Z. Angew. Math. Phys. (2021) 72:128

0044-2275/21/030001-18
published online May 28, 2021
https://doi.org/10.1007/s00033-021-01559-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

On stability of Boussinesq equations without thermal conduction

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Abstract. We investigate the stability of a specific stationary solution to Boussinesq equations without thermal conduction in the flat strip $\Omega = \mathbb{T} \times (0, 1)$. Explicit decay rates of the vorticity/velocity are given as well as the limit state of the temperature. Our method is based on time-weighted energy estimates.

Keywords. Boussinesq equations, Stationary solution, Stability, Decay rate.

1. Introduction

1.1. Presentation of the problem

In the mathematical study of fluid dynamics, the Boussinesq approximation is one of important models among various simplified ones to the Navier–Stokes–Fourier system. In such an approximation, both temperature and density of the flow are assumed to vary small so that the variation of temperature is (inversely) proportional to that of density. Hence, the fluid is assumed to be divergence free and only the action of gravity is considered. In a general 3D setting, the Boussinesq equations read as

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = g\vartheta \mathbf{e}_3, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t \vartheta + \mathbf{v} \cdot \nabla \vartheta - \kappa \Delta \vartheta = 0. \end{cases} \tag{1.1}$$

Here $\mathbf{v}, p$ and ϑ are the velocity, pressure and temperature of the fluid, respectively, while ν is the viscosity, κ is the thermal conductivity, g is the constant of gravity and $\mathbf{e}_3$ is the inverse direction of the gravity. Note that the term $g\vartheta\mathbf{e}_3$ on the right-hand side of momentum Eq. $(1.1)_1$ is nothing but the effect of gravity/buoyancy. In case of $\nu, \kappa > 0$, the Boussinesq equations is an elliptic–parabolic coupled nonlinear PDE system which plays a key role in the study of hydrodynamic instability problems, especially the Rayleigh-Bénard convection, see [3,10] among others. When the thermal conductivity is neglected, the temperature variation ϑ can be considered as the inverse of density variation. This explains the physical significance of the transport equation in (1.1), which now becomes an elliptic–parabolic–hyperbolic coupled system.

The present paper is concerned with Boussinesq equations

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta \mathbf{v} + \nabla p = \vartheta \mathbf{e}_2, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t \vartheta + \mathbf{v} \cdot \nabla \vartheta = 0 \end{cases} \tag{1.2}$$

in the two-dimensional domain $\Omega = \mathbb{T} \times (0, 1) \subset \mathbb{R}^2$. Here the vector field $\mathbf{v} = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t))$ is the velocity describing the motion of viscous fluid under the action of buoyancy force, while the temperature ϑ is transported by the fluid motion. The fluid viscosity ν and the constant of gravity g are assumed to be 1, which are irrelevant in the following analysis. We use $\mathbf{x}$ to denote the space variable $(x,y)\in\mathbb{R}^2$ and $\mathbf{e}_2 := \nabla y = (0,1)$ the direction of buoyancy force.

We supplement system (1.2) with the following initial and (slip) boundary conditions.

$$\begin{cases} \mathbf{v}(\mathbf{x},0)=\mathbf{v}_0(\mathbf{x}),\ \vartheta(\mathbf{x},0)=\vartheta_0(\mathbf{x}) \text{ in } \Omega, \\ (\mathbf{v}\cdot\mathbf{n})(\mathbf{x},t)=0,\ (\mathbb{S}(\nabla\mathbf{v})\mathbf{n}\cdot\tau)(\mathbf{x},t)=0 \text{ on } \partial\Omega,\ t>0, \end{cases} \tag{1.3}$$

where $\mathbf{n}$ (τ) is the outward unit normal (unit tangential direction) to $\partial\Omega=\mathbb{T}\times\{y=0,1\}$ and the stress tensor $\mathbb{S}(\nabla\mathbf{v})$ is the symmetric part of $\nabla\mathbf{v}$. Since the boundary $\partial\Omega$ is flat, $(1.3)_2$ is equivalent to

$$v_2(\mathbf{x},t)=0,\ \partial_2 v_1(\mathbf{x},t)=0 \text{ on } \partial\Omega,\ t>0. \tag{1.4}$$

By setting $\mathbf{v}=0$ in (1.2), we immediately obtain that

$$\mathbf{v}_s=0,\ p_s=p_s(y),\ \vartheta_s=p_s'(y),\ y\in[0,1] \tag{1.5}$$

is a stationary solution (hydrostatic equilibrium) to (1.2), where $p_s(\cdot)$ is an arbitrary smooth function. It is well known that when $\vartheta_s'(y_0)<0$ for some $y_0\in[0,1]$, such a stationary solution is unstable—the Rayleigh–Taylor instability happens. In the present work, we are interested in the opposite case $\vartheta_s'(y)>0$ for all $y\in[0,1]$, which implies that fluid with lower temperature (higher density) lies below the fluid with higher temperature (lower density). Specifically we choose $\vartheta_s(y)=y$ and succeed to show that this steady solution is stable in the sense of Lyapunov, meaning that the solution to (1.2)-(1.3) starting from initial data close to this stationary solution is close to it for all time $t>0$ in a suitable sense. Moreover, we obtain that the velocity $\mathbf{v}$ and $\partial_1\vartheta$ converge to zero in H^2 with explicit decay rates $(1+t)^{-1}$ as $t\to\infty$, see Theorem 1.1.

1.2. Related results on the Boussinesq equations

Boussinesq equations have rich physical background and mathematical features. In particular, the two-dimensional model keeps some key features of the 3D Navier–Stokes/Euler equations, see [20,23]. Over the past years, there have been many works devoted to Boussinesq equations. For the reason of brevity, we only review some related results on the two-dimensional Boussinesq system (1.2).

First result on global well-posedness of Cauchy problem to (1.2) has been obtained in [6] and [15] with arbitrary large initial data. Since then, other kinds of initial (boundary) value problems to (1.2) have been investigated by many authors under different settings, see [1,7,13,14,16,19], among others. It is then natural to study the large-time behavior of solutions to (1.2). In [17], N. Ju has obtained the global-in-time uniform boundedness of $\mathbf{v}$ in H^2 and the exponential growth e^{ct^2} of $\|\nabla\vartheta(t)\|_{L^2}$. It is then improved by I. Kukavica and W. Wang in [18] that the growth of $\|\nabla\vartheta(t)\|_{L^2}$ is at most e^{ct}. In [8], for general initial data it is shown that $\mathbf{v}$ converges in H^1 to zero and $\|\nabla^2\mathbf{v}(t)\|_{L^2}$ is uniformly bounded. Very recently, by using spectral analysis, the authors in [24] have investigated the global asymptotic stability of the specific hydrostatic equilibrium (1.6). Besides analysis for the linearized system, they also give explicit decay rate $(1+t)^{-1/2}$ of $\|\mathbf{v}(t)\|_{L^2}$ under certain assumptions on the solution $(\mathbf{v},\vartheta)$ to (1.2) on periodic domain $\mathbb{T}^2$. In the most relevant work [26], R. Wan has obtained asymptotic behavior and explicit decay rates for solutions to the perturbed system (1.7) in $\mathbb{R}^2$ by using spectral analysis. Finally we refer the reader to [4,5,25] for works on the global existence and stability of the 2D Boussinesq equations with a velocity damping term (without viscosity) near the hydrostatic equilibrium (1.6).

1.3. Main result

Choosing $\vartheta_s(y) = y$, we infer from (1.5) that

$$\mathbf{v}_s = 0,\ p_s = \frac{1}{2}y^2,\ \vartheta_s = y,\ y \in [0,1]. \tag{1.6}$$

Perturbing this specific stationary solution, we have

$$\mathbf{v} = \mathbf{u},\ p = q + \frac{1}{2}y^2,\ \vartheta = y + \theta.$$

Then $(\mathbf{u}, q, \theta)$ satisfies the perturbed equations as follows.

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla q = \theta\mathbf{e}_2, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = -u_2, \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.7}$$

with initial and boundary conditions

$$\begin{cases} (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}) \text{ in } \Omega, \\ u_2(\mathbf{x},t) = 0,\ \partial_2 u_1(\mathbf{x},t) = 0 \text{ on } \partial\Omega,\ t>0. \end{cases} \tag{1.8}$$

Since $\nabla\cdot\mathbf{u} = 0$, there exists a stream function ψ such that $\mathbf{u} = \nabla^\perp\psi = (\partial_2\psi, -\partial_1\psi)$. Here we choose ψ satisfying

$$\begin{cases} -\Delta\psi = \omega, \\ \psi|_{\partial\Omega} = 0, \end{cases} \tag{1.9}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$. By denoting $\psi = (-\Delta)^{-1}\omega$,

$$\mathbf{u} = \nabla^\perp(-\Delta)^{-1}\omega.$$

Hence, the equations for (ω,θ) are written as

$$\begin{cases} \partial_t\omega + \mathbf{u}\cdot\nabla\omega - \Delta\omega = \partial_1\theta, \\ \partial_t\theta + \mathbf{u}\cdot\nabla\theta = -u_2, \\ \mathbf{u} = \nabla^\perp(-\Delta)^{-1}\omega, \end{cases} \tag{1.10}$$

together with initial and boundary conditions

$$\begin{cases} (\omega,\theta)(\mathbf{x},0) = (\omega_0,\theta_0)(\mathbf{x}) \text{ in } \Omega, \\ \omega(\mathbf{x},t) = 0 \text{ on } \partial\Omega,\ t>0. \end{cases} \tag{1.11}$$

