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Torsion of a cracked elastic body by an embedded semi-infinite rigid cylinder

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Abstract. Torsion of a cracked elastic body by an embedded semi-infinite rigid cylinder is studied. A coaxial penny-shaped crack is situated in the plane of the end of the cylinder. The problem is reduced to a system of dual integral equations including Hankel and Weber–Orr transforms and then, by using ansatzs, to an integral equation of the second kind with Hankel integral operator given on a semi-infinite interval or to an equivalent infinite system of linear algebraic equations with a Hankel matrix. A detailed investigation allowed us to suggest efficient methods for solving equations for any size of the crack. In particular, accurate approximate formulas for the stress intensity factor and the contact stresses are derived, as well as an asymptotic formula for the stress intensity factor when a crack is large. Analytical estimations and calculations manifest a strong increase in the stress intensity factor and the contact stress at the end of the cylinder as the crack tip is very close to the cylinder surface.

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1. Introduction

In this article, we consider static torsion of a cracked elastic space with the semi-infinite cavity $r = a$, $0 \leq z < \infty$, $0 \leq \theta \leq 2\pi$, and the coaxial crack $z = 0$, $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$, twisted by an embedded semi-infinite rigid circular cylinder whose cylindrical surface is bonded to the surface of the cavity and whose flat end is not in contact with the elastic medium. The cylinder is rotated through an angle γ about the axis z.

The problem stated above describes the leading asymptotic term for the torsional stress–strain state of a semi-infinite elastic body deformed by rotation of a partially embedded long rigid cylinder (a pin, a single rigid cylindrical pile or a rigid cylindrical pier) that has induced a crack. Our goal is to get an efficient approximate solution by basing on rigorous analytical methods and estimations of accuracy for approximations.

If a crack is absent, various torsional problems describing torsional stress in the vicinity of the base of a rigid or elastic inclusion were studied in [4,5,8,18–22]. Note that the problem investigated in this article is a typical mixed boundary value problem, and therefore methods of the papers mentioned above cannot be applied in this case. Readers can find some mathematical similarity of the considered problem with the Reissner–Sagoci-type torsional contact problems for a half-space with a circular cylindrical hole or inclusion [1,3,15] and analogous axisymmetric contact problems [2,11–14,16].

In the dimensionless cylindrical coordinates $\rho = r/a$, θ, $\zeta = z/a$, the boundary conditions are:

Birkhäuser

$$u_\theta(1,\zeta) = a\gamma, \quad 0 \le \zeta < \infty, \tag{1}$$

$$u_\theta(\rho, 0+) = u_\theta(\rho, 0-), \ \rho \ge \beta, \tag{2}$$

$$\tau_{\theta z}(\rho, 0+) = \tau_{\theta z}(\rho, 0-), \ \rho \ge \beta, \tag{3}$$

$$\tau_{\theta z}(\rho, 0+) = 0, \ 1 \le \rho < \beta, \tag{4}$$

$$\tau_{\theta z}(\rho, 0-) = 0, \ 0 < \rho < \beta, \tag{5}$$

where $\beta = b/a > 1$.

The only nonvanishing displacement $u_\theta(\rho,\zeta)$ obeys the partial differential equation (Lurie [9])

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_\theta}{\partial\rho}\right) - \frac{u_\theta}{\rho^2} + \frac{\partial^2 u_\theta}{\partial\zeta^2} = 0, \tag{6}$$

and the non-trivial components of the stress tensor are expressed via $u_\theta(r,z)$ by the formulas

$$\tau_{\theta r} = \frac{G}{a}\rho\frac{\partial}{\partial\rho}\left(\frac{u_\theta}{\rho}\right), \quad \tau_{\theta z} = \frac{G}{a}\frac{\partial u_\theta}{\partial\zeta}, \tag{7}$$

where G is the shear moduli.

The following integral representations in the form of inverse Hankel and Weber–Orr transforms (Titchmarsh [23]) with integrands including the Bessel functions $J_\nu(\xi)$, $Y_\nu(\xi)$ and $H_\nu^{(1)}(\xi)$,

$$u_\theta(\rho,\zeta) = \begin{cases} \int\limits_0^\infty \exp(\xi\zeta) A(\xi) J_1(\xi\rho)\, \mathrm{d}\xi, & \zeta \le 0 \\ \frac{a\gamma}{\rho} + \int\limits_0^\infty \frac{\exp(-\xi\zeta)}{\left|H_1^{(1)}(\xi)\right|^2} B(\xi)\chi_1(\xi,\rho)\, \mathrm{d}\xi, & \zeta \ge 0 \end{cases}, \tag{8}$$

$$\chi_\nu(\xi,\rho) = J_1(\xi) Y_\nu(\xi\rho) - Y_1(\xi) J_\nu(\xi\rho), \tag{9}$$

satisfy the differential equation (6) and boundary condition (1). On inserting into the remaining boundary conditions (2),(3),(4) and (5), the above representations yield the system of the dual integral equations:

$$\int\limits_0^\infty \xi A(\xi) J_1(\xi\rho)\, \mathrm{d}\xi = 0, \quad 0 \le \rho < \beta, \tag{10}$$

$$\int\limits_0^\infty B(\xi) \frac{\xi\chi_1(\xi,\rho)}{\left|H_1^{(1)}(\xi)\right|^2} \mathrm{d}\xi = 0, \quad 1 \le \rho < \beta, \tag{11}$$

$$\int\limits_0^\infty \xi A(\xi) J_1(\xi\rho)\, \mathrm{d}\xi + \int\limits_0^\infty \frac{\xi B(\xi)\chi_1(\xi,\rho)}{\left|H_1^{(1)}(\xi)\right|^2} \mathrm{d}\xi = 0, \quad \rho \ge \beta, \tag{12}$$

$$\int\limits_0^\infty A(\xi) J_1(\xi\rho)\, \mathrm{d}\xi - \int\limits_0^\infty \frac{B(\xi)\chi_1(\xi,\rho)}{\left|H_1^{(1)}(\xi)\right|^2} \mathrm{d}\xi = \frac{a\gamma}{\rho}, \quad \rho \ge \beta. \tag{13}$$

In Sect. 2, by using ansatzs and discontinuous integrals, the system of the dual integral equations is reduced to an Abel-type integral equation and, after that, to an integral equation of the second kind on the infinite interval $[\beta,\infty)$ with an integral operator whose kernel depends on the sum of the arguments (a Hankel integral operator). This integral equation is shown to have in $L^2(\beta,\infty)$ a unique solution $\omega(x)$ such that $x\omega(x) \in C^\infty[\beta,\infty) \cap L^1(\beta,\infty)$ which provides a unique solution of the problem studied in the article.

