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Some regularity criteria for the 3D generalized Navier–Stokes equations

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Abstract. We show some regularity criteria (Prodi–Serrin type regularity) to weak solutions of the 3D generalized Navier–Stokes equations in viewpoint of the velocity vector u or the vorticity vector $\omega := \nabla \times u$ in Lorentz space. Moreover, we briefly mention some results for coupled equations with Navier–Stokes equation (see Remark 1.5 and 1.8).

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1. Introduction

Consider the following 3D generalized Navier–Stokes equations:

$$\begin{cases} u_t + (-\triangle)^\alpha u + (u \cdot \nabla)u + \nabla P = f, \\ \text{div } u = 0, \end{cases} \qquad \text{in } \; Q_T := \mathbb{R}^3 \times [0,\, T), \tag{1.1}$$

where $u : Q_T \to \mathbb{R}^3$ is the flow velocity vector and $P : Q_T \to \mathbb{R}$ is the magnetic pressure. We consider the initial value problem of (1.1), which requires initial conditions

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^3. \tag{1.2}$$

The fractional power of Laplace operator $(-\triangle)^\alpha$ is defined as in [28]

$$\widehat{(-\triangle)^\alpha} f(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

where $\hat{f}$ denotes the Fourier transform of f. For notational convenience, we write $(-\Delta)^{1/2}$ as Λ. Before we take a further look, let's recall the definition of Leray–Hopf (weak) solution to (1.1)–(1.2) which is given in [6].

Definition 1.1. Let $\alpha > 0$ and $u_0 \in L^2(\mathbb{R}^3)$ divergence-free. A Leray–Hopf solution is a distributional solution (u, P) of (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T)$ such that

I. $u \in L^\infty((0,T), L^2(\mathbb{R}^3)) \cap L^2((0,T), H^\alpha(\mathbb{R}^3))$,
II. P is the potential-theoretic solution of $-\Delta P = \text{divdiv } u \otimes u$,
III. For every $t \in (0,T)$, for $s = 0$ and for almost every $0 < s < t$ there holds the global energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_s^t \|(-\Delta)^{\alpha/2} u\|^2_{L^2(\mathbb{R}^3)}\, d\tau \le \|u(s)\|^2_{L^2(\mathbb{R}^3)}.$$

Recently, in the authors in [6, Theorem 2.2] construct the existence of Leray–Hopf solution in the sense of Definition 1.1 on $\mathbb{R}^3 \times (0,T)$ to (1.1)–(1.2). Let $u(x,t)$ be a solution to system (1.1), then $u_\lambda(x,t)$ with any $\lambda > 0$ is also a solution, where $u_\lambda(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t)$. It is worth pointing that $\dot{H}^{\frac{5-4\alpha}{2}}$ is a critical space, that is, $\dot{H}^{\frac{5-4\alpha}{2}}$ norm is scaling invariant. By Sobolevs embedding theorem, we also obtain $\dot{H}^{\frac{5-3\alpha}{2}} \hookrightarrow L^{\frac{6}{3-2\alpha}}$. Then

$$\|u\|_{L^4(0,T;L^{\frac{6}{3-2\alpha}}(\mathbb{R}^3))} \le C\|u\|^2_{L^\infty(0,T;\dot{H}^{\frac{5-4\alpha}{2}})}\|u\|^2_{L^2(0,T;\dot{H}^{\frac{5-2\alpha}{2}})}, \quad \frac{2\alpha}{4} + \frac{3}{\frac{6}{3\alpha-2}} = 2\alpha - 1.$$

It is well-known that the local and global-in time existence of strong solutions (i.e., pointwise sense solutions) to the system (1.1)–(1.2) were established for $\alpha \ge \frac{5}{4}$ using Galerkin approximation method with the energy estimate (see e.g., [35] and [18]). About the uniqueness, very recently, Colombo et al. [5] proved the ill-posedness in the case $\alpha < \frac{1}{5}$. After that, Rosa [9], proving the non-uniqueness of such solutions in the range $\alpha < \frac{1}{3}$. That is, there exist infinitely many Leray solutions u of (1.1) in $\mathbb{T}^3 \times [0,\infty)$ based on the convex integration theory in the limelight recently. As Rosa mentioned, the method given by Colombo et al. [5] also would give us infinitely many weak solutions bounded in $L^\infty(0,T;L^2(\mathbb{T}^3))$ in the range $\frac{1}{3} \le \alpha < \frac{1}{2}$.