In the following we focus on the analysis of (1.10)-(1.11) instead of considering (1.2)-(1.3) directly. Assume that

$$\omega_0 \in H^m(\Omega) \text{ and } \partial_2^n\omega_0 = 0 \text{ on } \partial\Omega,\ \text{ for } n = 0,2,\cdots,2[(m-1)/2], \tag{1.12}$$

$$\theta_0 \in H^{m+1}(\Omega) \text{ and } \partial_2^n\theta_0 = 0 \text{ on } \partial\Omega,\ \text{ for } n = 0,2,\cdots,2[m/2], \tag{1.13}$$

where $[m/2] = k$ if $m = 2k$ or $2k+1$, $k = 0,1,2,\cdots$. We remark that assumptions (1.12) and (1.13) on initial data are motivated by [5]. According to $(1.10)_2$ and $u_2(t)|_{\partial\Omega} = 0$, we get the transport equation

$$\partial_t\theta(t)|_{\partial\Omega} + u_1(t)\partial_1\theta(t)|_{\partial\Omega} = 0.$$

Then $\theta_0|_{\partial\Omega} = 0$ implies $\theta(t)|_{\partial\Omega} = 0$. Taking ∂_2 on $\nabla\cdot\mathbf{u} = 0$ and using $\partial_2 u_1(t)|_{\partial\Omega} = 0$ yield

$$\partial_2^2 u_2(t)|_{\partial\Omega} = -\partial_2\partial_1 u_1(t)|_{\partial\Omega} = 0.$$

Now operating ∂_2^2 on $(1.10)_2$ and restricting to the boundary, one gets

$$\partial_t\partial_2^2\theta(t)|_{\partial\Omega} + u_1(t)\partial_1\partial_2^2\theta(t)|_{\partial\Omega} + 2\partial_2 u_2(t)\partial_2^2\theta(t)|_{\partial\Omega} = 0.$$

This implies that $\partial_2^2\theta_0|_{\partial\Omega}=0$ is preserved in time. Furthermore, from the evolution Eq. (1.10)$_1$, it follows that

$$-\Delta\omega(t)|_{\partial\Omega}=\partial_1\theta(t)|_{\partial\Omega}-\partial_t\omega(t)|_{\partial\Omega}-u_1(t)\partial_1\omega(t)|_{\partial\Omega}-u_2(t)\partial_2\omega(t)|_{\partial\Omega}.$$

Then $\partial_2^2\omega(t)|_{\partial\Omega}=0$. By using this fact and the incompressibility condition for $\mathbf{u}$,

$$\partial_2^3u_1(t)|_{\partial\Omega}=0,\ \partial_2^4u_2(t)|_{\partial\Omega}=0.$$

Similarly, from $\partial_2^n\theta_0|_{\partial\Omega}=0$ for $n=2,4\cdots$, we conclude that the sufficiently smooth function $(\omega,\theta,\mathbf{u})$ satisfies

$$\partial_2^n\theta(t)|_{\partial\Omega}=0,\ \partial_2^{n+2}\omega(t)|_{\partial\Omega}=0,\ \partial_2^{n+3}u_1(t)|_{\partial\Omega}=0,\ \partial_2^{n+4}u_2(t)|_{\partial\Omega}=0. \tag{1.14}$$

Hence, we assume that the initial data (ω_0,θ_0) satisfies the special setting (1.12)-(1.13).

We now state the main result of this paper as follows.

Theorem 1.1. *Let $m\geq 5$ be a fixed integer and (ω_0,θ_0) satisfy (1.12)-(1.13). There exists a constant $\epsilon_0>0$ such that if*

$$\|\omega_0\|_{H^m}^2+\|\theta_0\|_{H^{m+1}}^2\leq\epsilon_0^2, \tag{1.15}$$

then (1.10)-(1.11) admits a unique global smooth solution

$$(\omega,\theta)\in C([0,\infty);H^m)\times C([0,\infty),H^{m+1})$$

satisfying

$$\|\omega(t)\|_{H^m}^2+\|\theta(t)\|_{H^{m+1}}^2\lesssim\epsilon_0^2,\ \textit{for all } t>0.$$

Moreover,

$$\|\omega(t)\|_{H^1}+\|\partial_1\omega(t)\|_{H^1}\lesssim(1+t)^{-1},\ \|\partial_1\theta(t)\|_{H^2}\lesssim(1+t)^{-1}.$$

Remark 1.1. It follows from Theorem 1.1 that the solution to (1.2)-(1.3) satisfies

$$\|\mathbf{v}(t)\|_{L^\infty}\lesssim\|\mathbf{v}(t)\|_{H^2}\lesssim(1+t)^{-1},$$

$$\|\vartheta(x,y,t)-\overline{\vartheta}(y,t)\|_{L^\infty}\lesssim\|\vartheta(x,y,t)-\overline{\vartheta}(y,t)\|_{H^2}\lesssim\|\partial_1\vartheta(x,y,t)\|_{H^2}\lesssim(1+t)^{-1}.$$

This means that $\overline{\vartheta}(y,t)$ determines the large-time asymptotics of $\vartheta(x,y,t)$. Furthermore, we deduce from (1.2)$_3$ that

$$\overline{\vartheta}(y,t)=\overline{\vartheta}_0(y)+\partial_2\int_0^t\overline{v_2\vartheta}(y,\tau)\mathrm{d}\tau,$$

which has been proved in [24]. Here the bar denotes the horizontal average, that is,

$$\overline{f}(y,t)=\int_{\mathbb{T}}f(x,y,t)\mathrm{d}x.$$

The method of the proof of Theorem 1.1 is motivated by [22]. To prove Theorem 1.1, it is necessary to show the global uniform estimates of the solution (ω,θ) to (1.10)-(1.11). However, we have to face the difficulty arising from the absence of thermal conduction. By making full use of the structure of (1.10), we obtain the only partial dissipation of θ. This partial dissipation implies that it is difficult to control the growth of $\mathbf{u}\cdot\nabla\theta$. In the estimate process of $\mathcal{E}(T)$ defined in (3.4), we find that the key point is to obtain L^1 estimate of $\|\partial\mathbf{u}(t)\|_{L^\infty}$ in time and L^1 estimate of $\|\partial_2^2u_2(t)\|_{L^\infty}$ in time. Using a carefully designed time-weighted energy $\mathcal{F}(T)$ defined in (3.5) and applying Poincaré inequality in the x-direction, we overcome these above challenges and prove uniform estimates of $\mathcal{E}(T)$ and $\mathcal{F}(T)$.

The structure of this paper is as follows. In Sect. 2, we give some notations and lemmas which are used in the sequel. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we present some remarks.

2. Preliminaries

The following notations and results will be used in the paper. For simplicity, we set $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and use $\langle \cdot, \cdot \rangle$ as the inner product in L^2. Here $A \lesssim B$ means that $A \leq CB$, where C is a generic constant. Throughout this paper, $\nabla = (\partial_1, \partial_2)$, $\Delta = \nabla \cdot \nabla = \partial_{11} + \partial_{22}$ is the 2D Laplacian operator. For $m \in \mathbb{N}$, the inhomogeneous Sobolev space with derivatives up to order m in $L^2(\Omega)$ is denoted by $H^m(\Omega)$. Moreover, we use $\dot{H}^m(\Omega)$ to denote the homogeneous Sobolev space with the mth-order derivatives in $L^2(\Omega)$. For $f \in H^m(\Omega)$, $\|f\|_{H^m(\Omega)} = \|f\|_{L^2(\Omega)} + \|f\|_{\dot{H}^m(\Omega)}$. In this paper, we may use L^p, $\dot{H}^m$ and H^m to stand for $L^p(\Omega)$, $\dot{H}^m(\Omega)$ and $H^m(\Omega)$, respectively, in some places.

In this section, we give some necessary lemmas.

Lemma 2.1. ([2]) *Let* $0 \leq m_1 \leq m \leq m_2$. *If* $f \in H^{m_2}(\Omega)$, *then*

$$\|f\|_{H^m(\Omega)} \lesssim \|f\|^s_{H^{m_1}(\Omega)} \|f\|^{1-s}_{H^{m_2}(\Omega)}, \text{ with } m = sm_1 + (1-s)m_2,\ 0 \leq s \leq 1. \tag{2.1}$$

The following estimates are classical, see [11,21], among others.

Lemma 2.2. *For* $m \in \mathbb{N}$, *we have*

- *If* $f, g \in H^m(\Omega) \cap L^\infty(\Omega)$, *then*

$$\|fg\|_{H^m(\Omega)} \lesssim \|f\|_{H^m(\Omega)} \|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \|g\|_{H^m(\Omega)}. \tag{2.2}$$

- *If* $f \in H^m(\Omega) \cap W^{1,\infty}(\Omega)$ *and* $g \in H^{m-1}(\Omega) \cap L^\infty(\Omega)$, *then for* $|\alpha| \lesssim m$,

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_{L^2(\Omega)} \lesssim \|f\|_{W^{1,\infty}(\Omega)} \|g\|_{H^{m-1}(\Omega)} + \|f\|_{H^m(\Omega)} \|g\|_{L^\infty(\Omega)}. \tag{2.3}$$

The following lemma related to Poincaré inequality in the x-direction is important. The proof of this lemma is standard, see [2,22], among others.

