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Analysis of a malaria epidemic model with age structure and spatial diffusion

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Abstract. This paper aims to provide the complete analysis on the threshold dynamics of an age-space structured malaria epidemic model. We formulate the model in a spatially bounded domain by assuming that: (i) the density of susceptible humans at space x stabilizes at $H(x)$; (ii) the force of infection between human population and mosquitoes is given by the mass action incidence. By appealing to the theory of fixed point problem and Picard sequences and iteration, the well-posedness of the model is shown by verifying that the solution exists globally and the model admits a global attractor. In the spatially homogeneous case, we establish the explicit formula for the basic reproduction number, which governs the malaria extinction and persistence. The local and global stability of equilibria is achieved by studying the distribution of characteristic roots of characteristic equation and constructing the suitable Lyapunov functions, respectively.

Mathematics Subject Classification. 35K57, 35J57, 35B40, 92D25.

Keywords. Age-space structured, Spatial heterogeneity, Basic reproduction number, Malaria model, Global stability.

1. Introduction

It is well known that malaria is a parasitic infections in humans, caused by the genus Plasmodium. In general, humans acquire malaria through effective biting by several species of infectious female anopheles mosquitoes [23,25]. Susceptible mosquitoes acquire malaria through effective bites of infectious human host. It has been reported in [15] that about more than one hundred countries are under prevalence of malaria. Every year, two billion people are at risk affected by Plasmodium falciparum.

Reaction–diffusion model frameworks have been proved to be a powerful tool to generalize the classical Ross–Macdonald malaria models [4,6,11,20–22,32,34]. For a spatial transmission dynamics of malaria, it is usually assumed that human and mosquitoes are confined in a bounded domain Ω. Laplacian operator $\partial/\partial x^2$, $x \in \Omega \subset \mathbb{R}^n (n \geq 1)$ are introduced to reflect the spatial random movement of humans and mosquitoes. Recent publications [13,14,21] demonstrated that the spatial heterogeneity is a more meaningful and important factor in disease transmission. In reality, as environmental conditions vary spatially, for example, temperature and humidity, etc., it comes natural to demonstrate spatial heterogeneity for disease transmission parameters. It is also argued in [26] that "the non-random distribution of humans and mosquitoes across the landscape can generate spatially heterogeneous biting patterns".

In typical and pioneering work of malaria transmission model [1,18–20,23,25], humans are categorized into susceptible and infected class. The adult female mosquitoes (termed as vector) are categorized into susceptible and infected mosquito classes. The mosquitoes have to breed in water, and the adult female mosquitoes emerge from aquatic mosquitoes. The adult female mosquitoes transmit malaria to human hosts. The infected mosquitoes may spend a incubation period of 10 to 14 days (termed as extrinsic

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incubation period (EIP)) to survive (during which the infected mosquitoes cannot transmit malaria to human hosts) and transmit malaria to human hosts [18]. Taking EIP and mobility of human and mosquitoes in a spatial domain into account, Lou and Zhao [21] generalized the model in [25] and proposed a nonlocal and time-delayed diffusive malaria model, for $x \in \Omega,\ t > 0$,

$$\begin{cases} \dfrac{\partial S_m(t,x)}{\partial t} - D_m \Delta S_m(t,x) = \mu(x) - \dfrac{b\beta(x)}{H(x)} S_m(t,x) I_h(t,x) - d_m S_m(t,x), \\ \dfrac{\partial I_m(t,x)}{\partial t} - D_m \Delta I_m(t,x) = e^{-d_m \tau} \displaystyle\int_\Omega \Gamma(\tau,x,y) \dfrac{b\beta(y)}{H(y)} S_m(t-\tau,y) I_h(t-\tau,y) \mathrm{d}y - d_m I_m(t,x), \\ \dfrac{\partial I_h(t,x)}{\partial t} - D_h \Delta I_h(t,x) = \dfrac{c\beta(x)}{H(x)} (H(x) - I_h) I_m(t,x) - (d_h + \rho) I_h(t,x), \end{cases} \tag{1.1}$$

with

$$\frac{\partial S_m(t,x)}{\partial n} = \frac{\partial I_m(t,x)}{\partial n} = \frac{\partial I_h(t,x)}{\partial n} = 0,\ \ x \in \partial\Omega, t > 0.$$

Here, $S_m(t,x)$, $I_m(t,x)$ and $I_h(t,x)$ denote the density of susceptible, infected adult female mosquitoes and infected humans at space x and time t, respectively, which are equipped with diffusion rates D_m, D_m and D_h, respectively. The total population stabilizes at $H(x)$. $\beta(x)$ and $\mu(x)$ stand for the space-dependent biting rate and recruitment rate of adult female mosquitoes, respectively. d_m represents the natural death rate of mosquitoes. b and c stand for the transmission probabilities per bite from I_h to S_m and from I_m to $H(x) - I_h$, respectively. d_h and ρ represent the death rate and recovery rate of humans. Γ is the Green function corresponding to the operator $D_m\Delta$ subject to the Neumann boundary condition. τ represents the fixed incubation period constant.

Since the latent mosquitoes in one location can fly around during EIP, and arrive at any location in the domain when they can transmit malaria to human hosts, the spatial movement of mosquitoes in EIP will result in non-local infection [3,13,14]. In model (1.1), the nonlocal time delay term is obtained by introducing an infection age variable a. At time t and space x, denote by $i_m(t,a,x)$ the density of the mosquitoes with infection age a. Suppose that τ is a fixed latent period. It then follows that $E_m(t,x) = \int_0^\tau i_m(t,a,x)\mathrm{d}a$ and $I_m(t,x) = \int_\tau^\infty i_m(t,a,x)\mathrm{d}a$ are the density of latent and infected mosquitoes, respectively. Then, the evolution of $i_m(t,a,x)$ satisfies

$$\begin{cases} \dfrac{\partial i_m(t,a,x)}{\partial t} + \dfrac{\partial i_m(t,a,x)}{\partial a} - D_m \Delta i_m(t,a,x) = -d_m i_m(t,a,x), & x \in \Omega,\ t > 0, a \geq 0, \\ i_m(t,0,x) = \dfrac{b\beta(x)}{H(x)} S_m I_h, & x \in \Omega,\ t > 0, \\ \dfrac{\partial i_m(t,a,x)}{\partial n} = 0, & x \in \partial\Omega,\ a \geq 0, \end{cases} \tag{1.2}$$

where $i_m(t,0,x)$ is the newly infected mosquitoes. The nonlocal time delay term in (1.1) comes from the key point that $i_m(t,\tau,x)$ can be determined by the integration along characteristics line $t - a = const.$,

$$i_m(t,a,x) = \begin{cases} \displaystyle\int_\Omega \Gamma_1(a,x,y) i_m(t-a,0,y) \mathrm{d}y \Pi(a), & t \geq a, \\ \displaystyle\int_\Omega \Gamma_1(t,x,y) i_{m0}(a-t,y) \mathrm{d}y \dfrac{\Pi(a)}{\Pi(a-t)}, & t < a, \end{cases} \tag{1.3}$$

where i_{m0} is initial condition, and $\Pi(a) = e^{-d_m a}$. Hence,

$$i_m(t,\tau,x) = \Pi(\tau) \int_\Omega \Gamma_1(\tau,x,y) i_m(t-\tau,0,y) \mathrm{d}y,\ \ \forall t \geq \tau.$$

From the standpoint of mathematical analysis, the authors in [21] confirmed that the threshold dynamics of (1.1) is governed by the sign of the principle eigenvalue of the associated linear and nonlocal eigenvalue problem. Subsequently, a typical feature of malaria termed as vector-bias, was introduced to describe the difference between the probability (denote by p and l) of a mosquito picking the human when he/she is infected and susceptible (see, e.g., [8,29,33,34]). To explore the seasonal patterns of malaria epidemics (may be caused by annual temperature and rainfall variation (see, e.g., [12])), Bai et al. [4] further extended the models in [21,33] by incorporating the seasonality, vector-bias, EIP and the spatial heterogeneity, and formulated a time-delayed periodic reaction-diffusion model. Their results suggest that the sharp threshold results can be achieved by defining the basic reproduction number. They also confirmed the existence of a periodic solution for the proposed models, which built an interesting biological implication: Seasonal patterns of malaria epidemics will occur. It is important to mention that recent studies on diffusive Zika epidemic models (see, e.g., [11,22,32]) can also be regarded as a generalization of the classical model in [25]. Like other vector-borne disease models, Zika epidemic models share the same cross-infection mechanism between humans and mosquitoes.

Our goal of this paper is to perform an original analysis of (1.1). A more complete understanding of (1.1) can help to get better understanding of the malaria transmission in regions. As in studies of Zika outbreak in Rio De Janerio, Bastos et al. [28] argued that compared to the total population, infected human density takes a fairly small number [28]. Subsequently, Fitzgibbon et al. [11], Magal et al. [22] and Wang and Chen [32] proposed the diffusive vector-borne disease model by assuming the density of susceptible humans to be $H(x)$, which depends only on space x, and will not be altered within a short time. On the other hand, in (1.1), the force of infection for humans and mosquitoes is described by adopting standard infection mechanism, $\frac{b\beta(x)}{H(x)}S_m I_h$ and $\frac{c\beta(x)}{H(x)}(H(x)-I_h)I_m$, respectively, where $H(x)-I_h$ represents the density of susceptible humans. It comes naturally to wonder what happens if standard incidence is replaced by the mass action. In this work, we continue to borrow the idea in [11,22,32] that the density of susceptible humans is $H(x)$ and adopt the mass action incidence, which result in the force of infection for humans and mosquitoes at space x and time t given by $b\beta(x)S_m I_h$ and $c\beta(x)H(x)I_m$, respectively.

This work is also inspired by some recent works on diffusive disease models with age structure (see, for example, [5,9,10,35,36], aiming to understand the effects of the spatial heterogeneity and infection age on disease transmission. For the standard age-space structured susceptible–infective–recovered (SIR) model, Chekroun and Kuniya [5] reformulated the model by a hybrid system of one diffusive equation and one Volterra integral equation, and studied the threshold dynamics for the disease extinction and persistence in one-dimensional domain. Further, the global stability problem of a constant equilibrium was achieved by constructing Lyapunov function. In another works, the existence of travelling wave solutions of age-space structured SIR model with or without birth and death processes was established in a spatially unbounded domain [9,10]. For a age-space structured SIR model with seasonality, Zhang and Wang [36] established the threshold dynamics that basic reproduction number more than one or less than one determines whether or not disease extinction. Yang et al. [35] made an attempt to extend the methods and ideas in [5] to propose a model for the spatial spreading of brucellosis in a continuous bounded domain. Some basic mathematical arguments, including the existence and uniqueness of the solution and threshold dynamics, were successfully addressed.

Unlike in [13,21] where disease transmission is modeled with a fixed incubation period in a spatial bounded domain, here we intend to incorporate an infection age to be a continuous variable and the fixed incubation period is ignored. This work can be considered as a continuation of the work [21]. For convenience, we shall adopt some notations used in [21], as proceeded below. With these considerations, we propose the following age-space-structured malaria model:

$$\begin{cases} \dfrac{\partial S_m(t,x)}{\partial t} - D_m\Delta S_m(t,x) = \mu(x) - \beta(x)S_m(t,x)I_h(t,x) - d_mS_m(t,x), & x\in\Omega, t>0,\\ \dfrac{\partial i_m(t,a,x)}{\partial t} + \dfrac{\partial i_m(t,a,x)}{\partial a} - D_m\Delta i_m(t,a,x) = -d_m i_m(t,a,x), & x\in\Omega, t>0, a\geq 0,\\ \dfrac{\partial I_h(t,x)}{\partial t} - D_h\Delta I_h(t,x) = H(x)\displaystyle\int_0^\infty \beta_1(a)i_m(t,a,x)\mathrm{d}a - (d_h+\rho)I_h(t,x), & x\in\Omega, t>0,\\ i_m(t,0,x) = \beta(x)S_mI_h, & x\in\Omega, t>0, \end{cases} \tag{1.4}$$

with the following initial condition

$$S_m(0,x) = \phi_1(x)\geq 0,\ i_m(0,a,x) = \phi_2(a,x)\in L^1_+(\mathbb{R}_+;C(\Omega)),\ I_h(0,x) = \phi_3(x)\geq 0,\ x\in\overline{\Omega}, \tag{1.5}$$

and boundary condition

$$\frac{\partial S_m(t,x)}{\partial n} = \frac{\partial i_m(t,a,x)}{\partial n} = \frac{\partial I_h(t,x)}{\partial n} = 0,\ x\in\partial\Omega,\ t>0. \tag{1.6}$$

Here, $\beta(x)$ and $\beta_1(a)\in L^\infty_+(\mathbb{R}_+)$ are the age-dependent disease transmission rate. All the location-dependent parameters are continuous, strictly positive and uniformly bounded functions on $\overline{\Omega}$.

