Z. Angew. Math. Phys. (2020) 71:45

0044-2275/20/020001-19
published online February 18, 2020
https://doi.org/10.1007/s00033-020-1269-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

Asymptotic profiles and convergence rate of the compressible fluid models of Korteweg type

Yinxia Wang and Yu-Zhu Wang

Abstract. In this paper, we consider the initial value problem for the compressible fluid models of Korteweg type in $\mathbb{R}^n (n \geq 3)$ and asymptotic profile of global solutions and the corresponding convergence rate are established. The structure of the nonlinear term plays a very important role in constructing asymptotic profile. The proof is based on the decay estimate of solutions operator, decay estimate and weighted decay estimate of global solutions.

Mathematics Subject Classification. 35B40, 35Q35.

Keywords. Compressible fluid models of Korteweg type, Asymptotic profile, Convergence rate.

1. Introduction

This paper concerns the initial value problem for the compressible fluid models of Korteweg type

$$\begin{cases} \partial_t \rho + \nabla \cdot m = 0, \\ \partial_t m + \nabla \cdot (\frac{m \otimes m}{\rho}) + \nabla P(\rho) - \mu \Delta \frac{m}{\rho} - (\mu + \lambda) \nabla \nabla \cdot (\frac{m}{\rho}) = \alpha \rho \nabla \Delta \rho \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0 : \rho = \rho_0(x), \quad m = m_0(x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

The variables are the density ρ and the momentum m. Furthermore, $p = p(\rho)$ is the pressure function satisfying $P'(\rho) > 0$ for $\rho > 0$. The viscosity coefficients satisfy $\mu > 0, 2\mu + n\lambda > 0$, $\alpha > 0$.

This compressible fluid model of Korteweg type describes the dynamics of a liquid–vapor mixture in the setting of the diffuse interface approach: between the two phases lies a thin region of continuous transition and the phase changes are described through the variations of the density, for example a Van der Waals pressure. The compressible fluid model of Korteweg type was derived rigorously by Dunn and Serrin [4] (see also [2,5]).

Let $u = \frac{m}{\rho}$, then (1.1) may be rewritten as

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\mu + \lambda) \nabla (\nabla \cdot u) = \alpha \rho \nabla \Delta \rho. \end{cases} \tag{1.3}$$

There are numerous works dedicated to the study of the compressible fluid models of Korteweg type, and lots of important results were established. For well-posedness results, we refer to [1,3,6,7,9,15]. Global existence of classical solutions in Sobolev space was established by [9]. Danchin and Desjardins [3] proved that global well-posedness in the critical Besov spaces for the initial data is close enough to stable equilibria. Moreover, local existence of solutions for initial densities bounded away from zero was also established. Bresch et al. [1] and Haspot [6] proved the global existence of weak solutions for the compressible Navier–Stokes–Korteweg system, respectively. Global strong solutions to the compressible Navier–Stokes–Korteweg system in two space dimensions have been proved in [7].

For decay estimate results of global solutions, we may refer to [12,20–22,27,28]. Wang and Tan [27] established the L^2 and $L^p(p \geq 2)$ decay rates for the classical solutions by the detailed study of the linear decay estimates and nonlinear energy estimates. For the global existence and decay estimate of strong solutions, please refer to [20]. Tan and Zhang [22] obtained the faster decay estimate by assuming suitable condition on the initial value. For more decay estimates results, we may refer to [12,21,28]. The optimal decay estimates of global mild solutions in the critical nonhomogeneous Besov spaces to the problem (1.1), (1.2) were established in [25], provided that the initial perturbations of density and velocity are small in the space $B^{\frac{n}{2}}_{2,1} \bigcap \dot{B}^0_{1,\infty}$ and $B^{\frac{n}{2}-1}_{2,1} \bigcap \dot{B}^0_{1,\infty}$.

Our main purpose of this paper is to investigate asymptotic profile of global solutions obtained by Hattori and Li [9] to the problem (1.1), (1.2) and the corresponding convergence rate in the sprit of [14]. The nonlinear term plays a very important role in asymptotic profile of global solutions obtained in this paper. For the details, we refer to the following Theorem 1.3. The proof is based mainly on the decay estimate of solutions operator in the low-frequency region, the high-frequency region, the structure of the nonlinear term and global solutions obtained in [9] and decay estimate of global solutions established by Wang and Tan [27].

To state our asymptotic profile and the corresponding convergence rate results, we firstly state global solution and decay estimate results established by Hattori and Li [9] and Wang and Tan [27], respectively, as follows:

Theorem 1.1. ([9]) *Let $n \geq 3$ and $s = [\frac{n}{2}] + 1$ be an integer. Assume that $\rho_0 - 1 \in H^{s+1}$, $m_0 \in H^s$. Put*

$$E_0 = \|\rho_0 - 1\|_{H^{s+1}} + \|m_0\|_{H^s}.$$

Then there is a positive constant δ_0 such that if $E_0 \leq \delta_0$, then the problem (1.3), (1.2) has a unique global solution (ρ, u) satisfying

$$\|(\rho - 1)(t)\|^2_{H^{s+1}} + \|u(t)\|^2_{H^{s+1}} + \int_0^t (\|\partial_x \rho(\tau)\|^2_{H^{s+1}} + \|\partial_x u(\tau)\|^2_{H^s}) d\tau \leq CE_0. \tag{1.4}$$

Theorem 1.2. ([27]) *Let $n \geq 3$ and $s = [\frac{n}{2}] + 2$ be an integer. Assume that $\rho_0 - 1 \in H^{s+1}$, $m_0 \in H^s$ and L^1 norm of $(\rho_0 - 1, m)$ is finite. Let (ρ, u) be the global solution to the problem (1.3), (1.2) obtained in Theorem 1.1. Then*

$$\begin{aligned} \|(\rho - 1, u)(t)\|_{L^p} \leq C(&\|\rho_0 - 1\|_{H^{s+1} \bigcap L^1} \\ &+ \|m_0\|_{H^s \bigcap L^1})(1+t)^{-\frac{n}{2}(1-\frac{1}{p})}, \quad \forall 2 \leq p \leq \infty. \end{aligned} \tag{1.5}$$

$$\begin{aligned} \|\partial_x(\rho - 1, u)(t)\|_{L^p} \leq C(&\|\rho_0 - 1\|_{H^{s+1} \bigcap L^1} \\ &+ \|m_0\|_{H^s \bigcap L^1})(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad \forall 2 \leq p \leq 6. \end{aligned} \tag{1.6}$$

and

$$\|\partial_x(\rho - 1)(t)\|_{H^s} + \|\partial_x u(t)\|_{H^{s-1}} \leq C(\|\rho_0 - 1\|_{H^{s+1} \bigcap L^1} + \|m_0\|_{H^s \bigcap L^1})(1+t)^{-\frac{n}{4}-\frac{1}{2}}. \tag{1.7}$$

Let $U = (\rho - 1, m)^\tau = (\sigma, m)^\tau$. We state our asymptotic profile of global solutions obtained by Hattori and Li [9] and the corresponding convergence rate as follows:

Theorem 1.3. *Let $n \geq 3$, $s = [n/2] + 2$ and $\kappa = \frac{1}{2}$ when $n = 3$ and $\kappa = 1$ when $n \geq 4$. Assume that $\rho_0 - 1 \in H^{s+1} \cap L^1_\kappa \bigcap L^2_\kappa$, $m_0 \in H^s \cap L^1_\kappa \bigcap L^2_\kappa$ and $\partial_x \rho_0 \in L^2_\kappa$. Put*

$$E_1 = \|\rho_0 - 1\|_{H^{s+1} \bigcap L^1} + \|m_0\|_{H^s \bigcap L^1}.$$

Let (ρ, m) be the global solution to the problem (1.1), (1.2) obtained in Theorem 1.1. If E_1 is suitable small, then it holds that

$$\Big\| U(t) - G(t) * U_0 - \sum_{i=1}^{n} \partial_{x_i} G_l(t) \int_0^\infty \int_{\mathbb{R}^n} \mathbb{F}_i(U) dy d\tau \Big\|_{L^2} \leq C \begin{cases} (1+t)^{-\frac{n}{4}-\frac{1}{2}}, \ n = 3, 4 \\ (1+t)^{-\frac{n}{4}-1}, \ n \geq 5 \end{cases} \tag{1.8}$$

where

$$\mathbb{F}_i(U) = \begin{pmatrix} 0 \\ \overline{F}_i(U) \end{pmatrix} \tag{1.9}$$

and $\overline{F}_i(U)$ is the i-th column vector of the matrix $\overline{F}_{ij}(U)$ defined by

$$\overline{F}_{ij}(U) = -\frac{m_i m_j}{\sigma + 1} - [P(1+\sigma) - P'(1)\sigma]\delta_{ij} - \frac{\alpha}{2}|\nabla\sigma|^2\delta_{ij} - \alpha \partial_{x_i}\sigma \partial_{x_j}\sigma \tag{1.10}$$

and $G_l(t)$ is defined by (2.8).