On the other hand, we list only some results relevant to the regularity. The authors in [37] and [14] showed the following the integral conditions, typically referred to as Serrin's condition for $\frac{3}{4} < \alpha \le \frac{3}{2}$

$$u \in L^q(0,\, T; L^p(\mathbb{R}^3)) \quad \text{with } \frac{3}{p} + \frac{2\alpha}{q} \le 2\alpha - 1, \quad \frac{3}{2\alpha - 1} < p \le \infty.$$

On the other hand, in [2] for a vorticity $w = \nabla \times u$, for $0 < \alpha < 2$,

$$w \in L^q(0,\, T; L^p(\mathbb{R}^3)) \quad \text{with } \frac{3}{p} + \frac{2}{q} \le \alpha, \quad \frac{6}{\alpha} < p \le \infty,$$

After that the authors in [14] shown that for $0 < \alpha < \frac{5}{4}$,

$$\nabla u \in L^{\frac{2r\alpha}{2r\alpha-3}}(0,T;L^r), \quad \frac{3}{2\alpha} < r \le \infty, \quad 0 < T < \infty.$$

Also, other types of regularity criteria can be referred to, for example, [10,13,21,23,30], and the related references therein. Our study is motivated by these direction; we obtain the regularity conditions for a local solution to 3D generalized Navier–Stokes equations (1.1)–(1.2) in Lorentz space.

Our results reads as follows:

Theorem 1.2. *Let $u_0 \in H^m(\mathbb{R}^3)$ with div $u_0 = 0$ and $m > \frac{5}{2}$ and $\frac{3}{4} < \alpha \le \frac{5}{4}$. There exists a sufficient constant $\epsilon > 0$ such that if u satisfies*

$$\|u\|_{L^{\frac{2r\alpha}{2r\alpha-r-3},\infty}(0,T;L^{r,\infty})} \le \epsilon, \quad \frac{3}{2\alpha - 1} < r \le \infty, \quad 0 < T < \infty. \tag{1.3}$$

Then the solution u can be extended beyond $T > 0$.

Theorem 1.3. *Let $u_0 \in H^m(\mathbb{R}^3)$ with div $u_0 = 0$ and $m > \frac{5}{2}$ and $0 < \alpha < \frac{5}{4}$. There exists a sufficient constant $\epsilon > 0$ such that if*

$$\|S\|_{L^{\frac{2r\alpha}{2r\alpha-3},\infty}(0,T;L^{r,\infty})} \le \epsilon, \quad \frac{3}{2\alpha} < r \le \infty, \quad 0 < T < \infty. \tag{1.4}$$

Then the solution u can be extended beyond $T > 0$. Here, $S = (S_{ij}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$.

Remark 1.4. The result in Theorem 1.3 is more weaker condition to that in [2] or [14]. More speaking, the strain tensor S in Theorem 1.3 is replaced by ∇u or $\omega := \nabla \times u$ due to

$$\|S\|_{L^z(\mathbb{R}^3)} \le \|\nabla u\|_{L^z(\mathbb{R}^3)} \le \|\omega\|_{L^z(\mathbb{R}^3)} \tag{1.5}$$

for $1 < z < \infty$.

Remark 1.5. The result in Theorem 1.2 and Theorem 1.3 don't hold for the 3D generalized MHD equations, which is Navier–Stokes equation coupled with the simplified Maxwell equations (3D MHD equations). Instead, the condition in Theorem (1.3) can be replaced by

$$u \in L^{\frac{2r\alpha}{2r\alpha-r-3}}(0,T;L^{r,\infty}) < \infty, \quad \frac{3}{2\alpha-1} < r \leq \infty, \quad 0 < T < \infty.$$

and the condition in Theorem (1.4) can be substituted by

$$S \in L^{\frac{2r\alpha}{2r\alpha-3}}(0,T;L^{r,\infty}) < \infty, \quad \frac{3}{2\alpha} < r \leq \infty, \quad 0 < T < \infty,$$

since we don't apply Lemma 2.2. However, according the argument in [20], the result in Theorem 1.2 and Theorem 1.3 may be hold for the 3D generalized micro-polar fluid, which are fluids with microstructure containing the velocity of rotation of the fluid particles. For these models, we refer to [11,13,17,18,25] for the fractional diffusion and [12,13,15,16,24,31,32,36,38] for the viscous fluids, and the related references therein.

Motivated by the papers [3,19], we give a geometric regularity condition for the volume of parallelepiped type in viewpoint of u and ω.

Theorem 1.6. *Let $\frac{3}{r}+\frac{2\alpha}{s} \leq 2\alpha-1$, $\frac{3}{2\alpha-1} < r \leq \infty$ and $1 \leq \alpha \leq \frac{3}{2}$. Suppose that u be a solution of 3D generalized Navier–Stokes equations (1.1)–(1.2) with initial condition $u_0 \in H^m(\mathbb{R}^3)$ with $m \geq \frac{5}{2}$. There exists a sufficient constant $\epsilon > 0$ such that if*

$$\left\| \left[\left(u \times \frac{\omega}{|\omega|} \right) \right] \cdot \frac{\nabla \times \omega}{|\nabla \times \omega|} \right\|_{L^{s,\infty}(0,T;L^{r,\infty}(\mathbb{R}^3))} < \epsilon.$$

then a regular solution u exists beyond T.