In Sect. 3, the expressions in terms of $\omega(x)$ for the stress intensity factor at the tip of the crack and the contact stress between the cylinder and elastic body are derived and certain lower bounds for the quantities mentioned above are obtained. In particular, the contact stress at the base of the cylinder is shown to be finite but large as the tip of the crack is very close to the cylinder surface.

Highly efficient analytic approximate formulas for $\omega(x)$ and a quantities of interests, as well as estimates of their errors, are obtained in Sect. 4 as $\beta \geq 1.01$ by basing on the little known in the research community formulas for approximate solutions of linear operator equations given in [10].

An approach for solving the integral equation, which is efficient also for small $\beta-1$, is given in Sect. 5. The solution $\omega(x)$ is represented in the form of an orthogonal series with coefficients to be a l_1-solution of an infinite system of linear algebraic equations whose matrix operator is a sum of the unit operator and a compact in l_2 Hankel matrix operator. Truncating the infinite system to a finite system by using finite-dimensional operators with upper anti-triangular matrix as approximations for the Hankel operator, we first find an approximate l_2-solution which then is used to construct an approximate solution of the problem. The estimates of errors for approximate quantities of interest found in such a way are given. For very small $\beta-1$, the order of finite systems is rather large. In this case, we suggest a novel highly efficient iterative procedure that allows us to get an approximate solution and estimate its error. The methods suggested in this section can be helpful in various problems of mechanics, mathematical physics and engineering.

Asymptotic formula for the stress intensity factor is derived in Sect. 6 for a large crack, $\beta \geq 1.7$, by analyzing terms of the integral equation Neumann series with asymptotic evaluating arising integrals.

Results of calculations of the stress intensity factor and the contact stress are discussed in Sect. 7. Some estimates for solutions of integral equations that are exploited in the article are obtained in the "Appendix."

2. Reducing the problem to an integral equation of the second kind

To solve the system of the dual integral equations obtained in the preceding section, we make use of the integrals

$$Q_n = \int_0^\infty \sqrt{\xi}\frac{\chi_1(\xi,\rho)\,\chi_{n+\frac{1}{2}}(\xi,t)}{\left|H_1^{(1)}(\xi)\right|^2}\mathrm{d}\xi = \int_0^\infty \sqrt{\xi}J_1(\xi\rho)\,J_{n+\frac{1}{2}}(\xi t)\,\mathrm{d}\xi$$
$$-\mathrm{Re}\int_0^\infty \sqrt{\xi}\frac{J_1(\xi)}{H_1^{(1)}(\xi)}H_1^{(1)}(\xi\rho)\,H_{n+\frac{1}{2}}^{(1)}(\xi t)\,\mathrm{d}\xi, \quad \rho>\alpha,\ t>\alpha. \tag{14}$$

On evaluating the first integral [17] and rotating the path of integration in the second integral on the right side of (14) to 90°,

$$Q_0 = \int_0^\infty \sqrt{\xi}J_1(\xi\rho)\,J_{\frac{1}{2}}(\xi t)\,\mathrm{d}\xi = \sqrt{\frac{2t}{\pi}}\frac{H(\rho-t)}{\rho\sqrt{\rho^2-t^2}}, \tag{15}$$

$$Q_1 = \int_0^\infty \sqrt{\xi}J_1(\xi\rho)\,J_{\frac{3}{2}}(\xi t)\,\mathrm{d}\xi - \frac{2}{\pi}\int_0^\infty \sqrt{\xi}\frac{I_1(\xi)}{K_1(\xi)}K_1(\xi\rho)\,K_{\frac{3}{2}}(\xi t)\,\mathrm{d}\xi$$
$$= \sqrt{\frac{2}{\pi t^3}}\frac{\rho H(t-\rho)}{\sqrt{t^2-\rho^2}} - \frac{2}{\pi}\int_0^\infty \sqrt{\xi}\frac{I_1(\xi)}{K_1(\xi)}K_1(\xi\rho)\,K_{\frac{3}{2}}(\xi t)\,\mathrm{d}\xi, \tag{16}$$

where $H(x)$ is the Heaviside unit step function and $I_\nu(\xi)$ and $K_\nu(\xi)$ are modified Bessel functions.

Represent $A(\xi)$ and $B(\xi)$ in the form of the ansatzs:

$$\frac{A(\xi)}{a\gamma} = \sqrt{\xi}\int_\beta^\infty t\psi(t)\, J_{\frac{3}{2}}(\xi t)\,\mathrm{d}t \tag{17}$$

$$= \frac{\beta\psi(\beta)}{\sqrt{\xi}} J_{\frac{1}{2}}(\xi\beta) + \int_\beta^\infty \frac{1}{\sqrt{\xi t}} J_{\frac{1}{2}}(\xi t)\,\mathrm{d}\left(t^{\frac{3}{2}}\psi(t)\right), \tag{18}$$

$$\frac{B(\xi)}{a\gamma} = -\sqrt{\xi}\int_\beta^\infty t\psi(t)\,\chi_{\frac{3}{2}}(\xi,t)\,\mathrm{d}t \tag{19}$$

$$= -\frac{\beta\psi(\beta)}{\sqrt{\xi}}\chi_{\frac{1}{2}}(\xi,\beta) - \frac{1}{\sqrt{\xi}}\int_\beta^\infty \frac{1}{\sqrt{t}}\chi_{\frac{1}{2}}(\xi,t)\,\mathrm{d}\left(t^{\frac{3}{2}}\psi(t)\right), \tag{20}$$

with $\psi(t)$ to be determined.

Insert (18) and (20) into (10), (11) and (12). Interchanging the order of integration, we ascertain by exploiting (15) that the above ansatzs satisfy identically the above-said equations.

