| \le [p|\nabla v_n| + p|\nabla\omega_0| + 2(p-1)|\nabla\underline{v}|]\,\chi_{\overline{\Omega_\epsilon^n}} \in L^p(\Omega) \tag{2.4}$$

and

$$|\nabla\xi_2| \le \left[\frac{pn^{p-1}}{\epsilon^{p-1}}|\nabla v_n| + \left(1+\frac{(p-1)n^p}{\epsilon^p}\right)|\nabla\omega_0|\right]\chi_{\overline{\Omega_\epsilon^n}} \in L^p(\Omega). \tag{2.5}$$

Moreover, for each $x \in \Omega$, there exist $\theta_1, \theta_2 \in (0,1)$ such that

$$0 \le \xi_1(x) \le \frac{p\left[\theta_1 v_n + (1-\theta_1)(\omega_0+\epsilon)\right]^{p-1}(v_n-\omega_0-\epsilon)^+}{\underline{v}^{p-1}} \le \frac{n^{p-1}p}{\epsilon^{p-1}}(v_n-\omega_0-\epsilon)^+ \in W_0^{1,p}(\Omega) \tag{2.6}$$

and

$$0 \le \xi_2(x) \le \frac{p\left[\theta_2 v_n + (1-\theta_2)(\omega_0+\epsilon)\right]^{p-1}(v_n-\omega_0-\epsilon)^+}{(\omega_0+\epsilon)^{p-1}} \le \frac{n^{p-1}p}{\epsilon^{p-1}}(v_n-\omega_0-\epsilon)^+ \in W_0^{1,p}(\Omega) \tag{2.7}$$

Therefore, we conclude from (2.4)–(2.7) that ξ_1 and $\xi_2 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Let $(\phi_k), (\psi_k) \subset C_c^\infty(\Omega)$ be sequences satisfying

$$\phi_k \to \xi_1 \text{ and } \psi_k \to \xi_2 \text{ in } W_0^{1,p}(\Omega)$$

and set $\tilde{\phi}_k = \min\{\xi_1, \phi_k^+\}$ and $\tilde{\psi}_k = \min\{\xi_2, \psi_k^+\}$. Then, $\tilde{\phi}_k, \tilde{\psi}_k \in W_0^{1,p}(\Omega)\cap L_c^\infty(\Omega)$ and exploring that $\underline{v}$ is a subsolution of (2.1) and the inequality (2.3), one has

$$\int_\Omega |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\tilde{\phi}_k \mathrm{d}x \le \int_\Omega \left(a(x)\underline{v}^{-\delta} + b(x)\underline{v}^\beta\right)\tilde{\phi}_k \mathrm{d}x \tag{2.8}$$

and

$$\int_\Omega |\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\tilde{\psi}_k \mathrm{d}x \ge \int_\Omega \left[a(x)(\omega_0+\epsilon)^{-\delta} + b(x)(\omega_0+\epsilon)^\beta\right]\tilde{\psi}_k \mathrm{d}x. \tag{2.9}$$

Since $\underline{v} \ge \epsilon$ in supp $\tilde{\phi}_k$, one obtains from Lebesgue theorem, (2.8) and (2.9) that

$$\int_\Omega |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\xi_1 \mathrm{d}x \le \int_\Omega \left(a(x)\underline{v}^{-\delta} + b(x)\underline{v}^\beta\right)\xi_1 \mathrm{d}x \tag{2.10}$$

and

$$\int_\Omega |\nabla\omega_0|^{p-2}\nabla\omega_0\nabla\xi_2 \mathrm{d}x \ge \int_\Omega \left[a(x)(\omega_0+\epsilon)^{-\delta} + b(x)(\omega_0+\epsilon)^\beta\right]\xi_2 \mathrm{d}x. \tag{2.11}$$

Therefore, by combining (2.10) and (2.11), we get

$$\begin{aligned}&\int_{[\underline{v}\le n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{\underline{v}^p+(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\mathrm{d}x+\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\mathrm{d}x\\&\quad-\int_{[\underline{v}\le n]} |\nabla \omega_0|^{p-2}\nabla \omega_0\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\mathrm{d}x-\int_{[\underline{v}>n]} |\nabla \omega_0|^{p-2}\nabla \omega_0\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\mathrm{d}x\\&\quad\le\int_\Omega a(x)\Big[\frac{\underline{v}^{-\delta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x\\&\quad+\int_\Omega b(x)\Big[\frac{\underline{v}^{\beta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x.\end{aligned}\tag{2.12}$$

Since

$$-\int_{[\underline{v}>n]} |\nabla \omega_0|^{p-2}\nabla \omega_0\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]\mathrm{d}x=\int_{[\underline{v}>n]} |\nabla \omega_0|^{p}\Big[1+\frac{n^p(p-1)}{(\omega_0+\epsilon)^{p}}\Big]\mathrm{d}x\ge 0,$$

by (2.12) and the classical Picones's inequality, we have

$$\begin{aligned}0&\le\int_{[\underline{v}\le n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\mathrm{d}x-\int_{[\underline{v}\le n]} |\nabla \omega_0|^{p-2}\nabla \omega_0\nabla\Big[\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big]^+\mathrm{d}x\\&\le-\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+\mathrm{d}x+\int_\Omega a(x)\Big[\frac{\underline{v}^{-\delta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x\\&\quad+\int_\Omega b(x)\Big[\frac{\underline{v}^{\beta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\Big][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x.\end{aligned}\tag{2.13}$$

Below, let us estimate the integrals in (2.13). For the last two integrals, we can deduce by the assumption either (B_1) or (B_2) that there exists $\epsilon'>0$ such that

$$\begin{aligned}&\int_\Omega a(x)\left[\frac{\underline{v}^{-\delta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{-\delta}}{(\omega_0+\epsilon)^{p-1}}\right][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x\\&\quad+\int_\Omega b(x)\left[\frac{\underline{v}^{\beta}}{\underline{v}^{p-1}}-\frac{(\omega_0+\epsilon)^{\beta}}{(\omega_0+\epsilon)^{p-1}}\right][v_n^p-(\omega_0+\epsilon)^p]^+\mathrm{d}x<-2\epsilon',\end{aligned}\tag{2.14}$$

for all $n>n_0$.

On the other hand, estimating the first integral in the second line, we have

$$\begin{aligned}-\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+ \mathrm{d}x &= n^p(p-1)\int_{[\underline{v}>n]} |\nabla \underline{v}|^p \underline{v}^{-p}\chi_{\overline{\Omega^n_\epsilon}}\mathrm{d}x\\ &\quad + p\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla \omega_0 \frac{(\omega_0+\epsilon)^{p-1}}{\underline{v}^{p-1}}\chi_{\overline{\Omega^n_\epsilon}}\mathrm{d}x\\ &\quad -(p-1)\int_{[\underline{v}>n]} |\nabla \underline{v}|^p \frac{(\omega_0+\epsilon)^p}{\underline{v}^p}\chi_{\overline{\Omega^n_\epsilon}}\mathrm{d}x\\ &\le (p-1)\int_{[\underline{v}>n]} |\nabla \underline{v}|^p\chi_{\mathscr{O}_\epsilon}\mathrm{d}x\\ &\quad + p\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-1}|\nabla\omega_0|\chi_{\mathscr{O}_\epsilon}\mathrm{d}x.\end{aligned}$$

Since $|\nabla \underline{v}| \in L^p(\mathscr{O}_\epsilon)$, for all $n > n_0$ large enough, one gets

$$-\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big[\frac{n^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big]^+ \mathrm{d}x \le (p-1)\int_{[\underline{v}>n]} |\nabla \underline{v}|^p\chi_{\mathscr{O}_\epsilon}\mathrm{d}x + p\int_{[\underline{v}>n]} |\nabla \underline{v}|^{p-1}|\nabla\omega_0|\chi_{\mathscr{O}_\epsilon}\mathrm{d}x < \epsilon'. \tag{2.15}$$

Hence, getting back to the inequality (2.13) and using (2.14) and (2.15), we obtain

$$0 \le \int_{[\underline{v}\le n]} |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\Big(\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{\underline{v}^{p-1}}\Big)\mathrm{d}x - \int_{[\underline{v}\le n]} |\nabla \omega_0|^{p-2}\nabla \omega_0\nabla\Big(\frac{\underline{v}^p-(\omega_0+\epsilon)^p}{(\omega_0+\epsilon)^{p-1}}\Big)\mathrm{d}x < 0,$$

which is an absurd. Therefore $|\Omega^n_\epsilon| = 0$ for all n, which implies $|\Omega_\epsilon| = 0$ and so $\underline{v} \le \omega_0+\epsilon \le \overline{v}+\epsilon$ a.e in Ω for all $\epsilon > 0$, whence $\underline{v} \le \overline{v}$ in Ω.