We point here that the main difficulty lies in that (1.4) is formulated in a non-uniform Banach space. The existence and positivity of the solution should be carefully verified, to which adopt the methods used in [5,35]. Comparing to [5,35], we need to turn the existence of global nonnegative classical solution into a fixed point problem by inserting the first and third equation to the second equation due to the cross-infection mechanism. On the other hand, the cross-infection mechanism brings us difficulties in obtaining the explicit formula for the basic reproduction number through defining the next generation operator. Finally, in a homogeneous case, the local stability of equilibria achieved by studying the distribution of characteristic roots of characteristic equation, the strong persistence result and the global stability of equilibria achieved by Lyapunov functions in different cases are indeed not trivial, as proceed below.

We proceed the paper as follows: In section 2, we present the preliminary results on (1.4) with (1.5) and (1.6), and the well-posedness of the model is shown by verifying that the solution exists globally and the model admits a global attractor. Section 3 is spent on giving the basic reproduction number for (1.4) by appealing to the theories of the next generation operator and renewal equations. In section 4, the local and global stability problem for space-independent equilibria of (1.4) is studied in a homogeneous case. A brief conclusion section ends the paper.

2. Preliminaries

Denote by $\mathbb{X} = C(\overline{\Omega},\mathbb{R})$ the Banach space endowed with the norm $\|\phi\|_{\mathbb{X}} = \sup_{x\in\overline{\Omega}}\{|\phi(x)|\}$, and let $\mathbb{X}_+$ be its positive cone. Denote by $\mathbb{Y} = L_1(\mathbb{R}_+,\mathbb{X})$ the space of Lebesgue integrable functions endowed with the norm $\|\varphi\|_{\mathbb{Y}} = \int_0^\infty |\varphi(a)|_{\mathbb{X}}\mathrm{d}a$. The positive cone of $\mathbb{Y}$ is $\mathbb{Y}_+$.

Suppose that $T_1(t), T_2(t): \mathbb{X}\to\mathbb{X},\ t\geq 0$ are the strongly continuous semigroups corresponding to the operators, $D_m\Delta$ and $D_h\Delta$, subject to (1.6), respectively. Clearly,

$$(T_i(t)\phi)(x) = \int_\Omega \Gamma_i(t,x,y)\phi(y)\mathrm{d}y,\ t\geq 0,\ \phi\in\mathbb{X}, \tag{2.1}$$

where $\Gamma_i(t,x,y), i=1,2$ is the Green function associated with $D_m\triangle$ and $D_h\triangle$ subject to (1.6). It then follows from [27, Corollary 7.2.3] (see also in [24, Theorem 1.5]) that $T_i(t):\mathbb{X}\to\mathbb{X},\ t\geq 0,\ i=1,2,$ is strongly positive and compact. Further, $T(t) = (T_1(t),T_1(t),T_2(t)):\mathbb{X}^3\to\mathbb{X}^3,\ t\geq 0$ forms a strongly continuous semigroup.

We place our problem on the state space $\mathbb{X}\times\mathbb{Y}\times\mathbb{X}$. For convenience, in what follows, we denote

$$\overline{\beta}_1 = ess.\sup_{a\in\mathbb{R}_+}\beta_1(a),\ \ \chi^* = \max_{x\in\overline{\Omega}}\chi(x) \text{ and } \chi_* = \min_{x\in\overline{\Omega}}\chi(x),$$

where $\chi = \mu, \beta, H$, respectively.

We next aim to prove that the solution of (1.4) exists globally and (1.4) admits a global attractor. For simplicity, we denote by $\tilde{\mathbb{X}} = \mathbb{X}\times\mathbb{Y}\times\mathbb{X}$ and $\tilde{\mathbb{X}}_+ = \mathbb{X}_+\times\mathbb{Y}_+\times\mathbb{X}_+$. We first present the main result.

Theorem 2.1. *For any $\phi = (\phi_1,\phi_2,\phi_3)\in\tilde{\mathbb{X}}$, (1.4) admits a unique global nonnegative classical solution. Further, the solution semiflow, i.e.,*

$$\Phi(t)\phi := u(t,x;\phi) = (S_m(t,x;\phi_1), i_m(t,a,x;\phi_2), I_h(t,x;\phi_3)),\ \ a\geq 0,$$

possesses a global attractor in $\tilde{\mathbb{X}}_+$.

We shall first introduce the following lemmas and then combine with them to prove Theorem 2.1. For convenience, we always denote

$$\mathcal{B}(t,x) := i_m(t,0,x) = \beta(x)S_mI_h. \tag{2.2}$$

For positive T, we define the space

$$\mathbb{Y}_T = C([0,T],\mathbb{X}) \text{ with } \|\psi\|_{\mathbb{Y}_T} = \sup_{0\leq t\leq T}\|\psi(t,\cdot)\|_{\mathbb{X}},\ \ \psi\in\mathbb{Y}_T.$$

Lemma 2.2. *For any $\phi\in\tilde{\mathbb{X}}_+$, (1.4) admits a unique nonnegative solution defined on $[0,T]\times\overline{\Omega}$ with $T>0$.*

Proof. In view of the S_m equation of (1.4), direct calculation yields

$$S_m = \mathbb{F}_1(t,x) + \int_0^t e^{-d_m(t-s)}\int_\Omega \Gamma_1(t-s,x,y)[\mu(y)-\mathcal{B}(s,y)]\mathrm{d}y\mathrm{d}s,\ \ (t,x)\in[0,T]\times\Omega, \tag{2.3}$$

where $\mathbb{F}_1(t,x) = e^{-d_mt}\int_\Omega\Gamma_1(t,x,y)\phi_1(y)\mathrm{d}y$. Similarly, in view of the I_h equation of (1.4), we get

$$I_h = \mathbb{F}_2(t,x) + \int_0^t e^{-(d_h+\rho)(t-s)}\int_\Omega\Gamma_2(t-s,x,y)H(y)\int_0^\infty\beta_1(a)i_m(s,a,y)\mathrm{d}a\mathrm{d}y\mathrm{d}s,\ \ (t,x)\in[0,T]\times\Omega, \tag{2.4}$$

where $\mathbb{F}_2(t,x) = e^{-(d_h+\rho)t}\int_\Omega\Gamma_2(t,x,y)\phi_3(y)\mathrm{d}y$. It follows from (1.3) that

$$\int_0^\infty\beta_1(a)i_m(t,a,x)\mathrm{d}a = \int_0^t\beta_1(a)\Pi(a)\int_\Omega\Gamma_1(a,x,y)\mathcal{B}(t-a,y)\mathrm{d}y\mathrm{d}a + \mathbb{F}_3(t,x), \tag{2.5}$$

where $\mathbb{F}_3(t,x) = \int_0^\infty\beta_1(a+t)\frac{\Pi(a+t)}{\Pi(a)}\int_\Omega\Gamma_1(t,x,y)\phi_2(a,y)\mathrm{d}y\mathrm{d}a$.

Plugging (2.5) into (2.4) and then substituting (2.3) and (2.4) in (2.2) yield

$$\begin{aligned}\mathcal{B}(t,x) &= \beta(x)\left(\mathbb{F}_1 + \int_0^t e^{-d_m(t-s)}\int_\Omega\Gamma_1(t-s,x,y)[\mu(y)-\mathcal{B}(s,y)]\mathrm{d}y\mathrm{d}s\right)\times(\mathbb{F}_2+\mathbb{F}_4\\ &\quad + \int_0^t e^{-(d_h+\rho)(t-s)}\int_\Omega\Gamma_2(t-s,x,y)H(y)\int_0^s\beta_1(a)\Pi(a)\int_\Omega\Gamma_1(a,y,z)\mathcal{B}(s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s\Bigg)\\ &:= \mathbb{F}(\mathcal{B})(t,x),\end{aligned} \tag{2.6}$$

where $\mathbb{F}_4(t,x) = \int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y)H(y) \int_0^\infty \beta_1(a+s)\frac{\Pi(a+s)}{\Pi(a)} \int_\Omega \Gamma_1(s,y,z)\phi_2(a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s$.

Next, we claim that the nonlinear operator $\mathbb{F}$ has a fixed point, which in turn ensures that (1.4) admits a unique solution on $[0,T]$. For ease of notations, we denote

$$\begin{cases} \overline{\mathbb{F}}_2(t,x) := \mathbb{F}_2(t,x) + \mathbb{F}_4(t,x), \\ \mathbb{G}_1(\mathcal{B}) := \displaystyle\int_0^t e^{-d_m(t-s)} \int_\Omega \Gamma_1(t-s,x,y)[\mu(y) - \mathcal{B}(s,y)]\mathrm{d}y\mathrm{d}s, \\ \mathbb{G}_2(\mathcal{B}) := \displaystyle\int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y)H(y) \int_0^s \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s. \end{cases} \tag{2.7}$$

It then follows that the nonlinear operator $\mathbb{F}$ can be rewritten as:

$$\mathbb{F}(\mathcal{B}) = \beta(x)(\mathbb{F}_1 + \mathbb{G}_1(\mathcal{B}))(\overline{\mathbb{F}}_2 + \mathbb{G}_2(\mathcal{B})).$$

By appealing the Banach–Picard fixed point theorem, we only need to verify that the nonlinear operator $\mathbb{F}$ is a strict contraction in $\mathbb{Y}_T$. To this end, we choose any $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{Y}_T$ and let $\widetilde{\mathcal{B}} = \mathcal{B}_1 - \mathcal{B}_2$ such that

$$\begin{aligned} \mathbb{F}(\mathcal{B}_1) - \mathbb{F}(\mathcal{B}_2) &= \beta(x)(\mathbb{F}_1\mathbb{G}_2(\widetilde{\mathcal{B}}) - \overline{\mathbb{F}}_2\widehat{\mathbb{G}}_1(\widetilde{\mathcal{B}}) + \mathbb{G}_1(\mathcal{B}_1)\mathbb{G}_2(\widetilde{\mathcal{B}}) - \mathbb{G}_2(\mathcal{B}_2)\widehat{\mathbb{G}}_1(\widetilde{\mathcal{B}})) \\ &\le \beta^* \left|\mathbb{F}_1\overline{\mathbb{G}}_2 - \overline{\mathbb{F}}_2\overline{\mathbb{G}}_1 + \mathbb{G}_1\overline{\mathbb{G}}_2 - \overline{\mathbb{G}}_1\mathbb{G}_2\right| \widetilde{\mathcal{B}}, \end{aligned}$$

where

$$\widehat{\mathbb{G}}_1(\mathcal{B}) := \int_0^t e^{-d_m(t-s)} \int_\Omega \Gamma_1(t-s,x,y)\mathcal{B}(s,y)\mathrm{d}y\mathrm{d}s,$$

$$\overline{\mathbb{G}}_1 = \int_0^t e^{-d_m(t-s)} \int_\Omega \Gamma_1(t-s,x,y)\mathrm{d}y\mathrm{d}s,$$

and

$$\overline{\mathbb{G}}_2 = \int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y)H(y) \int_0^s \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s.$$

Putting

$$\tilde{h}(T) = \beta^* \sup_{0\le s\le T} \left|\mathbb{F}_1(s,x)\overline{\mathbb{G}}_2(s,x) - \overline{\mathbb{F}}_2(s,x)\overline{\mathbb{G}}_1(s,x) + \mathbb{G}_1(s,x)\overline{\mathbb{G}}_2(s,x) + \overline{\mathbb{G}}_1(s,x)\mathbb{G}_2(s,x)\right|_{\mathbb{X}},$$

which leads to

$$|\mathbb{F}(\mathcal{B}_1) - \mathbb{F}(\mathcal{B}_2)| \le \tilde{h}(T)\,|\mathcal{B}_1 - \mathcal{B}_2|\,.$$

Choosing sufficiently small $T > 0$ such that $\tilde{h}(T) < 1$. We arrive at the conclusion that $\mathbb{F}$ is a strict contraction in $\mathbb{Y}_T$, that is, the nonlinear operator $\mathbb{F}$ has a unique fixed point. This proves Lemma 2.2. □

We next to prove that the local solution of (1.4) remains positive for any $\phi \in \tilde{\mathbb{X}}_+$.