On inserting (17) and (19) into (13) and interchanging the order of integration, we obtain by exploiting (15) and (16):

$$2\rho\int_\rho^\infty \sqrt{\frac{2}{\pi t}}\frac{\psi(t)\,\mathrm{d}t}{\sqrt{t^2-\rho^2}} = \int_\beta^\infty t\psi(t)\,\widetilde{L}(t,\rho)\,\mathrm{d}t + \frac{1}{\rho}, \quad \rho \ge \beta, \tag{21}$$

where

$$\widetilde{L}(t,\rho) = \frac{2}{\pi}\int_0^\infty \sqrt{\xi}\frac{I_1(\xi)}{K_1(\xi)} K_{\frac{3}{2}}(\xi t)\, K_1(\xi\rho)\,\mathrm{d}\xi.$$

The integral term on the left side of (21) is the well-known Abel integral operator [6]. Inverting the Abel operator and evaluating arising integrals

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{K_1(\xi\rho)}{\sqrt{\rho^2-x^2}}d\rho = -\sqrt{\frac{\pi\xi}{2x}}K_{\frac{3}{2}}(\xi x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{d\rho}{\rho\sqrt{\rho^2-x^2}} = -\frac{\pi}{2x^2},$$

yields the integral equation of the second kind on semi-infinite interval

$$\sqrt{x}\psi(x) = \mathbf{L}\left(\sqrt{t}\psi(t)\right) + \sqrt{\frac{\pi}{8}}\frac{1}{x}, \quad x \in [\beta,\infty), \tag{22}$$

$$\mathbf{L}(f(t)) = \int_\beta^\infty f(t)\, L(x,t)\,\mathrm{d}t,$$

$$L(x,t) = \frac{\sqrt{tx}}{\pi}\int_0^\infty \xi\frac{I_1(\xi)}{K_1(\xi)} K_{\frac{3}{2}}(\xi x)\, K_{\frac{3}{2}}(\xi t)\,\mathrm{d}\xi > 0. \tag{23}$$

Expressing $K_{3/2}(x)$ in terms of exponential function and integrating by parts, we have the estimate $0 < L(x,t) < \Theta (xt)^{-1/2}(x+t)^{-1}$, where Θ is a constant. Then, $\int\limits_{\beta}^{\infty}\int\limits_{\beta}^{\infty} L^2(x,s)\,dxds < \infty$, and therefore the self-adjoint integral operator $\mathbf{L}$ is a positive compact operator in the Hilbert space $L^2(\beta,\infty)$, which transforms functions from $L^2(\beta,\infty)$ into function belonging to $L^2(\beta,\infty)\cap C^{\infty}[\beta,\infty)$. Hence, any $L^2(\beta,\infty)$-solution $\sqrt{x}\psi(x)$ belongs to $L^2(\beta,\infty)\cap C^{\infty}[\beta,\infty)$ and can be written in the form

$$x^{\frac{3}{2}}\psi(x) = -\int\limits_{\beta}^{x} s\omega^*(s)\,\mathrm{d}s + \beta^{\frac{3}{2}}\psi(\beta), \quad x \geq \beta, \tag{24}$$

where $s\omega^*(s)$ is an integrable on $[\beta,\infty)$ function.

By inserting (24) into (22), interchanging the order of integration and differentiating, we obtain the integral equation of the second kind:

$$(\mathbf{I}+\mathbf{R})\,\omega^*(x) = \sqrt{\beta}\psi(\beta)\,R(x+\beta), \quad x \in [\beta,\infty), \tag{25}$$

in which $\mathbf{R}$ is a positive self-adjoint integral operator whose kernel is continuous on $[\beta,\infty)\times[\beta,\infty)$,

$$\mathbf{R}\omega(x) = \int\limits_{\beta}^{\infty} \omega(s)\,R(x+s)\,\mathrm{d}s,$$

$$R(s) = \frac{1}{2}\int\limits_{0}^{\infty} \frac{I_1(\xi)}{K_1(\xi)} e^{-\xi s}\mathrm{d}\xi. \tag{26}$$

It is readily seen by using asymptotic expansions of Laplace integrals and behavior of cylindrical functions at $\xi = 0$ that $R(x+s) = O\left(1/(x+s)^3\right)$ as $x+s \to \infty$. Therefore, $\mathbf{R}$ is a positive compact operator in $L^2(\beta,\infty)$, and if $\omega^*(x)$ is a $L^2(\beta,\infty)$, then $\omega^*(x) \in C^{\infty}[\beta,\infty)\cap L^2(\beta,\infty)$. Also, the estimate $|\mathbf{R}\omega^*(x)| \leq \|\omega^*(s)\|_{L^2}\|R(x+s)\|_{L^2}$ involves $x\omega^*(x) \in C^{\infty}[\beta,\infty)\cap L^1(\beta,\infty)$. Since $\mathbf{I}+\mathbf{R}$ is a coercive operator, such a $L^2(\beta,\infty)$-solution exists and is unique for any $\beta > 0$.

Now, note that $\psi(\beta) \neq 0$. Otherwise, it follows from (25) that $\omega^*(x) = 0$ and then (24) gives $\psi(x) \equiv 0$. But the latter is impossible because $\psi(x) = 0$ does not satisfies (22).

Thus, one can write

$$\omega^*(x) = \sqrt{\beta}\psi(\beta)\,\omega(x), \tag{27}$$

$$(\mathbf{I}+\mathbf{R})\,\omega(x) = R(x+\beta),\ x \in [\beta,\infty), \tag{28}$$

and solve (28) instead of (22).

To determine the unknown constant $\psi(\beta)$, one can note that (22) involves

$$\lim_{x\to\infty} x^{\frac{3}{2}}\psi(x) = \sqrt{\frac{\pi}{8}}.$$

Setting $x = \infty$ in (24) yields

$$\psi(\beta) = \sqrt{\frac{\pi}{8\beta}}\frac{1}{\beta - \int\limits_{\beta}^{\infty} s\omega(s)\,\mathrm{d}s}. \tag{29}$$

In conclusion of this section, we note that the aforementioned properties of $\psi(x)$ and $\omega(x)$ manifest that pertaining integrals in this section converge absolutely and uniformly. This validates all formal operations made in the process of deriving the above equations. Thus, there exists an unique solution of the problem if $\omega(x)$ is a $L^2(\beta,\infty)$-solution to (28). It will be seen in the next section that this solution also leads to physically correct results.

3. Stress intensity factor and contact stress

Once the function $\psi(t)$ is determined, the stress on the crack plane $z = 0$, $r \geq b$, is given by the relation:

$$\begin{aligned}\frac{\tau_{\theta z}(r,0)}{\gamma G} &= \frac{1}{a}\int_0^\infty \xi A(\xi) J_1(\xi\rho)\,\mathrm{d}\xi \\ &= \beta\psi(\beta)\int_0^\infty \sqrt{\xi} J_{\frac{1}{2}}(\xi\beta) J_1(\xi\rho)\,\mathrm{d}\xi \\ &\quad + \int_\beta^\infty \sqrt{\frac{\xi}{t}}\left(\int_0^\infty \sqrt{\xi} J_{\frac{1}{2}}(\xi t) J_1(\xi\rho)\,\mathrm{d}\xi\right) d\left(t^{3/2}\psi(t)\right).\end{aligned}$$

Then, for $r \in [b, \infty]$,

$$\frac{\tau_{\theta z}(r,0)}{a\gamma G} = \sqrt{\frac{2\beta}{\pi}}\frac{\psi(\beta)}{r}\left[\frac{b}{\sqrt{r^2-b^2}} - \int_\beta^{\frac{r}{a}} \frac{x\omega(x)\,dx}{\sqrt{(r/a)^2 - x^2}}\right],$$

where $\psi(\beta)$ is given by (29).