As mentioned in [33], geometric-analytic regularity criterion expressed as a balance between the vorticity direction and the vorticity magnitude, key geometric and analytic descriptors of the flow, respectively. For this, define a pointwise measure of the coherence of the vorticity direction in in turbulence by

$$\rho_\gamma(x,t) := \sup_{y \neq x} \frac{|\sin\ \varphi(\xi(x+y,t),\xi(x,t))|}{|y|^\gamma}.$$

(see refer to [27] for the coherence structure of the vorticity and [8] for effect of vorticity coherence).

In this connection, our result reads as follows:

Theorem 1.7. *Let $w \in C([0,T),L^p(\mathbb{R}^3))$ be a solution to the 3D generalized Navier–Stokes equations (1.1)–(1.2) for some $p > \frac{3}{2}$. Assume that w satisfies*

$$\int_0^T \Big[\int_{\mathbb{R}^3} \Big(\rho_\gamma(x,t)|w(x,t)|^a \Big)^p \, dx \Big]^{\frac{2}{p}} \, dt \leq \infty, \tag{1.6}$$

where the parameters γ, p and a conform to the scaling-invariant condition $p(\gamma+2a)-3 = \alpha p$. Then T is not a blow-up time.

Remark 1.8. The results in Theorem 1.6 and Theorem 1.7 don't hold for the 3D generalized MHD equations(or 3D generalized micro-polar fluid). In order to achieve these kind of results, it may need additional assumption for a magnetic field (or the velocity of rotation of the fluid particles).

2. Preliminaries

In this section, we introduce notations and definitions used throughout this paper. We also recall the well-known results for our analysis. For $1 \leq q \leq \infty$, $W^{k,q}(\mathbb{R}^3)$ indicates the usual Sobolev space with standard norm $\|\cdot\|_{k,q}$, i.e.,

$$W^{k,q}(\mathbb{R}^3) = \{u \in L^q(\mathbb{R}^3) \,:\, D^\alpha u \in L^q(\mathbb{R}^3), 0 \leq |\alpha| \leq k\}.$$

In case that $q = 2$, we write $W^{k,q}(\mathbb{R}^3)$ as H^k. All generic constants will be denoted by C, which may vary from line to line. In particular, $A \lesssim B$ means for $A \leq CB$.

2.1. Lorentz spaces

Let $m(\varphi, t)$ be the Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}$, *i.e.*,

$$m(\varphi, t) := m\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}.$$

We denote by the Lorentz space $L^{p,q}(\mathbb{R}^3)$ with $1 \leq p,\, q \leq \infty$ with the norm [29]

$$\|\varphi\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \left(\int_0^\infty t^q (m(\varphi,t))^{q/p}\, \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty, & \text{for } \ 1 \leq q < \infty, \\ \sup_{t \geq 0}\{t(m(\varphi,t))^{\frac{1}{p}}\} < \infty, & \text{for } \ q = \infty \end{cases}$$

Followed in [29], Lorentz space $L^{p,q}(\mathbb{R}^3)$ is defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3),\, L^{p_2}(\mathbb{R}^3))_{\alpha,q},$$

with $\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \quad 1 \leq p_1 < p < p_2 \leq \infty$. In particular, we note that

$$L^{\frac{2p}{p-1},2}(\mathbb{R}^3) = \Big(L^2(\mathbb{R}^3),\, L^6(\mathbb{R}^3)\Big)_{\frac{3}{2p},2}.$$

We mention the Hölder inequality in Lorentz spaces (see [22]).

Lemma 2.1. *Assume* $1 \leq p_1,\, p_2 \leq \infty$, $1 \leq q_1,\, q_2 \leq \infty$ *and* $u \in L^{p_1,q_1}(\mathbb{R}^3)$, $v \in L^{p_2,q_2}(\mathbb{R}^3)$. *Then* $uv \in L^{p_3,q_3}(\mathbb{R}^3)$ *with* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}$, *and moreover,*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3)} \leq C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|v\|_{L^{p_2,q_2}(\mathbb{R}^3)}$$

is valid.

Also, we recall the following nonlinear Gronwall-type inequality established in [26] (see also [1]).