To finish the proof, let us assume that $\underline{v}$, $\overline{v} \in W^{1,p}_0(\Omega)$ and (2.2) is satisfied for all $0 \le \varphi \in W^{1,p}_0(\Omega)$. By supposing $(\underline{v}-\overline{v})^+ \ne 0$, defining $\underline{v}^\epsilon_n(x) := \min\{\underline{v}(x)+\epsilon, n\}$, $\overline{v}^\epsilon_n(x) := \min\{\overline{v}(x)+\epsilon, n\}$ and testing the differential inequalities for $\underline{v}$ and $\overline{v}$ against

$$\xi_1 = \Big[(\underline{v}^\epsilon_n)^p - (\overline{v}^\epsilon_n)^p\Big]^+(\underline{v}^\epsilon_n)^{1-p} \text{ and } \xi_2 = \Big[(\underline{v}^\epsilon_n)^p - (\overline{v}^\epsilon_n)^p\Big]^+(\overline{v}^\epsilon_n)^{1-p},$$

respectively, we obtain

$$\begin{aligned}&\int_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\le n]}\Big(-|\nabla \underline{v}|^{p-2}\nabla\underline{v}\nabla\overline{v}\frac{(\overline{v}+\epsilon)^{p-1}p}{n^{p-1}} + |\nabla\overline{v}|^p + \frac{(p-1)n^p|\nabla\overline{v}|^p}{(\overline{v}+\epsilon)^p}\Big)\mathrm{d}x\\ &+\int_{[\overline{v}+\epsilon\le\underline{v}+\epsilon\le n]}\Big(|\nabla\underline{v}|^p - p\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^{p-1}|\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\overline{v} + (p-1)\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^p|\nabla\underline{v}|^p\\ &+ |\nabla\overline{v}|^p - p\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^{p-1}|\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\underline{v} + (p-1)\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^p|\nabla\overline{v}|^p\Big)\mathrm{d}x\\ &= \int_\Omega |\nabla\underline{v}|^{p-2}\nabla\underline{v}\nabla\xi_1\mathrm{d}x - \int_\Omega |\nabla\overline{v}|^{p-2}\nabla\overline{v}\nabla\xi_2\mathrm{d}x\\ &\le \int_\Omega a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}^\epsilon_n)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}^\epsilon_n)^{p-1}}\Big][(\underline{v}^\epsilon_n)^p - (\overline{v}^\epsilon_n)^p]^+\mathrm{d}x + \int_\Omega b\Big[\frac{\underline{v}^{\beta}}{(\underline{v}^\epsilon_n)^{p-1}} - \frac{\overline{v}^{\beta}}{(\overline{v}^\epsilon_n)^{p-1}}\Big][(\underline{v}^\epsilon_n)^p - (\overline{v}^\epsilon_n)^p]^+\mathrm{d}x.\end{aligned}$$

Denoting by

$$I = \int\limits_{[\overline{v}+\epsilon\le \underline{v}+\epsilon\le n]} \Big(|\nabla \underline{v}|^p - p\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^{p-1}|\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla \overline{v} + (p-1)\Big(\frac{\overline{v}+\epsilon}{\underline{v}+\epsilon}\Big)^{p}|\nabla \underline{v}|^p$$
$$+ |\nabla \overline{v}|^p - p\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^{p-1}|\nabla \overline{v}|^{p-2}\nabla \overline{v}\nabla \underline{v} + (p-1)\Big(\frac{\underline{v}+\epsilon}{\overline{v}+\epsilon}\Big)^{p}|\nabla \overline{v}|^p\Big)\mathrm{d}x,$$

and using the previous inequality along with the Picone's inequality, we have

$$0 \le I \le \int\limits_{[\underline{v}+\epsilon> n,\overline{v}+\epsilon\le n]} p|\nabla \underline{v}|^{p-1}|\nabla \overline{v}|\mathrm{d}x + \int\limits_{\Omega} a\left[\frac{\underline{v}^{-\delta}}{(\underline{v}_n^\epsilon)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}_n^\epsilon)^{p-1}}\right][(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p]^+\mathrm{d}x$$
$$+ \int\limits_{[\underline{v}+\epsilon> n,\overline{v}+\epsilon\le n]} b\Big[\frac{\underline{v}^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p - (\overline{v}+\epsilon)^p]^+\mathrm{d}x$$
$$+ \int\limits_{[\overline{v}+\epsilon\le \underline{v}+\epsilon\le n]} b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]^+\mathrm{d}x. \tag{2.16}$$

Let us consider each one of the integrals in (2.16). The dominated convergence theorem implies that

$$\int\limits_{[\underline{v}+\epsilon> n,\overline{v}+\epsilon\le n]} |\nabla \underline{v}|^{p-1}|\nabla \overline{v}|\mathrm{d}x \overset{n\to\infty}{\longrightarrow} 0. \tag{2.17}$$

By manipulating the second integral in (2.16), we obtain

$$\int\limits_{\Omega} a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}_n^\epsilon)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}_n^\epsilon)^{p-1}}\Big][(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p]^+\mathrm{d}x \le 0 \tag{2.18}$$

for all $n \in \mathbb{N}$ and $\epsilon > 0$. To the second last one, the dominated convergence theorem implies again

$$\int\limits_{[\underline{v}+\epsilon> n,\overline{v}+\epsilon\le n]} b\Big[\frac{\underline{v}^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p - (\overline{v}+\epsilon)^p]\mathrm{d}x$$
$$\le \int\limits_{[\underline{v}+\epsilon> n,\overline{v}+\epsilon\le n]} b\left[\underline{v}^\beta(\underline{v}+\epsilon) + \overline{v}^\beta(\underline{v}+\epsilon)\right]\mathrm{d}x \overset{n\to\infty}{\longrightarrow} 0. \tag{2.19}$$

For the last integral, since

$$b\left[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\right][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]^+ \le b\left[\underline{v}^\beta(\underline{v}+\epsilon) + \overline{v}^\beta(\overline{v}+\epsilon)\right] \in L^1(\Omega),$$

it follows from Fatou's lemma that

$$\limsup_{\epsilon\to 0} \int\limits_{[\overline{v}+\epsilon\le \underline{v}+\epsilon\le n]} b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]\mathrm{d}x$$
$$\le \int\limits_{[\overline{v}\le \underline{v}\le n]} b\Big[\frac{\underline{v}^\beta}{\underline{v}^{p-1}} - \frac{\overline{v}^\beta}{\overline{v}^{p-1}}\Big][\underline{v}^p - \overline{v}^p]\mathrm{d}x \le 0, \text{ for all } n \in \mathbb{N}. \tag{2.20}$$

Hence, going back to (2.16) and using (2.17)–(2.20), we get

$$0 \le \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty} I \le \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty}\Bigg(\int\limits_{\Omega} a\Big[\frac{\underline{v}^{-\delta}}{(\underline{v}_n^\epsilon)^{p-1}} - \frac{\overline{v}^{-\delta}}{(\overline{v}_n^\epsilon)^{p-1}}\Big][(\underline{v}_n^\epsilon)^p - (\overline{v}_n^\epsilon)^p]^+\mathrm{d}x$$

$$+ \int\limits_{[\underline{v}+\epsilon>n,\overline{v}+\epsilon\leq n]} b\Big[\frac{\underline{v}^\beta}{n^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][n^p - (\overline{v}+\epsilon)^p]\mathrm{d}x$$

$$+ \int\limits_{[\overline{v}+\epsilon\leq\underline{v}+\epsilon\leq n]} b\Big[\frac{\underline{v}^\beta}{(\underline{v}+\epsilon)^{p-1}} - \frac{\overline{v}^\beta}{(\overline{v}+\epsilon)^{p-1}}\Big][(\underline{v}+\epsilon)^p - (\overline{v}+\epsilon)^p]\mathrm{d}x\Bigg).$$