Lemma 2.3. *For any $\phi \in \tilde{\mathbb{X}}_+$, we have that $\forall t \in [0,T]$ and $x \in \Omega$,*

$$S_m(t,x) > 0,\ \mathcal{B}(t,x) > 0 \text{ and } I_h(t,x) > 0.$$

Proof. It is easily seen that due to the positivity of $\mu(x)$ and ϕ_1,

$$S_m(t,x) = \widehat{\mathbb{F}}_1 + \int_0^t e^{-\int_s^t \left(d_m + \frac{\mathcal{B}(\tau,y)}{S_m(\tau,y)}\right)d\tau} \mu(x) \int_\Omega \Gamma_1(t-s,x,y)\mathrm{d}y\mathrm{d}s > 0,$$

where $\widehat{\mathbb{F}}_1 = e^{-\int_{s-t}^{s}\left(d_m + \frac{\mathcal{B}(\tau,y)}{S_m(\tau,y)}\right)d\tau} \int_\Omega \Gamma_1(t,x,y)\phi_1(y)\mathrm{d}y$.

We next to prove the positivity of $\mathcal{B}(t,x)$ by the method of Picard sequences. We first set

$$\mathcal{B}_0(t,x) = \beta(x)S_m(t,x)\overline{\mathbb{F}}_2(t,x) > 0,$$

where $\overline{\mathbb{F}}_2(t,x)$ is defined in (2.7). Assume that $\mathcal{B}_n(t,x) > 0$ $(n \in \mathbb{N})$. We then directly get

$$\begin{aligned}\mathcal{B}_{n+1}(t,x) = \mathcal{B}_0(t,x) + \beta(x)S_m(t,x) \int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y)H(y) \\ \times \int_0^s \beta_1(s-a)\Pi(s-a) \int_\Omega \Gamma_1(s-a,y,z)\mathcal{B}_n(a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s \\ > 0.\end{aligned}$$

It remains to show that the sequence $\{\mathcal{B}_n(t,x)\}_0^\infty$ converges to $\mathcal{B}(t,x)$ in the sense that $\lim_{n\to\infty} \mathcal{B}_n(t,x) = \mathcal{B}(t,x)$. To achieve this, we introduce

$$\widehat{\mathcal{B}}_n(t,x) = e^{-\lambda t}\mathcal{B}_n(t,x), \text{ for } \lambda \in \mathbb{R}_+.$$

Hence,

$$\begin{aligned}\widehat{\mathcal{B}}_{n+1}(t,x) &= e^{-\lambda t}\mathcal{B}_0(t,x) + \beta(x)S_m(t,x) \int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y)H(y) \\ &\quad \times \int_0^s \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)e^{-\lambda t}\mathcal{B}_n(s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s \\ &= e^{-\lambda t}\mathcal{B}_0(t,x) + \beta(x)S_m(t,x) \int_0^t e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad \times \int_0^{t-s} \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)e^{-\lambda t}\mathcal{B}_n(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s \\ &= e^{-\lambda t}\mathcal{B}_0(t,x) + \beta(x)S_m(t,x) \int_0^t e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad \times \int_0^{t-s} \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)e^{-\lambda(a+s)}e^{-\lambda(t-a-s)}\mathcal{B}_n(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s \\ &= e^{-\lambda t}\mathcal{B}_0(t,x) + \beta(x)S_m(t,x) \int_0^t e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y)H(y)\end{aligned}$$

$$\times \int_0^{t-s} \beta_1(a)\Pi(a)e^{-\lambda a} \int_\Omega \Gamma_1(a,y,z)e^{-\lambda s}\widehat{\mathcal{B}}_n(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s.$$

Note that both sides of above equality always hold. It allows us to choose $\tilde{x} \in \overline{\Omega}$ such that $\mathcal{B}^*(t,\widetilde{x}) = \max_{t\in[0,T],x\in\overline{\Omega}} \widehat{\mathcal{B}}(t,x)$. Hence, for any $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\mathcal{B}^*_{n+1} - \mathcal{B}^*_n\right\|_\infty &\le \beta^* S^*_m \int_0^\infty e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad \times \int_0^\infty \beta_1(a)\Pi(a)e^{-\lambda a} \int_\Omega \Gamma_1(a,y,z)e^{-\lambda s}\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s \left\|\mathcal{B}^*_n - \mathcal{B}^*_{n-1}\right\|_\infty \\ &\le \frac{\beta^* S^*_m \overline{\beta}_1 H^*}{\lambda^2} \left\|\mathcal{B}^*_n - \mathcal{B}^*_{n-1}\right\|_\infty,\end{aligned}$$

where $H^* = \max_{x\in\Omega}\{H(x)\}$ and $S^*_m = \max_{t\in[0,T]} \|S_m(t,\cdot)\|_{\mathbb{X}}$. By repeating the iteration, it is easily seen that

$$\left\|\mathcal{B}^*_{n+1} - \mathcal{B}^*_n\right\|_\infty \le L_\lambda \left\|\mathcal{B}^*_n - \mathcal{B}^*_{n-1}\right\|_\infty \le L^n_\lambda \left\|\mathcal{B}^*_1 - \mathcal{B}^*_0\right\|_\infty,$$

where $L_\lambda = \frac{\beta^* S^*_m \overline{\beta}_1 H^*}{\lambda^2}$. Hence, for any $m, n \in \mathbb{N}$,

$$\|\mathcal{B}^*_m - \mathcal{B}^*_n\|_\infty \le \frac{L^n_\lambda}{1 - L_\lambda} \|\mathcal{B}^*_1 - \mathcal{B}^*_0\|_\infty.$$

Choose λ large enough that $L_\lambda < 1$. Consequently, as $n \to \infty$, $\|\mathcal{B}^*_m - \mathcal{B}^*_n\|_\infty \to 0$, implying that $\lim_{n\to\infty} \mathcal{B}_n(t,x) = \mathcal{B}(t,x)$, $\forall t \in [0,T]$ and $x \in \Omega$. The positivity of $\mathcal{B}_n(t,x)$ directly implies that $\mathcal{B}(t,x)$ is positive.

We next to show that $I_h(t,x) > 0$ by contradiction. Assume to the contrary that there exist $x_* \in \Omega$ and $t_1 \in (0,T)$ such that

$$\begin{cases} I_h(t,x) \ge 0, & t \in [0,t_1] \text{ and } x \in \Omega; \\ I_h(t,x_*) = 0, & t = t_1 \text{ and } x_* \in \Omega; \\ I_h(t+\epsilon,x_*) < 0, & t = t_1, x_* \in \Omega \text{ and } 0 < \epsilon \ll 1. \end{cases}$$

By (2.4), together with the positivity of $\mathcal{B}(t,x)$, we can easily obtain

$$\begin{aligned} I_h(t_1+\epsilon,x_*) &= \mathbb{F}_2 + \int_0^{t_1+\epsilon} e^{-(d_h+\rho)(t_1+\epsilon-s)} \int_\Omega \Gamma_2(t_1+\epsilon-s,x,y)H(y) \\ &\quad \times \int_0^s \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s + \mathbb{F}_4 > 0, \end{aligned}$$

which results in a contradiction with $I_h(t_1+\epsilon,x_*) < 0$. Hence, $I_h(t,x) > 0$ directly follows. This completes the proof. □

Next, we confirm that the solution of (1.4) exists globally by extending solution existence interval from $[0,T] \times \overline{\Omega}$ to $[0,+\infty) \times \overline{\Omega}$.

Lemma 2.4. *For any $\phi \in \tilde{\mathbb{X}}_+$, (1.4) admits a unique nonnegative solution defined on $[0,\infty) \times \overline{\Omega}$.*

Proof. In view of S_m equation of system (1.4), we know that S_m is governed by

$$\begin{cases} \dfrac{\partial \hat{S}_m}{\partial t} - D_m \Delta \hat{S}_m = \mu(x) - d_m \hat{S}_m, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \hat{S}_m}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.8}$$

According to [21, Lemma 1] and comparison principle, S_m is bounded above by the upper solution $\frac{\mu^*}{d_m}$, i.e., $S_m \le \frac{\mu^*}{d_m} = M_S$ for all $t > 0$ and $x \in \overline{\Omega}$.

We now claim that for all $t > 0$ and $x \in \Omega$, $\mathcal{B}(t, x) < \infty$ by contradiction. Assume to the contrary that there exist $t^* > 0$ and $x^* \in \Omega$ such that $\lim_{t \to t^*} \mathcal{B}(t, x^*) = +\infty$. In view of S_m equation of system (1.4), we get $\lim_{t \to t^*} \partial_t S_m(t, x^*) = -\infty$, i.e., $S_m(t, x^*) < 0$ in near of t^*. This leads to a contradiction to the positivity of S_m (see Lemma 2.3). Hence, $\mathcal{B}(t, x) < \infty$. Based on this fact, we assume that there exists $M_B > 0$ such that $\mathcal{B}(t, x) < M_B$.

Finally, we determine the boundedness of $I_h(t, x)$. Let $\widetilde{I} = \int_0^\infty i_m(t, a, x)\mathrm{d}a$, then it satisfies

$$\begin{aligned} \widetilde{I} &= \int_0^t \Pi(a) \int_\Omega \Gamma_1(a, x, y)\mathcal{B}(t - a, y)\mathrm{d}y\mathrm{d}a + \int_0^\infty \frac{\Pi(a + t)}{\Pi(a)} \int_\Omega \Gamma_1(t, x, y)\phi_2(a, y)\mathrm{d}y\mathrm{d}a \\ &\le \int_0^\infty \Pi(a) \int_\Omega \Gamma_1(a, x, y)\mathrm{d}y\mathrm{d}a M_B + \int_0^\infty \int_\Omega \Gamma_1(t, x, y)\phi_2(a, y)\mathrm{d}y\mathrm{d}a := M_I. \end{aligned}$$

It then follows that

$$\begin{aligned} I_h(t, x) &\le H^*\overline{\beta}_1 \int_0^\infty e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t - s, x, y)\widetilde{I}(t, y)\mathrm{d}y\mathrm{d}s + \int_\Omega \Gamma_2(t, x, y)\phi_3(y)\mathrm{d}y \\ &\le \frac{H^*\overline{\beta}_1 M_I}{d_h + \rho} + \|\phi_3\|_X := M_{I_h}. \end{aligned} \tag{2.9}$$

Therefore, the solution of (1.4) exists globally. This proves Lemma 2.4. □

By utilizing the previous lemmas, we shall give the proof of Theorem 2.1.

Proof of Theorem 2.1. By the assertions in Lemmas 2.2, 2.3 and 2.4, we know that the solution of (1.4) with initial condition $\phi = (\phi_1, \phi_2, \phi_3) \in \tilde{\mathbb{X}}_+$ exists globally. Let $\Phi(t) : \tilde{\mathbb{X}}_+ \to \tilde{\mathbb{X}}_+, t \ge 0$, be the semiflow generated by the solution of (1.4), i.e.,

$$\Phi(t)\phi := u(t, x; \phi) = (S_m(t, x; \phi_1), i_m(t, a, x; \phi_2), I_h(t, x; \phi_3)).$$

From Lemma 2.4, solution semiflow $\Phi(t)$ is ultimately bounded. Hence, we can apply [16, Theorem 2.4.6] to confirm that (1.4) possesses a global attractor. This proves Theorem 2.1.

Throughout of the paper, we define the following positively invariant set, at which the dynamics of (1.4) are confined.

Letting $\mathcal{M}(t) := \int_\Omega S_m \mathrm{d}x + \int_\Omega \widetilde{I}\mathrm{d}x$. Integrate the i_m equation from 0 to ∞, and integrating the S_m and i_m equation of system (1.4) over Ω and then adding them up, yield

$$\frac{d\mathcal{M}(t)}{\mathrm{d}t} \le \mu^*|\Omega| - d_m\mathcal{M}(t).$$

Hence, we obtain the a priori estimate

$$\mathcal{M}(t) \le \mathcal{M}(0)e^{-d_m t} + \frac{\mu^*|\Omega|}{d_m}(1 - e^{-d_m t}) \le \max\left\{\mathcal{M}(0), \frac{\mu^*|\Omega|}{d_m}\right\}.$$

This implies

$$\limsup_{t\to\infty}\mathcal{M}(t)\le\frac{\mu^*|\Omega|}{d_m}.$$

Hence, there exists $t_0>0$ such that for any $t>t_0$, $\mathcal{M}(t)\le\frac{\mu^*|\Omega|}{d_m}$.