Remark 1.4. The result in Theorem 1.3 implies that the solutions to the problem (1.1), (1.2) are asymptotic to a new asymptotic profile, which is given by the nonlinear term in $\overline{F}_{ij}(U)$. In fact, the corresponding decay rate of the other nonlinear term in F_{ij} (see (2.2)) is much faster.

The paper is organized as follows. We make the detail analysis for solution operators to (1.1), (1.2) in Sect. 2. In Sect. 3, weighted decay estimate of global solutions to the problem (1.3), (1.2) is established. Section 4 is devoted to derive the asymptotic profile of global solutions to the problem (1.1), (1.2) and the corresponding convergence rate.

Notations Let $\mathcal{F}[f]$ denote the Fourier transform of f defined by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x) dx.$$

We denote its inverse transform by $\mathcal{F}^{-1}$. $L^p = L^p(\mathbb{R}^n)(1 \leq p \leq \infty)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. $L^p_\kappa = L^p_\kappa(\mathbb{R}^n)(1 \leq p < \infty)$ denotes the weighted Lebesgue space with the norm

$$\|f\|_{L^p_\kappa} = \left(\int_{\mathbb{R}^n} |(1+|x|^\kappa) f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

The usual Sobolev space of order s is defined by $H^s = (I - \partial_x^2)^{-\frac{s}{2}} L^2$ with the norm $\|f\|_{H^s} = \|(I - \partial_x^2)^{\frac{s}{2}} f\|_{L^2}$.

For a nonnegative integer k, ∂_x^k denotes the totality of all the k-th order derivatives with respect to $x \in \mathbb{R}^n$. ∂_t^l denotes the totality of all the l-th order derivatives with respect to $t \in \mathbb{R}_+$. Also, for an interval I and a Banach space X, $C^k(I; X)$ denotes the space of k-times continuously differential functions on I with values in X.

2. Decay properties of solution operators

This section is devoted to derive the solution operators to the compressible fluid models of Korteweg type (1.1), (1.2). To do so, let $\sigma = \rho - 1$ and $P'(1) = 1$. (1.1) may be rewritten as

$$\begin{cases} \partial_t \sigma + \nabla \cdot m = 0, \\ \partial_t m + \nabla\sigma - \mu\Delta m - (\mu + \lambda)\nabla(\nabla \cdot m) - \alpha\nabla\Delta\sigma = \nabla \cdot F, \end{cases} \tag{2.1}$$

where F is the $n \times n$ matrix F_{ij} defined by

$$\begin{aligned} F_{ij} = & -\frac{m_i m_j}{\sigma+1} - \delta_{ij}\mu\nabla\cdot\left(\frac{\sigma m}{\sigma+1}\right) - (\mu+\lambda)\partial_{x_j}\left(\frac{\sigma m_i}{\sigma+1}\right) - \delta_{ij}[P(\sigma+1) - P'(1)\sigma] \\ & +\frac{\alpha}{2}(\Delta\sigma^2 - |\nabla\sigma|^2)\delta_{ij} - \alpha\partial_{x_i}\sigma\partial_{x_j}\sigma \end{aligned} \tag{2.2}$$

with the i-th component of $\nabla \cdot F$ given by $\sum_{j=1}^{n} \partial_{x_j} F_{ij}$.

Let $U = (\sigma, m)$, then (1.1) may be written as

$$\partial_t U - AU = \nabla \cdot \mathfrak{F}(U), \tag{2.3}$$

where

$$A = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla + \alpha\nabla\Delta & \mu\Delta + (\mu+\lambda)\nabla\nabla\cdot \end{pmatrix}$$

and $\mathfrak{F}(U) = (0, F(U))^\tau$.

Let $G(t)*$ be the solution operators to the compressible fluid models of Korteweg type (1.1). Then Fourier transform of $G(t,x)$ is given by

$$\widehat{G}(t,\xi) = \begin{pmatrix} \widehat{G}^{11} & \widehat{G}^{12} \\ \widehat{G}^{21} & \widehat{G}^{22} \end{pmatrix} \tag{2.4}$$

with

$$\begin{cases} \widehat{G}^{11} = \hat{\mathcal{H}}, \quad \widehat{G}^{12} = -i\xi^\tau\hat{\mathcal{G}}, \\ \widehat{G}^{21} = -i(1+|\xi|^2)\xi\widehat{\mathcal{G}}, \quad \widehat{G}^{22} = (2\mu+\lambda)\xi\xi^\tau\widehat{\mathcal{G}} + \widehat{\mathcal{H}}I_{n\times n}, \\ \widehat{\mathcal{G}}(\xi,t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \\ \widehat{\mathcal{H}}(\xi,t) = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \end{cases} \tag{2.5}$$

and

$$\lambda_\pm(\xi) = \frac{-(2\mu+\lambda)|\xi|^2 \pm \sqrt{(2\mu+\lambda)^2|\xi|^4 - 4(|\xi|^2 + \alpha|\xi|^4)}}{2}. \tag{2.6}$$

Due to Duhamel principle, the solution to the problem (1.1), (1.2) may expressed as

$$U(t) = G(t) * \begin{pmatrix} \sigma_0 \\ m_0 \end{pmatrix} + \int_0^t G(t-\tau) * \nabla\cdot\mathfrak{F}(U)(\tau)d\tau. \tag{2.7}$$

Let $\mathcal{F}\widehat{\phi} \in C^\infty(\mathbb{R}^n)$ such that

$$\mathcal{F}\widehat{\phi} = \begin{cases} 1, & |\xi| < \epsilon \\ 0, & |\xi| \geq 2\epsilon \end{cases}$$

where $0 < \epsilon < 1$ is a constant. Set

$$G_l = \phi * G, \quad G_h = G - \phi * G = G - G_l. \tag{2.8}$$

The above Fourier splitting frequency technique was early introduced in [13] and so on. Then (2.7) implies

$$U(t) = U_l(t) + U_h(t), \tag{2.9}$$

where

$$U_l(t) = G_l(t) * U_0 + \int_0^t G_l(t-\tau) * \nabla\cdot\mathfrak{F}(U)(\tau)d\tau \tag{2.10}$$

and

$$U_h(t) = G_h(t) * U_0 + \int_0^t G_h(t-\tau) * \nabla \cdot \mathfrak{F}(U)(\tau) d\tau. \tag{2.11}$$