Now, the stress intensity factor is given by the formula

$$K_{III} = \lim_{r\to b+0}\sqrt{2\pi(r-b)}\tau_{\theta z}(r,0) = \gamma G\sqrt{2a}\psi(\beta). \tag{30}$$

Then, by using the estimate for $\psi(\beta)$ obtained in the "Appendix," we have the lower estimate

$$K_{III} \geq \frac{\gamma G}{2\beta^2}\sqrt{\frac{\pi a}{2}}(1 + C(\beta)) > 0,$$

with

$$C(\beta) = C(\beta,\beta), C(s,\beta) = \int_0^\infty \frac{\xi\beta+1}{2\xi^2\beta}\frac{I_1(\xi)}{K_1(\xi)}e^{-\xi(\beta+s)}\mathrm{d}\xi > 0. \tag{31}$$

Thus, as it is expected, $K_{III} \to \infty$ if $\beta \to 1$.

The contact stress $\tau_{\theta r}(a,z)$, $z \geq 0$, is evaluated by making use of (20), (24), (27), (29) and (15):

$$\begin{aligned}\tau_{\theta r}(a,z) &= \frac{2\gamma G}{\pi a}\int_0^\infty \frac{B(\xi)\exp(-\xi\zeta)}{J_1^2(\xi)+Y_1^2(\xi)}\mathrm{d}\xi - 2\gamma G \\ &= \gamma G\left(\frac{\int_\beta^\infty x\omega(x)\Phi(x,z/a)\,\mathrm{d}x - \beta\Phi(\beta,z/a)}{\beta - \int_\beta^\infty x\omega(x)\,\mathrm{d}x} - 2\right) \qquad (32) \\ &= \frac{2\sqrt{\beta}K_{III}}{\sqrt{\pi a}}\left(\int_\beta^\infty x\omega(x)\Phi(x,z/a)\,\mathrm{d}x - \beta\Phi(\beta,z/a)\right) - 2\gamma G, \qquad (33)\end{aligned}$$

with

$$\Phi(x,\zeta)=\frac{1}{\sqrt{2\pi x}}\int_0^\infty \frac{e^{-\xi\zeta}\chi_{\frac{1}{2}}(\xi,x)\,\mathrm{d}\xi}{\sqrt{\xi}\left(J_1^2(\xi)+Y_1^2(\xi)\right)}=\frac{1}{\sqrt{2\pi x}}\mathrm{Im}\int_0^\infty \frac{H_{\frac{1}{2}}^{(1)}(\xi x)\,e^{-\xi\zeta}}{\sqrt{\xi}H_1^{(1)}(\xi)}\mathrm{d}\xi$$
$$=\frac{1}{2x}\int_0^\infty \frac{\exp(-\xi x)}{\xi K_1(\xi)}\cos(\xi\zeta)\,\mathrm{d}\xi,\ x\geq\beta, \tag{34}$$

where the integral $\Phi(x,\zeta)$ is transformed into the form (34) by rotating the path of integration to 90°, $\Phi(x,\zeta)=O\left(1/x^2\right)$ as $x\to\infty$, and $\Phi(x,\zeta)=O\left(1/z^2\right)$ as $z\to\infty$. Note that $k(\xi)=\exp(-\xi x)/(\xi K_1(\xi))$ is a bounded monotonically decreasing function that tends to zero as $\xi\to\infty$ and has a monotonically increasing negative derivative. Then, it is well known that the continuous function $2x\Phi(x,\zeta)$, a cosine Fourier transform of $k(\xi)$, is positive for $0\leq\zeta\leq\infty$.

At the point $z=0$, the contact stress can be written as

$$\frac{\tau_{\theta r}(a,0+)}{\gamma G}=\frac{\int_\beta^\infty x\omega(x)\left(\Phi(x,0)-\Phi(\beta,0)\right)\mathrm{d}x}{\beta-\int_\beta^\infty x\omega(s)\,\mathrm{d}x}-\Phi(\beta,0)-2.$$

According to the estimates given in the "Appendix," $\omega(x)$ and the denominator in the above formula are positive. Then, by noting $\Phi(x,0)-\Phi(\beta,0)<0$, one might see that all terms on the right side are negative, and therefore

$$-\frac{\tau_{\theta r}(a,0+)}{\gamma G}>\Phi(\beta,0)+2.$$

Now, by rewriting $\Phi(\beta,0)$ in the form

$$\Phi(\beta,0)=\frac{1}{2}\int_0^\infty\left(\frac{1}{zK_1(z)\,e^z}-e^{-\frac{z}{2}}I_0\left(\frac{z}{2}\right)\right)e^{-(\beta-1)z}dz+\frac{1}{2\sqrt{\beta^3(\beta-1)}},$$

where the integral converges absolutely and uniformly with respect to $\beta\geq1$, we infer that for very small $\beta-1$ the contact stress $\tau_{\theta r}(a,z)$ appears to be large at the end of the cylinder.

4. Approximate formulas for the stress intensity factor and contact stress

Because $\mathbf{I}+\mathbf{R}$ is a bounded coercive self-adjoint operator,

$$1\leq\frac{\langle(\mathbf{I}+\mathbf{R})\,\omega,\omega\rangle}{\|\omega\|_{L^2}^2}\leq1+\|\mathbf{R}\|_{L^2}\leq1+\rho_R,$$
$$\rho_R^2=\int_\beta^\infty\int_\beta^\infty R^2(x+s)\,\mathrm{d}x\mathrm{d}s,$$

an efficient approximate L^2-solution to (28), $\widetilde{\omega}(x)$, can be found by the formulas suggested by Malits [10]:

$$\widetilde{\omega}(x) = \mathbf{A}_n R(x+\beta),$$
$$\mathbf{A}_n = \frac{\eta}{1-q^2} P_n\left(\frac{1-\eta}{q}\mathbf{I} - \frac{\eta}{q}\mathbf{R}\right),$$
$$\eta = \frac{2}{2+\rho_R}, q = \frac{\rho_R}{2+\rho_R},$$

where the polynomial $P_n(x)$ is expressed in terms of the Chebyshev polynomials $T_n(x)$:

$$P_n(x) = \frac{\left[T_n(x)(cx-q) + T_{n+1}(x)\sqrt{1-q^2}\right]c^n - \left(1-q^2\right)q^{n-1}}{(qx-1)\,q^{n-1}},$$
$$c = 1-\sqrt{1-q^2}, P_0(x) = 1,\ P_1(x) = qx + \sqrt{1-q^2}, \tag{35}$$

and the error estimate is

$$\left\|\omega(x) - \widetilde{\omega}(x)\right\|_{L^2} \le \frac{q^{n+1}\left\|R(x+\beta)\right\|_{L^2}}{(1+\rho_R)(1-q)\left(1+\sqrt{1-q^2}\right)^n}. \tag{36}$$