Denote by $\widehat{I}_h(t)=\int_\Omega I_h(t,x)\mathrm{d}x$. We directly have

$$\frac{d\widehat{I}_h(t)}{\mathrm{d}t}\le\frac{H^*\overline{\beta}_1\mu^*|\Omega|}{d_m}-(d_h+\rho)\widehat{I}_h(t).$$

We integrate this differential inequality to obtain that for $t_1>t_0$,

$$\widehat{I}_h(t)\le\frac{H^*\overline{\beta}_1\mu^*|\Omega|}{d_m(d_h+\rho)},\ \text{for } t>t_1.$$

Consequently, we can define the following positively invariant set

$$\mathcal{D}=\left\{\phi\in\tilde{\mathbb{X}}_+|\ 0<\mathcal{M}(t)<\tfrac{\mu^*|\Omega|}{d_m},0<\widehat{I}_h(t)<\tfrac{H^*\overline{\beta}_1\mu^*|\Omega|}{d_m(d_h+\rho)}\right\}. \tag{2.10}$$

□

3. The basic reproduction number

Obviously, (1.4) has a disease-free steady state $E_0=(\frac{\mu(x)}{d_m},0,0)$. Assuming that both humans and mosquitos are near E_0. Following the standard procedures as those in [30] and [7], this section is spent on defining basic reproduction number of model (1.4).

Linearizing system (1.4) at E_0 for infectious components yields

$$\begin{cases}\dfrac{\partial i_m(t,a,x)}{\partial t}+\dfrac{\partial i_m(t,a,x)}{\partial a}-D_m\Delta i_m(t,a,x)=-d_m i_m(t,a,x), & x\in\Omega,\ a\ge0,\\ i_m(t,0,x)=\beta(x)\dfrac{\mu(x)}{d_m}I_h:=\overline{\mathcal{B}}(t,x), & x\in\Omega,\\ \dfrac{\partial I_h}{\partial t}-D_h\triangle I_h=H(x)\displaystyle\int_0^\infty\beta_1(a)i_m(t,a,x)\mathrm{d}a-(d_h+\rho)I_h, & x\in\Omega,\ a\ge0,\\ \dfrac{\partial i_m(t,a,x)}{\partial n}=\dfrac{\partial I_h}{\partial n}=0, & x\in\partial\Omega.\end{cases} \tag{3.1}$$

We integrate the i_m equation of (3.1) along the characteristic line $t-a=const.$

$$i_m(t,a,x)=\begin{cases}\displaystyle\int_\Omega\Gamma_1(a,x,y)\overline{\mathcal{B}}(t-a,y)\mathrm{d}y\Pi(a), & t>a,\ x\in\Omega,\\ \displaystyle\int_\Omega\Gamma_1(t,x,y)\phi_2(a-t,y)\mathrm{d}y\frac{\Pi(a)}{\Pi(a-t)}, & a\ge t,\ x\in\Omega.\end{cases} \tag{3.2}$$

We substitute (3.2) in I_h (defined in (2.4)) resulting in

$$\begin{aligned}\overline{\mathcal{B}}(t,x)=\beta(x)\frac{\mu(x)}{d_m}\Bigg(\mathbb{F}_2+\mathbb{F}_4+\int_0^t e^{-(d_h+\rho)(t-s)}\int_\Omega\Gamma_2(t-s,x,y)H(y)\\ \times\int_0^s\beta_1(a)\Pi(a)\int_\Omega\Gamma_1(a,y,z)\overline{\mathcal{B}}(s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s\Bigg)\end{aligned}$$

$$= \beta(x)\frac{\mu(x)}{d_m}\Bigg(\mathbb{F}_2 + \mathbb{F}_4 + \int_0^t e^{-(d_h+\rho)s}\int_\Omega \Gamma_2(s,x,y)H(y) \times \int_0^{t-s}\beta_1(a)\Pi(a)\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s\Bigg), \tag{3.3}$$

where $\mathbb{F}_2$ and $\mathbb{F}_4$ are defined in (2.4) and (2.6), respectively. Since (3.3) is a renewal equation, we can make Laplace transformation to (3.3)

$$\int_0^\infty e^{-\lambda t}\overline{\mathcal{B}}(t,x)\mathrm{d}t = \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-\lambda t}\int_0^t e^{-(d_h+\rho)s}\int_\Omega \Gamma_2(s,x,y)H(y) \times \int_0^{t-s}\beta_1(a)\Pi(a)\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s\mathrm{d}t.$$

Interchanging the order of integration (t and s) and letting $t-s=\tilde{t}$ (for simplicity, still denote $\tilde{t}$ by t) yield

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\overline{\mathcal{B}}(t,x)\mathrm{d}t &= \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_s^\infty e^{-\lambda t}\int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad\times \int_0^{t-s}\beta_1(a)\Pi(a)\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}t\mathrm{d}s \\ &= \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_0^\infty e^{-\lambda(t+s)}\int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad\times \int_0^{t}\beta_1(a)\Pi(a)\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Interchanging the order of integration (t and a) and letting $t-a=\bar{t}$ (for simplicity, still denote $\bar{t}$ by t) yield

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\overline{\mathcal{B}}(t,x)\mathrm{d}t &= \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad\times \int_a^\infty e^{-\lambda t}\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t-a,z)\mathrm{d}z\mathrm{d}t\mathrm{d}y\mathrm{d}a\mathrm{d}s \\ &= \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)e^{-\lambda a}\int_\Omega \Gamma_2(s,x,y)H(y) \\ &\quad\times \int_0^\infty e^{-\lambda t}\int_\Omega \Gamma_1(a,y,z)\overline{\mathcal{B}}(t,z)\mathrm{d}z\mathrm{d}t\mathrm{d}y\mathrm{d}a\mathrm{d}s.\end{aligned}$$

Consequently, interchanging the order of integration (t and z) yields

$$\int_0^\infty e^{-\lambda t}\overline{\mathcal{B}}(t,x)\mathrm{d}t = \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)e^{-\lambda a}\int_\Omega \Gamma_2(s,x,y)H(y) \times \int_\Omega \Gamma_1(a,y,z)\int_0^\infty e^{-\lambda t}\overline{\mathcal{B}}(t,z)\mathrm{d}t\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s. \tag{3.4}$$

Setting $\lambda = 0$ leads to

$$\int_0^\infty \overline{\mathcal{B}}(t,x)\mathrm{d}t = \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_2(s,x,y)H(y)\int_\Omega \Gamma_1(a,y,z)\int_0^\infty \overline{\mathcal{B}}(t,z)\mathrm{d}t\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s. \tag{3.5}$$

Hence, the following operator $\mathcal{L}$ is termed as the next generation operator

$$\mathcal{L}[\psi](x) = \beta(x)\frac{\mu(x)}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_2(s,x,y)H(y)\int_\Omega \Gamma_1(a,y,z)\psi(z)\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s. \tag{3.6}$$

The following result indicates that $\mathcal{L}$ defined in (3.6) is strictly positive and compact.

Lemma 3.1. *Let $\mathcal{L}$ be defined in (3.6). Then, $\mathcal{L}$ is strictly positive and compact.*

Proof. It is easily seen that the operator $\mathcal{L}$ is positive. Due to the properties of Γ_1 and Γ_2, we can select a bounded sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ such that for $\mathbb{M} > 0$ and $\forall x \in \Omega$,

$$|\phi_n| \leq \mathbb{M}, \text{ where } \mathbb{M} > 0.$$

Hence,

$$\mathcal{L}[\phi_n](x) \leq \frac{\mathbb{M}H^*\beta^*\mu^*}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_2(s,x,y)\int_\Omega \Gamma_1(a,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s,$$

that is, $\mathcal{L}$ is uniformly bounded. According to the Arzelà–Ascoli theorem, it remains to confirm that $\mathcal{L}$ is equicontinuous. In fact, for any $x, \bar{x} \in \Omega$ with $|x - \bar{x}| < \delta$ and $y \in \overline{\Omega}$,

$$\mathcal{L}[\phi_n](x) - \mathcal{L}[\phi_n](\bar{x}) \leq \frac{\mathbb{M}H^*\beta^*\mu^*}{d_m}\int_0^\infty e^{-(d_h+\rho)s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega |\Gamma_2(s,x,y) - \Gamma_2(s,\bar{x},y)|\int_\Omega \Gamma_1(a,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s. \tag{3.7}$$

Due to the compactness of the operator Δ and the uniform continuity of $\Gamma_2(s,x,y)$, finding $\varepsilon_0 > 0$ ensures

$$|\Gamma_2(a,x,y) - \Gamma_2(a,\bar{x},y)| \leq \frac{d_m\varepsilon_0}{\mathbb{M}H^*\beta^*\mu^*\mathcal{M}},$$

where $\mathcal{M} = |\Omega|\int_0^\infty e^{-(d_h+\rho)s}ds\int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a$ and $|\Omega|$ is the volume of Ω. Using this δ and ϵ_0, we know that $|\mathcal{L}[\phi](x) - \mathcal{L}[\phi](\bar{x})| < \epsilon_0$, for all $|x - \bar{x}| < \delta$. Consequently, the compactness of $\mathcal{L}$ directly follows (as $\mathcal{L}$ is uniformly bounded and equicontinuous). This proves Lemma 3.1. □

According to the general results as those in [30] and [7], we define the basic reproduction number as:

$$\Re_0 = r(\mathcal{L}),$$

where $r(\cdot)$ is the spectral radius of the operator $\mathcal{L}$. As usual, it is difficult to obtain spectral radius of the next-generation operator $\mathcal{L}$, if not impossible, so that we cannot get further information on dynamical properties of (1.4). To proceed further, we will consider the homogeneous case that

$$\mu(x) \equiv \mu, \beta(x) \equiv \beta, H(x) \equiv H.$$

4. Global stability problem of space-independent steady states

This section is spent on exploring the global stability problem of the space-independent equilibria of the following system:

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m = \mu - \beta S_m I_h - d_m S_m, & x \in \Omega,\ t > 0, \\ \dfrac{\partial i_m(t,a,x)}{\partial t} + \dfrac{\partial i_m(t,a,x)}{\partial a} - D_m \Delta i_m(t,a,x) = -d_m i_m(t,a,x), & x \in \Omega,\ t > 0,\ a \geq 0, \\ \dfrac{\partial I_h}{\partial t} - D_h \Delta I_h(t,x) = H \displaystyle\int_0^\infty \beta_1(a) i_m(t,a,x) \mathrm{d}a - (d_h + \rho) I_h, & x \in \Omega,\ t > 0, \\ i_m(t,0,x) = \beta S_m I_h, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S_m}{\partial n} = \dfrac{\partial i_m(t,a,x)}{\partial n} = \dfrac{\partial I_h}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{4.1}$$

Note that the assertions of model (1.4) in previous sections, including the existence and uniqueness of the solution, existence of global attractor, and definition of the basic reproduction number, still hold for (4.1). Obviously, (4.1) has a space-independent disease-free equilibrium $\tilde{E}_0 = (S_m^0, 0, 0)$, where $S_m^0 = \frac{\mu}{d_m}$. Similar to (1.3) and (3.2), $i_m(t,a,x)$ takes the following form:

$$i_m(t,a,x) = \begin{cases} \displaystyle\int_\Omega \Gamma_1(a,x,y) \mathcal{B}(t-a,y) \mathrm{d}y \Pi(a), & t - a > 0,\ x \in \Omega, \\ \displaystyle\int_\Omega \Gamma_1(t,x,y) \phi_2(a-t,y) \mathrm{d}y \frac{\Pi(a)}{\Pi(a-t)}, & a - t \geq 0,\ x \in \Omega, \end{cases} \tag{4.2}$$

with $\mathcal{B}(t-a,y) := i_m(t-a,0,y) = \beta S_m(t-a,y) I_h(t-a,y)$.