Since $(\underline{v}-\overline{v})^+ \neq 0$ and $a+b>0$ holds, we obtain from the previous inequality that

$$0 \leq \limsup_{\epsilon\to 0^+}\liminf_{n\to\infty} I < 0,$$

which is an absurd. Therefore $(\underline{v}-\overline{v})^+ = 0$ and this ends the proof. □

3. Unbounded connected for a local very-singular problem

In this section, let us consider the local problem

$$(Q_\lambda)\ \begin{cases} -\Delta_p u = \lambda\left(u^{-\delta} + u^\beta\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega,\ u > 0 \text{ in } \Omega, \end{cases}$$

where $\delta > 0$, $0 < \beta \leq p-1$ and $\lambda > 0$ is a real parameter. The main goal is to establish results about existence, non-existence and uniqueness of $W^{1,p}_{\mathrm{loc}}(\Omega)$-solutions and their λ-behavior.

This kind of problem has been widely considered in the literature to answer issues about existence and uniqueness of solutions by using different techniques in a variety of environments, principally in the context of finite energy. The interested reader may consult, for instance, [2,3,8–11,17] and references therein for more information on these issues. In particular, Rabinowitz et al. [10] considered Problem (Q_λ) for the case $p=2$, and recently the last two authors in [18] complemented some results of [10] for the case $p>1$ with solutions in loc-sense, but excluding nonlinearities that have $(p-1)$-asymptotically linear behaviors.

To deal problem (Q_λ) for arbitrary $\delta > 0$, we are going to approach it via the approximate ϵ-problems

$$(Q_{\lambda,\epsilon})\ \begin{cases} -\Delta_p u = \lambda\left[(u+\epsilon)^{-\delta} + (u+\epsilon)^{p-1}\right] \text{ in } \Omega, \\ \qquad u|_{\partial\Omega} = 0 \text{ on } \partial\Omega;\ u > 0 \text{ in } \Omega, \end{cases}$$

$\epsilon > 0$.

In order to get an unbounded continuum $\Sigma \subset R \times C(\overline{\Omega})$ of positive solution of (Q_λ), we will prove in this section the existence of an unbounded ϵ-continuum of $W^{1,p}_0(\Omega)$-solutions, denoted by Σ_ϵ, for the problem $(Q_{\lambda,\epsilon})$ and establish the limit behavior of Σ_ϵ as ϵ goes to zero.

Proposition 3.1. *For each $\epsilon > 0$, the problem $(Q_{\lambda,\epsilon})$ admits an unbounded continuum $\Sigma_\epsilon \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of $W^{1,p}_0(\Omega)$-solutions emanating from $(0,0)$.*

Proof. Let $\epsilon > 0$. We know from the classical theory of existence and regularity for elliptic equations that

$$\begin{cases} -\Delta_p u = \lambda\left[(|v|+\epsilon)^{-\delta} + (|v|+\epsilon)^{p-1}\right] \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

admits a unique solution $u \in C^{1,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, for any $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ fixed. Thus, the operator $T\colon \mathbb{R}^+ \times C(\overline{\Omega}) \to C(\overline{\Omega})$, which associates each pair $(\lambda, v) \in \mathbb{R}^+ \times C(\overline{\Omega})$ to the unique weak solution of (3.1), is well-defined.

By following standard arguments, we are able to prove that T is a compact operator so that Theorem 3.2 of [20] can be applied to produce an unbounded ϵ-continuum $\Sigma_\epsilon \subset \mathbb{R}^+ \times C(\overline{\Omega})$ of solutions of $T(\lambda, u) =$

u. Moreover, since $T(0,v)=0$ for all $v\in C(\overline{\Omega})$ and $T(\lambda,0)=0$ implies $\lambda=0$, we can conclude that $\Sigma_\epsilon\setminus\{(0,0)\}$ consists of nontrivial solutions of $(Q_{\lambda,\epsilon})$.

Besides this, by using that $0<\lambda\left((|v|+\epsilon)^{-\delta}+|v|+\epsilon)^{\beta}\right)\in L^\infty(\Omega)$, for each $v\in C(\overline{\Omega})$, and the classical strong maximum principle, we obtain that $T\left((\mathbb{R}^+\setminus\{(0,0)\})\times C(\overline{\Omega})\right)\subset C(\overline{\Omega})_+$, where $C(\overline{\Omega})_+=\{u\in C(\overline{\Omega})\colon u>0 \text{ in } \Omega\}$. Therefore, Σ_ϵ is an unbounded ϵ-continuum which consists of nontrivial solutions of $(Q_{\lambda,\epsilon})$, for any $\epsilon>0$ fixed. □

Lemma 3.1. *Let* $\epsilon>0$. *Then* $\mathrm{Proj}_{\mathbb{R}^+}\Sigma_\epsilon\subset(0,\lambda_1)$.

Proof. Let $(\lambda,u_\epsilon)\in\Sigma_\epsilon$. Since $\varphi_1,u_\epsilon\in C(\overline{\Omega})$, there exists $c>0$ such that

$$\int_\Omega((c\varphi_1+\epsilon)^p-(u_\epsilon+\epsilon)^p)^+>0.$$

Defining

$$\psi_1=\frac{[(c\varphi_1+\epsilon)^p-(u_\epsilon+\epsilon)^p]^+}{(c\varphi_1+\epsilon)^{p-1}}\text{ and }\psi_2=\frac{[(c\varphi_1+\epsilon)^p-(u_\epsilon+\epsilon)^p]^+}{(u_\epsilon+\epsilon)^{p-1}},$$

we have that ψ_1,ψ_2 are non-trivial functions and belongs to $W_0^{1,p}(\Omega)\cap C(\overline{\Omega})$. Since

$$\begin{cases}-\Delta_p u_\epsilon=\lambda\left[(u_\epsilon+\epsilon)^{-\delta}+(u_\epsilon+\epsilon)^{p-1}\right]\text{ in }\Omega,\\ -\Delta_p(c\varphi_1)\le\lambda_1(c\varphi_1+\epsilon)^{p-1}\text{ in }\Omega,\\ c\varphi_1,u>0\text{ in }\Omega;\ c\varphi_1,u=0\text{ on }\partial\Omega,\end{cases}$$

we obtain from Picone's inequality that

$$\begin{aligned}0&\le\int_\Omega|\nabla(c\varphi_1)|^{p-2}\nabla(c\varphi_1)\nabla\psi_1\mathrm{d}x-\int_\Omega|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\nabla\psi_2\mathrm{d}x\\&<(\lambda_1-\lambda)\int_\Omega((c\varphi_1+\epsilon)^p-(u_\epsilon+\epsilon)^p)^+\mathrm{d}x.\end{aligned}$$

Therefore, $\lambda<\lambda_1$, which ends the proof. □

The next result completes the description of Σ_ϵ.

Lemma 3.2. *For each* $\epsilon>0$, Σ_ϵ *bifurcates from infinity at* $\lambda=\lambda_1$.