Lemma 2.3. *Let* $m \in \mathbb{N}$. *If* $f \in H^{1+m}(\Omega)$,

$$\left\| f(x,y) - \int_{\mathbb{T}} f(x,y)\mathrm{d}x \right\|_{H^m(\Omega)} \lesssim \|\partial_1 f\|_{H^m(\Omega)}. \tag{2.4}$$

Furthermore, if f *satisfies* $\int_{\mathbb{T}} f(x,y)\mathrm{d}x = 0$, *then*

$$\|f\|_{H^m(\Omega)} \lesssim \|\partial_1 f\|_{H^m(\Omega)}. \tag{2.5}$$

The following result is well known, see [12], among others.

Lemma 2.4. *Let* $m \in \mathbb{N}$. *Consider the elliptic equations*

$$\begin{cases} -\Delta U = f, \\ U|_{\partial\Omega} = 0. \end{cases}$$

If $f \in H^m(\Omega)$, *then*

$$\|U\|_{H^{m+2}(\Omega)} \lesssim \|f\|_{H^m(\Omega)}.$$

From (1.9) and Lemma 2.4, we obtain

Corollary 2.1. *Let* $m \in \mathbb{N}$. *Assume that* $\mathbf{v} \in H^m(\Omega)$, $\nabla \cdot \mathbf{v} = 0$ *in* Ω *and* $\mathbf{v} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$. *Then*

$$\|\mathbf{v}\|_{H^m(\Omega)} \lesssim \|\nabla \times \mathbf{v}\|_{H^{m-1}(\Omega)}.$$

3. Proof of the main result

The local existence of solution to (1.10)-(1.11) for general smooth initial data in a bounded domain Ω can be proved by the classical methods, see [5,21] and references therein. We omit the proof of the following result for brevity.

Proposition 3.1. *Let $m \geq 2$ be an integer and (ω_0, θ_0) satisfy* (1.12)-(1.13). Then there exists $T_0 > 0$ such that (1.10)-(1.11) admits a unique solution

$$(\omega, \theta) \in C([0,T_0]; H^m(\Omega)) \times C([0,T_0]; H^{m+1}(\Omega))$$

satisfying

$$\partial_2^s \omega|_{\partial\Omega} = 0, \text{ for } s = 0, 2, ..., 2[(m-1)/2], \tag{3.1}$$

$$\partial_2^n \theta|_{\partial\Omega} = 0, \text{ for } n = 0, 2, ..., 2[m/2]. \tag{3.2}$$

Moreover, if T^* is the lifespan to the solution (ω, θ) and $T^* < \infty$, then

$$\int_0^{T^*} (\|\nabla \mathbf{u}(t)\|_{L^\infty} + \|\nabla\theta(t)\|_{L^\infty})\,\mathrm{d}t = \infty. \tag{3.3}$$

Based on Proposition 3.1, it is enough to show global a priori bounds for the smooth solution (ω, θ). To this end, we define for $m \in \mathbb{N}$,

$$\mathcal{E}(T) = \sup_{0\leq t\leq T} \left(\|\theta(t)\|_{H^{m+1}}^2 + \|\omega(t)\|_{H^m}^2\right) + \int_0^T \|\partial_1\theta(t)\|_{H^{m-1}}^2 \mathrm{d}t + \int_0^T \|\omega(t)\|_{H^{m+1}}^2 \mathrm{d}t, \tag{3.4}$$

$$\begin{aligned}\mathcal{F}(T) = &\sup_{0\leq t\leq T} (1+t)^2 \left(\|\partial_1\theta(t)\|_{H^2}^2 + \|\partial_1\omega(t)\|_{H^1}^2 + \|\omega(t)\|_{H^1}^2\right)\\ &+ \int_0^T (1+t)^2 \left(\|\partial_{11}\theta(t)\|_{L^2}^2 + \|\partial_1\omega(t)\|_{H^2}^2 + \|\omega(t)\|_{H^2}^2\right) \mathrm{d}t.\end{aligned} \tag{3.5}$$

The estimates of $\mathcal{E}(T)$ and $\mathcal{F}(T)$ will be given, respectively, in the following two sections.

3.1. A priori estimate of $\mathcal{E}(T)$

To begin, we set

$$\mathcal{E}_1(T) = \sup_{0\leq t\leq T} \left(\|\theta(t)\|_{H^{m+1}}^2 + \|\omega(t)\|_{H^m}^2\right) + \int_0^T \|\omega(t)\|_{H^{m+1}}^2 \mathrm{d}t, \tag{3.6}$$

$$\mathcal{E}_2(T) = \int_0^T \|\partial_1\theta(t)\|_{H^{m-1}}^2 \mathrm{d}t. \tag{3.7}$$

Then $\mathcal{E}(T) = \mathcal{E}_1(T) + \mathcal{E}_2(T)$.

We start with controlling the bound of $\mathcal{E}_1(T)$ by the combination of energies defined in (3.4)-(3.5).

Lemma 3.1. *Let $m \geq 2$. Then*

$$\mathcal{E}_1(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.8}$$

Proof. Multiplying $(1.10)_1$ by ω and testing $(1.10)_2$ by $-\Delta\theta$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\theta\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) + \|\nabla\omega\|_{L^2}^2 = -\langle\nabla\mathbf{u}\cdot\nabla\theta, \nabla\theta\rangle + \langle -\nabla u_2, \nabla\theta\rangle + \langle\partial_1\theta, \omega\rangle. \tag{3.9}$$

Due to $u_2 = -\partial_1(-\Delta)^{-1}\omega$, we use integrations by parts to obtain

$$\langle -\nabla u_2, \nabla\theta\rangle + \langle\partial_1\theta, \omega\rangle = \langle\nabla\partial_1(-\Delta)^{-1}\omega, \nabla\theta\rangle + \langle\partial_1\theta, \omega\rangle = 0. \tag{3.10}$$

By integrating (3.9) in time from 0 to T and using (3.10),

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|\nabla\theta(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2\right)+\int_0^T\|\nabla\omega(t)\|_{L^2}^2\mathrm{d}t\\
&\lesssim\|\nabla\theta_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\int_0^T\|\nabla\mathbf{u}(t)\|_{L^\infty}\mathrm{d}t\sup_{0\le t\le T}\|\nabla\theta(t)\|_{L^2}^2\\
&\lesssim\|\nabla\theta_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2+\left(\int_0^T(1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T(1+t)^{-2}\mathrm{d}t\right)^{\frac12}\sup_{0\le t\le T}\|\theta(t)\|_{H^1}^2\\
&\lesssim\mathcal{E}(0)+\mathcal{F}^{\frac12}(T)\mathcal{E}(T)\lesssim\mathcal{E}(0)+\mathcal{E}^{\frac32}(T)+\mathcal{F}^{\frac32}(T),
\end{aligned}\tag{3.11}$$

where Young's inequality is used in the last inequality. Moreover, we get for $m\ge1$,

$$\langle\partial_t\partial^m\omega,\partial^m\omega\rangle+\langle\partial^m(\mathbf{u}\cdot\nabla\omega),\partial^m\omega\rangle-\langle\Delta\partial^m\omega,\partial^m\omega\rangle=\langle\partial^m\partial_1\theta,\partial^m\omega\rangle,\tag{3.12}$$

$$\langle\partial_t\partial^m\nabla\theta,\partial^m\nabla\theta\rangle+\langle\partial^m\nabla(\mathbf{u}\cdot\nabla\theta),\partial^m\nabla\theta\rangle=\langle-\partial^m\nabla u_2,\partial^m\nabla\theta\rangle.\tag{3.13}$$

Adding (3.12) to (3.13) yields

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta\|_{\dot H^{m+1}}^2+\|\omega\|_{\dot H^m}^2\right)+\|\omega\|_{\dot H^{m+1}}^2=I_1+I_2+I_3\tag{3.14}$$

with

$$\begin{aligned}
I_1&=-\langle\partial^m(\mathbf{u}\cdot\nabla\omega),\partial^m\omega\rangle,\\
I_2&=-\langle\partial^m\nabla(\mathbf{u}\cdot\nabla\theta),\partial^m\nabla\theta\rangle,\\
I_3&=\langle-\partial^m\nabla u_2,\partial^m\nabla\theta\rangle+\langle\partial^m\partial_1\theta,\partial^m\omega\rangle.
\end{aligned}$$

According to (2.3) in Lemma 2.2, together with Corollary 2.1 and the Sobolev imbedding theorem, we have

$$\begin{aligned}
I_1&=-\left\langle\partial^m(\mathbf{u}\cdot\nabla\omega)-\mathbf{u}\cdot\nabla\partial^m\omega,\partial^m\omega\right\rangle=-\left\langle\partial^m\mathrm{div}(\mathbf{u}\omega)-\mathbf{u}\cdot\nabla\partial^m\omega,\partial^m\omega\right\rangle\\
&\lesssim\|\mathbf{u}\|_{W^{1,\infty}}\|\omega\|_{H^m}^2+\|\omega\|_{L^\infty}\|\mathbf{u}\|_{H^{m+1}}\|\omega\|_{H^m}\lesssim\|\omega\|_{H^2}\|\omega\|_{H^m}^2,
\end{aligned}$$

where we use the fact that $\langle\mathbf{u}\cdot\nabla\partial^m\omega,\partial^m\omega\rangle=0$. Hence, for $m\ge2$,

$$\int_0^T|I_1(t)|\mathrm{d}t\lesssim\sup_{0\le t\le T}\|\omega(t)\|_{H^m}\int_0^T\|\omega(t)\|_{H^m}^2\mathrm{d}t\lesssim\mathcal{E}^{\frac32}(T).\tag{3.15}$$