Lemma 2.2. *Let* $T > 0$ *and* $\varphi \in L_{loc}([0,T))$ *be nonnegative function. Assume further that*

$$\varphi(t) \leq C_0 + C_1 \int_0^t \mu(s)\varphi(s)\,\mathrm{d}s + \kappa \int_0^t \lambda(s)^{1-\epsilon}\varphi(s)^{1+A(\epsilon)}\,\mathrm{d}s, \quad \forall\ 0 < \epsilon < \epsilon_0.$$

Where $\kappa, \epsilon_0 > 0$ *are constants,* $\mu \in L^1(0,T)$ *and* $A(\epsilon) > 0$ *satisfies* $\lim_{\epsilon \to 0} \frac{A(\epsilon)}{\epsilon} = c_0 > 0$. *Then* φ *is bounded on* $[0,T]$ *if* $\|\lambda\|_{L^{1,\infty}(0,T)} < c_0^{-1}\kappa^{-1}$.

To control the fractional diffusion term, we recall the following result (see e.g., [4] or [34]).

Lemma 2.3. *With* $0 < \alpha < 2$, $v, \Lambda^\alpha v \in L^p(\mathbb{R}^3)$ *with* $p = 2k$, $k \in \mathbb{N}$, *we obtain*

$$\int |v|^{p-2} v \Lambda^\alpha v\,\mathrm{d}x \geq \frac{1}{p}\int |\Lambda^{\frac{\alpha}{2}} v^{\frac{p}{2}}|^2\,\mathrm{d}x.$$

3. Proofs of Theorems

Proof of Theorem 1.2. A. Case $1 \le \alpha \le \frac{3}{2}$: Multiplying $-\Delta u$ to the first equation of (1.1), integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 \le \Big|\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x\Big|. \tag{3.1}$$

Using Hölder inequality, Lemma 2.1, interpolation inequality for Lorentz space and Young's inequality, the right-hand side term in (3.1) is estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}^3}|u||\nabla u||\nabla^2 u|\,\mathrm{d}x &\lesssim \|u\|_{L^{r,\infty}}\|\nabla u\|_{L^{m,2}}\|\nabla^2 u\|_{L^{q,2}}\\ &\lesssim \|u\|_{L^{r,\infty}}\|\nabla u\|^{\theta}_{L^{2,2}}\|\Lambda^{\alpha+1}u\|^{1-\theta}_{L^{2,2}}\|\nabla u\|^{\delta}_{L^{2,2}}\|\Lambda^{\alpha+1}u\|^{1-\delta}_{L^{2,2}}\\ &\le \|u\|^{\frac{2}{\theta+\delta}}_{L^{r,\infty}}\|\nabla u\|^2_{L^2}+\frac{1}{4}\left\|\Lambda^{\alpha+1}u\right\|^2_{L^2},\end{aligned}$$

where for $1 < s, q < \infty$ and $0 \le \theta, \delta \le 1$

$$\begin{aligned}&\frac{1}{r}+\frac{1}{m}+\frac{1}{q}=1,\\ &\frac{1}{m}-\frac{1}{3}=\theta\left(\frac{1}{2}-\frac{1}{3}\right)+(1-\theta)\Big(\frac{1}{2}-\frac{\alpha+1}{3}\Big),\end{aligned}$$

and

$$\frac{1}{q}-\frac{2}{3}=\delta\left(\frac{1}{2}-\frac{1}{3}\right)+(1-\delta)\Big(\frac{1}{2}-\frac{\alpha+1}{3}\Big).$$

This implies $\theta = 1+\frac{3}{m\alpha}-\frac{3}{2\alpha}$, $\delta = 1+\frac{3}{q\alpha}-\frac{5}{2\alpha}$. Thus, we set $m=\frac{6r}{2r-3}$ and $q=\frac{6r}{4r-3}$, and then, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \left\|\Lambda^{\alpha+1}u\right\|^2_{L^2} \lesssim \|u\|^{\frac{2r\alpha}{2r\alpha-r-3}}_{L^{r,\infty}}\|\nabla u\|^2_{L^2}$$

Note that

$$\frac{3}{r}+\frac{2\alpha}{s}=\frac{3}{r}+\alpha(\theta+\delta)=2\alpha-1,$$

and $\alpha \le \frac{3}{2}$ implies $q > \frac{3}{2}$.

Now, we use an argument similar to the one used in the work of Bosia et al. [1]. For $\epsilon > 0$, Choose $s_\epsilon = s+\epsilon(\frac{4\alpha}{2\alpha-1}-\alpha s)$ and $r_\epsilon := \frac{3s+3\epsilon(\frac{4\alpha}{2\alpha-1}-\alpha s)}{(2\alpha-1)(s+\epsilon(\frac{4\alpha}{2\alpha-1}-\alpha s))-2\alpha}$ with $\frac{3}{r_\epsilon}+\frac{2\alpha}{s_\epsilon}=2\alpha-1$. Then we have

$$\|u\|^{s_\epsilon}_{r_\epsilon,\infty} \le C(\epsilon)\|u\|^{s(1-\alpha\epsilon)}_{r,\infty}\|\nabla u\|^{4\alpha\epsilon}_{L^2}, \tag{3.2}$$

where we use the interpolation and Sobolev inequalities. Then, we know

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le C(\epsilon)\|u\|^{s(1-\epsilon)}_{r,\infty}\phi(t)^{1+2\epsilon}, \quad \phi(t) := \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$$