Proof. As a consequence of Proposition 3.1 and Lemma 3.1, there must exist $\lambda^*\in(0,\lambda_1]$ and a sequence $(\lambda_n,u_n)\subset\Sigma_\epsilon$ such that

$$\begin{cases}\lambda_n\to\lambda^*\\ \|u_n\|_\infty\to\infty.\end{cases}$$

By contradiction, assume $\lambda^*<\lambda_1$. In this case, by taking an $\eta>0$ such that $\lambda^*+\eta<\lambda_1$ and testing $(Q_{\lambda_n,\epsilon})$ against u_n, we obtain

$$\int_\Omega|\nabla u_n|^p\mathrm{d}x\le(\lambda^*+\eta)\int_\Omega\left[(u_n+\epsilon)^{-\delta}u_n+(u_n+\epsilon)^p\right]\mathrm{d}x,$$

for all n large enough. Since $\lambda_1>0$ is the first eigenvalue, we get

$$[\lambda_1-(\lambda^*+\eta)]\|u_n\|_p^p\le(\lambda^*+\eta)\int_\Omega\left[(u_n+\epsilon)^{-\delta}u_n+(u_n+\epsilon)^p-u_n^p\right]\mathrm{d}x$$

$$\leq (\lambda^* + \eta) \int_\Omega \left[(u_n + \epsilon)^{-\delta} u_n + p\epsilon (u_n + \epsilon)^{p-1} \right] \mathrm{d}x$$
$$\leq c_\epsilon \left(\|u_n\|_p + \|u_n\|_p^{p-1} + 1 \right),$$

which implies $\|u_n\|_p \leq c_1$, for some $c_1 > 0$ independent of n. By combining this fact with Lemma A.1 in [19], we conclude that $\|u_n\|_\infty \leq c_2$, for some $c_2 > 0$ independent of n, which contradicts $\|u_n\|_\infty \to \infty$. This ends the proof. □

3.1. Proof of Theorem 1.1

Here, we are going to complete the proof of Theorem 1.1 by using the results proved in the previous section and some results of [18].

Proof of Theorem 1.1-completed. Proof of (i) The existence of the unbounded connected Σ with the properties stated is a consequence of Theorem 1.1 in [18] and Theorem 2.1. It remains to prove estimate (1.1). For this purpose, let us consider $(\lambda, u) \in \Sigma$ and construct appropriate sub- and supersolutions for the problem (Q_λ) to obtain the desired estimates by using Theorem 2.1 . To construct a such subsolution, let us set $\underline{u}_1 = \gamma\varphi_1$ with $\gamma = (\lambda/\lambda_1)^{\frac{1}{p-1+\delta}}$ to infer that

$$\gamma^{p-1}\lambda_1 \|\varphi_1\|_\infty^{p-1+\delta} = \gamma^{p-1}\lambda_1 = \gamma^{-\delta}\lambda,$$

whence

$$-\Delta_p \underline{u}_1 = \gamma^{p-1}\lambda_1 \varphi_1^{p-1} \leq \lambda\gamma^{-\delta}\varphi_1^{-\delta} = \lambda \underline{u}_1^{-\delta} \leq \lambda \left(\underline{u}_1^{-\delta} + \underline{u}_1^{\beta} \right),$$

that is, $\underline{u}_1$ is a subsolution of Problem (Q_λ), which implies by Theorem 2.1 that

$$\left(\frac{\lambda}{\lambda_1} \right)^{\frac{1}{p-1+\delta}} \varphi_1 \leq u \text{ in } \Omega. \tag{3.2}$$

Proceeding in a similar way, we are able to prove that

$$\underline{u}_2 = \left(\frac{\lambda}{\lambda_1} \right)^{\frac{1}{p-1-\beta}} \varphi_1$$

is a subsolution of the problem (Q_λ) as well. Thus, again by Theorem 2.1 we obtain

$$\left(\frac{\lambda}{\lambda_1} \right)^{\frac{1}{p-1-\beta}} \varphi_1 \leq u \text{ in } \Omega, \tag{3.3}$$

which leads to the first inequality in (1.1) after gathering the information in (3.2) and (3.3).

To show the second inequality in (4.7), we will prove that $\overline{u} = K\lambda^\tau \phi^t$ is a supersolution of the problem (Q_λ) for appropriate values of $K > 0$ and $t \in (0, 1)$. Indeed, $K\lambda^\tau \phi^t$ is a supersolution of (Q_λ) if

$$\int_\Omega (K\lambda^\tau)^{p-1} \phi^{(t-1)(p-1)} t^{p-1} |\nabla\phi|^{p-2} \nabla\phi \nabla\varphi \geq \lambda \int_\Omega \left[(K\lambda^\tau)^{-\delta} \phi^{-t\delta} + (K\lambda^\tau)^\beta \phi^{t\beta} \right] \varphi,$$

for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\Omega)$, which is equivalent to

$$(K\lambda^\tau t)^{p-1} \int_\Omega \left[\varphi \phi^{(t-1)(p-1)} - (t-1)(p-1) \phi^{(t-1)(p-1)-1} |\nabla\phi|^p \varphi \right]$$
$$\geq \lambda \int_\Omega \left[(K\lambda^\tau)^{-\delta} \phi^{-t\delta} + (K\lambda^\tau)^\beta \phi^{t\beta} \right] \varphi, \text{ for all } 0 \leq \varphi \in C_c^\infty(\Omega).$$

By taking $t = (p-1)/(p-1+\delta) \in (0,1)$, a sufficient condition for the above inequality to occur is

$$(K\lambda^\tau t)^{p-1}\phi^{(t-1)(p-1)} \geq \lambda\left[(K\lambda^\tau)^{-\delta}\phi^{-t\delta} + (K\lambda^\tau)^\beta\phi^{t\beta}\right],$$

and, for this inequality being true, it suffices that

$$(K\lambda^\tau)^{p-1}t^{p-1} \geq \lambda\left[(K\lambda^\tau)^{-\delta} + (K\lambda^\tau)^\beta\|\phi^{t(\beta+\delta)}\|_\infty\right]. \tag{3.4}$$

Let us consider two cases. If $\lambda \leq 1$, by choosing $\tau = 1/(p-1+\delta)$, the inequality (3.4) becomes

$$(Kt)^{p-1} \geq K^{-\delta} + \lambda^{\frac{\beta+\delta}{p-1+\delta}}K^\beta\|\phi^{t(\beta+\delta)}\|_\infty,$$

while (3.4) turns into

$$(Kt)^{p-1} \geq \lambda^{-\frac{\beta+\delta}{p-1+\delta}}K^{-\delta} + K^\beta\|\phi^{t(\beta+\delta)}\|_\infty$$

if $\lambda > 1$ and $\tau = 1/(p-1-\beta)$.

In any case, by taking K as the unique solution of

$$(Kt)^{p-1} = K^{-\delta} + K^\beta\|\phi^{t(\beta+\delta)}\|_\infty$$

and setting

$$\tau = \begin{cases} 1/(p-1+\delta) & \text{if } \lambda \leq 1, \\ 1/(p-1-\beta) & \text{if } \lambda > 1, \end{cases}$$

we obtain that

$$\overline{u} = K\lambda^\tau\phi^t$$

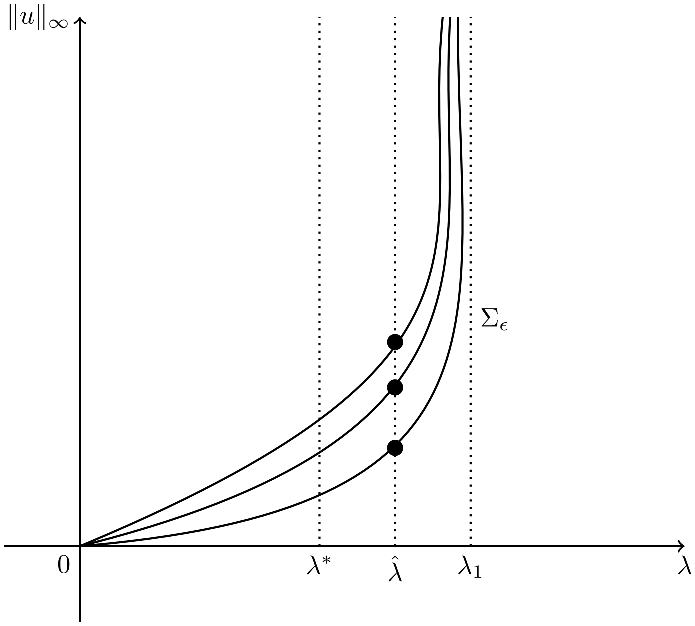
is a supersolution of (Q_λ). Again, as a consequence of Theorem 2.1, we conclude $u \leq \overline{u}$, whence the second inequality in (1.1) is true.