Lemma 3.1, together with Krein–Rutman theorem (see, e.g., [2, Theorem 3.2]), implies that the basic reproduction number is the only positive eigenvalue of $\mathcal{L}$, with a positive eigenvector. Substituting $\phi(x) \equiv \mathbf{1} > 0$ in (3.6) and using $\int_\Omega \Gamma_i(\cdot,x,y)\mathrm{d}y = 1 (i = 1,2)$, one gets

$$\mathcal{L}[\mathbf{1}] = \beta \frac{\mu}{d_m} H \int_0^\infty e^{-(d_h+\rho)s} \int_0^\infty \beta_1(a) \Pi(a) \mathrm{d}a \mathrm{d}s [\mathbf{1}].$$

In this setting, the basic reproduction number of (4.1), $[\Re_0]$, is

$$[\Re_0] = \beta \frac{\mu}{d_m} H \int_0^\infty e^{-(d_h+\rho)s} ds \int_0^\infty \beta_1(a) \Pi(a) \mathrm{d}a. \tag{4.3}$$

Denote by $E^* = (S_m^*, i_m^*(\cdot), I_h^*)$ the space-independent endemic equilibrium of (4.1), if it exists. Then, it satisfies

$$\begin{cases} \mu - \beta S_m^* I_h^* - d_m S_m^* = 0, \\ \dfrac{\partial i_m^*(a)}{\partial a} = -d_m i_m^*(a), \\ i_m^*(0) = \beta S_m^* I_h^*, \\ H \int\limits_0^\infty \beta_1(a) i_m^*(a) \mathrm{d}a - (d_h + \rho) I_h^* = 0. \end{cases} \tag{4.4}$$

Direct calculation gives

$$S_m^* = \frac{d_h + \rho}{\beta H K}, \ i_m^*(a) = i_m^*(0)\Pi(a), \text{ and } I_h^* = \frac{K H i_m^*(0)}{d_h + \rho}, \tag{4.5}$$

where $K = \int\limits_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a$ and $i_m^*(0) = \frac{d_m(d_h+\rho)([\Re_0]-1)}{HK\beta}$.

Obviously, we have the following assertion.

Lemma 4.1. *If* $[\Re_0] > 1$, (4.1) *has a space-independent endemic equilibrium* $E^* = (S_m^*, i_m^*(0)\Pi(a), I_h^*)$, *which is unique and defined by* (4.5).

4.1. Local dynamics

This subsection is spent on the local stability problem of $\tilde{E}_0$ and E^*. To this end, letting $E = (\bar{S}_m, \bar{i}_m(\cdot), \bar{I}_h)$ be any equilibrium of (4.1). We linearize (4.1) around E

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m = -\beta \bar{S}_m I_h(t,x) - \beta S_m(t,x)\bar{I}_h - d_m S_m(t,x), \\ \dfrac{\partial i_m(t,a,x)}{\partial t} + \dfrac{\partial i_m(t,a,x)}{\partial a} - D_m \Delta i_m(t,a,x) = -d_m i_m(t,a,x), \\ i_m(t,0,x) = \beta \bar{S}_m I_h + \beta S_m \bar{I}_h, \\ \dfrac{\partial I_h}{\partial t} - D_h \Delta I_h = H \int\limits_0^\infty \beta_1(a) i_m(t,a,x)\mathrm{d}a - (d_h + \rho) I_h, \\ \dfrac{\partial S_m}{\partial n} = \dfrac{\partial i_m(t,a,x)}{\partial n} = \dfrac{\partial I_h}{\partial n} = 0. \end{cases} \tag{4.6}$$

We proceed to determine the characteristic equation of E. Since the linear system contains Laplacian term, we introduce the related theory from [6]. Denote by $\lambda_j (j = 1, 2, \dots)$ the eigenvalues of operator $-\Delta$ on a bounded set Ω with boundary condition (1.6), that is, $\Delta\nu(x) = -\zeta_i \nu(x)$. Hence,

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

corresponding to which there is the space of eigenfunctions in $C^1(\Omega)$, denoted by $E(\lambda_i)$.

Denote by $\{\phi_{ij} |\ j = 1, 2, \dots, \dim E(\lambda_i)\}$ the orthogonal basis of $E(\lambda_i)$. Further, let $\mathbb{X}_{ij} = \{c\phi_{ij} |\ c \in \mathbb{R}^3\}$ such that

$$\tilde{\mathbb{X}} = \bigoplus_{i=0}^{\infty} \mathbb{X}_i, \text{ where } \mathbb{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbb{X}_{ij}.$$

It is well known that the parabolic problem $\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x)$ with $\frac{\partial u(t,x)}{\partial n} = 0$, admits the exponential solution $u(t,x) = e^{\eta t}\nu(x)$, where $\nu(x) \in \mathbb{X}_i$. Substituting $(S_m, i_m(t,a,x), I_h) = e^{\eta t}(\psi(x), \xi(a,x), \phi(x))$ in

(4.1) gets

$$\begin{cases} \eta\psi(x) + D_m\lambda_i\psi(x) = -\beta\bar{S}_m\phi(x) - \beta\bar{I}_h\psi(x) - d_m\psi(x), \\ \eta\xi(a,x) + \dfrac{\partial\xi(a,x)}{\partial a} + D_m\lambda_i\xi(a,x) = -d_m\xi(a,x), \\ \xi(0,x) = \beta\bar{S}_m\phi(x) + \beta\bar{I}_h\psi(x), \\ \eta\phi(x) + D_h\lambda_i\phi(x) = H\int_0^\infty \beta_1(a)\xi(a,x)\mathrm{d}a - (d_h+\rho)\phi(x). \end{cases} \tag{4.7}$$

Combined with the second and third equation of (4.7), we get that

$$\xi(a,x) = \xi(0,x)e^{-\eta a}\tilde{\Pi}(a), \text{ where } \tilde{\Pi}(a) = e^{-D_m\lambda_i a}\Pi(a).$$

We prove the following claim.

Claim. $\eta \neq -(D_m\lambda_i + d_m)$ and $\eta \neq -(D_h\lambda_i + d_h + \rho)$.

In fact, if $\eta = -(D_m\lambda_i + d_m)$, together with the first equation of (4.7), implies that $\xi(0,x) = 0$. Hence, from the fourth equation of (4.7), we directly have $\eta = -(D_h\lambda_i + d_h + \rho)$, which results in a contradiction. $\eta \neq -(D_h\lambda_i + d_h + \rho) < 0$ can be proved in a similar way.

This claim together with the first and fourth equation of (4.7) implies that

$$\psi(x) = -\frac{\xi(0,x)}{\eta + D_m\lambda_i + d_m} \text{ and } \phi(x) = \frac{\xi(0,x)f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}, \tag{4.8}$$

where $f(\eta) = H\int_0^\infty \beta_1(a)e^{-\eta a}\tilde{\Pi}(a)da$. Plugging (4.8) into the third equation of (4.7) and canceling $\xi(0,x)$, we get

$$\left(1 + \frac{1}{\eta + D_m\lambda_i + d_m}\beta\bar{I}_h\right) = \beta\bar{S}_m\left(\frac{f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}\right). \tag{4.9}$$

Hence, (4.9) indeed admits a principal eigenvalue η^* (see, for example, [17, Lemma 2.2]).

The following result indicates that both $\tilde{E}_0$ and E^* are locally asymptotically stable (LAS) under threshold conditions, which is achieved by studying the distribution of characteristic roots of (4.9).

Theorem 4.2. *Let $[\Re_0]$ be defined by (4.3).*

(i) *If $[\Re_0] < 1$, then $\tilde{E}_0$ is LAS;*
(ii) *If $[\Re_0] > 1$, then E^* is LAS.*

Proof. Let us first prove (i). For $\tilde{E}_0$, that is, $\bar{S}_m = \frac{\mu}{d_m}$ and $\bar{I}_h = 0$, (4.9) can be reduced to

$$1 = \beta\frac{\mu}{d_m}\frac{f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}. \tag{4.10}$$

Suppose, by contradiction, (4.10) admits a real root $\eta > 0$. We estimate (4.10)

$$1 = \left|\frac{\mu}{d_m}\frac{\beta f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}\right| \leq \beta\frac{\mu}{d_m(d_h+\rho)}H\int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a = [\Re_0],$$

which leads to a contradiction with $[\Re_0] < 1$. Hence, all the real roots of (4.10) are negative. On the other hand, if (4.10) admits a pair of complex roots, denote by $\eta = c \pm di$ with $c \geq 0$ and $d > 0$, we then have

$$1 = \beta\frac{\mu}{d_m}H\left(\frac{(c + D_h\lambda_i + d_h + \rho)\int_0^\infty \beta_1(a)e^{-ca}\cos(da)\tilde{\Pi}(a)da - d\int_0^\infty \beta_1(a)e^{-ca}\sin(da)\tilde{\Pi}(a)da}{(c + D_h\lambda_i + d_h + \rho)^2 + d^2}\right) \leq [\Re_0], \tag{4.11}$$

again a contradiction with $[\Re_0] < 1$. This proves (i).

We next prove (ii). For E^*, that is, $\bar{S}_m = S_m^*$ and $\bar{I}_h = I_h^*$, (4.9) can be reduced to

$$\left(1 + \frac{1}{\eta + D_m\lambda_i + d_m}\beta I_h^*\right) = \beta S_m^* \left(\frac{f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}\right). \tag{4.12}$$

With $\eta = c + di$ with $c \geq 0$, we estimate the left-hand side (4.12),

$$\left|1 + \frac{1}{\eta + D_m\lambda_i + d_m}\beta I_h^*\right| = \frac{\sqrt{(c + D_m\lambda_i + d_m + \beta I_h^*)^2 + d^2}}{\sqrt{(c + D_m\lambda_i + d_m)^2 + d^2}} > 1. \tag{4.13}$$

The right-hand side (4.12) can be estimated as

$$\beta S_m^* \left|\frac{f(\eta)}{\eta + D_h\lambda_i + d_h + \rho}\right| = \beta S_m^* \frac{|f(\eta)|}{|\eta + D_h\lambda_i + d_h + \rho|} < \beta S_m^* \frac{HK}{d_h + \rho} = 1, \tag{4.14}$$

which is a contradiction. Consequently, if $[\Re_0] > 1$, all roots of (4.12) have negative real parts. This proves (ii). □

4.2. Global dynamics

This subsection is spent on the global stability problem of $\tilde{E}_0$ and E^*. Combined with local asymptotic stability (in subsection (4.1)) and global attractivity of equilibria, we shall confirm that both $\tilde{E}_0$ and E^* are globally asymptotically stable (GAS).

Theorem 4.3. *If $[\Re_0] < 1$ and $\phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{D}$, then $\tilde{E}_0$ is GAS.*

Proof. It is well known that function

$$G(u, v) = u - v - v \ln \frac{u}{v} \geq 0, \text{ for } u, v \in \mathbb{R}_+, \tag{4.15}$$

and $G(u, u) = 0$. By using this function, we define

$$\mathbb{L}[S_m, i_m, I_h](t) = \int_\Omega [\mathbb{L}_{S_m}(t, x) + \mathbb{L}_{i_m}(t, x) + \mathbb{L}_{I_h}(t, x)]\mathrm{d}x,$$

where $\mathbb{L}_{S_m} = G(S_m(t, x), S_m^0)$, $\mathbb{L}_{i_m} = \int_0^\infty \Theta_1(a) i_m(t, a, x)\mathrm{d}a$, $\mathbb{L}_{I_h} = \frac{S_m^0\beta}{d_h+\rho} I_h(t, x)$ and $\Theta_1(a) \in L_1(\infty)$ will be determined later.