By Lemma 2.2 with $\lambda(s) := \|u\|^{\frac{2r\alpha}{2r\alpha-r-3}}_{L^{r,\infty}(\mathbb{R}^3)}$, $\mu(s)=0$ and $C_0=0$ under the assumption (1.3), we obtain the desired result (see e.g., [20] for a detail proof).

B. Case $\frac{3}{4} < \alpha \le 1$: Note that

$$\int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x = \int_{\mathbb{R}^3} \nabla\cdot(u\otimes u)\cdot\Delta u\,\mathrm{d}x = \int_{\mathbb{R}^3} \Lambda^{1-\alpha}\nabla\cdot(u\otimes u)\cdot\Lambda^{1+\alpha}u\,\mathrm{d}x. \tag{3.3}$$

From (3.1) with (3.3), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 \le -\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,\mathrm{d}x.\\
&= -\int_{\mathbb{R}^3}\Lambda^{1-\alpha}\nabla\cdot(u\otimes u)\cdot\Lambda^{1+\alpha}u\,\mathrm{d}x \le \|\Lambda^{2-\alpha}(u\otimes u)\|_{L^2}\|\Lambda^{1+\alpha}u\|_{L^2}\\
&\lesssim \|u\|_{L^r}\|\Lambda^{2-\alpha}u\|_{L^{\frac{2r}{r-2}}}\|\Lambda^{1+\alpha}u\|_{L^2}\\
&\lesssim \|u\|_{L^r}\|\nabla u\|_{L^2}^{1-\theta}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta}\|\Lambda^{1+\alpha}u\|_{L^2}\\
&\lesssim \|u\|_{L^r}^{\frac{2}{1-\theta}}\|\nabla u\|_{L^2}^2 + \frac{1}{8}\|\Lambda^{1+\alpha}u\|_{L^2}^2, \quad \theta = \frac{1-\alpha+\frac{3}{r}}{\alpha}.
\end{aligned}$$

Hence, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 \lesssim \|u\|_{L^r}^{\frac{2r\alpha}{2r\alpha-r-3}}\|\nabla u\|_{L^2}^2.$$

In the same manner as the previous technique (the proof in case A), we obtain the desired result. □

Following [7], the symmetric part S, which we will refer to as the strain tensor and he anti-symmetric part A will be given by

$$S = (S_{ij}) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\Big), \quad \text{and} \quad A = (A_{ij}) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\Big),$$

respectively. Then, we rewrite the vorticity equation:

$$\omega_t + \nu(-\Delta)^{\alpha}\omega + (u\cdot\nabla)\omega = S\omega. \tag{3.4}$$

The vortex stretching term $S\omega$ is often written $(\omega\cdot\nabla)u$, but it is clear from (1.7) that $A\omega = 0$; therefore,

$$(\omega\cdot\nabla)u = (S+A)\omega = S\omega.$$

Proof of Theorem 1.3. Multiplying ω to the first equation of (3.4), integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\Lambda^{\alpha}\omega\|_{L^2}^2 \le \int_{\mathbb{R}^3}|S||\omega|^2\,\mathrm{d}x \tag{3.5}$$

Using Hölder inequality, Lemma 2.1, interpolation inequality for Lorentz space and Young's inequality, the right-hand side term of (3.5) is estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^3}|S||\omega|^2\,\mathrm{d}x \le \|S\|_{L^{r,\infty}}\|\omega\|_{L^{\frac{2r}{r-1},1}}^2\\
&\lesssim \|S\|_{L^{r,\infty}}\|\omega\|_{L^{2,2}}^{2(1-\frac{3}{2r\alpha})}\|\Lambda^{\alpha}\omega\|_{L^{2,2}}^{\frac{3}{r\alpha}} \lesssim \|S\|_{L^{r,\infty}(\mathbb{R}^3)}^{\frac{2r\alpha}{2r\alpha-3}}\|\omega\|_{L^2}^2 + \frac{1}{8}\|\Lambda^{\alpha}\omega\|_{L^2}^2,
\end{aligned}$$

and thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|w|^2\,\mathrm{d}x + \|\Lambda^{\alpha}w\|_{L^2}^2 \lesssim \|S\|_{L^{r,\infty}}^{\frac{2r\alpha}{2r\alpha-3}}\|\omega\|_{L^2}^2 \lesssim \|S\|_{L^{r,\infty}}^{\frac{2r\alpha}{2r\alpha-3}(1-\epsilon)}\|\omega\|_{L^2}^{2(1+\frac{r\alpha}{2r\alpha-3})},$$

where we use (1.5). By Lemma 2.2 under the assumption (1.4), we obtain the desired result. □