The number ρ_R can be evaluated as the following:

$$\rho_R^2 = \frac{1}{4}\int_\beta^\infty\int_\beta^\infty\left(\int_0^\infty \frac{I_1(\xi)}{K_1(\xi)}e^{-\xi(x+s)}\mathrm{d}\xi \int_0^\infty \frac{I_1(\upsilon)}{K_1(\upsilon)}e^{-\upsilon(x+s)}d\upsilon\right)dtds$$
$$= \frac{1}{4}\int_0^\infty\int_0^\infty \frac{I_1(\xi)}{K_1(\xi)}\frac{I_1(\upsilon)}{K_1(\upsilon)}\frac{e^{2(\xi+\upsilon)\beta}}{(\xi+\upsilon)^2}\mathrm{d}\xi d\upsilon.$$

Using the identity

$$\frac{1}{\xi+\upsilon} = \frac{2}{\left[1-\left(\frac{\xi-1}{\xi+1}\right)\left(\frac{\upsilon-1}{\upsilon+1}\right)\right](\upsilon+1)(\xi+1)} \tag{37}$$
$$= \frac{2}{(\upsilon+1)(\xi+1)}\sum_{m=0}^\infty\left(\frac{\xi-1}{\xi+1}\right)^m\left(\frac{\upsilon-1}{\upsilon+1}\right)^m, \tag{38}$$

we have as $0 < \ \xi, \upsilon < \infty$,

$$\frac{1}{(\xi+\upsilon)^2} = \frac{4}{(\xi+1)^2(\upsilon+1)^2}\sum_{m=0}^\infty (m+1)\left(\frac{\xi-1}{\xi+1}\right)^m\left(\frac{\upsilon-1}{\upsilon+1}\right)^m. \tag{39}$$

Finally, on inserting (39) and integrating term by term, we obtain

$$\rho_R = \sqrt{\sum_{m=0}^\infty (m+1)\,c_m^2}, \tag{40}$$

where

$$c_m = \int_0^\infty \frac{I_1(\xi)}{K_1(\xi)}\left(\frac{\xi-1}{\xi+1}\right)^m\frac{e^{-2\xi\beta}}{(\xi+1)^2}\mathrm{d}\xi. \tag{41}$$

Integrating by parts shows that

$$c_m = O\left(\frac{1}{m^3}\right) \text{ as } m\to\infty. \tag{42}$$

In a similar manner, (38) involves

$$\|R(x+s)\|_{L^2} = \sqrt{\sum_{k=0}^{\infty} f_k^2(s)}, \tag{43}$$

$$f_k(s) = \frac{1}{\sqrt{2}} \int_0^{\infty} \frac{I_1(\xi)}{K_1(\xi)} \left(\frac{\xi-1}{\xi+1}\right)^k \frac{e^{-\xi(\beta+s)}}{\xi+1} \mathrm{d}\xi, \tag{44}$$

$$f_k(s) = O\left(\frac{1}{k^3}\right) \text{ as } k \to \infty. \tag{45}$$

The values of ρ_R and $\|R(x+s)\|_{L^2}$ versus β are given in Table 1.

An approximate solution $\omega_{ap}(x) \in C^{\infty}[\beta,\infty)$ can be determined by the formula

$$\omega_{ap}(x) = R(x+\beta) - \mathbf{R}\widetilde{\omega}(x) \tag{46}$$

whose error is

$$\begin{aligned} |\omega(x) - \omega_{ap}(x)| &\leq \left| \int_{\beta}^{\infty} R(x+s)\left(\omega(s) - \widetilde{\omega}(s)\right) \mathrm{d}s \right| \\ &\leq \|\omega(s) - \widetilde{\omega}(s)\|_{L^2} \|R(s+x)\|_{L^2} \\ &\leq \frac{q^{n+1} \|R(s+x)\|_{L^2} \|R(s+\beta)\|_{L^2}}{(1+\rho_R)(1-q)\left(1+\sqrt{1-q^2}\right)^n}. \end{aligned} \tag{47}$$

Using the approximate solution (46) yields an approximate formula for $\psi(\beta)$,

$$\psi(\beta) \approx \sqrt{\frac{\pi}{8\beta}} \frac{1}{\beta - \int_{\beta}^{\infty} s\omega_{ap}(s)\,\mathrm{d}s}, \tag{48}$$

with

$$\frac{1}{\beta} \int_{\beta}^{\infty} s\omega_{ap}(s)\,\mathrm{d}t = C(\beta) - \int_{\beta}^{\infty} C(s,\beta)\,\widetilde{\omega}(s)\,\mathrm{d}s,$$

whose error estimate can be found by means of (36) and

$$\left| \int_{\beta}^{\infty} s\omega(s)\,\mathrm{d}s - \int_{\beta}^{\infty} s\omega_{ap}(s)\,\mathrm{d}s \right| \leq \left| \beta \int_{\beta}^{\infty} C(s,\beta)\left(\omega(s) - \widetilde{\omega}(s)\right) \mathrm{d}s \right| \tag{49}$$

$$\leq \frac{\beta q^{n+1} \|C(s,\beta)\|_{L^2} \|R(s+\beta)\|_{L^2}}{(1+\rho_R)(1-q)\left(1+\sqrt{1-q^2}\right)^n}. \tag{50}$$

The values of $\|C(s,\beta)\|_{L^2}$ again can be found by using (38):

$$\|C(s,\beta)\|_{L^2} = \sqrt{\sum_{k=0}^{\infty} C_k^2(\beta)},$$

$$C_k(\beta) = \frac{1}{\sqrt{2}} \int_0^{\infty} \frac{I_1(\xi)}{K_1(\xi)} \left(\frac{\xi-1}{\xi+1}\right)^k \frac{(\xi\beta+1)\,e^{-2\beta\xi}\mathrm{d}\xi}{\xi^2(\xi+1)\,\beta}.$$