Proof of (ii) Consider the case $\beta = p-1$. It follows from the results proved in the previous section and a similar argument as done in the proof of Theorem 1.1 in [18] that there exists an unbounded connected Σ of positive solutions of (Q_λ) such that $(0,0) \in \overline{\Sigma}$. Moreover, such Σ is obtained as a result of a limit process with ϵ converging to zero in $(Q_{\lambda,\epsilon})$. As consequences of this process and Lemma 3.1, we obtain that $\mathrm{Proj}_{\mathbb{R}^+}\Sigma \subset (0,\lambda_1)$, because $\mathrm{Proj}_{\mathbb{R}^+}\Sigma_\epsilon \subset (0,\lambda_1)$ for any $\epsilon > 0$.

Next, let us prove that Σ bifurcates from infinity at $\lambda = \lambda_1$. Assume by contradiction that Σ bifurcated at some $0 \leq \lambda^* < \lambda_1$. So, we could take a $\hat{\lambda} \in (\lambda^*,\lambda_1)$ and find a pair $(\hat{\lambda}, u_{\hat{\lambda},\epsilon}) \in \Sigma_\epsilon$, for each $\epsilon > 0$. That is, $(\hat{\lambda}, u_{\hat{\lambda},\epsilon})$ is such that

$$\begin{cases} -\Delta_p u_{\hat{\lambda},\epsilon} = \hat{\lambda}\left[(u_{\hat{\lambda},\epsilon}+\epsilon)^{-\delta} + (u_{\hat{\lambda},\epsilon}+\epsilon)^{p-1}\right] \text{ in } \Omega, \\ u_{\hat{\lambda},\epsilon}|_{\partial\Omega} = 0 \text{ on } \partial\Omega,\ u_{\hat{\lambda},\epsilon} > 0 \text{ in } \Omega, \end{cases} \tag{3.5}$$

for each $\epsilon > 0$ (see the figure below).



As a consequence of (3.5), we have that $u_{\hat{\lambda},\epsilon}$ is a supersolution of

$$\begin{cases} -\Delta_p u = \hat{\lambda}(u+1)^{-\delta} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{cases} \tag{3.6}$$

for any $\epsilon \in (0,1)$. Besides this, $\underline{u} = \hat{\lambda}^t \varphi_1$ is a subsolution of (3.6) as long as $t \in \mathbb{R}$ satisfies

$$\|\varphi_1\|_\infty^{p-1} \hat{\lambda}^{t(p-1)-1} \lambda_1 \left[\hat{\lambda}^t \|\varphi_1\|_\infty + 1\right]^\delta \leq 1,$$

which permits us to infer that $\hat{\lambda}^t \varphi_1 \leq u_{\hat{\lambda},\epsilon}$ in Ω by reducing $t \in \mathbb{R}$, if it is necessary.

Finally, we claim that

$$u_{\hat{\lambda},\epsilon} + \epsilon \leq u_{\hat{\lambda},1} + 1 \text{ in } \Omega,$$

for all $\epsilon \in (0,1)$. In fact, to prove this statement let us suppose, by contradiction, that

$$\left|\{x \in \Omega \colon u_{\hat{\lambda},\epsilon} + \epsilon > u_{\hat{\lambda},1} + 1\}\right| > 0$$

and define

$$\psi_1 = \frac{\left[(u_{\hat{\lambda},\epsilon} + \epsilon)^p - (u_{\hat{\lambda},1} + 1)^p\right]^+}{(u_{\hat{\lambda},\epsilon} + \epsilon)^{p-1}} \text{ and } \psi_2 = \frac{\left[(u_{\hat{\lambda},\epsilon} + \epsilon)^p - (u_{\hat{\lambda},1} + 1)^p\right]^+}{(u_{\hat{\lambda},1} + 1)^{p-1}}.$$

Since $0 < u_{\hat{\lambda},\epsilon}, u_{\hat{\lambda},1} \in C_0^1(\overline{\Omega})$, we obtain that $\psi_1, \psi_2 \in W_0^{1,p}(\Omega)$. Hence, by testing $(Q_{\hat{\lambda},\epsilon})$ and $(Q_{\hat{\lambda},1})$ against ψ_1, ψ_2, respectively, applying Picone's inequality and using the contradiction assumption, we get

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla u_{\hat{\lambda},\epsilon}|^{p-2} \nabla u_{\hat{\lambda},\epsilon} \nabla \Psi_1 \mathrm{d}x - \int_\Omega |\nabla u_{\hat{\lambda},1}|^{p-2} \nabla u_{\hat{\lambda},1} \nabla \Psi_2 \mathrm{d}x \\ &= \hat{\lambda} \int_\Omega \left[\frac{1}{(u_{\hat{\lambda},\epsilon} + \epsilon)^{p-1+\delta}} - \frac{1}{(u_{\hat{\lambda},1} + 1)^{p-1+\delta}}\right] \left[(u_{\hat{\lambda},\epsilon} + \epsilon)^p - (u_{\hat{\lambda},1} + 1)^p\right]^+ \mathrm{d}x < 0, \end{aligned}$$

which is impossible. Therefore, $u_{\hat{\lambda},\epsilon} + \epsilon \leq u_{\hat{\lambda},1} + 1$ in Ω, whence

$$\hat{\lambda}^t \varphi_1 \leq u_{\hat{\lambda},\epsilon} \leq u_{\hat{\lambda},1} + 1 \in L^\infty(\Omega), \tag{3.7}$$

for all $\epsilon \in (0,1)$.

After (3.7) and following the same approach as used in the proof of Theorem 1.1 in [21], we conclude that $u_{\hat{\lambda},\epsilon} \to v_{\hat{\lambda}}$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ as $\epsilon \to 0$, where $v_{\hat{\lambda}}$ is the unique solution of $(Q_{\hat{\lambda}})$. However, as we are assuming that Σ bifurcates from infinity at $\lambda^* < \hat{\lambda}$, it is possible to find some $\lambda < \lambda^* < \hat{\lambda}$ such that $\|v_\lambda\|_\infty > \|v_{\hat{\lambda}}\|_\infty$, which contradicts the monotonicity of $\lambda \mapsto v_\lambda$ that comes from Theorem 2.1. Therefore, $\hat{\lambda} = \lambda_1$.

To end the proof, let us show (1.2). So, define $\underline{u} = (\lambda_1 - \lambda)^{\tau_1} \lambda^{\tau_2} \varphi_1$, where τ_1 and τ_2 will be chosen in such a way that $\underline{u}$ happens to be a subsolution of (Q_λ). For this, it is enough that the following inequality holds

$$-\Delta_p \underline{u} = (\lambda_1 - \lambda)^{\tau_1(p-1)} \lambda^{\tau_2(p-1)} \lambda_1 \varphi_1^{p-1} \leq \lambda \left[((\lambda_1 - \lambda)^{\tau_1} \lambda^{\tau_2} \varphi_1)^{-\delta} + ((\lambda_1 - \lambda)^{\tau_1} \lambda^{\tau_2} \varphi_1)^{p-1} \right],$$

that is,

$$\varphi_1^{p-1} \lambda^{\tau_2(p-1)} (\lambda_1 - \lambda)^{\tau_1(p-1)+1} \leq \varphi_1^{-\delta} \lambda^{1-\tau_2\delta} (\lambda_1 - \lambda)^{-\delta\tau_1}$$

or equivalently

$$\varphi_1^{p-1+\delta} \lambda^{\tau_2(p-1+\delta)-1} (\lambda_1 - \lambda)^{\tau_1(p-1+\delta)+1} \leq 1 \tag{3.8}$$

Since $\|\varphi_1\|_\infty = 1$, by taking $\tau_2 = 1/(p-1+\delta) = -\tau_1$ the inequality (3.8) holds, whence

$$\underline{u} = (\lambda_1 - \lambda)^{\frac{-1}{p-1+\delta}} \lambda^{\frac{1}{p-1+\delta}} \varphi_1$$

is a subsolution of (Q_λ). Hence, by Theorem 2.1 we obtain $(\lambda_1 - \lambda)^{\frac{-1}{p-1+\delta}} \lambda^{\frac{1}{p-1+\delta}} \varphi_1 \leq v_\lambda$ in Ω. This ends the proof of Theorem 1.1. $\square$

4. Proof of non-local results

We will begin this section by establishing a connection between the solutions of the local problem

$$(Q_\lambda) \quad \begin{cases} -\Delta_p u = \lambda \left(u^{-\delta} + u^\beta \right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{cases}$$

for $\lambda \in \mathrm{Proj}_{\mathbb{R}}\Sigma$, and solutions of the non-local problem (P_μ), where Σ is the unbounded connected of $W^{1,p}_{\mathrm{loc}}(\Omega)$-solutions of (Q_λ). Hereafter, we will denote by (λ, v_λ) the unique solution of (Q_λ), for each $\lambda \in \mathrm{Proj}_{\mathbb{R}}\Sigma$, assured by Theorem 1.1.