Proof of Theorem 1.6. For a proof, $\|u\|_{L^{r,\infty}(\mathbb{R}^3)}$ in the proof of Theorem 1.2 is only replaced by $\left\|\left[\left(u\times\frac{\omega}{|\omega|}\right)\right]\cdot\frac{\nabla\times\omega}{|\nabla\times\omega|}\right\|_{L^{r,\infty}(\mathbb{R}^3)}$. Indeed, from (3.1), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2 + \left\|\Lambda^{\alpha+1}u\right\|_{L^2}^2 &\le \int_{\mathbb{R}^3}|u||\nabla u||\nabla^2 u|\,\mathrm{d}x\\
&\le \int_{\mathbb{R}^3}\left|\left(u\times\frac{\omega}{|\omega|}\right)\cdot\frac{\nabla\times\omega}{|\nabla\times\omega|}\right||\nabla u||\nabla^2 u|\\
&\le \left\|\left[\left(u\times\frac{\omega}{|\omega|}\right)\right]\cdot\frac{\nabla\times\omega}{|\nabla\times\omega|}\right\|_{L^{r,\infty}}\|\nabla u\|_{L^{m,2}}\|\nabla^2 u\|_{L^{q,2}}\\
&\lesssim \left\|\left[\left(u\times\frac{\omega}{|\omega|}\right)\right]\cdot\frac{\nabla\times\omega}{|\nabla\times\omega|}\right\|_{L^{r,\infty})}\|\nabla u\|_{L^{2,2}}^{\theta}\|\Lambda^{\alpha+1}\omega\|_{L^{2,2}}^{1-\theta}\|\nabla u\|_{L^{2,2}}^{\delta}\|\Lambda^{\alpha+1}u\|_{L^{2,2}}^{1-\delta}\\
&\lesssim \left\|\left[\left(u\times\frac{\omega}{|\omega|}\right)\right]\cdot\frac{\nabla\times\omega}{|\nabla\times\omega|}\right\|_{L^{r,\infty}}^{\frac{2r\alpha}{2r\alpha-r-3}}\|\nabla u\|_{L^2}^2.
\end{aligned}$$

In the same manner as the proof(case A) of Theorem 1.2, we obtain the desired result. □

Remark 3.1. Comparing to the result of Theorem 1.2, the range of α in Theorem 1.6 restrict to $1\le\alpha\le\frac{3}{2}$ since the argument in Theorem 1.2 hold for $\frac{3}{4}\le\alpha\le 1$, in particular, do not work in this case.

Lastly, we consider the vorticity equation:

$$\omega_t + \nu(-\Delta)^{\alpha}\omega + (u\cdot\nabla)w = (w\cdot\nabla)u. \tag{3.6}$$

Proof of Theorem 1.7. Multiplying the equations (3.6) by $|w|^{p-2}w$ and integrating over the whole space yields

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int|w|^p\,\mathrm{d}x + \int(\sqrt{-\Delta})^{2\alpha}w\cdot|w|^{p-2}w\,\mathrm{d}x = \int(w\cdot\nabla)u\cdot|w|^{p-2}w\,\mathrm{d}x = \int\alpha|w|^p,$$

where we use the fact the pointwise identity

$$(w\cdot\nabla)u = \alpha|w|^2.$$

Here, following [7], the strain matrix $S(x,t)$ is given

$$S(x,t)\equiv\frac{1}{2}\Big(\nabla u(x,t)+\nabla u(x,t)^T\Big) := \frac{3}{4\pi}P.V.\int M(\hat{y},w(x+y))\frac{\mathrm{d}y}{|y|^3},\quad \hat{y}=\frac{y}{|y|},$$

where $M(\hat{y},w)=\frac{1}{2}\Big[\hat{y}\otimes(\hat{y}\times w)+(\hat{y}\times w)\otimes\hat{y}\Big]$ with $(a\otimes b)_{ij}=a_ib_j$. Denote

$$a(x) := S(x)\xi(x)\cdot\xi(x),\quad \xi(x)=\frac{w(x)}{|w(x)|}.$$

By Lemma 2.3 implies a lower bound on the fractional diffusion term follow as

$$\int(\sqrt{-\Delta})^{2\alpha}w\cdot|w|^{p-2}w\,\mathrm{d}x \ge \frac{2}{p}\int\left|(\sqrt{-\Delta})^{\alpha}|w|^{\frac{p}{2}}\right|^2\mathrm{d}x \ge c_\alpha\,\|w(t)\|_{L^{\frac{3p}{3-2\alpha}}}^p$$