TABLE 1. *Quantities pertaining to approximate formulas and estimates*

β	1.0004	1.01	1.02	1.1	1.2	1.5	2	2.5
ρ_R	0.326	0.187	0.153	0.075	0.048	0.021	0.009	0.006
q	0.403	0.086	0.071	0.036	0.023	0.010	0.094	0.003
$\left\|C(x,\beta)\right\|_{L^2}$	0.361	0.339	0.325	0.259	0.212	0.138	0.085	0.059
$\left\|R(x+\beta)\right\|_{L^2}$	5.506	0.958	0.619	0.207	0.092	0.030	0.011	0.005
Φ	7.202	1.644	1.453	1.025	0.852	0.644	0.505	0.433

As given in Table 1, the simplest approximate inverse operator $\mathbf{A}_0$ can be taken in a wide interval of the parameter β. Then,

$$\omega_{ap}(x) = R(x+\beta) - \frac{2+\rho_R}{2+2\rho_R}\int_\beta^\infty R(x+s)\,R(s+\beta)\,\mathrm{d}s + R(x+\beta)$$

$$= R(x+\beta) - \frac{2+\rho_R}{2(1+\rho_R)}\sum_{k=0}^\infty f_k(x)\,f_k(\beta)\,. \tag{51}$$

This leads to the approximate formula for the stress intensity factor at $r=b$:

$$K_{III} \approx \frac{\sqrt{\pi a}\gamma G}{2\beta^{3/2}}\left(1 - C(\beta) + \frac{(2+\rho_R)}{2(1+\rho_{R-})}\sum_{k=0}^\infty C_k(\beta)\,f_k(\beta)\right)^{-1}. \tag{52}$$

Evaluations show that a relative error of (52) monotonically decreases, is less than 1.45% as $\beta \geq 1.01$, is less than 0.75% as $\beta \geq 1.02$ and is less than 0.18% as $\beta \geq 1.1$.

To derive an approximate formula for the contact stress at the end of the cylinder $\tau_{\theta r}(a,0+)$, we take $\omega(x) \approx \widetilde{\omega}(x) = \mathbf{A}_0 R(x+\beta)$ and again exploit (38). Then,

$$\tau_{\theta r}(a,0+) \approx \widehat{\tau} = \frac{2\sqrt{\beta}K_{III}}{\sqrt{\pi a}}\left(\int_\beta^\infty x\widetilde{\omega}(x)\,\Phi(x,0)\,\mathrm{d}x - \beta\Phi(\beta,0)\right) - 2\gamma G$$

$$= \frac{2\sqrt{\beta}K_{III}}{\sqrt{\pi a}}\left(\frac{\eta}{1-q^2}\sum_{k=0}^\infty f_k(\beta)\,\Phi_k(\beta) - \beta\Phi(\beta,0)\right) - 2\gamma G,$$

$$\Phi_k(\beta) = \frac{1}{\sqrt{2}}\int_0^\infty \frac{e^{-\xi\beta}}{\xi K_1(\xi)}\left(\frac{\xi-1}{\xi+1}\right)^k \frac{\mathrm{d}\xi}{\xi+1}, \tag{53}$$

with an error

$$\frac{\left|\tau_{\theta r}(a,0+) - \widehat{\tau}\right|\sqrt{a}}{K_{III}} \leq 2\sqrt{\frac{\beta}{\pi}}\left\|\omega(x) - \widetilde{\omega}(x)\right\|_{L^2}\left\|x\Phi(x,0)\right\|_{L^2}$$

$$\leq \frac{2q\sqrt{\beta}\left\|R(x+\beta)\right\|_{L^2}\left\|x\Phi(x,0)\right\|_{L^2}}{\sqrt{\pi}(1+\rho_R)(1-q)},$$

$$\left\|x\Phi(x,0)\right\|_{L^2} = \sqrt{\sum_{k=0}^\infty \Phi_k^2(\beta)} < \Phi,$$

$$\Phi = \int_0^\infty \frac{e^{-\xi\beta}\mathrm{d}\xi}{(2\xi)^{3/2} K_1(\xi)} = \frac{e^{\frac{\beta-1}{\pi}}}{2\sqrt{\pi}} K_0\left(\frac{\beta-1}{\pi}\right) + \\ + \frac{1}{\sqrt{8}} \int_0^\infty \left(\frac{1}{\xi K_1(\xi) e^{\xi}} - \frac{\sqrt{\frac{2}{\pi}}}{\sqrt{\xi + \frac{2}{\pi}}} \right) \frac{e^{-(\beta-1)\xi}}{\sqrt{\xi}} \mathrm{d}\xi.$$

Evaluations of the above coarse estimate manifest that a relative error of (53) monotonically decreases and is less than 1.06% as $\beta \geq 1.01$, is less than 0.22% as $\beta \geq 1.1$ and is less than 0.01% as $\beta \geq 1.5$.

5. Reducing the problem to an infinite system of linear algebraic equations

The solution based on employing iterations of the integral operator suggested in the preceding section becomes inconvenient for very small $\beta - 1$ because of difficulties to ensure accurate calculating iterations as well as computing time which is needed. In this section, we will develop a more robust approach by reducing the integral equation to an infinite system of linear algebraic equations.

Let us represent a solution of the integral equation in the form of a series expansion in the orthonormal basis $\{\mathfrak{L}_m(s)\}$,

$$\omega(s) = \sum_{m=0}^\infty \omega_m \mathfrak{L}_m(x), \tag{54}$$

$$\mathfrak{L}_m(s) = \sqrt{2} e^{\beta - x} L_m(2(x-\beta)), \tag{55}$$

where $L_m(x)$ are Laguerre polynomials, and by exploiting the integral

$$\int_\beta^\infty \mathfrak{L}_m(s) e^{-\xi x} \mathrm{d}x = \frac{e^{-\xi\beta}(\xi-1)^n}{\sqrt{2}(\xi+1)^{n+1}},$$

expand $R(x+s)$ into $\mathfrak{L}_m(x)$ series

$$R(x+s) = \sum_{m=0}^\infty f_m(s) \mathfrak{L}_m(x), \tag{56}$$

where $f_m(s)$ is defined by (44). The above series converges absolutely and uniformly on $[\beta, \infty)$ because both $\omega(x)$ and $R(x+s)$ belong to $C^\infty[\beta,\infty) \cap L^1[\beta,\infty)$.