Note that

$$\begin{aligned}
I_2&=-\int_\Omega\partial^m\partial_1(\mathbf{u}\cdot\nabla\theta)\partial^m\partial_1\theta\mathrm{d}x\mathrm{d}y-\int_\Omega\partial^m\partial_2(\mathbf{u}\cdot\nabla\theta)\partial^m\partial_2\theta\mathrm{d}x\mathrm{d}y\\
&=:I_{21}+I_{22}.
\end{aligned}$$

For I_{21}, using the fact that $\langle\mathbf{u}\cdot\nabla\partial^m\partial_1\theta,\partial^m\partial_1\theta\rangle=0$ and applying integration by parts together with the boundary conditions (1.14),

$$\begin{aligned}
I_{21}=&-\sum_{1\le\alpha\le m+1}C_{m+1}^\alpha\int_\Omega\partial^\alpha\mathbf{u}\cdot\nabla\partial^{m+1-\alpha}\theta\partial^m\partial_1\theta\mathrm{d}x\mathrm{d}y\\
=&\sum_{2\le\alpha\le m+1}C_{m+1}^\alpha\int_\Omega\partial^{\alpha+1}\mathbf{u}\cdot\nabla\partial^{m+1-\alpha}\theta\partial^{m-1}\partial_1\theta\mathrm{d}x\mathrm{d}y\\
&+\sum_{2\le\alpha\le m+1}C_{m+1}^\alpha\int_\Omega\partial^\alpha\mathbf{u}\cdot\nabla\partial^{m+2-\alpha}\theta\partial^{m-1}\partial_1\theta\mathrm{d}x\mathrm{d}y\\
&-(m+1)\int_\Omega\partial\mathbf{u}\cdot\nabla\partial^m\theta\partial^m\partial_1\theta\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

where $\partial^m\partial_1$ is denoted by ∂^{m+1} which has at least one derivative on x. Then

$$\begin{aligned} I_{21} &\lesssim \|\mathbf{u}\|_{H^{m+2}}\|\theta\|_{H^{m+1}}\|\partial_1\theta\|_{H^{m-1}} \\ &\quad +\|\mathbf{u}\|_{H^{m+1}}\|\theta\|_{H^{m+1}}\|\partial_1\theta\|_{H^{m-1}} + \|\partial\mathbf{u}\|_{L^\infty}\|\theta\|_{H^{m+1}}^2 \\ &\lesssim \|\omega\|_{H^{m+1}}\|\theta\|_{H^{m+1}}\|\partial_1\theta\|_{H^{m-1}} + \|\omega\|_{H^2}\|\theta\|_{H^{m+1}}^2. \end{aligned}$$

With the help of Hölder inequality and Young's inequality,

$$\begin{aligned} \int_0^T |I_{21}(t)|\mathrm{d}t &\lesssim \left(\int_0^T \|\omega(t)\|_{H^{m+1}}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|\partial_1\theta(t)\|_{H^{m-1}}^2\mathrm{d}t\right)^{\frac{1}{2}}\sup_{0\le t\le T}\|\theta(t)\|_{H^{m+1}} \\ &\quad + \left(\int_0^T (1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T (1+t)^{-2}\mathrm{d}t\right)^{\frac{1}{2}}\sup_{0\le t\le T}\|\theta(t)\|_{H^{m+1}}^2 \\ &\lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{1}{2}}(T)\mathcal{E}(T) \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \end{aligned} \tag{3.16}$$

For I_{22}, we analyze the case, namely ∂^m has only derivative on y. Other cases can be estimated by the method as employed in the proof of I_{21}. Since $\langle \mathbf{u}\cdot\nabla\partial^m\partial_2\theta, \partial^m\partial_2\theta\rangle = 0$,

$$\begin{aligned} I_{22} &= -\sum_{1\le\alpha\le m+1} C_{m+1}^{\alpha}\int_\Omega \partial_2^{\alpha}\mathbf{u}\cdot\nabla\partial_2^{m+1-\alpha}\theta\partial_2^{m+1}\theta\mathrm{d}x\mathrm{d}y \\ &= I_{221} + I_{222} \end{aligned}$$

with

$$\begin{aligned} I_{221} &= -\sum_{1\le\alpha\le m+1} C_{m+1}^{\alpha}\int_\Omega \partial_2^{\alpha}u_1\partial_1\partial_2^{m+1-\alpha}\theta\partial_2^{m+1}\theta\mathrm{d}x\mathrm{d}y, \\ I_{222} &= -\sum_{1\le\alpha\le m+1} C_{m+1}^{\alpha}\int_\Omega \partial_2^{\alpha}u_2\partial_2^{m+2-\alpha}\theta\partial_2^{m+1}\theta\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Note that

$$\begin{aligned} I_{221} &= -\sum_{2\le\alpha\le m+1} C_{m+1}^{\alpha}\int_\Omega \partial_2^{\alpha}u_1\partial_2^{m+1-\alpha}\partial_1\theta\partial_2^{m+1}\theta\mathrm{d}x\mathrm{d}y \\ &\quad -(m+1)\int_\Omega \partial_2u_1\partial_2^{m}\partial_1\theta\partial_2^{m+1}\theta\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Similarly,

$$\begin{aligned} I_{221} &\lesssim \|\mathbf{u}\|_{H^{m+1}}\|\partial_1\theta\|_{H^{m-1}}\|\theta\|_{H^{m+1}} + \|\partial_2u_1\|_{L^\infty}\|\theta\|_{H^{m+1}}^2 \\ &\lesssim \|\omega\|_{H^m}\|\partial_1\theta\|_{H^{m-1}}\|\theta\|_{H^{m+1}} + \|\omega\|_{H^2}\|\theta\|_{H^{m+1}}^2. \end{aligned}$$

Then

$$\int_0^T |I_{221}(t)|\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{1}{2}}(T)\mathcal{E}(T) \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.17}$$

Using the fact that $\partial_2 u_2 = -\partial_1 u_1$ and integrating by parts yield

$$\begin{aligned}
I_{222} = &- \sum_{3\le\alpha\le m+1} C_{m+1}^{\alpha} \int_\Omega \partial_2^{\alpha-1} u_1 \partial_2^{m+2-\alpha} \partial_1 \theta \partial_2^{m+1} \theta \mathrm{d}x\mathrm{d}y \\
&- \sum_{3\le\alpha\le m+1} C_{m+1}^{\alpha} \int_\Omega \partial_2^{\alpha-1} u_1 \partial_2^{m+2-\alpha} \theta \partial_1 \partial_2^{m+1} \theta \mathrm{d}x\mathrm{d}y \\
&-(m+1)\int_\Omega \partial_2 u_2 \partial_2^{m+1}\theta \partial_2^{m+1}\theta \mathrm{d}x\mathrm{d}y - C_{m+1}^2 \int_\Omega \partial_2^2 u_2 \partial_2^m \theta \partial_2^{m+1}\theta \mathrm{d}x\mathrm{d}y \\
= &- \sum_{3\le\alpha\le m+1} C_{m+1}^{\alpha} \int_\Omega \partial_2^{\alpha-1} u_1 \partial_2^{m+2-\alpha} \partial_1 \theta \partial_2^{m+1} \theta \mathrm{d}x\mathrm{d}y \\
&- \sum_{3\le\alpha\le m+1} C_{m+1}^{\alpha} \int_\Omega \sum_{0\le\beta\le 2} C_2^\beta \partial_2^{\alpha-1+\beta} u_1 \partial_2^{m+2-\alpha+2-\beta} \theta \partial_1 \partial_2^{m-1} \theta \mathrm{d}x\mathrm{d}y \\
&+(m+1)\int_\Omega \partial_1 u_1 \partial_2^{m+1}\theta \partial_2^{m+1}\theta \mathrm{d}x\mathrm{d}y + C_{m+1}^2 \int_\Omega \partial_2 \partial_1 u_1 \partial_2^m \theta \partial_2^{m+1}\theta \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Furthermore, one gets

$$\begin{aligned}
I_{222} &\lesssim \|\mathbf{u}\|_{H^m} \|\partial_1\theta\|_{H^{m-1}} \|\theta\|_{H^{m+1}} + \|\mathbf{u}\|_{H^{m+2}} \|\partial_1\theta\|_{H^{m-1}} \|\theta\|_{H^{m+1}} \\
&\quad + \|\partial_1 u_1\|_{L^\infty} \|\theta\|_{H^{m+1}}^2 + \|\partial_2\partial_1 u_1\|_{L^\infty} \|\theta\|_{H^{m+1}}^2 \\
&\lesssim \|\omega\|_{H^{m+1}} \|\partial_1\theta\|_{H^{m-1}} \|\theta\|_{H^{m+1}} + \|\partial_1\omega\|_{H^2} \|\theta\|_{H^{m+1}}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^T |I_{222}(t)|\mathrm{d}t &\lesssim \mathcal{E}^{\frac{3}{2}}(T) \\
&\quad + \left(\int_0^T (1+t)^2 \|\partial_1\omega(t)\|_{H^2}^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T (1+t)^{-2}\mathrm{d}t\right)^{\frac{1}{2}} \sup_{0\le t\le T} \|\theta(t)\|_{H^{m+1}}^2 \\
&\lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{1}{2}}(T)\mathcal{E}(T) \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T).
\end{aligned} \tag{3.18}$$