Let us define the map $\Phi_\mu : \mathrm{Proj}_{\mathbb{R}}\Sigma \to \mathbb{R}^+$ by

$$\Phi_\mu(\lambda) = \begin{cases} \lambda^{\frac{q}{p-1-\beta}} \displaystyle\int_\Omega \Psi^q & \text{if } \mu = 0 \text{ and } 0 < \beta < p-1, \\ \mu^{\frac{q}{\beta+\delta}} \displaystyle\int_\Omega v^q_{\lambda\mu^{-(p-1-\beta)/(\beta+\delta)}} & \text{if } \mu > 0 \text{ and } 0 < \beta \leq p-1, \end{cases} \tag{4.1}$$

where $(\lambda\mu^{-(p-1-\beta)/(\beta+\delta)}, v^q_{\lambda\mu^{-(p-1-\beta)/(\beta+\delta)}}) \in \Sigma$ and $\Psi \in W^p_0(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta_p u = u^\beta \text{ in } \Omega, \\ u > 0 \text{ in}, \ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Lemma 4.1. *Let $\mu \geq 0$. Then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of the problem (P_μ) if, and only if, there exists a $\lambda = \lambda_\mu \in \mathrm{Proj}_{\mathbb{R}}\Sigma$ such that*

$$g\left(\Phi_\mu(\lambda)\right) = \frac{1}{\lambda}, \tag{4.3}$$

and

$$u := u_{\lambda,\mu} = \begin{cases} \lambda^{\frac{1}{p-1-\beta}}\Psi & \text{if } \mu = 0 \text{ and } 0 < \beta < p-1, \\ \mu^{\frac{1}{\beta+\delta}} v_{\lambda\mu^{-(p-1-\beta)/(\beta+\delta)}} & \text{if } \mu > 0 \text{ and } 0 < \beta \leq p-1. \end{cases} \tag{4.4}$$

Proof. Let $\mu > 0$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of (P_μ). Then $g\left(\int_\Omega u^q\right) \neq 0$ and so

$$g\left(\int_\Omega u^q\right) = \frac{1}{\lambda}, \tag{4.5}$$

for some $\lambda = \lambda_\mu > 0$. In particular, we have that u is solution of the local problem

$$\begin{cases} -\Delta_p u = \lambda\left(\mu u^{-\delta} + u^\beta\right) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.6}$$

So, by rewriting (4.6), we get that u solves the problem

$$\begin{cases} -\Delta_p\left(\mu^{\frac{-1}{\beta+\delta}}u\right) = \alpha\left[\left(\mu^{\frac{-1}{\beta+\delta}}u\right)^{-\delta} + \left(\mu^{\frac{-1}{\beta+\delta}}u\right)^{\beta}\right], \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\alpha := \lambda\mu^{-\frac{p-1-\beta}{\beta+\delta}}$, which implies by Theorem 1.1 that $(\alpha, \mu^{\frac{-1}{\beta+\delta}}u) \in \Sigma$, that is,

$$\lambda\mu^{-\frac{p-1-\beta}{\beta+\delta}} \in \mathrm{Proj}_{\mathbb{R}}\Sigma \quad \text{and} \quad \mu^{-\frac{1}{\beta+\delta}}u = v_\alpha,$$

where v_α is the unique solution of (Q_α).

Hence, going back to (4.5), we obtain

$$g\left(\mu^{\frac{q}{\beta+\delta}}\int_\Omega v^q_{\lambda\mu^{-\frac{p-1-\beta}{\beta+\delta}}}\right) = \frac{1}{\lambda}.$$

For the case $\mu = 0$, problem (4.6) reduces to

$$\begin{cases} -\Delta_p u = \lambda u^\beta \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases}$$

whose unique $W^{1,p}_0(\Omega)$-solution is $u = \lambda^{\frac{1}{p-1-\beta}}\Psi$ whenever $0 < \beta < p-1$, where $\Psi \in W^{1,p}_0(\Omega)$ is the unique solution of problem (4.2). Thus, by definition of Φ_μ and (4.5), relation (4.3) follows again.

On the other hand, let $\mu \geq 0$ and suppose $\{\mu \geq 0 \text{ and } 0 < \beta < p-1\}$ or $\{\mu > 0 \text{ and } \beta = p-1\}$. Moreover, assume that there exists a $\lambda = \lambda_\mu \in \mathrm{Proj}_{\mathbb{R}}\Sigma$ satisfying (4.3). So, by defining u as in (4.4), we are able to show that u is a solution of (P_μ). This ends the proof. □

As a consequence of Lemma and Theorem 1.1, we have that for $\mu > 0$, the problem

$$\begin{cases} -\Delta_p u = \mu u^{-\delta} + u^\beta \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega \end{cases}$$

admits a $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$-solution, say u, if, and only if, $u = \mu^{\frac{1}{\beta+\delta}} v_{\mu^{-(p-1-\beta)/(\beta+\delta)}}$ and $\mu^{-\frac{p-1-\beta}{\beta+\delta}} \in \mathrm{Proj}_{\mathbb{R}}\Sigma$, that is, $\mu > 0$ if $0 < \beta < p-1$ and $\lambda_1 > 1$ if $\beta = p-1$.

In the next result, we established a crucial information about Φ_μ.

Lemma 4.2. *Assume $\delta > 0$ and $0 < \beta \leq p-1$. Then:*
(a) *the maps $\lambda \mapsto \Phi_\mu(\lambda)$ and $\mu \mapsto \Phi_\mu(\lambda)$ are increasing.*
(b) *for $\lambda, \mu > 0$ and $0 < \beta < p-1$, we have*

$$\|\varphi_1\|_q^q \max\left\{\left(\frac{\lambda\mu}{\lambda_1}\right)^{\frac{q}{p-1+\delta}}, \left(\frac{\lambda}{\lambda_1}\right)^{\frac{q}{p-1-\beta}}\right\} \leq \Phi_\mu(\lambda) \leq K^q\|\phi^t\|_q^q \max\left\{(\lambda\mu)^{\frac{q}{p-1+\delta}}, \lambda^{\frac{q}{p-1-\beta}}\right\}, \tag{4.7}$$

where $t = (p-1)/(p-1+\delta)$ and K is the unique solution of the equation

$$t^{p-1}K^{p-1} = K^{-\delta} + \|\phi\|_\infty^{t(\beta+\delta)}K^\beta.$$

Proof. of (a) As noted above, $u_{\lambda,\mu}$ defined in (4.4) is a solution of (4.6). So the monotonicity claimed follows directly from Theorem 2.1 by noting that for $\lambda' < \lambda''$ and $\mu' < \mu''$ the function $u_{\lambda',\mu}$ is a subsolution of (4.6) with $\lambda = \lambda''$ and $u_{\lambda,\mu'}$ is a subsolution of (4.6) with $\mu = \mu''$.
Proof of (b) Since $v_\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is the unique solution of (Q_α) with $\alpha = \lambda\mu^{-\frac{p-1-\beta}{\beta+\delta}}$, the result follows from definition of Φ_μ and estimates (1.1) by replacing λ with $\alpha = \lambda\mu^{-\frac{p-1-\beta}{\beta+\delta}}$. □

Now, let us define the application $R\colon \mathrm{Proj}_{\mathbb{R}}\Sigma \to \mathbb{R}$ given by

$$R(\lambda) = \int_\Omega v_\lambda^q \mathrm{d}x,$$

where $v_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ is the unique solution of the problem (Q_λ). The estimate (1.1) and Lemma 3.3 in [21] provide the following result.