$\widehat{\mathcal{G}}_l(t,\xi)$ and $\widehat{\mathcal{H}}_l(t,\xi)$ may be written

$$\widehat{\mathcal{G}}_l(t,\xi) = \frac{1}{|\xi|\vartheta(\xi)} e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \sin(|\xi|\vartheta(\xi)t) \tag{2.12}$$

and

$$\widehat{\mathcal{H}}_l(t,\xi) = \frac{(2\mu+\lambda)|\xi|}{4\vartheta(\xi)} e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \sin(|\xi|\vartheta(\xi)t) + e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \cos(|\xi|\vartheta(\xi)t), \tag{2.13}$$

respectively, where

$$\vartheta(\xi) = \sqrt{1 + (\alpha - \frac{(2\mu+\lambda)^2}{4})|\xi|^2}.$$

Due to Taylor formula, it is not difficult to find that

$$\widehat{\mathcal{G}}_l(t,\xi) = \frac{(1+O(|\xi|^2))}{|\xi|} e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \sin(|\xi| + O(|\xi|^3))t \tag{2.14}$$

and

$$\begin{aligned} \widehat{\mathcal{H}}_l(t,\xi) = & \frac{(2\mu+\lambda)|\xi|}{4}(1+O(|\xi|^2)) e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \sin(|\xi| + O(|\xi|^3))t \\ & + e^{-\frac{2\mu+\lambda}{2}|\xi|^2 t} \cos(|\xi| + O(|\xi|^3))t. \end{aligned} \tag{2.15}$$

We state the pointwise estimates for the solution operators $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ to the generalized Boussinesq equation (see [23] and [26]) as follows, which comes from [25].

Lemma 2.1. *Let $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{H}}$ be given by (2.5). Then we have the pointwise estimates*

$$\begin{aligned} & |\widehat{\mathcal{G}}(t,\xi)| \le C|\xi|^{-1}(1+|\xi|^2)^{-\frac{1}{2}} e^{-c|\xi|^2 t}, \\ & |\widehat{\mathcal{H}}(t,\xi)| \le C e^{-c|\xi|^2 t}, \\ & |\widehat{\partial_t \mathcal{G}}(t,\xi)| \le C e^{-c|\xi|^2 t}, \\ & |\widehat{\partial_t H}(t,\xi)| \le C|\xi|(1+|\xi|^2)^{\frac{1}{2}} e^{-c|\xi|^2 t} \end{aligned} \tag{2.16}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$.

From the above results in Lemma 2.1, it is not difficult to find that

$$|\nabla_\xi^k \widehat{\mathcal{G}}_h| \le C|\xi|^{-k-2} e^{-ct}, \quad |\nabla_\xi^k \widehat{\mathcal{H}}_h| \le C|\xi|^{-k} e^{-ct}. \tag{2.17}$$

By Lemma 2.1 and the Plancherel theorem, the following decay estimates of solution operators can be established.

Lemma 2.2. *Let $1 \le q \le 2$, and let k, j and l be nonnegative integers. Then we have*

$$\|\partial_x^k G(t) * U_0\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k-j}{2}} \|\partial_x^j U_0\|_{L^q} + Ce^{-ct}\|\partial_x^{k+l} U_0\|_{L^2} \tag{2.18}$$

and

$$\|\partial_x^k \partial_t G(t) * U_0\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}} \|\partial_x^j U_0\|_{L^q} + Ce^{-ct}\|\partial_x^{k+l+1} U_0\|_{L^2}. \tag{2.19}$$

Lemma 2.3. *Let k, j be nonnegative integers. Then* *(i) If $2 \le p \le \infty$, it holds that*

$$\|\partial_t^j \partial_x^k G_l(t)\|_{L^p} \le C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k+j}{2}}. \tag{2.20}$$

(ii) If $1 \le p < 2$, it holds that

$$\|\partial_t^j \partial_x^k G_l(t)\|_{L^p} \le C \begin{cases} (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{k+j}{2}}, & n \ge 3 \text{ and } n \text{ is odd} \\ (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n}{4}(1-\frac{2}{p})-\frac{k+j}{2}}, & n \ge 2 \text{ and } n \text{ is even} \end{cases} \tag{2.21}$$

Proof. We may refer to [10,11,16–18] and [24] for the proof. The details are omitted. □

From Lemma 2.2 and (2.8), (2.17), we get immediately the decay properties of $G_h(t,x)$.

Lemma 2.4. *Let k be a nonnegative integer. Then*

$$\|\partial_x^k (G(t) * U_0 - G_l(t) * U_0)\|_{L^2} \le Ce^{-ct}\|\partial_x^k U_0\|_{L^2}. \tag{2.22}$$

To establish the weighted decay estimate of global solutions, we need the following decay properties of solution operators in low-frequency parts.

Lemma 2.5. *Let $\kappa = 0, \frac{1}{2}, 1$ and let k be nonnegative integer. Then we have*

$$\||x|^\kappa \partial_x^k G_l(t) * U_0\|_{L^2} \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\kappa}\|U_0\|_{L^1} + (1+t)^{-\frac{n}{4}-\frac{k}{2}}\||x|^\kappa U_0\|_{L^1}. \tag{2.23}$$

Proof. When $\kappa = 0$, by the Plancherel theorem, it is not difficult to see that (2.23) holds. Owing to the properties of Fourier transform and (2.14), (2.15), (2.5), we have

$$\begin{aligned} \||x|\partial_x^k G_l(t)\|_{L^2}^2 &\le C \int\limits_{|\xi|\le 2r} \left|\nabla_\xi\left((i\xi)^k \widehat{G}_l\right)\right|^2 d\xi \\ &\le C \int\limits_{|\xi|\le 2r} |\xi|^{2(k-1)} e^{-c|\xi|^2 t} d\xi + Ct^2 \int\limits_{|\xi|\le 2r} |\xi|^{2k} e^{-c|\xi|^2 t} d\xi \\ &\le C(1+t)^{-\frac{n}{2}-k+2}. \end{aligned} \tag{2.24}$$

From Hölder inequality and (2.24), we have

$$\begin{aligned} \||x|^{\frac{1}{2}}\partial_x^k G_l(t)\|_{L^2} &\le \||x|\partial_x^k G_l(t)\|_{L^2}^{\frac{1}{2}} \|\partial_x^k G_l(t)\|_{L^2}^{\frac{1}{2}} \\ &\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\frac{1}{2}}. \end{aligned} \tag{2.25}$$

When $\kappa = \frac{1}{2}, 1$, it follows from Young inequality and (2.24), (2.25) that

$$\begin{aligned} \||x|^\kappa \partial_x^k G_l(t) * U_0\|_{L^2} &= \||x|^\kappa \partial_x^k G_l(t) * U_0\|_{L^2} \\ &\le \||x|^\kappa \partial_x^k G_l(t)\|_{L^2}\|U_0\|_{L^1} + \|\partial_x^k G_l(t)\|_{L^2}\||x|^\kappa U_0\|_{L^1} \\ &\le C(1+t)^{-\frac{n}{4}-\frac{k}{2}+\kappa}\|U_0\|_{L^1} + (1+t)^{-\frac{n}{4}-\frac{k}{2}}\||x|^\kappa U_0\|_{L^1}. \end{aligned}$$

The lemma is proved. □

Lemma 2.6. *Assume that $U_0 \in L_2^1(\mathbb{R}^n)$. Then we have*

$$\begin{aligned} &\Big\|G_l(t) * U_0 - G_l(t)\int\limits_{\mathbb{R}^n} U_0(y)dy + \sum_{|\alpha|=1} \partial_x^\alpha G_l(t)\int\limits_{\mathbb{R}^n} U_0(y)y^\alpha dy\Big\|_{L^2} \\ &\le C(1+t)^{-\frac{n}{4}-1}\|U_1\|_{L_2^1}. \end{aligned} \tag{2.26}$$

Proof. The definition of convolution, Taylor formula, Young inequality and (2.20) entails that

$$\begin{aligned}&\Big\|G_l(t)*U_0-G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)dy+\sum_{|\alpha|=1}\partial_x^\alpha G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)y^\alpha dy\Big\|_{L^2}\\&=\Big\|\int\limits_{\mathbb{R}^n}\Big(G_l(t,x-y)-G_l(t,x)+\sum_{|\alpha|=1}\partial_x^\alpha G_l(t,x)y^\alpha\Big)U_0(y)dy\Big\|_{L^2}\\&=\Big\|\int\limits_{\mathbb{R}^n}\sum_{|\alpha|=2}\partial_x^\alpha G_l(t,x-\theta y))U_0(y)y^\alpha dy\Big\|_{L^2}\\&\le\|\partial_x^2G_l(t)\|_{L^2}\||x|^2U_0\|_{L^1}\\&\le C(1+t)^{-\frac{n}{4}-1}\|U_0\|_{L^1_2},\end{aligned}$$

where $\theta\in(0,1)$. We complete the proof of Lemma 2.6. □

To derive the weighted decay estimate of solution operators in high-frequency part, we need the following lemma, which comes from [14].