Inserting (54) and (44) into (28), on integrating and equating coefficients of like elements of the basis $\{\mathfrak{L}_m(x)\}$, we obtain an infinite system of algebraic equations:

$$\begin{aligned} &\overrightarrow{\omega} = -\mathbf{\Omega}\overrightarrow{\omega} + \overrightarrow{f}, \\ &\overrightarrow{\omega} = (\omega_0, \omega_1, \omega_2, \ldots),\ \overrightarrow{f} = (f_0, f_1, f_2, \ldots), \\ f_m = f_m(\beta),\quad &\mathbf{\Omega}\overrightarrow{\omega} = \sum_{n=0}^\infty \omega_n c_{n+m}, \end{aligned} \tag{57}$$

where the matrix elements of the Hankel matrix operator c_{n+m} are given by (41) and $f_m(\beta)$ by (44). Here, the operator $\mathbf{\Omega}$, as a matrix representation of the operator $\mathbf{R}$, is compact in l_2, positive and

$$\sum_{m,n=0}^\infty c_{n+m}^2 = \rho_R^2.$$

Consequently, the operator $\mathbf{I}+\mathbf{\Omega}$ is coercive: for $\overrightarrow{z} \in l_2$,

$$\|\overrightarrow{z}\|_{l_2}^2 \leq ((\mathbf{I}+\mathbf{\Omega})\,\overrightarrow{z},\overrightarrow{z}) \leq (1+\rho_R)\,\|\overrightarrow{z}\|_{l_2}^2, \tag{58}$$

and therefore (57) has in l_2 a unique solution for any $\beta > 1$. By virtue of (45) and (42), a l_2-solution belongs to l_1 also.

The compact matrix operator $\mathbf{\Omega}$ can be approximated by some finite-dimensional operators $\mathbf{\Omega}_N$ obtained with certain truncations of $\mathbf{\Omega}$ under the condition that a sequence of operators $\mathbf{\Omega}_N$ converges to $\mathbf{\Omega}$ in l_2 as $N \to \infty$. We take the operators with upper anti-triangular matrices

$$\mathbf{\Omega}_N\left(\overrightarrow{\omega}\right) = \begin{cases} \sum_{n=0}^{2N-m} \omega_n c_{n+m}, & m \leq 2N \\ 0,\ m > 2N \end{cases}$$

as such operators. Then, $\|\mathbf{\Omega}_N\|_{l_2} \leq \rho_N$,

$$\rho_N^2 = \sum_{m=0}^{2N}\sum_{n=0}^{2N-m} c_{n+m}^2 = \sum_{m=0}^{2N}(m+1)\,c_m^2,$$

$$\|\mathbf{\Omega}-\mathbf{\Omega}_N\|_{l_2}^2 \leq \sum_{m=2N+1}^{\infty}(m+1)\,c_m^2 = \rho_R^2 - \rho_N^2, \tag{59}$$

and therefore $\|\mathbf{\Omega}-\mathbf{\Omega}_N\|_{l_2} \to 0$ as $N \to \infty$.

Thus, the approximation of coefficients ω_n in l_2 can be found by determining a vector $\overrightarrow{\widetilde{\omega}} = (\widetilde{\omega}_0, \widetilde{\omega}_1, \widetilde{\omega}_2, \ldots)$ from the system

$$\widetilde{\omega}_m + \sum_{n=0}^{2N-m}\widetilde{\omega}_n c_{n+m} = f_m, \quad m = 0,1,2,\ldots,2N, \tag{60}$$

$$\widetilde{\omega}_m = f_m, \quad m \geq 2N+1. \tag{61}$$

The matrix operator $\mathbf{\Omega}_N$ can be represented as a sum:

$$\mathbf{\Omega}_N = \widetilde{\mathbf{\Omega}}_N + \widehat{\mathbf{\Omega}}_N,\ \widetilde{\mathbf{\Omega}}_N\overrightarrow{\omega} = \begin{cases} \sum\limits_{n=0}^{N}\omega_n c_{n+m}, & m \leq N \\ 0,\ m > N \end{cases},$$

where the Hankel operator $\widetilde{\mathbf{\Omega}}_N$ is positive in l_2 and in the Euclidean space $\mathbb{R}^n$, $n \geq 2N+1$,

$$\left(\overrightarrow{\omega},\ \widetilde{\mathbf{\Omega}}_N\overrightarrow{\omega}\right) = \int_0^{\infty}\frac{I_1(\xi)}{K_1(\xi)}\frac{e^{-2\beta\xi}}{(\xi+1)^2}\left(\sum_{n=0}^{N}\omega_n\left(\frac{\xi-1}{\xi+1}\right)^n\right)^2 \mathrm{d}\xi,$$

while

$$\left\|\widehat{\mathbf{\Omega}}_N\right\|^2 \leq \widehat{\rho}_N^2 = 2\sum_{m=0}^{N} m c_{N+m}^2,$$

$$\widehat{\rho}_N = \sqrt{\sum_{m=0}^{N} O\left(\frac{m}{(m+N)^6}\right)} = O\left(\frac{1}{N^2}\right) \text{ as } N \gg 1.$$

Now, it follows from

$$(1-\widehat{\rho}_N)\,\|\overrightarrow{z}\|^2 \leq ((\mathbf{I}+\mathbf{\Omega}_N)\,\overrightarrow{z},\overrightarrow{z}) \leq (1+\rho_N)\,\|\overrightarrow{z}\|^2, \tag{62}$$

that the operator $\mathbf{I}+\mathbf{\Omega}_N$ is coercive at least for sufficiently large N. In such a case, the system (60) has a unique solution.

To estimate an error of the approximate solution $\overrightarrow{\widetilde{\omega}}$, we note that the equality

$$(\mathbf{I}+\mathbf{\Omega})\,\overrightarrow{\omega}=\overrightarrow{f}=(\mathbf{I}+\mathbf{\Omega}_N)\,\overrightarrow{\widetilde{\omega}}$$

involves

$$\left((\mathbf{I}+\mathbf{\Omega})\left(\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right),\left(\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right)\right)=\left((\mathbf{\Omega}_N-\mathbf{\Omega})\,\overrightarrow{\widetilde{\omega}},\left(\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right)\right).$$

Hence, due to (58)

$$\left\|\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right\|_{l_2}^2\leq\|\mathbf{\Omega}_N-\mathbf{\Omega}\|_{l_2}\left\|\overrightarrow{\widetilde{\omega}}_n\right\|_{l_2}\left\|\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right\|_{l_2},$$

and then by virtue of (59) the error estimate is

$$\left\|\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right\|_{l_2}\leq\sqrt{\rho_R^2-\rho_N^2}\left\|\overrightarrow{\widetilde{\omega}}_n\right\|_{l_2}.$$