We take the derivative of $\mathbb{L}_{S_m}$,

$$\frac{\partial \mathbb{L}_{S_m}}{\partial t} = D_m \left(1 - \frac{S_m^0}{S_m}\right) \Delta S_m - d_m \frac{(S_m - S_m^0)^2}{S_m} + S_m^0 \beta I_h - \beta I_h S_m. \tag{4.16}$$

With the help of (4.2), we rewrite $\mathbb{L}_{i_m}$

$$\mathbb{L}_{i_m} = \int_0^t \Theta_1(t-a) \int_\Omega \Gamma_1(t-a, x, y)\mathcal{B}(a, y)\mathrm{d}y\Pi(t-a)\mathrm{d}a + \int_0^\infty \Theta_1(a+t) \int_\Omega \Gamma_1(t, x, y)\phi_2(a, y)\mathrm{d}y\Pi(t)\mathrm{d}a. \tag{4.17}$$

We take the derivative of $\mathbb{L}_{i_m}$

$$\frac{\partial \mathbb{L}_{i_m}}{\partial t} = \Theta_1(0) \int_\Omega \Gamma_1(0, x, y)\mathcal{B}(t, y)\mathrm{d}y + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(t-a) \int_\Omega \Gamma_1(t-a, x, y)\mathcal{B}(a, y)\mathrm{d}y\Pi(t-a)\mathrm{d}a$$

$$
\begin{aligned}
&+\int_0^t \Theta_1(t-a)\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t-a,x,y)\mathcal{B}(a,y)\mathrm{d}y\Pi(t-a)\mathrm{d}a\\
&-d_m\int_0^t \Theta_1(t-a)\int_\Omega \Gamma_1(t-a,x,y)\mathcal{B}(a,y)\mathrm{d}y\Pi(t-a)\mathrm{d}a\\
&+\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Theta_1(a+t)\int_\Omega \Gamma_1(t,x,y)\phi_2(a,y)\mathrm{d}y\Pi(t)\mathrm{d}a\\
&+\int_0^\infty \Theta_1(a+t)\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t,x,y)\phi_2(a,y)\mathrm{d}y\Pi(t)\mathrm{d}a\\
&-d_m\int_0^\infty \Theta_1(a+t)\int_\Omega \Gamma_1(t,x,y)\phi_2(a,y)\mathrm{d}y\Pi(t)\mathrm{d}a.
\end{aligned} \tag{4.18}
$$

Collecting the terms with the form of (4.17), together with the fact that $\frac{\partial\Gamma_1(t,x,y)}{\partial t} = D_m\Delta\Gamma_1(t,x,y)$ a.e. for $x\in\Omega$, we get

$$
\frac{\partial \mathbb{L}_{i_m}}{\partial t} = \Theta_1(0)\mathcal{B}(t,x) + \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}a}\Theta_1(a) - (d_m - D_m\Delta)\Psi(a)\right] i_m(t,a,x)\mathrm{d}a. \tag{4.19}
$$

We take the derivative of $\mathbb{L}_{I_h}(t,x)$

$$
\frac{\partial \mathbb{L}_{I_h}}{\partial t} = \beta\frac{S_m^0}{d_h+\rho}D_h\Delta I_h + \beta\frac{S_m^0}{d_h+\rho}H\int_0^\infty \beta_1(a)i_m(t,a,x)\mathrm{d}a - S_m^0\beta I_h. \tag{4.20}
$$

With the help of (4.16), (4.19) and (4.20), we get the derivation of $\mathbb{L}(t)$

$$
\begin{aligned}
\frac{\partial \mathbb{L}(t)}{\partial t} &= \int_\Omega D_m\left(1-\frac{S_m^0}{S_m}\right)\Delta S_m\mathrm{d}x - d_m\int_\Omega \frac{(S_m-S^0)^2}{S_m}\mathrm{d}x + \int_\Omega (\Theta_1(0)-1)\mathcal{B}(t,x)\mathrm{d}x\\
&\quad + \int_\Omega\int_0^\infty \left[\beta\frac{S_m^0}{d_h+\rho}H\beta_1(a) + \frac{\mathrm{d}}{\mathrm{d}a}\Theta_1(a) - (d_m-D_m\Delta)\Theta_1(a)\right] i_m(t,a,x)\mathrm{d}a\mathrm{d}x.
\end{aligned} \tag{4.21}
$$

Setting

$$
\Theta_1(a) = \frac{1}{\Pi(a)}\int_a^\infty \beta\frac{S_m^0}{d_h+\rho}H\beta_1(\theta)\Pi(\theta)d\theta.
$$

Then, it satisfies

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}a}\Theta_1(a) = -\beta\frac{S_m^0}{d_h+\rho}H\beta_1(a) + d_m\Theta_1(a),\\
\Theta_1(0) = [\Re_0],
\end{cases}
$$

Hence, (4.21) becomes

$$
\frac{\partial \mathbb{L}(t)}{\partial t} = -D_m\int_\Omega \frac{|\nabla S_m|^2}{S_m^2}\mathrm{d}x - \int_\Omega d_m\frac{(S_m-S^0)^2}{S_m}\mathrm{d}x + ([\Re_0]-1)\int_\Omega \mathcal{B}(t,x)\mathrm{d}x.
$$

Hence, the global attractiveness of $\tilde{E}_0$ holds in $\mathcal{D}$ when $[\Re_0] < 1$ (see, e.g., [31, Theorem 4.2]). This together with Theorem 4.2 indicates that the $\tilde{E}_0$ is GAS. This proves Theorem 4.3. □

Define

$$\mathcal{D}_0 = \left\{ (\phi_1, \phi_2, \phi_3) \in \tilde{\mathbb{X}}_+ |\ \phi_3 > 0, \text{for some } x \in \overline{\Omega} \right\}. \tag{4.22}$$

In the following, we pay attention to the persistence problem of system (4.1) when $[\Re_0] > 1$. We first give the following lemma.

Lemma 4.4. *Let $\mathcal{D}_0$ be defined by (4.22). If $[\Re_0] > 1$, then there exists $\epsilon_1 > 0$ such that*

$$\limsup_{t\to\infty} |\mathcal{B}(t,\cdot)|_X > \epsilon_1,\ \ \forall \phi \in \mathcal{D}_0.$$

Proof. Due to $[\Re_0] > 1$, we can select $\epsilon_1 > 0$ such that

$$\beta \frac{(\mu - \epsilon_1)}{d_m} H \int_0^\infty e^{-(d_h+\rho)s} ds \int_0^\infty \beta_1(a)\Pi(a)\mathrm{d}a > 1. \tag{4.23}$$

We will prove the assertion by contradiction. Assume, by contradiction, that there exists $T_1 > 0$ such that for all $t \geq T_1$ and $x \in \Omega$, we have $\mathcal{B}(t,x) \leq \epsilon_1$. By the inequality in (4.23), choosing $T_2 > T_1$ (denote by $h = T_2 - T_1$) and $\lambda > 0$ small enough that

$$\odot = \beta \frac{(\mu - \epsilon_1)}{d_m} H(1 - e^{-d_m h}) \int_0^\infty e^{-(d_h+\rho)s} e^{-\lambda s} ds \int_0^\infty \beta_1(a)\Pi(a) e^{-\lambda a} da > 1, \tag{4.24}$$

Note that S_m satisfies

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m \geq \mu - \epsilon_1 - d_m S_m, & x \in \Omega,\ t \geq T_2, \\ \dfrac{\partial S_m}{\partial n} = 0, & x \in \partial\Omega,\ t \geq T_2. \end{cases} \tag{4.25}$$

Hence, direct integrating (4.25) yields

$$S_m \geq \frac{\mu - \epsilon_1}{d_m}(1 - e^{-d_m(t-T_1)}) \geq \frac{\mu - \epsilon_1}{d_m}(1 - e^{-d_m h}),\ \forall t \geq T_2.$$

Further from (2.4),

$$\begin{aligned} I_h &\geq H \int_0^t e^{-(d_h+\rho)(t-s)} \int_\Omega \Gamma_2(t-s,x,y) \int_0^\infty \beta_1(a) i_m(s,a,y) \mathrm{d}a\mathrm{d}y\mathrm{d}s \\ &\geq H \int_0^t e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y) \int_0^{t-s} \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s. \end{aligned}$$

Combined with the above two inequalities, we have

$$\begin{aligned} &\mathcal{B}(t,x) \\ &\quad \geq \beta \frac{\mu - \epsilon_1}{d_m} H(1 - e^{-d_m h}) \int_0^t e^{-(d_h+\rho)s} \int_\Omega \Gamma_2(s,x,y) \int_0^{t-s} \beta_1(a)\Pi(a) \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{4.26}$$

Note that $\int_0^\infty e^{-\lambda t}\mathcal{B}(t,x)\mathrm{d}t < \infty$ for all $x \in \Omega$. Choose $\widetilde{x} \in \Omega$ that $\mathcal{B}(t,\widetilde{x}) = \min_{x\in\Omega}\mathcal{B}(t,x)$. By (4.26), we have

$$\int_0^\infty e^{-\lambda t}\mathcal{B}(t,\widetilde{x})\mathrm{d}t \geq \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m h})\int_0^\infty e^{-\lambda t}\int_0^t e^{-(d_h+\rho)s}\int_\Omega \Gamma_2(s,\widetilde{x},y)\int_0^{t-s}\beta_1(a)\Pi(a) \times \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}s\mathrm{d}t.$$

Interchanging the order of integration (t and s) and letting $t-s=\bar{t}$ (for simplicity, still denote $\bar{t}$ by t) yield

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\mathcal{B}(t,\widetilde{x})\mathrm{d}t &\geq \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m h})\int_0^\infty e^{-(d_h+\rho)s}\int_s^\infty e^{-\lambda t}\int_\Omega \Gamma_2(s,\widetilde{x},y)\int_0^{t-s}\beta_1(a)\Pi(a)\\ &\quad\times \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-s-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}t\mathrm{d}s\\ &= \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m T_1})\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty e^{-\lambda t}\int_\Omega \Gamma_2(s,\widetilde{x},y)\int_0^t\beta_1(a)\Pi(a)\\ &\quad\times \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Interchanging the order of integration (t and a) and letting $t-a=\bar{t}$ (for simplicity, still denote $\bar{t}$ by t) yield

$$\begin{aligned}\int_0^\infty e^{-\lambda t}B(t,\widetilde{x})\mathrm{d}t &\geq \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m h})\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)\int_\Omega \Gamma_2(s,\widetilde{x},y)\int_a^\infty e^{-\lambda t}\\ &\quad\times \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t-a,z)\mathrm{d}z\mathrm{d}t\mathrm{d}y\mathrm{d}a\mathrm{d}s\\ &= \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m h})\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)e^{-\lambda a}\int_\Omega \Gamma_2(s,\widetilde{x},y)\int_0^\infty e^{-\lambda t}\\ &\quad\times \int_\Omega \Gamma_1(a,y,z)\mathcal{B}(t,z)\mathrm{d}z\mathrm{d}t\mathrm{d}y\mathrm{d}a\mathrm{d}s.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\mathcal{B}(t,\widetilde{x})\mathrm{d}t &\geq \beta\frac{\mu-\epsilon_1}{d_m}H(1-e^{-d_m h})\int_0^\infty e^{-(d_h+\rho)s}e^{-\lambda s}\int_0^\infty \beta_1(a)\Pi(a)e^{-\lambda a}\int_\Omega \Gamma_2(s,\widetilde{x},y)\\ &\quad\times \int_\Omega \Gamma_1(a,y,z)\int_0^\infty e^{-\lambda t}\mathcal{B}(t,z)\mathrm{d}t\mathrm{d}z\mathrm{d}y\mathrm{d}a\mathrm{d}s\\ &\geq \bigodot\int_0^\infty e^{-\lambda t}\mathcal{B}(t,\widetilde{x})\mathrm{d}t,\end{aligned}$$

which results in a contradiction with (4.24). This proves Lemma 4.4. □

Lemma 4.5. *If* $[\Re_0] > 1$, *then* $\liminf_{t\to\infty} |\mathcal{B}(t,\cdot)|_{\mathbb{X}} > \epsilon_1$, $\forall \phi \in \mathcal{D}_0$.

Proof. Suppose, by contradiction, that $\liminf_{t\to\infty} |\mathcal{B}(t,\cdot)|_{\mathbb{X}} < \epsilon_1$. This together with Lemma 4.4 allows us to choose increasing sequences $\{t_{1j}\}_{j=1}^{\infty}$, $\{t_{2j}\}_{j=1}^{\infty}$, $\{t_{3j}\}_{j=1}^{\infty}$ and a decreasing sequence $\{t_{4j}\}_{j=1}^{\infty}$ such that $t_{1j} > t_{2j} > t_{3j}$, $\lim\limits_{j\to+\infty} t_{4j} = 0$ that

$$\begin{cases} |\mathcal{B}(t,\cdot)|_{\mathbb{X}} > \epsilon_1, & t = t_{3j}; \\ |\mathcal{B}(t,\cdot)|_{\mathbb{X}} = \epsilon_1, & t = t_{2j}; \\ |\mathcal{B}(t,\cdot)|_{\mathbb{X}} < t_{4j} < \epsilon_1, & t = t_{1j}; \end{cases} \tag{4.27}$$

It is easy to see that $|\mathcal{B}(t,\cdot)|_{\mathbb{X}} < \epsilon_1$, for all $t \in (t_{2j}, t_{1j})$. When $t = t_{2j}$, we denote by $\{S_{mj}\}_{j=1}^{\infty}$, $\{\mathcal{B}_j\}_{j=1}^{\infty}$ and $\{I_{hj}\}_{j=1}^{\infty}$ be functional sequences in $\mathbb{X}$, which satisfies $S_{mj} = S_m(t_{2j},\cdot) \in \mathbb{X}$, $\mathcal{B}_j = \mathcal{B}(t_{2j},\cdot) \in \mathbb{X}$ and $I_{hj} = I_h(t_{2j},\cdot) \in \mathbb{X}$, respectively. From the expressions in (2.2)-(2.4), we can conclude that there exist $(\breve{S}_m, \breve{\mathcal{B}}, \breve{I}_h) \in \mathbb{X}^+ \times \mathbb{Y}^+ \times \mathbb{X}^+$ such that

$$\lim_{j\to+\infty} S_{mj} = \breve{S}_m, \ \lim_{j\to+\infty} \mathcal{B}_j = \breve{\mathcal{B}}, \ \lim_{j\to+\infty} I_{hj} = \breve{I}_h.$$