Lemma 2.7. *Let $\partial_\xi^k\widehat{f}\in L^\infty$ for $k=0,1$, and let $(1+|x|^{\frac12})g\in L^2$. Then we have*

$$\||x|^{\frac12}f*g\|_{L^2}\le C(\|\widehat{f}\|_{L^\infty}+\|\partial_\xi\widehat{f}\|_{L^\infty})(\|g\|_{L^2}+\||x|^{\frac12}g\|_{L^2}). \tag{2.27}$$

Lemma 2.8. *Let $\kappa=0,\frac12,1$ and let k be nonnegative integer. Then we have*

$$\||x|^\kappa\partial_x^kG_h(t)*U_0\|_{L^2}\le Ce^{-ct}(\|\partial_x^kU_0\|_{L^2}+\||x|^\kappa\partial_x^kU_0\|_{L^2}) \tag{2.28}$$

and

$$\||x|^\kappa\partial_x^{k+1}G_h^{12}(t)*u_0\|_{L^2}\le Ce^{-ct}(\|\partial_x^ku_0\|_{L^2}+\||x|^\kappa\partial_x^ku_0\|_{L^2}). \tag{2.29}$$

Proof. When $\kappa=0,1$, (2.28) immediately follows from the Plancherel theorem. When $\kappa=\frac12$, making use of (2.27) and (2.5), (2.17), we deduce that

$$\begin{aligned}\Big\||x|^\kappa\partial_x^kG_h(t)*U_0\Big\|_{L^2}&=\Big\||x|^\kappa G_h(t)*\partial_x^kU_0\Big\|_{L^2}\\&\le\Big(\Big\|\partial_\xi\widehat{G}_h(t)\Big\|_{L^\infty}+\Big\|\widehat{G}_h(t)\Big\|_{L^\infty}\Big)\left(\|\partial_x^kU_0\|_{L^2}+\||x|^\kappa\partial_x^kU_0\|_{L^2}\right)\\&\le Ce^{-ct}\left(\|\partial_x^kU_0\|_{L^2}+\||x|^\kappa\partial_x^kU_0\|_{L^2}\right).\end{aligned}$$

Similarly, we may prove (2.29). Then the proof of Lemma 2.8 is completed. □

3. Weighted decay estimate of global solutions

The purpose of this section is to establish weighted decay estimate of global solutions obtained in [9]. To this end, let $\sigma=\rho-1$, $u=\frac{m}{\rho}$, $P'(1)=1$ and $V=(\sigma,u)^\tau$,$V_0=(\sigma_0,u_0)^\tau=(\sigma_0,\frac{m_0}{\sigma_0+1})^\tau$. Then (1.3) may be rewritten as

$$\begin{cases}\partial_t\sigma+\nabla\cdot u=\widetilde{F}_1(V),\\\partial_tu+\nabla\sigma-\mu\Delta u-(\mu+\lambda)\nabla(\nabla\cdot u)-\alpha\nabla\Delta\sigma=\widetilde{F}_2(V),\end{cases} \tag{3.1}$$

The initial value becomes

$$t=0:\quad V=V_0, \tag{3.2}$$

where

$$\widetilde{F}_1(V)=-\nabla\cdot(\sigma u) \tag{3.3}$$

and

$$\widetilde{F}_2(V) = -u\cdot\nabla u - \mu\frac{\sigma}{\sigma+1}\Delta u - (\mu+\lambda)\frac{\sigma}{\sigma+1}\nabla(\nabla\cdot u) - \frac{1}{\sigma+1}[\nabla(P(1+\sigma) - P'(1)\sigma)]. \tag{3.4}$$

The problem (3.1), (3.2) may be rewritten as

$$\begin{cases} \partial_t V - AV = \widetilde{F}(V), \\ t=0: \quad V = V_0, \end{cases} \tag{3.5}$$

where $\widetilde{F}(V) = (\widetilde{F}_1(V), \widetilde{F}_2(V))^\tau$.
Noting that (2.8), then the solution to the problem (3.5) is given by

$$\begin{aligned} V(t) &= G(t)*V_0 + \int_0^t G(t-\tau)*\widetilde{F}(V)(\tau)d\tau \\ &= G_l(t)*V_0 + G_h(t)*V_0 + \int_0^t G_l(t-\tau)*\widetilde{F}(V)(\tau)d\tau \\ &\quad + \int_0^t G_h(t-\tau)*\widetilde{F}(V)(\tau)d\tau. \end{aligned} \tag{3.6}$$

We state the weighted decay estimate of global solutions obtained in [9] as follows.

Theorem 3.1. *Let $n \geq 3$ and $\kappa = \frac{1}{2}$ when $n=3$ and $\kappa = 1$ when $n \geq 4$. Assume that the conditions of Theorem 1.2 hold. Moreover, assume that $V_0 \in L^1_\kappa \bigcap L^2_\kappa$ and $\partial_x\sigma_0 \in L^2_\kappa$. The solutions V to the problem (3.1), (3.2) satisfy*

$$\||x|^\kappa V(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}+\kappa} \tag{3.7}$$

and

$$\||x|^\kappa \partial_x\sigma(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}+\kappa}. \tag{3.8}$$

Proof. Let

$$X(t) = \sup_{0\leq\tau\leq t}\Big\{(1+\tau)^{\frac{n}{4}-\kappa}(\||x|^\kappa V(\tau)\|_{L^2} + \||x|^\kappa\partial_x\sigma(\tau)\|_{L^2})\Big\}. \tag{3.9}$$

Thanks to (3.6) and Minkowski inequality, we obtain

$$\begin{aligned} \||x|^\kappa V(\tau)\|_{L^2} &\leq \||x|^\kappa G_l(t)*V_0\|_{L^2} + \||x|^\kappa G_h(t)*V_0\|_{L^2} \\ &\quad + \int_0^t \||x|^\kappa G_1(t-\tau)*\widetilde{F}(V)(\tau)\|_{L^2}d\tau \\ &\quad + \int_0^t \||x|^\kappa G_h(t-\tau)*\widetilde{F}(V)(\tau)\|_{L^2}d\tau \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.10}$$

By virtue of (2.23), we can get

$$\begin{aligned} I_1 &\leq C(1+t)^{-\frac{n}{4}+\kappa}(\|V_0\|_{L^1} + \||x|^\kappa V_0\|_{L^1}) \\ &\leq C(1+t)^{-\frac{n}{4}+\kappa}\|V_0\|_{L^1_\kappa}. \end{aligned} \tag{3.11}$$

(2.28) entails that

$$\begin{aligned} I_2 &\le Ce^{-ct}(\|V_0\|_{L^2}+\||x|^\kappa V_0\|_{L^2}) \\ &\le C(1+t)^{-\frac{n}{4}+\kappa}\|V_0\|_{L^2_\kappa}. \end{aligned} \tag{3.12}$$