Putting (3.17) and (3.18) together,

$$\int_0^T |I_{22}(t)|\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.19}$$

Moreover, the preceding estimates (3.16) and (3.19) show that

$$\int_0^T |I_{2}(t)|\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.20}$$

To handle the last term, substituting $u_2 = -\partial_1(-\Delta)^{-1}\omega$ into I_3 and then integrating by parts give

$$I_3 = \langle \partial^m \nabla \partial_1 (-\Delta)^{-1}\omega, \partial^m \nabla\theta\rangle + \langle \partial^m \partial_1\theta, \partial^m\omega\rangle = 0. \tag{3.21}$$

Integrating (3.14) in time from 0 to T and summing up (3.15), (3.20), one has

$$\sup_{0\le t\le T} \left(\|\theta(t)\|_{\dot H^{m+1}}^2 + \|\omega(t)\|_{\dot H^m}^2\right) + \int_0^T \|\omega(t)\|_{\dot H^{m+1}}^2 \mathrm{d}t \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.22}$$

Finally adding (3.11) to (3.22), we get by Poincaré inequality that

$$\mathcal{E}_1(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.23}$$

□

In the following lemma, we proceed to deal with the estimate of $\mathcal{E}_2(T)$.

Lemma 3.2. *Let $m \geq 2$. Then*

$$\mathcal{E}_2(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.24}$$

Proof. Applying ∂^{m-1} on $(1.10)_1$ and taking inner product with $\partial_1\partial^{m-1}\theta$, one gets

$$\begin{aligned}\|\partial_1\theta\|_{\dot{H}^{m-1}}^2 &= \langle \partial^{m-1}\partial_t\omega, \partial_1\partial^{m-1}\theta\rangle + \langle \partial^{m-1}(\mathbf{u}\cdot\nabla\omega), \partial_1\partial^{m-1}\theta\rangle - \langle \partial^{m-1}\Delta\omega, \partial_1\partial^{m-1}\theta\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \partial^{m-1}\omega, \partial_1\partial^{m-1}\theta\rangle - \langle \partial^{m-1}\omega, \partial_1\partial^{m-1}\partial_t\theta\rangle \\ &\quad + \langle \partial^{m-1}(\mathbf{u}\cdot\nabla\omega), \partial_1\partial^{m-1}\theta\rangle - \langle \partial^{m-1}\Delta\omega, \partial_1\partial^{m-1}\theta\rangle \\ &=: N_1 + N_2 + N_3 + N_4. \end{aligned} \tag{3.25}$$

We need estimate $N_j, j = 1, 2, 3, 4$, respectively. For N_1, by Hölder inequality,

$$\int_0^T |N_1(t)|\mathrm{d}t \lesssim \sup_{0\leq t\leq T} \|\theta(t)\|_{H^m} \sup_{0\leq t\leq T} \|\omega(t)\|_{H^{m-1}} \lesssim \mathcal{E}_1(T). \tag{3.26}$$

Substituting $\partial_t\theta = -\mathbf{u}\cdot\nabla\theta - u_2$ into N_2 and using (2.2) in Lemma 2.2 give

$$\begin{aligned} N_2 &= \langle \partial^{m-1}\omega, \partial_1\partial^{m-1}(\mathbf{u}\cdot\nabla\theta)\rangle + \langle \partial^{m-1}\omega, \partial_1\partial^{m-1}u_2\rangle \\ &\lesssim \|\omega\|_{H^{m+1}}^2\|\theta\|_{H^{m+1}} + \|\omega\|_{H^{m+1}}^2. \end{aligned}$$

Then

$$\int_0^T |N_2(t)|\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \int_0^T \|\omega(t)\|_{H^{m+1}}^2\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{E}_1(T). \tag{3.27}$$

Similarly for N_3,

$$\int_0^T |N_3(t)|\mathrm{d}t \lesssim \int_0^T \|\omega(t)\|_{H^{m+1}}^2\|\partial_1\theta(t)\|_{H^{m-1}}\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T). \tag{3.28}$$

By Hölder inequality and Young's inequality,

$$N_4 \leq C_\varepsilon\|\omega\|_{H^{m+1}}^2 + \varepsilon\|\partial_1\theta\|_{\dot{H}^{m-1}}^2,$$

where C_ε is a constant depending on ε. Thus,

$$\begin{aligned}\int_0^T |N_4(t)|\mathrm{d}t &\leq C_\varepsilon\int_0^T \|\omega(t)\|_{H^{m+1}}^2\mathrm{d}t + \varepsilon\int_0^T \|\partial_1\theta(t)\|_{\dot{H}^{m-1}}^2\mathrm{d}t \\ &\leq C_\varepsilon\mathcal{E}_1(T) + \varepsilon\int_0^T \|\partial_1\theta(t)\|_{\dot{H}^{m-1}}^2\mathrm{d}t. \end{aligned} \tag{3.29}$$

Integrating (3.25) in time from 0 to T, summing up (3.26)-(3.29) and then taking ε small enough, we infer from Lemma 3.1 that

$$\int_0^T \|\partial_1\theta(t)\|_{\dot{H}^{m-1}}^2\mathrm{d}t \lesssim \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{E}_1(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.30}$$

Similarly,

$$\int_0^T \|\partial_1\theta(t)\|_{L^2}^2\mathrm{d}t \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.31}$$

Finally, the desired estimate (3.24) follows from (3.30) and (3.31). □

From Lemma 3.1-3.2, we obtain

Lemma 3.3. *Let $m \geq 2$. Then*

$$\mathcal{E}(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.32}$$

3.2. A priori estimate of $\mathcal{F}(T)$

Here we set

$$\mathcal{F}_1(T) = \sup_{0\le t\le T}(1+t)^2\left(\|\partial_1\theta(t)\|_{H^2}^2 + \|\partial_1\omega(t)\|_{H^1}^2\right) + \int_0^T (1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t, \tag{3.33}$$

$$\mathcal{F}_2(T) = \int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t, \tag{3.34}$$

$$\mathcal{F}_3(T) = \sup_{0\le t\le T}(1+t)^2\|\omega(t)\|_{H^1}^2 + \int_0^T (1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t. \tag{3.35}$$

To obtain a priori estimate of $\mathcal{F}(T)$, we shall estimate $\mathcal{F}_1(T)$, $\mathcal{F}_2(T)$ and $\mathcal{F}_3(T)$, respectively.

Lemma 3.4. *Let $m\ge 5$. Then*

$$\mathcal{F}_1(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T). \tag{3.36}$$

Proof. Taking $\partial\partial_1$ on $(1.10)_1$ and testing by $\partial\partial_1\omega$ yield

$$\langle\partial_t\partial\partial_1\omega,\partial\partial_1\omega\rangle + \langle\partial\partial_1(\mathbf{u}\cdot\nabla\omega),\partial\partial_1\omega\rangle - \langle\Delta\partial\partial_1\omega,\partial\partial_1\omega\rangle = \langle\partial\partial_{11}\theta,\partial\partial_1\omega\rangle. \tag{3.37}$$

Applying $\partial\partial_1\nabla$ to $(1.10)_2$ and taking inner product with $\partial\partial_1\nabla\theta$ give

$$\langle\partial_t\partial\partial_1\nabla\theta,\partial\partial_1\nabla\theta\rangle + \langle\partial\partial_1\nabla(\mathbf{u}\cdot\nabla\theta),\partial\partial_1\nabla\theta\rangle = \langle-\partial\partial_1\nabla u_2,\partial\partial_1\nabla\theta\rangle. \tag{3.38}$$

By adding (3.37) to (3.38) and multiplying the time weight $(1+t)^2$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^2\left(\|\partial_1\theta\|_{\dot H^2}^2 + \|\partial_1\omega\|_{\dot H^1}^2\right) + (1+t)^2\|\partial_1\omega\|_{\dot H^2}^2 = J_1+J_2+J_3+J_4, \tag{3.39}$$

where

$$\begin{aligned}
J_1 &= -(1+t)^2\langle\partial\partial_1(\mathbf{u}\cdot\nabla\omega),\partial\partial_1\omega\rangle,\\
J_2 &= -(1+t)^2\langle\partial\partial_1\nabla(\mathbf{u}\cdot\nabla\theta),\partial\partial_1\nabla\theta\rangle,\\
J_3 &= -(1+t)\left(\|\partial_1\theta\|_{\dot H^2}^2 + \|\partial_1\omega\|_{\dot H^1}^2\right),\\
J_4 &= -(1+t)^2\left(\langle\partial\partial_{11}\theta,\partial\partial_1\omega\rangle + \langle-\partial\partial_1\nabla u_2,\partial\partial_1\nabla\theta\rangle\right).
\end{aligned}$$