In this setting, for all $a \geq 0$ and $x \in \Omega$, denote by $(S_m^\diamond, i_m^\diamond(t,a,x), I_h^\diamond)$ the solution of (4.1) with

$$\phi_1(x) = \breve{S}_m(x), \ \phi_2(a,x) = \int\limits_{\Omega} \Gamma_2(a,x,y)\breve{\mathcal{B}}(y)\mathrm{d}y\Pi(a), \ \phi_3(x) = \breve{I}_h(x).$$

Here, the choice of ϕ_2 is based on (1.3). According to Lemma 4.4, there exists $\tau^* > 0$ and $\ell > 0$ such that

$$|\mathcal{B}^\diamond(\tau^*,\cdot)|_{\mathbb{X}} > \epsilon_1, \text{ and } |\mathcal{B}^\diamond(t,\cdot)|_{\mathbb{X}} > \ell, \ \forall t \in (0,\tau^*).$$

Denote by $\mathcal{B}_j^\diamond(t,\cdot) = \mathcal{B}(t+\theta_j,\cdot)$. The above inequalities imply that for sufficiently large j that

$$\begin{cases} |\mathcal{B}_j^\diamond(t,\cdot)|_{\mathbb{X}} > \epsilon_1, & t = \tau^*; \\ |\mathcal{B}_j^\diamond(t,\cdot)|_{\mathbb{X}} > \ell > e_j, & t \in (0,\tau^*). \end{cases} \tag{4.28}$$

Corresponding to this, however, by letting $\widetilde{t}_j := t_{1j} - t_{2j}$, (4.27) indicates that

$$\begin{cases} |\mathcal{B}_j^\diamond(t,\cdot)|_{\mathbb{X}} < e_j < \epsilon_1, & t = \widetilde{t}_j; \\ |\mathcal{B}_j^\diamond(t,\cdot)|_{\mathbb{X}} < \epsilon_1, & t \in (0,\widetilde{t}_j). \end{cases} \tag{4.29}$$

For the cases where $\tau^* < \widetilde{t}_j$ and $\tau^* > \widetilde{t}_j$, it is easily seen that (4.28) and (4.29) contradict each other. This proves Lemma 4.5. □

Combined with Lemmas 4.4 and 4.5, we directly have:

Theorem 4.6. *If* $[\Re_0] > 1$, *system (4.1) is uniformly strongly persistent, that is, there exists* $\epsilon_0 > 0$ *such that for* $x \in \Omega$ *and* $a \geq 0$,

$$\liminf_{t\to\infty, x\in\Omega} U > \epsilon_0, \tag{4.30}$$

where $U = S_m, i_m(t,a,x), I_h$, *respectively.*

Proof. For all $a \geq 0, x \in \Omega$, Lemma 4.5 implies that there exist $\epsilon_1 > 0$ and $t_2 > 0$ such that $i_m(t,a,x) > \epsilon_1\Pi(a) := \epsilon_{i_m}$ for all $t > t_2$. By using this estimate, the I_h equation of (4.1) satisfies

$$\begin{cases} \dfrac{\partial I_h}{\partial t} > D_h \Delta I_h + \epsilon_1 HK - (d_h + \rho) I_h, & x \in \Omega, \ t \geq t_2, \\ \dfrac{\partial I_h}{\partial n} = 0, & x \in \partial\Omega, \ t \geq t_2. \end{cases}$$

By a comparison principle, we have

$$\liminf_{t\to\infty, x\in\Omega} I_h(t,x) > \frac{\epsilon_1 HK}{d_h+\rho} := \epsilon_{I_h}.$$

where $\frac{\epsilon_2 HK}{d_h+\rho}$ is the unique positive steady state of the following system

$$\begin{cases} \dfrac{\partial I_h}{\partial t} = D_h \Delta I_h + \epsilon_1 HK - (d_h+\rho) I_h, & x\in\Omega,\ t\ge t_2, \\ \dfrac{\partial I_h}{\partial n} = 0, & x\in\partial\Omega,\ t\ge t_2. \end{cases}$$

In view of S_m equation of (4.1) and (2.9), there exists $t_3>0$ such that

$$\begin{cases} \dfrac{\partial S_m}{\partial t} \ge\ D_m\Delta S_m + \mu - (\beta M_{I_h} + d_m) S_m & x\in\Omega,\ t\ge t_3, \\ \dfrac{\partial S_m}{\partial n} = 0, & x\in\partial\Omega,\ t\ge t_3. \end{cases}$$

Again, from comparison principle, we have

$$\liminf_{t\to\infty, x\in\Omega} S_m(t,x) > \frac{\mu}{\beta M_{I_h} + d_m} := \epsilon_{S_m}.$$

Consequently, the theorem is proved by letting $\epsilon_0 = \min\{\epsilon_{S_m}, \epsilon_{i_m}, \epsilon_{I_h}\}$. □

Theorem 4.7. *If $[\Re_0] > 1$, then E^* is GAS for $\phi\in\mathcal{D}_0$.*

Proof. Let us define

$$\mathbb{L}[S_m, i_m, I_h](t) = \int_\Omega [\mathbb{L}_{S_m}(t,x) + \mathbb{L}_{i_m}(t,x) + \mathbb{L}_{I_h}(t,x)] \mathrm{d}x,$$

where

$$\begin{cases} \mathbb{L}_{S_m} = G[S_m, S_m^*], \\ \mathbb{L}_{i_m} = \int_0^\infty \Psi_1(a) G[i_m(t,a,x), i_m^*(a)] \mathrm{d}a, \\ \mathbb{L}_{I_h} = \frac{S_m^*\beta}{d_h+\rho} G[I_h, I_h^*)], \end{cases}$$

and $G(u,v) = u - v - v\ln\frac{u}{v}$, for $u,v>0$, and $\Psi_1(a)$ will be determined later. We next wish to calculate the derivative of $\mathbb{L}_{S_m}$, $\mathbb{L}_{i_m}$, and $\mathbb{L}_{I_h}$, respectively.

We derivative $\mathbb{L}_{S_m}$ along the solution of (4.1)

$$\begin{aligned} \frac{\partial \mathbb{L}_{S_m}}{\partial t} &= \left(1 - \frac{S_m^*}{S_m}\right)(D_m\Delta S_m + \mu - d_m S_m - i_m(t,0,x)) \\ &= \left(1 - \frac{S_m^*}{S_m}\right) D_m \Delta S_m - \frac{d_m}{S_m}(S_m - S_m^*)^2 + i_m^*(0) - i_m(t,0,x) \\ &\quad - i_m^*(0)\frac{S_m^*}{S_m} + i_m(t,0,x)\frac{S_m^*}{S_m}. \end{aligned} \tag{4.31}$$

Here, the equality, $\mu = d_m S_m^* + i_m^*(0)$, is used. Using (4.2), rewriting $\mathbb{L}_{i_m}$ as

$$\begin{aligned} \mathbb{L}_{i_m} &= \int_0^t \Psi_1(t-r) G\left[\int_\Omega \Gamma_1(t-r,x,y) i_m(r,0,y)\mathrm{d}y \Pi(t-r), i_m^*(t-r)\right]\mathrm{d}r \\ &\quad + \int_0^\infty \Psi_1(t+r) G\left[\int_\Omega \Gamma_1(t,x,y)\phi_2(r,y)\mathrm{d}y\Pi(t), i_m^*(t+r)\right]\mathrm{d}r. \end{aligned}$$

For convenience, let

$$u_1 = \int_\Omega \Gamma_1(t-r,x,y)i_m(r,0,y)\mathrm{d}y\Pi(t-r),\ v_1 = i_m^*(t-r)$$

and

$$u_2 = \int_\Omega \Gamma_1(t,x,y)\phi_2(r,y)\mathrm{d}y\Pi(t),\ v_2 = i_m^*(t+r).$$

It follows from the relations $i^*(a) = i^*(0)e^{-d_m a}$ and $\Pi(0) = 1$ that

$$\begin{aligned}\frac{\partial \overline{\mathbb{L}}_{i_m}}{\partial t} &= \Psi_1(0)G\left[\int_\Omega \Gamma_1(0,x,y)i_m(t,0,y)\mathrm{d}y, i_m^*(0)\right]\\ &+ \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\Psi_1(t-r)G\left[u_1,v_1\right]\mathrm{d}r + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Psi_1(t+r)G\left[u_2,v_2\right]dr\\ &+ \int_0^t \Psi_1(t-r)\bigg\{\left[\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t-r,x,y)i_m(r,0,y)\mathrm{d}y\Pi(t-r) - d_m u_1\right]\frac{\partial}{\partial u_1}G\left[u_1,v_1\right]\\ &- d_m i_m^*(t-r)\frac{\partial}{\partial v_1}G\left[u_1,v_1\right]\bigg\}dr\\ &+ \int_0^\infty \Psi_1(t+r)\bigg\{\left[\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t,x,y)\phi(r,y)\mathrm{d}y\Pi(t) - d_m u_2\right]\frac{\partial}{\partial u_2}G\left[u_2,v_2\right]\\ &- d_m i_m^*(t+r)\frac{\partial}{\partial v_2}G\left[u_2,v_2\right]\bigg\}dr.\end{aligned}$$

With the equality that $u\frac{\partial}{\partial u}G[u,v] + v\frac{\partial}{\partial v}G[u,v] = G[u,v]$, we have

$$\begin{aligned}\frac{\partial \overline{\mathbb{L}}_{i_m}}{\partial t} &= \Psi_1(0)G\left[\int_\Omega \Gamma_1(0,x,y)i_m(t,0,y)\mathrm{d}y, i_m^*(0)\right]\\ &+ \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}a}\Psi_1(a) - d_m\Psi(a)\right]G[i_m(t,a,x), i_m^*(a)]\mathrm{d}a\\ &+ \int_0^t \Psi_1(t-r)\left[\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t-r,x,y)i_m(r,0,y)\mathrm{d}y\Pi(t-r)\right]\frac{\partial}{\partial u_1}G\left[u_1,v_1\right]dr\\ &+ \int_0^\infty \Psi_1(t+r)\left[\int_\Omega \frac{\partial}{\partial t}\Gamma_1(t,x,y)\phi(r,y)\mathrm{d}y\Pi(t)\right]\frac{\partial}{\partial u_2}G\left[u_2,v_2\right]dr.\end{aligned}$$

Note that $\frac{\partial}{\partial t}\Gamma_1 = D_m\Delta\Gamma_1$, $\frac{\partial}{\partial u}G[u,v] = 1 - \frac{v}{u}$ and semigroup $(T(0)[\phi])(x) = \int_\Omega \Gamma_1(0,x,y)\phi(y)\mathrm{d}y$ is an unit semigroup. It follows that

$$\frac{\partial \overline{\mathbb{L}}_{i_m}}{\partial t} = \Psi_1(0)G\left[i_m(t,0,x), i_m^*(0)\right] + \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}a}\Psi_1(a) - d_m\Psi(a)\right]G[i_m(t,a,x), i_m^*(a)]\mathrm{d}a$$

$$+\int_0^\infty \Psi_1(a) D_m \Delta i_m(t,a,x)\left[1-\frac{i_m^*(a)}{i_m(t,a,x)}\right]\mathrm{d}a. \tag{4.32}$$

Now we let

$$\Psi_1(a)=\frac{HS_m^*\beta}{d_h+\rho}\frac{1}{\Pi(a)}\int_a^\infty \beta_1(\theta)e^{-d_m\theta}d\theta. \tag{4.33}$$

Then, it satisfies

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}a}\Psi_1(a)=-\frac{HS_m^*\beta}{d_h+\rho}\beta_1(a)+d_m\Psi_1(a),\\ \Psi_1(0)=\beta\frac{HS_m^*}{d_h+\rho}K=S_m^*\frac{[\Re_0]}{S_m^0}=1.\end{cases}$$