It follows from (2.23) that

$$\begin{aligned} I_3 &\le C\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}\|\widetilde{F}(V)(\tau)\|_{L^1}d\tau \\ &\quad +C\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}\||x|^\kappa\widetilde{F}(V)(\tau)\|_{L^1}d\tau \\ &\le C\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}\|V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^2}+\|\partial_x^2 V(\tau)\|_{L^2})d\tau \\ &\quad +C\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}\||x|^\kappa V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^2}+\|\partial_x^2 V(\tau)\|_{L^2})d\tau \\ &\le CE_1^2\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}(1+\tau)^{-\frac{n}{2}-\frac{1}{2}}d\tau \\ &\quad +CE_1X(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}+\kappa}(1+\tau)^{-\frac{n}{2}-\frac{1}{2}+\kappa}d\tau \\ &\le CE_1^2(1+t)^{-\frac{n}{4}+\kappa}+CE_1X(t)(1+t)^{-\frac{n}{4}+\kappa}. \end{aligned} \tag{3.13}$$

Owing to (2.28), we arrive at

$$\begin{aligned} I_4 &\le C\int_0^t e^{-c(t-\tau)}\|\widetilde{F}(V)(\tau)\|_{L^2}d\tau + C\int_0^t e^{-c(t-\tau)}\||x|^\kappa\widetilde{F}(V)(\tau)\|_{L^2}d\tau \\ &\le C\int_0^t e^{-c(t-\tau)}\|V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^\infty}+\|\partial_x^2 u(\tau)\|_{L^\infty})d\tau \\ &\quad +C\int_0^t e^{-c(t-\tau)}\||x|^\kappa V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^\infty}+\|\partial_x^2 u(\tau)\|_{L^\infty})d\tau \\ &\le C\int_0^t e^{-c(t-\tau)}\|V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{H^{s-1}}+\|\partial_x u(\tau)\|_{H^s})d\tau \\ &\quad +C\int_0^t e^{-c(t-\tau)}\||x|^\kappa V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{H^{s-1}}+\|\partial_x u(\tau)\|_{H^s})d\tau \\ &\le C\left(\int_0^t e^{-2c(t-\tau)}(\|V(\tau)\|_{L^2}^2+\||x|^\kappa V(\tau)\|_{L^2}^2)d\tau\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_0^t (\|\partial_x V(\tau)\|_{H^{s-1}}^2+\|\partial_x u(\tau)\|_{H^s}^2)d\tau\right)^{\frac{1}{2}}\\
&\le CE_1^2\left(\int_0^t e^{-2c(t-\tau)}(1+t)^{-\frac{n}{2}}d\tau\right)^{\frac{1}{2}}\\
&\quad +CE_1X(t)\left(\int_0^t e^{-2c(t-\tau)}(1+t)^{-\frac{n}{2}+2\kappa}d\tau\right)^{\frac{1}{2}}\\
&\le CE_1^2(1+t)^{-\frac{n}{4}}+CE_1X(t)(1+t)^{-\frac{n}{4}+\kappa}.
\end{aligned}\tag{3.14}
$$

Making use of (3.6) and Minkowski inequality, we obtain

$$
\begin{aligned}
\||x|^\kappa\partial_x\sigma(\tau)\|_{L^2} \le\ & \||x|^\kappa\partial_x G_l^{11}(t)*\sigma_0\|_{L^2}+\||x|^\kappa\partial_x G_h^{11}(t)*\sigma_0\|_{L^2}\\
\le\ & \||x|^\kappa\partial_x G_l^{12}(t)*u_0\|_{L^2}+\||x|^\kappa\partial_x G_h^{12}(t)*u_0\|_{L^2}\\
&+\int_0^t \||x|^\kappa\partial_x G_1^{11}(t-\tau)*\widetilde{F}_1(V)(\tau)\|_{L^2}d\tau\\
&+\int_0^t \||x|^\kappa\partial_x G_h^{11}(t-\tau)*\widetilde{F}_1(V)(\tau)\|_{L^2}d\tau\\
&+\int_0^t \||x|^\kappa\partial_x G_l^{12}(t-\tau)*\widetilde{F}_2(V)(\tau)\|_{L^2}d\tau\\
&+\int_0^t \||x|^\kappa\partial_x G_h^{12}(t-\tau)*\widetilde{F}_2(V)(\tau)\|_{L^2}d\tau\\
=:\ & J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8.
\end{aligned}\tag{3.15}
$$

Thanks to (2.23), we arrive at

$$
\begin{aligned}
J_1 &\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}(\|\sigma_0\|_{L^1}+\||x|^\kappa\sigma_0\|_{L^1})\\
&\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|\sigma_0\|_{L^1_\kappa}.
\end{aligned}\tag{3.16}
$$

Applying (2.28), this gives

$$
\begin{aligned}
J_2 &\le Ce^{-ct}(\|\partial_x\sigma_0\|_{L^2}+\||x|^\kappa\partial_x\sigma_0\|_{L^2})\\
&\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|\partial_x\sigma_0\|_{L^2_\kappa}.
\end{aligned}\tag{3.17}
$$

By using (2.23), we get

$$
\begin{aligned}
J_3 &\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}(\|u_0\|_{L^1}+\||x|^\kappa u_0\|_{L^1})\\
&\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|u_0\|_{L^1_\kappa}.
\end{aligned}\tag{3.18}
$$

(2.29) entails that

$$
\begin{aligned}
J_4 &\le Ce^{-ct}(\|u_0\|_{L^2}+\||x|^\kappa u_0\|_{L^2})\\
&\le C(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|u_0\|_{L^2_\kappa}.
\end{aligned}\tag{3.19}
$$

By a similar calculation to (3.13), it follows from (2.23) that

$$
\begin{aligned}
J_5 &\le C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|\widetilde{F}_1(V)(\tau)\|_{L^1}d\tau \\
&\quad +C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\||x|^\kappa\widetilde{F}_1(V)(\tau)\|_{L^1}d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\|V(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{L^2}d\tau \\
&\quad +C\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}\||x|^\kappa V(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{L^2}d\tau \\
&\le CE_1^2\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}(1+\tau)^{-\frac{n}{2}-\frac{1}{2}}d\tau \\
&\quad +CE_1X(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}+\kappa}(1+\tau)^{-\frac{n}{2}-\frac{1}{2}+\kappa}d\tau \\
&\le CE_1^2(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}+CE_1X(t)(1+t)^{-\frac{n}{4}-\frac{1}{2}+\kappa}.
\end{aligned}
\tag{3.20}
$$

Due to (2.28), it holds that

$$
\begin{aligned}
J_6 &\le C\int_0^t e^{-c(t-\tau)}\|\widetilde{F}_1(V)(\tau)\|_{L^2}d\tau + C\int_0^t e^{-c(t-\tau)}\||x|^\kappa\widetilde{F}_1(V)(\tau)\|_{L^2}d\tau \\
&\le C\int_0^t e^{-c(t-\tau)}\Big(\|\sigma(\tau)\|_{L^2}\|\partial_x^2 u(\tau)\|_{L^\infty}+\|\partial_x\sigma(\tau)\|_{L^2}\|\partial_x u(\tau)\|_{L^\infty} \\
&\quad +\|u(\tau)\|_{L^2}\|\partial_x^2\sigma(\tau)\|_{L^\infty}\Big)d\tau \\
&\quad +C\int_0^t e^{-c(t-\tau)}(\||x|^\kappa\sigma(\tau)\|_{L^2}\|\partial_x^2 u(\tau)\|_{L^\infty}+\||x|^\kappa\partial_x\sigma(\tau)\|_{L^2}\|\partial_x u(\tau)\|_{L^\infty} \\
&\quad +\||x|^\kappa u(\tau)\|_{L^2}\|\partial_x^2\sigma(\tau)\|_{L^\infty})d\tau \\
&\le C\int_0^t e^{-c(t-\tau)}(\|V(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{H^s}+\|\partial_x V(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{H^{s-1}})d\tau \\
&\quad +C\int_0^t e^{-c(t-\tau)}(\||x|^\kappa V(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{H^s}+\||x|^\kappa\partial_x\sigma(\tau)\|_{L^2}\|\partial_x V(\tau)\|_{H^{s-1}})d\tau \\
&\le C\left(\int_0^t e^{-2c(t-\tau)}(\|V(\tau)\|_{L^2}^2+\|\partial_x V(\tau)\|_{L^2}^2+\||x|^\kappa V(\tau)\|_{L^2}^2+\||x|^\kappa\partial_x\sigma(\tau)\|_{L^2}^2)d\tau\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_0^t (\|\partial_x V(\tau)\|_{H^{s-1}}^2+\|\partial_x V(\tau)\|_{H^s}^2)d\tau\right)^{\frac12}\\
&\le CE_1^2\left(\int_0^t e^{-2c(t-\tau)}(1+t)^{-\frac n2}d\tau\right)^{\frac12}\\
&\quad+CE_1X(t)\left(\int_0^t e^{-2c(t-\tau)}(1+t)^{-\frac n2+2\kappa}d\tau\right)^{\frac12}\\
&\le CE_1^2(1+t)^{-\frac n4}+CE_1X(t)(1+t)^{-\frac n4+\kappa}.
\end{aligned}
\tag{3.21}
$$