Estimate of J_1. Note that

$$\begin{aligned}
J_1 = &-(1+t)^2\langle\partial\partial_1\mathbf{u}\cdot\nabla\omega,\partial\partial_1\omega\rangle - (1+t)^2\langle\partial\mathbf{u}\cdot\nabla\partial_1\omega,\partial\partial_1\omega\rangle\\
&-(1+t)^2\langle\partial_1\mathbf{u}\cdot\nabla\partial\omega,\partial\partial_1\omega\rangle - (1+t)^2\langle\mathbf{u}\cdot\nabla\partial\partial_1\omega,\partial\partial_1\omega\rangle.
\end{aligned}$$

By the incompressibility condition for $\mathbf{u}$,

$$\langle\mathbf{u}\cdot\nabla\partial\partial_1\omega,\partial\partial_1\omega\rangle = 0.$$

Then the Sobolev imbedding theorem implies that

$$\begin{aligned}
J_1 &\lesssim (1+t)^2\|\partial\partial_1\mathbf{u}\|_{L^2}\|\nabla\omega\|_{L^\infty}\|\partial_1\omega\|_{H^1}\\
&\quad+(1+t)^2\|\partial_1\omega\|_{H^1}^2\|\partial\mathbf{u}\|_{L^\infty} + (1+t)^2\|\partial_1\mathbf{u}\|_{L^\infty}\|\omega\|_{H^2}\|\partial_1\omega\|_{H^1}\\
&\lesssim (1+t)^2\|\partial_1\omega\|_{H^1}^2\|\omega\|_{H^m}.
\end{aligned}$$

Thus, by Young's inequality,

$$\begin{aligned}
\int_0^T|J_1(t)|\mathrm{d}t &\lesssim \int_0^T(1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t\sup_{0\le t\le T}\|\omega(t)\|_{H^m}\\
&\lesssim \mathcal{F}(T)\mathcal{E}^{\frac{1}{2}}(T) \lesssim \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{3}{2}}(T).
\end{aligned} \tag{3.40}$$

Estimate of J_2. We divide J_2 into six parts as follows.

$$J_2=-(1+t)^2\langle\partial^2\partial_1(\mathbf{u}\cdot\nabla\theta),\partial^2\partial_1\theta\rangle=J_{21}+J_{22}+J_{23}+J_{24}+J_{25}+J_{26}$$

with

$$\begin{aligned}&J_{21}=-(1+t)^2\langle\partial^2\partial_1\mathbf{u}\cdot\nabla\theta,\partial^2\partial_1\theta\rangle,\ J_{22}=-(1+t)^2\langle\partial^2\mathbf{u}\cdot\nabla\partial_1\theta,\partial^2\partial_1\theta\rangle,\\&J_{23}=-2(1+t)^2\langle\partial\partial_1\mathbf{u}\cdot\nabla\partial\theta,\partial^2\partial_1\theta\rangle,\ J_{24}=-(1+t)^2\langle\partial_1\mathbf{u}\cdot\nabla\partial^2\theta,\partial^2\partial_1\theta\rangle,\\&J_{25}=-2(1+t)^2\langle\partial\mathbf{u}\cdot\nabla\partial\partial_1\theta,\partial^2\partial_1\theta\rangle,\ J_{26}=-(1+t)^2\langle\mathbf{u}\cdot\nabla\partial^2\partial_1\theta,\partial^2\partial_1\theta\rangle.\end{aligned}$$

For J_{21}, we calculate that

$$\begin{aligned}J_{21}&=-(1+t)^2\langle\partial^2\partial_1u_1\partial_1\theta,\partial^2\partial_1\theta\rangle-(1+t)^2\langle\partial^2\partial_1u_2\partial_2\theta,\partial^2\partial_1\theta\rangle\\&=-(1+t)^2\langle\partial^2\partial_1u_1\partial_1\theta,\partial^2\partial_1\theta\rangle+(1+t)^2\langle\partial^2u_2\partial_2\partial_1\theta,\partial^2\partial_1\theta\rangle+(1+t)^2\langle\partial^2u_2\partial_2\theta,\partial^2\partial_{11}\theta\rangle\\&=:J_{211}+J_{212}+J_{213}.\end{aligned}$$

From Lemma 2.3, we find that

$$J_{211}\lesssim(1+t)^2\|\partial_1\omega\|_{H^1}\|\partial_1\theta\|_{L^2}\|\theta\|_{H^5}\lesssim(1+t)^2\|\partial_1\omega\|_{H^1}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.41}$$

Integrating by parts and using the fact that $\mathbf{u}=\nabla^\perp\psi=\nabla^\perp(-\Delta)^{-1}\omega$ yield

$$\begin{aligned}J_{212}&=(1+t)^2\int_\Omega\sum_{0\le\beta\le2}\partial^{2+\beta}u_2\partial^{2-\beta}\partial_2\partial_1\theta\partial_1\theta\mathrm{d}x\mathrm{d}y\\&=-(1+t)^2\int_\Omega\sum_{0\le\beta\le2}\partial^{2+\beta}\partial_1\psi\partial^{2-\beta}\partial_2\partial_1\theta\partial_1\theta\mathrm{d}x\mathrm{d}y.\end{aligned}$$

From Lemma 2.3-2.4, it follows that

$$J_{212}\lesssim(1+t)^2\|\partial_1\psi\|_{H^4}\|\partial_1\theta\|_{L^2}\|\theta\|_{H^5}\lesssim(1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.42}$$

In a similar way,

$$\begin{aligned}J_{213}&=(1+t)^2\int_\Omega\sum_{0\le\beta\le2}\partial^{2+\beta}u_2\partial^{2-\beta}\partial_2\theta\partial_{11}\theta\mathrm{d}x\mathrm{d}y\\&=-(1+t)^2\int_\Omega\sum_{0\le\beta\le2}\partial^{2+\beta}\partial_1\psi\partial^{2-\beta}\partial_2\theta\partial_{11}\theta\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then

$$J_{213}\lesssim(1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.43}$$

Summing up (3.41)-(3.43),

$$J_{21}\lesssim(1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.44}$$

Note that

$$\begin{aligned}J_{22}&=-(1+t)^2\langle\partial^2u_1\partial_{11}\theta,\partial^2\partial_1\theta\rangle-(1+t)^2\langle\partial^2u_2\partial_2\partial_1\theta,\partial^2\partial_1\theta\rangle\\&=-(1+t)^2\langle\partial^2u_1\partial_{11}\theta,\partial^2\partial_1\theta\rangle+(1+t)^2\langle\partial_2\partial^2u_2\partial_1\theta,\partial^2\partial_1\theta\rangle+(1+t)^2\langle\partial^2u_2\partial_1\theta,\partial^2\partial_2\partial_1\theta\rangle.\end{aligned}$$

This implies that

$$J_{22}\lesssim(1+t)^2\|\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.45}$$

Similarly, we obtain

$$J_{23}\lesssim(1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5},\tag{3.46}$$

$$J_{24}\lesssim(1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5},\tag{3.47}$$

and

$$J_{25}\lesssim(1+t)^2\|\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.\tag{3.48}$$

Moreover, the incompressibility condition for $\mathbf{u}$ implies that

$$J_{26} = 0. \tag{3.49}$$

We eventually deduce from (3.44)-(3.49) that

$$J_2 \lesssim (1+t)^2\|\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5} + (1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2}\|\theta\|_{H^5}.$$

Consequently, by Hölder inequality and Young's inequality,

$$\begin{aligned}\int_0^T |J_2(t)|\mathrm{d}t &\lesssim \left(\int_0^T (1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t\right)^{\frac12}\sup_{0\le t\le T}\|\theta(t)\|_{H^m}\\ &\quad + \left(\int_0^T (1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t\right)^{\frac12}\sup_{0\le t\le T}\|\theta(t)\|_{H^m}\\ &\lesssim \mathcal{F}(T)\mathcal{E}^{\frac12}(T) \lesssim \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac32}(T).\end{aligned} \tag{3.50}$$

Estimate of J_3. By the interpolation inequality (2.1) in Lemma 2.1,

$$\begin{aligned}J_3 &\lesssim (1+t)\|\partial_1\theta\|_{H^2}^2 + (1+t)\|\partial_1\omega\|_{H^2}^2\\ &\lesssim (1+t)\|\partial_1\theta\|_{L^2}\|\partial_1\theta\|_{H^4} + (1+t)\|\partial_1\omega\|_{H^2}^2.\end{aligned}$$

Hence, for $m \ge 5$,

$$\begin{aligned}\int_0^T |J_3(t)|\mathrm{d}t &\lesssim \left(\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T \|\partial_1\theta(t)\|_{H^{m-1}}^2\mathrm{d}t\right)^{\frac12}\\ &\quad + \left(\int_0^T (1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T \|\omega(t)\|_{H^{m+1}}^2\mathrm{d}t\right)^{\frac12}\\ &\lesssim \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T).\end{aligned} \tag{3.51}$$