Using Holder and Young's inequalities, the RHS is estimated as follows,

$$\begin{aligned} c_{p,\alpha}\Big|\int \alpha(x,t)|w(x,t)|^2\,\mathrm{d}x\Big| &\le c_{p,\alpha}\Big|\int \Big(\rho_\gamma||w(x,t)|^2\Big)\Big(\frac{1}{|y|^{3-\gamma}}||w(x+y,t)|\,\mathrm{d}y\Big)\mathrm{d}x\Big| \\ &\le c_{p,\alpha}\left\|\rho_\gamma|w|^2\right\|_{L^{p,\infty}}\left\|\frac{1}{|y|^{3-\gamma}}*|w|\right\|_{L^{\frac{p}{p-1},1}} \\ &\le c_{p,\alpha}\left\|\rho_\gamma|w|^2\right\|_{L^{p,\infty}}\|w\|_{L^{s,1}}, \quad \frac{1}{p_2}+1=\frac{3-\gamma}{3}+\frac{1}{s} \\ &\le c_{p,\alpha}\left\|\rho_\gamma|w|^2\right\|_{L^{p,\infty}}\|w\|^{\alpha}_{L^{2,2}}\|w\|^{1-\alpha}_{L^{\frac{6}{3-2\alpha},\frac{6}{3-2\alpha}}}, \quad \frac{1}{s}=\alpha\frac{1}{2}+(1-\alpha)\frac{3-2\alpha}{6} \\ &\le c_{p,\alpha}\left\|\rho_\gamma|w|^2\right\|^{\frac{2}{\alpha}}_{L^p}\|w\|^{\alpha}_{L^p}+\frac{c_{p,\alpha}}{8}\|w\|^{1-\alpha}_{L^{\frac{3p}{3-2\alpha}}}, \end{aligned}$$

where we use $\left\|\frac{1}{|y|^{3-\gamma}}*|w|\right\|_{L^{\frac{p}{p-1}}} \le \|w\|_{L^s}$ in third inequality and Lemma 2.1 in fourth inequality. It yields the final form of our differential inequality on (0, T),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{L^2}+c_\alpha\|w(t)\|^2_{L^{\frac{6}{3-2\alpha}}} \le C\left\|\rho_\gamma|w|^2\right\|^{\frac{2}{\alpha}}_{L^{p,\infty}}\|w(t)\|^2_{L^2} \tag{3.7}$$

Applying the Gronwall's inequality to (3.7) under the assumption (1.6), we have the desired result. □

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References

[1] Bosia, S., Pata, V., Robinson, J.C.: A weak-L^p Prodi–Serrin type regularity criterion for the Navier–Stokes equations. J. Math. Fluid Mech. **16**, 721–725 (2014)
[2] Chae, D.: On the regularity conditions for the Navier–Stokes and related equations. Rev. Mat. Iberoam. **23**, 371–384 (2007)
[3] Chae, D., Lee, J.: On the geometric regularity conditions for the 3D Navier–Stokes equations. Nonlinear Anal. **151**, 265–273 (2017)
[4] Códorba, A., Códorba, D.: A maximum principle applied to quasi-geostrophic equations. Commun. Math. Phys. **249**, 511–528 (2004)
[5] Colombo, M., De Lellis, C., De Rosa, L.: Ill-posedness of Leray solutions for the hypodissipative Navier–Stokes equations. Commun. Math. Phys. **362**, 659–688 (2017)
[6] Colombo, M., Haffter, S.: Global regularity for the hyperdissipative Navier–Stokes equation below the critical order. Journal of Differential Equations. **75,** 815–836 (2021)
[7] Constantin, P.: Geometric statistics in turbulence. SIAM Rev. **36**, 73 (1994)
[8] Dascaliuc, R., Grujić, Z., Jolly, M.S.: Effect of vorticity coherence on energy-enstrophy bounds for the 3D Navier–Stokes equations. J. Math. Fluid Mech. **17**, 393–410 (2015)
[9] De Rosa, L.: Infinitely many Leray–Hopf solutions for the fractional Navier–Stokes equations. Commun. Partial Differ. Equ. **44**, 335–365 (2019)
[10] Fan, J., Ozawa, T.: On the regularity criteria for the generalized Navier–Stokes equations and Lagrangian averaged Euler equations. Differ. Integral Equ. **21**, 443–457 (2008)
[11] Fan, J., Jiang, S., Nakamura, G., Zhou, Y.: Logarithmically improved regularity criteria for the Navier–Stokes and MHD equations. J. Math. Fluid Mech. **13**, 557–571 (2011)

[12] Fan, J., Zhou, Y.: Uniform local well-posedness for the density-dependent magnetohydrodynamic equations. Appl. Math. Lett. **24**, 1945–1949 (2011)