We estimate J_7 as follows by (2.23)

$$
\begin{aligned}
J_7 &\le C\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}\|\widetilde{F}_2(V)(\tau)\|_{L^1}d\tau\\
&\quad+C\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}\||x|^\kappa\widetilde{F}_2(V)(\tau)\|_{L^1}d\tau\\
&\le C\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}\|V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^2}+\|\partial_x^2 V(\tau)\|_{L^2})d\tau\\
&\quad+C\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}\||x|^\kappa V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^2}+\|\partial_x^2 V(\tau)\|_{L^2})d\tau\\
&\le CE_1^2\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}(1+\tau)^{-\frac n2-\frac12}d\tau\\
&\quad+CE_1X(t)\int_0^t (1+t-\tau)^{-\frac n4-\frac12+\kappa}(1+\tau)^{-\frac n2-\frac12+\kappa}d\tau\\
&\le CE_1^2(1+t)^{-\frac n4-\frac12+\kappa}+CE_1X(t)(1+t)^{-\frac n4-\frac12+\kappa}.
\end{aligned}
\tag{3.22}
$$

Making use of (2.29), we see that

$$
\begin{aligned}
J_8 &\le C\int_0^t e^{-c(t-\tau)}\|\widetilde{F}_2(V)(\tau)\|_{L^2}d\tau+C\int_0^t e^{-c(t-\tau)}\||x|^\kappa\widetilde{F}_2(V)(\tau)\|_{L^2}d\tau\\
&\le C\int_0^t e^{-c(t-\tau)}\|V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^\infty}+\|\partial_x^2 u(\tau)\|_{L^\infty})d\tau\\
&\quad+C\int_0^t e^{-c(t-\tau)}\||x|^\kappa V(\tau)\|_{L^2}(\|\partial_x V(\tau)\|_{L^\infty}+\|\partial_x^2 u(\tau)\|_{L^\infty})d\tau
\end{aligned}
$$

$$\leq C \int_0^t e^{-c(t-\tau)} \|V(\tau)\|_{L^2} (\|\partial_x V(\tau)\|_{H^{s-1}} + \|\partial_x u(\tau)\|_{H^s}) d\tau$$
$$+C \int_0^t e^{-c(t-\tau)} \||x|^\kappa V(\tau)\|_{L^2} (\|\partial_x V(\tau)\|_{H^{s-1}} + \|\partial_x u(\tau)\|_{H^s}) d\tau$$
$$\leq C \left(\int_0^t e^{-2c(t-\tau)} (\|V(\tau)\|_{L^2}^2 + \||x|^\kappa V(\tau)\|_{L^2}^2) d\tau \right)^{\frac{1}{2}}$$
$$\times \left(\int_0^t (\|\partial_x V(\tau)\|_{H^{s-1}}^2 + \|\partial_x u(\tau)\|_{H^s}^2) d\tau \right)^{\frac{1}{2}}$$
$$\leq C E_1^2 \left(\int_0^t e^{-2c(t-\tau)} (1+t)^{-\frac{n}{2}} d\tau \right)^{\frac{1}{2}}$$
$$+C E_1 X(t) \left(\int_0^t e^{-2c(t-\tau)} (1+t)^{-\frac{n}{2}+2\kappa} d\tau \right)^{\frac{1}{2}}$$
$$\leq C E_1^2 (1+t)^{-\frac{n}{4}} + C E_1 X(t) (1+t)^{-\frac{n}{4}+\kappa}. \tag{3.23}$$

Inserting (3.11)–(3.14) into (3.10) and (3.16)–(3.23) into (3.15) yields

$$X(t) \leq C(\|V_0\|_{L^1_\kappa} + \|V_0\|_{L^2_\kappa} + \|\partial_x \sigma_0\|_{L^2_\kappa}) + C E_1^2 + C E_1 X(t), \tag{3.24}$$

which implies

$$X(t) \leq C(\|V_0\|_{L^1_\kappa} + \|V_0\|_{L^2_\kappa} + \|\partial_x \sigma_0\|_{L^2_\kappa}) + C E_1^2,$$

provided that E_1 is suitably small. The proof of Theorem 3.1 is completed. □

4. Asymptotic profile of global solutions

In this section, our main goal is to establish asymptotic profile of global solutions and the corresponding convergence rate. To this end, we state the L^2 decay rate of global solutions obtained in Theorem 1.1 in high-frequency part as follows.

Lemma 4.1. *Under the conditions of Theorem 1.3, $U_h(t,x)$ satisfies*

$$\|U_h(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{4}-1}. \tag{4.1}$$

Proof. (4.1) immediately follows from Theorem 1.1, 1.2 and the decay property of $G_h(x,t)$. Here we omit the details. □

In what follows, we give the proof of Theorem 1.3.

Proof. From (2.7), (2.9) and (2.10), we arrive at

$$
\begin{aligned}
&U(t)-G(t)*U_0-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int_0^{\infty}\int_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\\
&=U_h(t)+G_l(t)*U_0-G(t)*U_0+\int_0^t G_l(t-\tau)*\nabla\cdot\mathfrak{F}(U)(\tau)d\tau\\
&\quad-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int_0^{\infty}\int_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\\
&=U_h(t)+(G_l(t)*U_0-G(t)*U_0)\\
&\quad+\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}\sum_{i=1}^{n}\Big(\partial_{x_i}G_l(t-\tau,x-y)-\partial_{x_i}G_l(t,x)\Big)\mathbb{F}_i(U)dyd\tau\\
&\quad+\int_0^{\frac{t}{2}}\sum_{i=1}^{n}\partial_{x_i}G_l(t-\tau)*\Big(\mathfrak{F}_i(U)-\mathbb{F}_i(U)\Big)(\tau)d\tau\\
&\quad+\int_{\frac{t}{2}}^{t}\sum_{i=1}^{n}\partial_{x_i}G_l(t-\tau)*\mathfrak{F}_i(U)(\tau)d\tau\\
&\quad-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau.
\end{aligned}
\tag{4.2}
$$

Then (4.2) and Minkowski inequality give

$$
\begin{aligned}
&\Big\|U(t)-G(t)*U_0-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int_0^{\infty}\int_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\Big\|_{L^2}\\
&\le\|U_h(t)\|_{L^2}+\Big\|G_l(t)*U_0-G(t)*U_0\Big\|_{L^2}\\
&\quad+\Bigg\|\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}\sum_{i=1}^{n}\Big(\partial_{x_i}G_l(t-\tau,x-y)-\partial_{x_i}G_l(t,x)\Big)\mathbb{F}_i(U)dyd\tau\Bigg\|_{L^2}\\
&\quad+\Bigg\|\int_0^{\frac{t}{2}}\sum_{i=1}^{n}\partial_{x_i}G_l(t-\tau)*\Big(\mathfrak{F}_i(U)-\mathbb{F}_i(U)\Big)(\tau)d\tau\Bigg\|_{L^2}\\
&\quad+\Bigg\|\int_{\frac{t}{2}}^{t}\sum_{i=1}^{n}\partial_{x_i}G_l(t-\tau)*\mathfrak{F}_i(U)(\tau)d\tau\Bigg\|_{L^2}\\
&\quad+\Bigg\|\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int_{\frac{t}{2}}^{\infty}\int_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\Bigg\|_{L^2}\\
&=:K_1+K_2+K_3+K_4+K_5+K_6.
\end{aligned}
\tag{4.3}
$$