Estimate of J_4. Substituting $u_2 = -\partial_1(-\Delta)^{-1}\omega$ into J_4 and integrating by parts give

$$J_4 = 0. \tag{3.52}$$

Integrating (3.39) in time from 0 to T and putting (3.40), (3.50), (3.51), (3.52) together, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}(1+t)^2\left(\|\partial_1\theta(t)\|_{\dot H^2}^2 + \|\partial_1\omega(t)\|_{\dot H^1}^2\right) + \int_0^T (1+t)^2\|\partial_1\omega(t)\|_{\dot H^2}^2\mathrm{d}t\\ &\quad\lesssim \mathcal{F}(0) + \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T).\end{aligned} \tag{3.53}$$

Similarly,

$$\begin{aligned}&\sup_{0\le t\le T}(1+t)^2\left(\|\nabla\partial_1\theta(t)\|_{L^2}^2 + \|\partial_1\omega(t)\|_{L^2}^2\right) + \int_0^T (1+t)^2\|\partial_1\nabla\omega(t)\|_{L^2}^2\mathrm{d}t\\ &\quad\lesssim \mathcal{F}(0) + \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T).\end{aligned} \tag{3.54}$$

Hence, summing up (3.53)-(3.54) and using Poincaré inequality give

$$\mathcal{F}_1(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T).$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *Let* $m \ge 5$. *Then*

$$\mathcal{F}_2(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T). \tag{3.55}$$

Proof. Taking ∂_1 on $(1.10)_1$ and testing by $\partial_{11}\theta$ yield

$$\begin{aligned}\|\partial_{11}\theta\|_{L^2}^2 &= \langle \partial_t\partial_1\omega, \partial_{11}\theta\rangle + \langle \partial_1(\mathbf{u}\cdot\nabla\omega), \partial_{11}\theta\rangle - \langle \Delta\partial_1\omega, \partial_{11}\theta\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_1\omega, \partial_{11}\theta\rangle - \langle \partial_1\omega, \partial_{11}\partial_t\theta\rangle + \langle \partial_1(\mathbf{u}\cdot\nabla\omega), \partial_{11}\theta\rangle - \langle \Delta\partial_1\omega, \partial_{11}\theta\rangle. \end{aligned} \tag{3.56}$$

Multiplying (3.56) by $(1+t)^2$, we have

$$\begin{aligned}(1+t)^2\|\partial_{11}\theta\|_{L^2}^2 =& \frac{\mathrm{d}}{\mathrm{d}t}(1+t)^2\langle \partial_1\omega, \partial_{11}\theta\rangle - 2(1+t)\langle \partial_1\omega, \partial_{11}\theta\rangle - (1+t)^2\langle \partial_1\omega, \partial_{11}\partial_t\theta\rangle \\ &+ (1+t)^2\langle \partial_1(\mathbf{u}\cdot\nabla\omega), \partial_{11}\theta\rangle - (1+t)^2\langle \Delta\partial_1\omega, \partial_{11}\theta\rangle \\ =:& K_1 + K_2 + K_3 + K_4 + K_5. \end{aligned} \tag{3.57}$$

We now estimate $K_1, K_2, \cdots, K_5$ one by one. For K_1, by Hölder inequality,

$$\int_0^T |K_1(t)|\mathrm{d}t \lesssim \sup_{0\le t\le T}(1+t)\|\partial_1\theta(t)\|_{H^2} \sup_{0\le t\le T}(1+t)\|\partial_1\omega(t)\|_{H^1} \lesssim \mathcal{F}_1(T). \tag{3.58}$$

For K_2, one gets

$$\int_0^T |K_2(t)|\mathrm{d}t \lesssim \left(\int_0^T \|\partial_1\omega(t)\|_{L^2}^2\mathrm{d}t\right)^{\frac12}\left(\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t\right)^{\frac12} \lesssim \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T). \tag{3.59}$$

By substituting $\partial_t\theta = -\mathbf{u}\cdot\nabla\theta - u_2$ into K_3,

$$\begin{aligned}K_3 &= (1+t)^2\langle \partial_1\omega, \partial_{11}(\mathbf{u}\cdot\nabla\theta)\rangle + (1+t)^2\langle \partial_1\omega, \partial_{11}u_2\rangle \\ &\lesssim (1+t)^2\|\omega\|_{H^2}^2\|\theta\|_{H^3} + (1+t)^2\|\partial_1\omega\|_{H^2}^2.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T |K_3(t)|\mathrm{d}t &\lesssim \mathcal{E}^{\frac12}(T)\mathcal{F}(T) + \int_0^T (1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t \\ &\lesssim \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{F}_1(T). \end{aligned} \tag{3.60}$$

Note that

$$K_4 \lesssim (1+t)^2\|\partial_{11}\theta\|_{L^2}\|\partial_1\omega\|_{H^2}\|\omega\|_{H^1}.$$

Then

$$\int_0^T |K_4(t)|\mathrm{d}t \lesssim \mathcal{E}^{\frac12}(T)\mathcal{F}(T) \lesssim \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T). \tag{3.61}$$

For the last term, by Young's inequality,

$$\begin{aligned}K_5 &\lesssim (1+t)^2\|\partial_1\omega\|_{H^2}\|\partial_{11}\theta\|_{L^2} \\ &\le C_\varepsilon(1+t)^2\|\partial_1\omega\|_{H^2}^2 + \varepsilon(1+t)^2\|\partial_{11}\theta\|_{L^2}^2.\end{aligned}$$

Then

$$\begin{aligned}\int_0^T |K_5(t)|\mathrm{d}t &\le C_\varepsilon\int_0^T (1+t)^2\|\partial_1\omega(t)\|_{H^2}^2\mathrm{d}t + \varepsilon\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t \\ &\le C_\varepsilon\mathcal{F}_1(T) + \varepsilon\int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t. \end{aligned} \tag{3.62}$$

Finally, integrating (3.57) in time from 0 to T and summing up (3.58)-(3.62), we take ε small enough to obtain

$$\begin{aligned}\mathcal{F}_2(T) &\lesssim \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{F}_1(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T) \\ &\lesssim \mathcal{F}(0) + \mathcal{E}^{\frac32}(T) + \mathcal{F}^{\frac32}(T) + \mathcal{E}^{\frac12}(T)\mathcal{F}^{\frac12}(T),\end{aligned}$$

where we apply (3.36) in Lemma 3.4. □

Lemma 3.6. *Let $m \geq 5$. Then*

$$\mathcal{F}_3(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T). \tag{3.63}$$

Proof. Testing $(1.10)_1$ by $-\Delta\omega$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\dot{H}^1}^2 + \|\Delta\omega\|_{L^2}^2 = -\langle \mathbf{u}\cdot\nabla\omega, \Delta\omega\rangle - \langle \partial_1\theta, \Delta\omega\rangle. \tag{3.64}$$

Furthermore, multiplying (3.64) by $(1+t)^2$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(1+t)^2\|\omega\|_{\dot{H}^1}^2 + (1+t)^2\|\Delta\omega\|_{L^2}^2 = R_1 + R_2 + R_3, \tag{3.65}$$

where

$$\begin{aligned} R_1 &= -(1+t)^2\langle \mathbf{u}\cdot\nabla\omega, \Delta\omega\rangle, \\ R_2 &= -(1+t)^2\langle \partial_1\theta, \Delta\omega\rangle,\ R_3 = (1+t)\|\omega\|_{\dot{H}^1}^2. \end{aligned}$$

By Hölder inequality and Young's inequality,

$$\begin{aligned} \int_0^T |R_1(t)|\mathrm{d}t &\lesssim \int_0^T (1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t \sup_{0\leq t\leq T}\|\omega(t)\|_{H^m} \\ &\lesssim \mathcal{F}(T)\mathcal{E}^{\frac{1}{2}}(T) \lesssim \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{3}{2}}(T). \end{aligned} \tag{3.66}$$

Similarly for R_2, we deduce from Lemma 2.3 that

$$\begin{aligned} \int_0^T |R_2(t)|\mathrm{d}t &\leq C_\varepsilon \int_0^T (1+t)^2\|\partial_{11}\theta(t)\|_{L^2}^2\mathrm{d}t + \varepsilon\int_0^T (1+t)^2\|\Delta\omega(t)\|_{L^2}^2\mathrm{d}t \\ &\leq C_\varepsilon\mathcal{F}_2(T) + \varepsilon\int_0^T (1+t)^2\|\Delta\omega(t)\|_{L^2}^2\mathrm{d}t. \end{aligned} \tag{3.67}$$

For R_3, we find that

$$\begin{aligned} \int_0^T |R_3(t)|\mathrm{d}t &\lesssim \left(\int_0^T (1+t)^2\|\omega(t)\|_{H^2}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|\omega(t)\|_{H^1}^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &\lesssim \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T). \end{aligned} \tag{3.68}$$

Integrating (3.65) in time from 0 to T and summing up (3.66)-(3.68), we take ε small enough to yield

$$\begin{aligned} &\sup_{0\leq t\leq T}(1+t)^2\|\omega(t)\|_{\dot{H}^1}^2 + \int_0^T (1+t)^2\|\Delta\omega(t)\|_{L^2}^2\mathrm{d}t \\ &\quad\lesssim \mathcal{F}(0) + \mathcal{F}_2(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T). \end{aligned}$$

Finally, we use Poincaré inequality, Lemma 2.4 and (3.55) in Lemma 3.5 to obtain

$$\begin{aligned} \mathcal{F}_3(T) &\lesssim \mathcal{F}(0) + \mathcal{F}_2(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T) \\ &\lesssim \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T). \end{aligned}$$

□

The next lemma gives estimate of $\mathcal{F}(T)$.