Hence, (4.32) becomes

$$\frac{\partial\overline{\mathbb{L}}_{i_m}}{\partial t}=\frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a)\left[i_m^*(a)-i_m(t,a,x)+i_m^*(a)\ln\frac{i_m(t,a,x)}{i_m^*(a)}\right]\mathrm{d}a \tag{4.34}$$

$$+G[i_m(t,0,x),i_m^*(0)]+\int_0^\infty \Psi_1(a)\left[1-\frac{i_m^*(a)}{i_m(t,a,x)}\right]D_m\Delta i_m(t,a,x)\mathrm{d}a.$$

Taking derivative of $\overline{\mathbb{L}}_{I_h}$ yields

$$\frac{\partial\overline{\mathbb{L}}_{I_h}}{\partial t}=\left[1-\frac{I_h^*}{I_h}\right]\left[\frac{S_m^*\beta}{d_h+\rho}D_h\Delta I_h+\frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a)i_m(t,a,x)\mathrm{d}a-S_m^*\beta I_h\right]. \tag{4.35}$$

For simplicity, we let

$$\widetilde{\mathbb{L}}(t,x)=\overline{\mathbb{L}}_{S_m}(t,x)+\overline{\mathbb{L}}_{i_m}(t,x)+\overline{\mathbb{L}}_{I_h}(t,x).$$

With the help of (4.31), (4.34) and (4.35), we then have

$$\frac{\partial\widetilde{\mathbb{L}}}{\partial t}=\mathbb{W}+i_m^*(0)-i_m(t,0,x)-i_m^*(0)\frac{S_m^*}{S_m}+i_m(t,0,x)\frac{S_m^*}{S_m}$$
$$+G[i_m(t,0,x),i_m^*(0)]+\frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a)\left[i_m^*(a)-i_m(t,a,x)+i_m^*(a)\ln\frac{i_m(t,a,x)}{i_m^*(a)}\right]\mathrm{d}a$$
$$+\left[1-\frac{I_h^*}{I_h}\right]\left[\frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a)i_m(t,a,x)\mathrm{d}a-S_m^*\beta I_h\right],$$

where

$$\mathbb{W}=\left[1-\frac{S_m^*}{S_m}\right]D_m\Delta S_m-\frac{d_m}{S_m}(S_m-S_m^*)^2$$
$$+\int_0^\infty \Psi_1(a)\left[1-\frac{i_m^*(a)}{i_m(t,a,x)}\right]D_m\Delta i_m(t,a,x)\mathrm{d}a+\left[1-\frac{I_h^*}{I_h}\right]\frac{S_m^*\beta}{d_h+\rho}D_h\Delta I_h.$$

Collecting terms with $i_m^*(0)$, $i_m(t,0,x)$ and $\frac{HS_m^*\beta}{d_h+\rho}$, respectively,

$$\frac{\partial\widetilde{\mathbb{L}}}{\partial t}=\mathbb{W}+i_m^*(0)\left[1-\frac{S_m^*}{S_m}\right]-i_m(t,0,x)\left[1-\frac{S_m^*}{S_m}\right]$$

$$
\begin{aligned}
&+ G[i_m(t,0,x), i_m^*(0)] + \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a)\left[i_m^*(a) - i_m(t,a,x) + i_m^*(a)\ln\frac{i_m(t,a,x)}{i_m^*(a)}\right]\mathrm{d}a \\
&+ \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \left[1-\frac{I_h^*}{I_h}\right]\beta_1(a) i_m(t,a,x)\mathrm{d}a - i_m(t,0,x)\frac{S_m^*}{S_m} + i_m^*(0).
\end{aligned}
$$

Here, we have used the fact that $\left[1-\frac{I_h^*}{I_h}\right]S_m^*\beta I_h = i_m(t,0,x)\frac{S_m^*}{S_m} - i_m^*(0)$. Canceling the zero terms, together with $i_m^*(0) = \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a) i_m^*(a)\mathrm{d}a$, yields

$$
\begin{aligned}
\frac{\partial \overline{\mathbb{L}}}{\partial t} &= \mathbb{W} + i_m^*(0)\left[1-\frac{S_m^*}{S_m} - \ln\frac{i_m(t,0,x)}{i_m^*(0)}\right] \\
&\quad + \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a) i_m^*(a)\left[1+\ln\frac{i_m(t,a,x)}{i_m^*(a)} - \frac{I_h^* i_m(t,a,x)}{I_h i_m^*(a)}\right]\mathrm{d}a \\
&= \mathbb{W} + \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a) i_m^*(a)\left[2+\ln\frac{i_m(t,a,x)}{i_m^*(a)} - \frac{I_h^* i_m(t,a,x)}{I_h i_m^*(a)} - \frac{S_m^*}{S_m} - \ln\frac{i_m(t,0,x)}{i_m^*(0)}\right]\mathrm{d}a.
\end{aligned}
$$

By a zero trick that $1 - \frac{S_m I_h i_m^*(0)}{S_m^* I_h^* i_m(t,0,x)} = 0$, we have

$$
\frac{\partial \overline{\mathbb{L}}}{\partial t} = \mathbb{W} - \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a) i_m^*(a)\left[g\left(\frac{S_m^*}{S_m}\right) + g\left(\frac{I_h^* i_m(t,a,x)}{I_h i_m^*(a)}\right) + g\left(\frac{S_m I_h i_m^*(0)}{S_m^* I_h^* i_m(t,0,x)}\right)\right]\mathrm{d}a,
$$

where $g(\alpha) = \alpha - 1 - \ln\alpha, \alpha\in\mathbb{R}_+$ possesses the properties that $g(\alpha)\geq 0$ when $\alpha \geq 1$ and $g(1)=0$. Hence,

$$
\begin{aligned}
\frac{\partial \overline{\mathbb{L}}}{\partial t} &= -D_m S_m^*\int_\Omega \frac{|\nabla S_m|^2}{S_m^2}\mathrm{d}x - \int_0^\infty D_m\Psi_1(a) i_m^*(a)\int_\Omega \frac{|\nabla i_m(t,a,x)|^2}{i_m^2(t,a,x)}\mathrm{d}x\mathrm{d}a \\
&\quad - \frac{D_h S_m^*\beta}{d_h+\rho}\int_\Omega \frac{|\nabla I_h|^2}{I_h^2}\mathrm{d}x - \int_\Omega \frac{d_m}{S_m}(S_m - S_m^*)^2\mathrm{d}x \\
&\quad - \int_\Omega \frac{HS_m^*\beta}{d_h+\rho}\int_0^\infty \beta_1(a) i_m^*(a)\left[g\left(\frac{S_m^*}{S_m}\right) + g\left(\frac{I_h^* i_m(t,a,x)}{I_h i_m^*(a)}\right) + g\left(\frac{S_m I_h i_m^*(0)}{S_m^* I_h^* i_m(t,0,x)}\right)\right]\mathrm{d}a\mathrm{d}x \\
&\leq 0.
\end{aligned}
$$

Hence, with the help of the property of g, $\frac{\partial\overline{\mathbb{L}}(t,x)}{\partial t} = 0$ if and only if $S_m = S_m^*$, $i_m(t,a,x) = i_m^*(a)$ and $I_h = I_h^*$. According to the invariance principle (see, e.g., [31, Theorem 4.2]), the global attractivity of E^* directly follows. This together with the local stability of E^*, as stated in Theorem 4.2, implies that E^* is GAS for all $a\geq 0, x\in\Omega$. This proves Theorem. □

5. Conclusion and discussion

This paper provides the complete analysis on the threshold dynamics of an age-space structured malaria epidemic model. Unlike in [21] where the spatial movement of mosquitoes in EIP will result in non-local infection, here we intend to incorporate an infection age to be a continuous variable and the fixed incubation period is ignored. We conducted a complete analysis of model (1.4) by adopting the density of susceptible population to be $H(x)$ and the mass action incidence, which can be regarded as a continuous work of [21]. In Lemma 2.2, we confirmed that (1.4) admits a unique local solution through investigating the fixed point problem, which is defined in (2.6). It is also proved that the local solution of (1.4) is positive for initial conditions (see Lemma 2.3). The positivity of $i_m(t,a,x)$ is implied by the positivity of $\mathcal{B}(t,x)$, which is proved by the methods of Picard sequences and iteration. The proof of which is indeed not trivial, as $\mathcal{B}(t,x)$ involves the product of S_m and I_h, as demonstrated in (2.6). In addition, we have extended the interval of existence of the local solution to $[0,+\infty)$. To achieve this, we verified that in a finite time interval, the solution does not blow up (see Lemma 2.4). Consequently, the solution semiflow generated by the solution of (1.4) possesses a global attractor in $\tilde{\mathbb{X}}_+$.

By introducing the renewal equation and Laplace transformation, we identified that the next-generation operator $\mathcal{L}$ defined in (3.6) is strictly positive and compact (see Lemma 3.1), which allow us to define $\Re_0$. It is important to mention that it is difficult to obtain spectral radius of $\mathcal{L}$, if not impossible, so that we cannot get further information on dynamical properties of (1.4). To proceed further, we considered the special case where parameters are all independent of x. In such a setting, the Krein–Rutman theorem ensures that $[\Re_0]$ can be explicitly obtained if the positive eigenvector (corresponding to $[\Re_0]$) is a constant. It is also identified that if $[\Re_0]<1$, then space-independent disease-free equilibrium $\tilde{E}_0=(S_m^0,0,0)$, where $S_m^0=\frac{\mu}{d_m}$, is GAS (see Theorem 4.3); if $[\Re_0]>1$, then the model is uniformly persistent (see Theorem 4.6) and the space-independent endemic equilibrium $E^*=(S_m^*,i_m^*(0)\Pi(a),I_h^*)$ is GAS (see Theorem 4.7). The local stability of E_0 and E^* is achieved by studying the distribution of characteristic roots of characteristic equation (4.9) (see Theorem 4.2). The strong persistence result is implied by the weak persistence (see Lemmas 4.4 and 4.5), which is achieved by making the Laplace transformation and the way of contraction. Both $\tilde{E}_0$ and E^* are proved to be GAS by Lyapunov functions in different cases (see Theorem 4.3 and 4.7).

If the average incubation period is fixed as τ, and

$$\beta_1(a)=\begin{cases}0, & 0\le a<\tau;\\ \beta^*, & A\ge\tau,\end{cases}$$

as standard arguments in [13,21], our model (1.4) can be reformulated into, for $x\in\Omega,\ t>0$,

$$\begin{cases}\dfrac{\partial S_m}{\partial t}-D_m\Delta S_m=\mu(x)-\beta_2(x)S_mI_h-d_mS_m,\\ \dfrac{\partial I_m}{\partial t}-D_m\Delta I_m=e^{-d_m\tau}\displaystyle\int_\Omega\Gamma_2(D_m\tau,x,y)\beta_2(y)S_m(t-\tau,y)I_h(t-\tau,y)\mathrm{d}y-d_mI_m,\\ \dfrac{\partial I_h}{\partial t}-D_h\Delta I_h=H(x)\beta^*I-(d_h+\rho)I_h,\end{cases}\tag{5.1}$$

with

$$\frac{\partial S_m}{\partial n}=\frac{\partial I_m}{\partial n}=\frac{\partial I_h}{\partial n}=0,\ x\in\partial\Omega,\ t>0,$$

where $I_m=\int_\tau^\infty i_m(t,a,x)\mathrm{d}a$. As studied in [21, Section 4], with additional conditions, the global attractivity of endemic equilibrium in homogeneous case can be proved by using a fluctuation method.

If we further take seasonality into account, by using the standard arguments as those in [4,36], our model (1.4) can be reformulated by following system, for $x \in \Omega,\ t > 0$,

$$\begin{cases} \dfrac{\partial S_m}{\partial t} - D_m \Delta S_m = \mu(t,x) - \beta_2(t,x) S_m I_h - d_m(t,x) S_m, \\ \dfrac{\partial I_m}{\partial t} - D_m \Delta I_m = e^{-d_m \tau} \int\limits_{\Omega} \Gamma_2(D_m \tau, x, y) \beta_2(t,y) S_m(t-\tau, y) I_h(t-\tau, y) \mathrm{d}y - d_m(t,x) I_m, \\ \dfrac{\partial I_h}{\partial t} - D_h \Delta I_h = H(x) \beta^* I - (d_h + \rho) I_h, \end{cases} \tag{5.2}$$

with

$$\frac{\partial S_m}{\partial n} = \frac{\partial I_m}{\partial n} = \frac{\partial I_h}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0.$$

Here, $\mu(t,x)$, $\beta_2(t,x)$ and $d_m(t,x)$ are the recruitment rate, biting rate and mortality rate involving the seasonality, respectively. The dynamics of (5.1) and (5.2) will be left for future investigation.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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