By virtue of (4.1), it holds that

$$K_1 \le C(1+t)^{-\frac{n}{4}-1}. \tag{4.4}$$

Thanks to (2.22), we have

$$K_2 \le Ce^{-ct}\|U_0\|_{L^2}. \tag{4.5}$$

K_3 may be rewritten as

$$\begin{aligned}
K_3 &= \left\| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \Big(\partial_{x_i} G_l(t-\tau, x-y) - \partial_{x_i} G_l(t,x) \Big) \mathbb{F}_i(U) dy d\tau \right\|_{L^2} \\
&\le \left\| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \Big(\partial_{x_i} G_l(t-\tau, x-y) - \partial_{x_i} G_l(t-\tau,x) \Big) \mathbb{F}_i(U) dy d\tau \right\|_{L^2} \\
&\quad + \left\| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \Big(\partial_{x_i} G_l(t-\tau, x) - \partial_{x_i} G_l(t,x) \Big) \mathbb{F}_i(U) dy d\tau \right\|_{L^2} \\
&=: K_{31} + K_{32}.
\end{aligned}$$

In what follows, we estimate K_{31} for $n=3$ and $n \ge 4$, respectively. When $n=3$, mean value theorem, Young inequality, Hölder inequality and (2.20), (3.7), (3.8) entail that

$$\begin{aligned}
K_{31} &= \left\| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_x \partial_{x_i} G_l(t-\tau, x-\theta_1 y) y \mathbb{F}_i(U) dy d\tau \right\|_{L^2} \\
&\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-1} \|x\mathbb{F}_i(U)\|_{L^1} d\tau \\
&\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-1} \Big(\||x|^{\frac{1}{2}} U(\tau)\|_{L^2}^2 + \||x|^{\frac{1}{2}} \partial_x \sigma(\tau)\|_{L^2}^2 \Big) d\tau \\
&\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{4}-1} (1+\tau)^{-\frac{1}{2}} d\tau \\
&\le (1+t)^{-\frac{5}{4}},
\end{aligned} \tag{4.6}$$

where $\theta_1 \in (0,1)$.

When $n \ge 4$, using mean value theorem, Young inequality, Hölder inequality and (2.20), (1.5), (1.6), (3.7), (3.8), the similar estimate of (4.6) leads to

$$\begin{aligned}
K_{31} &= \left\| \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^n \partial_x \partial_i G_l(t-\tau, x-\theta_2 y) y \mathbb{F}_i(U) dy d\tau \right\|_{L^2} \\
&\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{4}-1} \|x\mathbb{F}_i(U)\|_{L^1} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}\Big(\||x|U(\tau)\|_{L^2}\|U(\tau)\|_{L^2}+\||x|\partial_x\sigma(\tau)\|_{L^2}\|\partial_x\sigma(\tau)\|_{L^2}\Big)d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}(1+\tau)^{-\frac{n}{2}+1}d\tau\\
&\le C\begin{cases}(1+t)^{-\frac{n}{4}-1}\log(1+t), & n=4\\ (1+t)^{-\frac{n}{4}-1}, & n\ge 5\end{cases}
\end{aligned}
\tag{4.7}
$$

where $\theta_2\in(0,1)$.

Making use of mean value theorem, Young inequality, Hölder inequality and (2.20), (1.5), (1.6), we have

$$
\begin{aligned}
K_{32}&=\left\|\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}\sum_{i=1}^n\partial_t\partial_{x_i}G_l(t-\theta_3\tau,x)\tau\mathbb{F}_i(U)dyd\tau\right\|_{L^2}\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}\tau\|\mathbb{F}_i(U)\|_{L^1}d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}\tau\Big(\|U(\tau)\|_{L^2}^2+\|\partial_x\sigma(\tau)\|_{L^2}^2\Big)d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}\tau(1+\tau)^{-\frac{n}{2}}d\tau\\
&\le C\begin{cases}(1+t)^{-\frac{n}{4}-\frac{1}{2}}, & n=3\\ (1+t)^{-\frac{n}{4}-1}\log(1+t), & n=4\\ (1+t)^{-\frac{n}{4}-1}, & n\ge 5\end{cases}
\end{aligned}
\tag{4.8}
$$

where $\theta_3\in(0,1)$.

It follows from (2.23) with $\kappa=0$ and (1.5) that

$$
\begin{aligned}
K_4&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}\|U(\tau)\|_{L^2}^2d\tau+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{3}{2}}\|\sigma(\tau)\|_{L^2}^2d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-1}(1+\tau)^{-\frac{n}{2}}d\tau\\
&\le C(1+t)^{-\frac{n}{4}-1}.
\end{aligned}
\tag{4.9}
$$

Young inequality, (2.21), (1.5), (1.6) entail that

$$
\begin{aligned}
K_5\le C\int_{\frac{t}{2}}^{t}&\Big(\|\partial_x\mathscr{G}_l(t-\tau)\|_{L^1}(\||U|^2(\tau)\|_{L^2}+\||\partial_x\sigma|^2(\tau)\|_{L^2})\\
&+\|\partial_x^2\mathscr{G}_l(t-\tau)\|_{L^1}\||U|^2(\tau)\|_{L^2}+\|\partial_x^3\mathscr{G}_l(t-\tau)\|_{L^1}\|\sigma(\tau)\|_{L^2}\Big)d\tau
\end{aligned}
$$

$$\leq C\begin{cases}\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{n-1}{4}-\frac{1}{2}}(\|U(\tau)\|_{L^\infty}\|U\|_{L^2}+\|\partial_x\sigma(\tau)\|_{L^4}^2)d\tau, \, n\geq 3 \text{ and } n \text{ is odd}\\ \int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{n}{4}-\frac{1}{2}}(\|U(\tau)\|_{L^\infty}\|U\|_{L^2}+\|\partial_x\sigma(\tau)\|_{L^4}^2)d\tau, \quad n\geq 4 \text{ and } n \text{ is even}\end{cases}$$

$$\leq C\begin{cases}\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{n-1}{4}-\frac{1}{2}}(1+\tau)^{-\frac{3n}{4}}d\tau, \, n\geq 3 \text{ and } n \text{ is odd}\\ \int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{n}{4}-\frac{1}{2}}(1+\tau)^{-\frac{3n}{4}}d\tau, \quad n\geq 4 \text{ and } n \text{ is even}\end{cases}$$

$$\leq C\begin{cases}(1+t)^{-\frac{n}{4}-\frac{1}{2}}, & n=3,4\\ (1+t)^{-\frac{n}{4}-1}, & n\geq 5\end{cases} \tag{4.10}$$

Owing to (2.20) and (1.5), (1.6), we deduce, for $n\geq 3$

$$\begin{aligned}K_6 &\leq \sum_{i=1}^{n}\|\partial_{x_i}G_l(t)\|_{L^2}\int\limits_{\frac{t}{2}}^{\infty}\int\limits_{\mathbb{R}^n}|\mathbb{F}_i(U)|dyd\tau\\ &\leq C(1+t)^{-\frac{n}{4}-\frac{1}{2}}\int\limits_{\frac{t}{2}}^{\infty}(\|U(\tau)\|_{L^2}^2+\|\partial_x\sigma(\tau)\|_{L^2}^2)d\tau\\ &\leq C(1+t)^{-\frac{n}{4}-\frac{1}{2}}\int\limits_{\frac{t}{2}}^{\infty}(1+\tau)^{-\frac{n}{2}}d\tau\\ &\leq C(1+t)^{-\frac{n}{4}-1}.\end{aligned} \tag{4.11}$$