[13] Fan, J., Fukumoto, Y., Zhou, Y.: Logarithmically improved regularity criteria for the generalized Navier–Stokes and related equations. Kinet. Relat. Models **6**, 545–556 (2013)

[14] Fan, J., Alsaedi, A., Hayat, T., Nakamura, G., Zhou, Y.: A regularity criterion for the 3D generalized MHD equations. Math. Phys. Anal. Geom. **17**, 333–340 (2014)

[15] Fan, J., Jia, X., Nakamura, G., Zhou, Y.: On well-posedness and blowup criteria for the magnetohydrodynamics with the Hall and ion-slip effects. Z. Angew. Math. Phys. **66**, 1695–1706 (2015)

[16] Fan, J., Zhang, Z., Zhou, Y.: Local well-posedness for the incompressible full magneto-micropolar system with vacuum. Z. Angew. Math. Phys. **71**, 1–11 (2020)

[17] Jiang, Z., Wang, Y., Zhou, Y.: On regularity criteria for the 2D generalized MHD system. J. Math. Fluid Mech. **18**, 331–341 (2016)

[18] Jiang, Z., Ma, C., Zhou, Y.: Commutator estimates with fractional derivatives and local existence for the generalized MHD equations. Z. Angew. Math. Phys. 111 (2021)

[19] Lee, J.: Notes on the geometric regularity criterion of 3D Navier–Stokes system. J. Math. Phys. **53**, 073103 (2012)

[20] Loayza, M., Rojas-Medar, M.A.: A weak-Lp Prodi–Serrin type regularity criterion for the micropolar fluid equations. J. Math. Phys. **57**, 021512 (2016)

[21] Luo, Y.: On the regularity of generalized MHD equations. J. Math. Anal. Appl. **365**, 806–808 (2010)

[22] O'Neil, R.: Convolution operators and $L(p, q)$ spaces. Duke Math. J. **30**, 129–142 (1963)

[23] Nakai, K.: Direction of vorticity and a refined regularity criterion for the Navier–Stokes equations with fractional Laplacian. J. Math. Fluid Mech. **21**, 1–8 (2019)

[24] Ni, L., Guo, Z., Zhou, Y.: Some new regularity criteria for the 3D MHD equations. J. Math. Anal. Appl. **396**, 108–118 (2012)

[25] Pan, N., Zhu, M.: A new regularity criterion for the 3D generalized Hall-MHD system with $\beta \in (\frac{1}{2}, 1]$. J. Math. Anal. Appl. **445**, 604–611 (2017)

[26] Pineau, B., Yu, X.: On Prodi–Serrin type conditions for the 3D Navier–Stokes equations. Nonlinear Anal. **190**, 111612 (2020)

[27] Roulstone, I., Banos, B., Gibbon, J.D., Roubtsov, V.N.A.: Geometric interpretation of coherent structures in Navier–Stokes flows. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **465**, 2015–2021 (2009)

[28] Stein, E.M.: Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970)

[29] Triebel, H.: Theory of Function Spaces. Birkhäuser Verlag, Basel (1983)

[30] Yamazaki, K.: Remarks on the regularity criteria of generalized MHD and Navier–Stokes systems. J. Math. Phys. **54**, 011502 (2013)

[31] Wan, R., Zhou, Y.: Global well-posedness for the 3D incompressible Hall-magnetohydrodynamic equations with Fujita–Kato type initial data. J. Math. Fluid Mech. **21**, 1–16 (2019)

[32] Wan, R., Zhou, Y.: Global well-posedness, BKM blow-up criteria and zero h limit for the 3D incompressible Hall-MHD equations. J. Differ. Equ. **267**, 3724–3747 (2019)

[33] Wang, J.: Balance of the vorticity direction and the vorticity magnitude in 3D fractional Navier–Stokes equations. arXiv:2001.04792

[34] Wu, J.: Lower bounds for an integral involving fractional Laplacians and the generalized Navier–Stokes equations in Besov spaces. Commun. Math. Phys. **263**, 803–831 (2006)

[35] Wu, J.: Generalized MHD equations. J. Differ. Equ. **195**, 284–312 (2003)

[36] Zhang, Z., Wang, W., Yong, Z.: Global regularity criterion for the Navier–Stokes equations based on the direction of vorticity. Math. Methods Appl. Sci. **42**, 7126–7134 (2019)

[37] Zhou, Y.: Regularity criteria for the generalized viscous MHD equations. Ann. Inst. H. Poincare Anal. Non Lineaire **24**, 491–505 (2007)

[38] Zhou, Y., Fan, J.: Logarithmically improved regularity criteria for the 3D viscous MHD equations. Forum Math. **24**, 691–708 (2012)

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