Lemma 3.7. *Let $m \geq 5$. Then*

$$\mathcal{F}(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T). \tag{3.69}$$

Proof. From Lemma 3.4-3.6, it follows that

$$\mathcal{F}(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}^{\frac{1}{2}}(T)\mathcal{F}^{\frac{1}{2}}(T).$$

By virtue of Young's inequality,

$$\mathcal{F}(T) \lesssim \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + C_\varepsilon \mathcal{E}(T) + \varepsilon \mathcal{F}(T).$$

Then taking ε small enough and using (3.32) in Lemma 3.3 give

$$\mathcal{F}(T) \lesssim \mathcal{E}(0) + \mathcal{F}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) + \mathcal{E}(T) \lesssim \mathcal{E}(0) + \mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T),$$

where we use the fact $\mathcal{F}(0) \lesssim \mathcal{E}(0)$. Hence, the proof of Lemma 3.7 is completed. □

3.3. Proof of theorem 1.1

From Lemma 3.3 and Lemma 3.7, there exists a constant $C_0 > 0$ such that

$$\mathcal{E}(T) + \mathcal{F}(T) \le C_0 \mathcal{E}(0) + C_0 \left(\mathcal{E}^{\frac{3}{2}}(T) + \mathcal{F}^{\frac{3}{2}}(T) \right). \tag{3.70}$$

By denoting $\mathcal{G}(T) = \mathcal{E}(T) + \mathcal{F}(T)$, one deduces from (3.70) that

$$\mathcal{G}(T) \le C_0 \mathcal{E}(0) + C_0 \mathcal{G}^{\frac{3}{2}}(T). \tag{3.71}$$

Assume that

$$\|\omega_0\|_{H^m}^2 + \|\theta_0\|_{H^{m+1}}^2 \le \epsilon_0^2, \tag{3.72}$$

with $\epsilon_0 \in (0,1)$ to be determined later. Then there exists a constant $C_1 > 0$ such that

$$\begin{aligned} C_0 \mathcal{E}(0) + \mathcal{G}(0) &= C_0 \mathcal{E}(0) + \mathcal{E}(0) + \mathcal{F}(0) \\ &\le C_0 \epsilon_0^2 + 3\epsilon_0^2 \le C_1 \epsilon_0^2. \end{aligned} \tag{3.73}$$

According to Proposition 3.1, there exists a positive time $T_0 < T^*$ such that

$$\mathcal{G}(T) \le 4C_1 \epsilon_0^2, \ \text{for all } T \in [0, T_0]. \tag{3.74}$$

Since T^* is the life span to the solution (ω, θ), we only need to show $T^* = \infty$ while completing the proof of Theorem 1.1. Otherwise, if $T^* < \infty$, the solution (ω, θ) satisfies (3.3). Then we define

$$\tilde{T} \triangleq \sup\{T < T^* : \mathcal{G}(T) \le 4C_1 \epsilon_0^2\}. \tag{3.75}$$

Moreover, we take ϵ_0 small enough to yield $8C_0 C_1^{\frac{1}{2}} \epsilon_0 < 1$. From (3.71) and (3.73), it follows that

$$\mathcal{G}(T) \le C_0 \epsilon_0^2 + 8C_0 C_1^{\frac{3}{2}} \epsilon_0^3 \le C_1 \epsilon_0^2 + 8C_0 C_1^{\frac{3}{2}} \epsilon_0^3 < 2C_1 \epsilon_0^2, \ \text{for all } t \in [0, \tilde{T}).$$

By using a continuity argument,

$$\mathcal{G}(T) \le 2C_1 \epsilon_0^2, \ \text{for all } t \in [0, T^*),$$

which gives a contradiction with (3.75) if $\tilde{T} < T^* < \infty$. This in turn implies that $\tilde{T} = T^* = \infty$. Thus, we finish the proof of Theorem 1.1.

4. Concluding remarks

In this article, we show stability of the specific stationary solution $\omega_s = 0$, $\vartheta_s = y$ to Boussinesq equations without thermal conduction in the two-dimensional domain $\mathbb{T}\times(0,1)$. For the vorticity/velocity, we obtain asymptotic stability and explicit decay rate while for the temperature only the stability in the sense of Lyapunov is given. One may expect the temperature ϑ converges to ϑ_s as time goes to infinity. However, this is not true in general, once we realize that the stationary solution

$$\tilde{\omega}_s = 0,\ \tilde{\vartheta}_s = y + \epsilon \sin 2\pi y$$

is a small perturbation of (ω_s, ϑ_s) if ϵ is small. Note that the existence of such a small perturbation is due to the choice of our underlying domain $\mathbb{T} \times (0,1)$, which is periodic in the horizontal direction. In a future paper [9], we will consider the underlying domain $\Omega = \mathbb{R} \times (0,1)$, in which case it seems possible to give the asymptotic stability of the temperature under suitable setting.

Acknowledgements

The author thanks Prof. Yongzhong Sun for guidance and useful discussions. The research is partially supported by NSFC under Grant No. 11771395, 12071211.

References

[1] Abidi, H., Hmidi, T.: On the global well-posedness for Boussinesq system. J. Differ. Equ. **233**, 199–220 (2007)
[2] Adams, R.A., Fournier, J.F.: Sobolev spaces, 2nd edition, Pure and Applied Mathematics (Amsterdam), vol. 140. Elsevier/Academic Press, Amsterdam (2003)
[3] Bergé, P., Dubois, M.: Rayleigh-Bénard Convection. Contemp. Phys. **25**, 535–582 (1984)
[4] Bianchini, R., Natalini, R.: Asymptotic behavior of 2D stably stratified fluids with a damping term in the velocity equation, arXiv:2009.01578
[5] Castro, Á., Córdoba, D., Lear, D.: The asymptotic stability of stratified solutions for the 2D Boussinesq equations with a velocity damping term. Math. Models Methods Appl. Sci. **29**, 1227–1277 (2019)
[6] Chae, D.: Global regularity for the 2D Boussinesq equations with partial viscosity terms. Adv. Math. **203**, 497–513 (2006)
[7] Danchin, R., Paicu, M.: Les théorèmes de Leray et de Fujita-Kato pour le système de Boussinesq partiellement visqueux. Bull. Soc. Math. France **136**, 261–309 (2008)
[8] Doering, C.R., Wu, J., Zhao, K., Zheng, X.: Long time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion. Phys. D. **376**(377), 144–159 (2018)
[9] Dong, L., Sun, Y.: Asymptotic stability of the 2D Boussinesq equations without thermal conduction, preprint, (2021)
[10] Drazin, P.G.: Introduction to hydrodynamic stability, Cambridge Texts in Applied Mathematics, (2002)
[11] Ferrari, A.B.: On the blow-up of solutions of the 3-D Euler equations in a bounded domain. Comm. Math. Phys. **155**, 277–294 (1993)
[12] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin (1983)
[13] He, L.: Smoothing estimates of 2d incompressible Navier-Stokes equations in bounded domains with applications. J. Funct. Anal. **262**, 3430–3464 (2012)
[14] Hmidi, T., Keraani, S.: On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity. Adv. Differ. Equ. **12**, 461–480 (2007)
[15] Hou, T., Li, C.: Global well-posedness of the viscous Boussinesq equations. Discrete Contin. Dyn. Syst. **12**, 1–12 (2005)
[16] Hu, W., Kukavica, I., Ziane, M.: On the regularity for the Boussinesq equations in a bounded domain, J. Math. Phys., **54**, 081507 10 pp. (2013)
[17] Ju, N.: Global regularity and long-time behavior of the solutions to the 2D Boussinesq equations without diffusivity in a bounded domain. J. Math. Fluid Mech. **19**, 105–121 (2017)

[18] Kukavica, I., Wang, W.: Long time behavior of solutions to the 2D Boussinesq equations with zero diffusivity. J. Dyn. Diff. Equat. **32**, 2061–2077 (2020)
[19] Lai, M., Pan, R., Zhao, K.: Initial boundary value problem for two-dimensional viscous Boussinesq equations. Arch. Ration. Mech. Anal. **199**, 739–760 (2011)
[20] Majda, A.J.: Introduction to PDEs and Waves for the Atmosphere and Ocean, Courant Lect. Notes Math., Vol. 9, AMS/CIMS, (2003)
[21] Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge University Press, Cambridge, UK (2002)
[22] Pan, R., Zhou, Y., Zhu, Y.: Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes. Arch. Ration. Mech. Anal. **227**, 637–662 (2018)
[23] Pedlosky, J.: Geophysical Fluid Dynamics. Springer-Verlag, New York (1987)
[24] Tao, L., Wu, J., Zhao, K., Zheng, X.: Stability near hydrostatic equilibrium to the 2D Boussinesq equations without thermal diffusion. Arch. Ration. Mech. Anal. **237**, 585–630 (2020)
[25] Wan, R.: Global well-posedness for the 2D Boussinesq equations with a velocity damping term. Discrete Contin. Dyn. Syst. **39**, 2709–2730 (2019)
[26] Wan, R.: Global existence and long time behavior of smooth solution for the 2D Boussinesq-Navier-Stokes equations, preprint

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(Received: October 19, 2020; revised: March 23, 2021; accepted: May 15, 2021)