Lemma 4.3. *Assume $\delta > 0$ and $0 < \beta < p-1$. Then R is an increasing and continuous application satisfying $R(\lambda) \to 0$ as $\lambda \to 0$ and $R(\lambda) \to \infty$ as $\lambda \to \infty$.*

To state the next result, let us define the functions $G_\mu : \mathrm{Proj}_{\mathbb{R}}\Sigma \to \mathbb{R}$ by

$$G_\mu(\lambda) = g\left(\Phi_\mu(\lambda)\right) - \frac{1}{\lambda} \tag{4.8}$$

and $\lambda_\mu\colon (0,\infty) \to \mathrm{Proj}_{\mathbb{R}}\Sigma$ by

$$\lambda_\mu(s) = \begin{cases} \mu^{\frac{p-1-\beta}{\beta+\delta}} R^{-1}\left(s\mu^{\frac{-q}{\beta+\delta}}\right) & \text{if } \mu > 0 \text{ and } \beta \leq p-1, \\ \left(\frac{s}{\int_\Omega \Psi^q}\right)^{\frac{p-1-\beta}{q}} & \text{if } \mu = 0 \text{ and } \beta < p-1, \end{cases} \tag{4.9}$$

where $\Psi \in W_0^{1,p}(\Omega)$ stands for the unique positive solution of problem (4.2) and the application Φ_μ was defined in (4.1). For each $s > 0$, $\lambda_\mu(s) \in \mathrm{Proj}_{\mathbb{R}}\Sigma$ is such that

$$\Phi_\mu(\lambda) = s \text{ if, and only if, } \lambda = \lambda_\mu(s),$$

that is, $\Phi_\mu(\lambda_\mu(s)) = s$ for $s > 0$. Since $\mu \mapsto \Phi_\mu(\lambda)$ and $\lambda \mapsto \Phi_\mu(\lambda)$ are increasing, $\mu \mapsto \lambda_\mu(s)$ is necessarily decreasing.

Remark 4.1. Denoting by $s = \Phi_\mu(\lambda)$, relation (4.8) can be rewritten as

$$G_\mu(\lambda_\mu(s)) = g(s) - \frac{1}{\lambda_\mu(s)}, \quad s > 0, \tag{4.10}$$

that is, the problem (P_μ) admits a solution satisfying $\|u\|_q^q \in (s_1, s_2)$ if, and only if, $G_\mu(\lambda_\mu(s)) = 0$ can be solved, for some $s \in (s_1, s_2)$.

Below, we will establish an essential result to prove the existence of a positive solution for the problem (P_μ), by solving the equation $G_\mu(\lambda) = 0$.

Proposition 4.1. *Assume $\delta > 0$ and $0 < \beta \le p-1$. Then,*

(i) *the map G_μ is continuous,*
(ii) *$G_\mu(\lambda_\mu(s_i)) < 0$, for $i = 1, 2$, if (g_1) or (g_2) holds,*
(iii) *$G_\mu(\lambda_\mu(s_*)) > 0$ for all $0 \le \mu < (s_*/R(1))^{(\beta+\delta)/q}$ if we assume (1.4) and $0 < \beta < p-1$,*
(iv) *there exists $\lambda' \in (\lambda_{\mu'}(s_1), \lambda_{\mu'}(s_2))$ such that $G_{\mu'}(\lambda') > 0$ if $\mu' < \mu''$ and $G_{\mu''}(\lambda'') > 0$ for some $\lambda'' \in (\lambda_{\mu''}(s_1), \lambda_{\mu''}(s_2))$.*

Proof. of (i). For $\mu = 0$ and $0 < \beta < p-1$, the result is obvious. For $\mu > 0$ and $0 < \beta \le p-1$, we just need to observe that

$$G_\mu(\lambda) = g\left(\mu^{\frac{q}{\beta+\delta}} R\left(\lambda \mu^{-\frac{p-1-\beta}{\beta+\delta}}\right)\right) - \frac{1}{\lambda},\ \lambda > 0.$$

Thus, the continuity follows from the continuity of g and R, claimed in Lemma 4.3.
Proof of (ii) The proof of this item follows directly from the assumptions (g_1) or (g_2) and (4.10).
Proof of (iii) Let us consider initially $\mu = 0$ and use (4.9) and (4.10) to conclude that

$$G_0(\lambda_0(s_*)) = g(s_*) - \left(\frac{\int_\Omega \Psi^q}{s_*}\right)^{\frac{p-1-\beta}{q}}.$$

Since Ψ is a positive subsolution of

$$-\Delta_p u = u^\beta \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

if follows from Theorem 1.1 and 2.1 that $\Psi \le K\phi^t$, which implies by hypothesis (1.4) that $G_0(\lambda_0(s_*)) > 0$.

For $0 < \mu < (s_*/R(1))^{(\beta+\delta)/q}$, we obtain from the monotonicity of R that $R^{-1}\left(s_*\mu^{\frac{-q}{\beta+\delta}}\right) > 1$, whence

$$\alpha = \mu^{-\frac{p-1-\beta}{\beta+\delta}} \lambda_\mu(s_*) = \mu^{-\frac{p-1-\beta}{\beta+\delta}} \mu^{\frac{p-1-\beta}{\beta+\delta}} R^{-1}\left(s_*\mu^{\frac{-q}{\beta+\delta}}\right) = R^{-1}\left(s_*\mu^{\frac{-q}{\beta+\delta}}\right) > 1$$

and from the second inequality in (4.7) we get

$$s_* = \mu^{\frac{q}{\beta+\delta}} \int_\Omega v^q_{\lambda_\mu(s_*)\mu^{-(p-1-\beta)/(\beta+\delta)}} \le K^q \|\phi^t\|_q^q \left(\lambda_\mu(s_*)\right)^{\frac{q}{p-1-\beta}},$$

that is,

$$\lambda_\mu(s_*) \ge \left(\frac{s_*}{K^q \|\phi^t\|_q^q}\right)^{\frac{p-1-\beta}{q}}. \tag{4.11}$$

Therefore, by (4.11) and hypothesis (1.4) one gets

$$G_\mu(\lambda_\mu(s_*)) = g\left(\mu^{\frac{q}{\beta+\delta}} \int_\Omega v^q_{\lambda_\mu(s_*)\mu^{-(p-1-\beta)/(\beta+\delta)}}\right) - \frac{1}{\lambda_\mu(s_*)} = \frac{g(s_*)\lambda_\mu(s_*) - 1}{\lambda_\mu(s_*)} > 0,$$

which concludes the proof of this item.
Proof of (iv) Denote by

$$\tilde{s} = (\mu'')^{\frac{q}{\beta+\delta}} \int_\Omega v^q_{\lambda''(\mu'')^{-(p-1-\beta)/(\beta+\delta)}} \in (s_1, s_2)$$

and choose

$$\lambda' = \lambda_{\mu'}(\tilde{s}) = (\mu')^{\frac{p-1-\beta}{\beta+\delta}} R^{-1}\left(\tilde{s}(\mu')^{\frac{-q}{\beta+\delta}}\right).$$

So, it follows from item a) of Lemma 4.2 that $\lambda' > \lambda''$. Moreover,

$$G_{\mu'}(\lambda') = g(\tilde{s}) - \frac{1}{\lambda'} > g(\tilde{s}) - \frac{1}{\lambda''} = G_{\mu''}(\lambda'') > 0,$$

as claimed, where the last inequality follows by assumption. This ends the proof of the Proposition. □

Now we are ready to conclude the proof of Theorems 1.2 and 1.3 .

Proof of Theorem 1.2. Suppose $0 < \beta < p-1$. For each $\mu \geq 0$ and $\lambda > 0$ given, let us denote by $s = \Phi_\mu(\lambda)$. According to the first estimate obtained in (4.7), we get

$$\|\varphi_1\|_q^q \left(\frac{\lambda}{\lambda_1}\right)^{\frac{q}{p-1-\beta}} \leq s,$$

which together with assumption (1.3) implies in

$$G_\mu(\lambda) = \frac{g(s)\lambda - 1}{\lambda} < \frac{g(s)s^{(p-1-\beta)/q}\|\varphi_1\|_q^{-(p-1-\beta)}\lambda_1 - 1}{\lambda} < 0.$$

So, it is not possible to solve $G_\mu(\lambda) = 0$ for any $\mu \geq 0$, that is, (P_μ) does not admit solution.