Substituting (4.4)-(4.11) into (4.3) yields (1.8). This concludes Theorem 1.3. □

Remark 4.2. Assume that the conditions of Theorem 1.3 hold. Furthermore, assume that $U_0\in L_2^1$. Then from (2.22) and (2.26), we have

$$\begin{aligned}&\Big\|G(t)*U_0-G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)dy+\sum_{|\alpha|=1}\partial_x^\alpha G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)y^\alpha dy\Big\|_{L^2}\\ &\quad\leq C(1+t)^{-\frac{n}{4}-1}(\|U_0\|_{L_2^1}+\|U_0\|_{L^2}).\end{aligned} \tag{4.12}$$

Therefore, (1.8) and (4.12) and Minkowski inequality imply that

$$\begin{aligned}&\Big\|u(t)-G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)dy+\sum_{|\alpha|=1}\partial_x^\alpha G_l(t)\int\limits_{\mathbb{R}^n}U_0(y)y^\alpha dy\\ &\quad-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int\limits_0^\infty\int\limits_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\Big\|_{L^2}\\ &\leq\Big\|u(t)-G(t)*U_0-\sum_{i=1}^{n}\partial_{x_i}G_l(t)\int\limits_0^\infty\int\limits_{\mathbb{R}^n}\mathbb{F}_i(U)dyd\tau\Big\|_{L^2}\end{aligned}$$

$$+\Big\|G(t)*U_0-G_l(t)\int_{\mathbb{R}^n}U_0(y)dy+\sum_{|\alpha|=1}\partial_x^\alpha G_l(t)\int_{\mathbb{R}^n}U_0(y)y^\alpha dy\Big\|_{L^2}$$
$$\leq C\begin{cases}(1+t)^{-\frac{n}{4}-\frac{1}{2}}, & n=3,4\\ (1+t)^{-\frac{n}{4}-1}, & n\geq 5.\end{cases} \tag{4.13}$$

Acknowledgements

The work is partially supported by the NNSF of China (Grant No. 11871212) and the Basic Research Project of Key Scientific Research Project Plan of Universities in Henan Province (Grant No. 20ZX002).

References

[1] Bresch, D., Desjardins, B., Lin, C.-K.: On some compressible fluid models: Korteweg, lubrication, and shallow water system. Comm. Part. Differ. Equ. **28**, 843–868 (2003)
[2] Cahn, J., Hilliard, J.: Free energy of a nonuniform system, I. Interfacial free energy. J. Chem. Phys. **28**, 258–267 (1998)
[3] Danchina, R., Desjardins, B.: Existence of solutions for compressible fluid models of korteweg type. Ann. Inst. Henri Poincaré Anal. Nonlinear **18**, 97–133 (2001)
[4] Dunn, J., Serrin, J.: On the thermomechanics of interstitial working. Arch. Ration. Mech. Anal. **88**, 95–133 (1985)
[5] Gurtin, M., Poligone, D., Vinals, J.: Two-phases binary fluids and immiscible fluids described by an order parameter. Math. Models Methods Appl. Sci. **6**, 815–831 (1996)
[6] Haspot, B.: Existence of global weak solution for compressible fluid models of Korteweg type. J. Math. Fluid Mech. **13**, 223–249 (2011)
[7] Haspot, B.: Existence of global strong solution for the compressible Navier-Stokes system and the Korteweg system in two-dimension. Methods Appl. Anal. **20**, 141–164 (2013)
[8] Hattori, H., Li, D.: Solutions for two-dimensional system for materials of Korteweg type. SIAM J. Math. Anal. **25**, 85–98 (1994)
[9] Hattori, H., Li, D.: Global solutions of a high-dimensional system for Korteweg materials. J. Math. Anal. Appl. **198**, 84–97 (1996)
[10] Hoff, D., Zumbrun, K.: Multidimensional diffusion waves for the Navier–Stokes equations of compressible flow. Indiana Univ. Math. J. **44**, 604–676 (1995)
[11] Hoff, D., Zumbrun, K.: Pointwise decay estimates for multidimensional Navier–Stokes diffusion waves. Z. Angew. Math. Phys. **48**, 597–614 (1997)
[12] Gao, J., Zou, Y., Yao, Z.: Long-time behavior of solution for the compressible Navier–Stokes-Korteweg equations in $\mathbb{R}^3$. Appl. Math. Lett. **48**, 30–35 (2015)
[13] Kawashima, S., Matsumura, A., Nishida, T.: On the fluid dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation. Commun. Math. Phys. **70**, 97–124 (1979)
[14] Kagei, Y., Okita, M.: Asymptotic profile for the compressible Navier–Stokes equations. J. Math. Anal. Appl. **445**, 297–317 (2017)
[15] Kotschote, M.: Strong solutions for a compressible fluid model of Korteweg type. Ann. Inst. H. Poincare Anal. Non Lineaire **25**, 679–696 (2008)
[16] Liu, T., Wang, W.: The pointwise estimates of diffusion waves for the Navier–Stokes equations in odd multi-dimensions. Commun. Math. Phys. **196**, 145–173 (1998)
[17] Shibata, Y.: On the rate of decay of solutions to linear viscoelastic equation. Math. Meth. Appl. Sci. **23**, 203–226 (2000)
[18] Kobayashi, T., Shibata, Y.: Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations. Pac. J. Math. **207**, 199–234 (2002)
[19] Shibata, Y., Enomoto, Y.: Global existence of classical solutions and optimal decay rate for compressible flows via the theory of semigroups. Handbook of mathematical analysis in mechanics of viscous fluids, 2085–2181, Springer, Cham (2018)
[20] Tan, Z., Wang, H., Xu, J.: Global existence and optimal L^2 decay rate for the strong solutions to the compressible fluid models of Korteweg type. J. Math. Anal. Appl. **390**, 181–187 (2012)

[21] Tan, Z., Wang, Y.: Large time behavior of solutions to the isentropic compressible fluid models of Korteweg type in $\mathbb{R}^3$. Commun. Math. Sci. **10**, 1207–1223 (2012)
[22] Tan, Z., Zhang, R.: Optimal decay rates of the compressible fluid models of Korteweg type. Z. Angew. Math. Phys. **65**, 279–300 (2014)
[23] Wang, S.B., Xu, H.Y.: On the asymptotic behavior of solution for the generalized IBq equation with hydrodynamical damped term. J. Differ. Equ. **252**, 4243–4258 (2012)
[24] Wang, Y.Z., Wei, C.H.: Asymptotic profile of global solutions to the generalized double dispersion equation via the nonlinear term. Z. Angew. Math. Phys. **69**(34), 1–17 (2018)
[25] Wang, Y.Z., Wang, Y.: Optimal decay estimate of mild solutions to the compressible Navier-Stokes-Korteweg system in the critical Besov space. Math. Meth. Appl. Sci. **41**, 9592–9606 (2018)
[26] Wang, Y.X.: On the Cauchy problem for one dimension generalized Boussinesq equation. Int. J. Math. **26**, 1550023 (2015)
[27] Wang, Y., Tan, Z.: Optimal decay rates for the compressible fluid models of Korteweg type. J. Math. Anal. Appl. **379**, 256–271 (2011)
[28] Zhang, X., Tan, Z.: Decay estimates of the non-isentropic compressible fluid models of Korteweg type in $\mathbb{R}^3$. Commun. Math. Sci. **12**, 1437–1456 (2014)

Yinxia Wang and Yu-Zhu Wang
School of Mathematics and statistics
North China University of Water Resources and Electric Power
Zhengzhou 450011
China
e-mail: yinxia117@126.com

Yu-Zhu Wang
e-mail: yuzhu108@163.com

(Received: July 11, 2019; revised: February 4, 2020)