Now, assume $\beta = p-1$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a solution of (P_μ). So, it follows from Lemma 4.1 that there exists a $\lambda_\mu \in \mathrm{Proj}_{\mathbb{R}}\Sigma$ such that (4.3) holds, that is,

$$g\left(\Phi_\mu(\lambda_\mu)\right) = \frac{1}{\lambda_\mu} > \frac{1}{\lambda_1}.$$

However, this is impossible under assumption (1.3); hence, the problem (P_μ) does not admit any solution. □

Proof of Theorem 1.3. First of all, let us define

$$\mu_1^* = \sup\left\{\mu > 0\colon \max_{[\lambda_\mu(s_1),\lambda_\mu(s_2)]} G_\mu(\lambda) \geq 0\right\}$$

both for $0 < \beta < p-1$ and $\beta = p-1$.

We begin by proving the item a). It follows from definition of μ_1^* and Proposition 4.1(iii) that

$$\mu_1^* \geq (s_*/R(1))^{(\beta+\delta)/q} > 0.$$

Now, let us prove that

$$\mu_1^* \leq \left(\frac{s_2}{\|\varphi_1\|_q^q}\right)^{\frac{p-1+\delta}{q}} g_1^*\lambda_1, \tag{4.12}$$

where

$$g_1^* := \max\left\{g(s)\colon s \in [s_1, s_2]\right\} = g(s_1^*).$$

Indeed, by fixing

$$\mu > \left(\frac{s_2}{\|\varphi_1\|_q^q}\right)^{\frac{p-1+\delta}{q}} g_1^*\lambda_1,$$

and applying the first inequality in (1.1) with $\lambda = \mu^{-\frac{p-1-\beta}{\beta+\delta}}/g_1^* \in \mathrm{Proj}_{\mathbb{R}}\Sigma = (0,\infty)$, we obtain

$$s_2\mu^{\frac{-q}{\beta+\delta}} < \|\varphi_1\|_q^q \left(\frac{\mu^{-\frac{(p-1-\beta)}{(\beta+\delta)}}}{g_1^*\lambda_1}\right)^{\frac{q}{p-1+\delta}} \leq R\left(\frac{\mu^{-\frac{p-1-\beta}{\beta+\delta}}}{g_1^*}\right),$$

whence

$$R^{-1}\left(s_2\mu^{\frac{-q}{\beta+\delta}}\right) < \frac{\mu^{-\frac{(p-1-\beta)}{\beta+\delta}}}{g_1^*}.$$

Thus,

$$\lambda_\mu(s_2) = \mu^{\frac{p-1-\beta}{\beta+\delta}} R^{-1}\left(s_2 \mu^{\frac{-q}{\beta+\delta}}\right) < \frac{1}{g_1^*},$$

which implies

$$G_\mu(\lambda) = g\left(\mu^{\frac{q}{\beta+\delta}} \int_\Omega v^q_{\lambda\mu^{-(p-1-\beta)/(\beta+\delta)}}\right) - \frac{1}{\lambda} \le g_1^* - \frac{1}{\lambda} < g_1^* - \frac{1}{\lambda_\mu(s_2)} < g_1^* - g_1^* = 0,$$

for all $\lambda \in (\lambda_\mu(s_1), \lambda_\mu(s_2))$ and this proves (4.12).

To finish the proof, we note that Proposition 4.1(iv) implies

$$\max\{G_\mu(\lambda): \lambda \in [\lambda_\mu(s_1), \lambda_\mu(s_2)]\} > 0,$$

for any $\mu \in (0, \mu_1^*)$ given. Since Proposition 4.1(ii) implies $G_\mu(\lambda_\mu(s_1)), G_\mu(\lambda_\mu(s_2)) < 0$, we obtain from the continuity of G_μ that there exist $\tilde{\lambda}, \hat{\lambda} \in (\lambda_\mu(s_1), \lambda_\mu(s_2))$, with $\tilde{\lambda} \neq \hat{\lambda}$, such that $G_\mu(\tilde{\lambda}) = G_\mu(\hat{\lambda}) = 0$. Hence, from Lemma 4.1 we conclude that (P_μ) admits at least two solutions satisfying $\|u\|_q^q \in (s_1, s_2)$.

For $\mu > \mu_1^*$, no solution satisfying $\|u\|_q^q \in (s_1, s_2)$ is obtained, because $G_\mu(\lambda) < 0$ for all $\lambda \in [\lambda_\mu(s_1), \lambda_\mu(s_2)]$. Finally, for $\mu = \mu_1^*$ the problem (P_μ) admits at least one solution satisfying $\|u\|_q^q \in (s_1, s_2)$, because the $\max_{[\lambda_\mu(s_1), \lambda_\mu(s_2)]} G_{\mu_1^*}(\lambda) \ge 0$.

Now we consider the item b). First, we will show that $\mu_1^* > 0$. We know from definition of $\lambda_\mu(s)$, properties of the application R and assumption (1.5), that

$$\liminf_{\mu\to 0}\left(g\left(\mu^{\frac{q}{\beta+\delta}}\int_\Omega v^q_{\lambda_\mu(s_1^*)}\mathrm{d}x\right) - \frac{1}{\lambda_\mu(s_1^*)}\right) = \lim_{\mu\to 0}\left(g\left(s_1^*\right) - \frac{1}{\lambda_\mu(s_1^*)}\right) = g_1^* - \frac{1}{\lambda_1} > 0,$$

where

$$g_1^* = \max\{g(s): s \in [s_1, s_2]\} = g(s_1^*).$$

So, there exists a $\mu_* > 0$ such that

$$G_{\mu_*}(\lambda_{\mu_*}(s_1^*)) = g\left(\mu_*^{\frac{q}{\beta+\delta}}\int_\Omega v^q_{\lambda_{\mu_*}(s_1^*)}\mathrm{d}x\right) - \frac{1}{\lambda_\mu(s_1^*)} > 0,$$

whence we infer that $\mu_1^* > 0$ because $\lambda_{\mu_*}(s_1^*) \in [\lambda_{\mu_*}(s_1), \lambda_{\mu_*}(s_2)]$.

To prove that $\mu_1^* < \infty$, we just need observe that if μ is large enough such that $\lambda_\mu(s_2) < 1/g_1^*$, then for all $\lambda \in [\lambda_\mu(s_1), \lambda_\mu(s_2)]$ one has $G_\mu(\lambda) \le g_1^* - \frac{1}{\lambda} < 0$, so $\mu_1^* < \infty$.

Finally, by following the same approach as done in proof of Proposition 4.1(iv), we can check that for any $\mu \in (0, \mu_1^*)$ we have the $\max\{G_\mu(\lambda): \lambda \in [\lambda_\mu(s_1), \lambda_\mu(s_2)]\} > 0$. Besides this, we know from Proposition 4.1 that $G_\mu(\lambda_\mu(s_1)), G_\mu(\lambda_\mu(s_2)) < 0$. Following the arguments as done in item a), we conclude the proof of this item and the proof of Theorem 1.3. □

Remark 4.2. Some consequences of the proofs:

(i) if $0 < \beta < p-1$, then

$$(s_*/R(1))^{(\beta+\delta)/q} \le \mu_1^* \le \left(\frac{s_2}{\|\varphi_1\|_q^q}\right)^{\frac{p-1+\delta}{q}} g_1^*\lambda_1,$$

(ii) if $g: [0, \infty) \to \mathbb{R}$ is a continuous function which is increasing in $[s_1, s_1^*]$, decreasing in $(s_1^*, s_2]$ and satisfies $g(s_1) = g(s_2) = 0$, then there exist just two solutions for $0 < \mu < \mu_1^*$, exactly one solution for $\mu = \mu_1^*$ and no solution for $\mu > \mu_1^*$. (This is consequence of the geometry of g and the monotonicity properties of $\mu \mapsto \lambda_\mu(s)$ and $s \mapsto \lambda_\mu(s)$.)

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