The series

$$\widetilde{\omega}\,(x)=\sum_{m=0}^{\infty}\widetilde{\omega}_m\mathfrak{L}_m\,(x) \tag{63}$$

will give L_2 approximations of $\omega\,(x)$. Hence, by using Parseval's equation

$$\int_{\beta}^{\infty}x\widetilde{\omega}\,(x)\,\Phi\,(x,0)\,\mathrm{d}x=\frac{1}{2}\sum_{m=0}^{\infty}\widetilde{\omega}_m\Phi_k\,(\beta)\,,$$

$$\begin{aligned}\left|\int_{\beta}^{\infty}x\,(\omega\,(x)-\widetilde{\omega}\,(x))\,\Phi\,(x,0)\,\mathrm{d}x\right|&\leq\frac{1}{2}\sum_{m=0}^{\infty}|\omega_m-\widetilde{\omega}_m|\,|\Phi_k\,(\beta)|\\&\leq\frac{\sqrt{\rho_R^2-\rho_N^2}}{2}\left\|\overrightarrow{\widetilde{\omega}}_n\right\|_{l_2}\|\Phi\,(x,0)\|_{L^2}.\end{aligned}$$

Inserting the (63) series in (25) yields an accurate approximate solution in $C[\beta,\infty)\cap L^1\,(\beta,\infty)$,

$$\omega\,(x)\approx\phi\,(x)=-\sum_{m=0}^{\infty}\widetilde{\omega}_m f_m\,(x)+R\,(x+\beta)\,,$$

$$\begin{aligned}|\omega\,(x)-\phi\,(x)|&\leq\sum_{m=0}^{\infty}|\omega_m-\widetilde{\omega}_m|\,|f_m\,(x)|\leq\left\|\overrightarrow{\omega}-\overrightarrow{\widetilde{\omega}}\right\|_{l_2}\|f_m\,(x)\|_{l_2}\\&\leq\sqrt{\rho_R^2-\rho_N^2}\left\|\overrightarrow{\widetilde{\omega}}_n\right\|_{l_2}\|R\,(x+\beta)\|_{L^2}\,,\end{aligned} \tag{64}$$

and then

$$\frac{1}{\beta}\int_{\beta}^{\infty}s\omega\,(s)\,\mathrm{d}s\approx C\,(\beta)-\sum_{m=0}^{\infty}\widetilde{\omega}_m C_m\,(\beta)\,,$$

$$\begin{aligned}\left|\int_{\beta}^{\infty}s\omega\,(s)\,\mathrm{d}s-\int_{\beta}^{\infty}s\phi\,(s)\,\mathrm{d}s\right|&\leq\beta\sum_{m=0}^{\infty}|\omega_m-\widetilde{\omega}_m|\,|C_m\,(\beta)|\\&\leq\beta\sqrt{\rho_R^2-\rho_N^2}\left\|\overrightarrow{\widetilde{\omega}}_n\right\|_{l_2}\|C\,(s,\beta)\|_{L^2}\,.\end{aligned} \tag{65}$$

Some additional simplification can be achieved by truncating infinite series in (64) and (65) by using (61).

When $\beta - 1$ is very small, evaluations show that matrix element c_{n+m} are decreasing very slowly and then $\sqrt{\rho_R^2 - \rho_N^2}$ becomes small only if $2N$ is rather large. For example, $\sqrt{\rho_R^2 - \rho_{200}^2} = 0.0\,412$ for $\beta = 1.0004$. It is important in this situation that for evaluation of matrix elements of $\mathbf{\Omega}_N$ one should evaluate $2N+1$ integrals $c_0, c_1, \ldots, c_{2N}$ only. Another important point in such a case is that for achieving good accuracy it is preferable to solve a finite system of linear algebraic equation by some iterative algorithm which is rapidly convergent in $\mathbb{R}^{2N+1}$ norm. Taking into account that self-adjoint operator $\mathbf{\Omega}_N$ might be chosen to be a coercive operator, one can again employ results of the paper by Malits [10] for constructing an iterative procedure. Then, an approximate solution has the form

$$\begin{aligned} \overrightarrow{Z} &\approx \frac{\eta}{1-q^2} P_M \left(\frac{1-\eta}{q} \mathbf{I} - \frac{\eta}{q} \mathbf{\Omega}_N \right) \overrightarrow{F}, \\ \eta &= \frac{2}{2 + \rho_N - \widehat{\rho}_N}, \quad q = \frac{\rho_N + \widehat{\rho}_N}{2 + \rho_N - \widehat{\rho}_N}, \end{aligned} \tag{66}$$

where $\overrightarrow{Z} = (\widetilde{\omega}_0, \widetilde{\omega}_1, \ldots, \widetilde{\omega}_{2N})$, $\overrightarrow{F} = (f_0, f_1, \ldots, f_{2N})$ and the polynomial $P_n(x)$ is defined by (35).

Using the recurrent formula for Chebyshev polynomials

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), T_0(x) = 1, \quad T_1(x) = x,$$

one derives from (35)

$$\begin{aligned} P_n(x) &= P_{n-1}(x) + q\lambda^{n-1} \left(T_n(x) - \lambda T_{n-1}(x) \right), \\ \lambda &= \frac{q}{1 + \sqrt{1-q^2}}, \quad P_0(x) = 1. \end{aligned}$$

Hence, we obtain the following iterative algorithm:

$$\begin{aligned} \overrightarrow{Z} &\approx \frac{\eta}{1-q^2} \overrightarrow{\mathcal{P}}_M, \\ \overrightarrow{\mathcal{P}}_k &= \overrightarrow{\mathcal{P}}_{k-1} + q\lambda^{k-1} \left(\overrightarrow{\mathcal{G}}_k - \lambda \overrightarrow{\mathcal{G}}_{k-1} \right), \\ \overrightarrow{\mathcal{G}}_k &= 2 \left(\frac{1-\eta}{q} \mathbf{I} - \frac{\eta}{q} \mathbf{\Omega}_N \right) \overrightarrow{\mathcal{G}}_{k-1} - \overrightarrow{\mathcal{G}}_{k-2}, \\ \overrightarrow{\mathcal{P}}_0 &= \overrightarrow{\mathcal{G}}_0 = \overrightarrow{F}, \quad \overrightarrow{\mathcal{G}}_1 = \left(\frac{1-\eta}{q} \mathbf{I} - \frac{\eta}{q} \mathbf{\Omega}_N \right) \overrightarrow{F}, \end{aligned}$$

whose error estimate

$$\left\| \overrightarrow{Z} - \frac{\eta}{1-q^2} \overrightarrow{\mathcal{P}}_M \right\|_{\mathbb{R}^{2N+1}} \leq \varepsilon_M = \frac{q\lambda^M \left\| \overrightarrow{F} \right\|_{\mathbb{R}^{2N+1}}}{(1+\rho_N)(1-q)}$$

is less than error estimates of other algorithms based on using (explicitly or implicitly) M iterations of operator $\mathbf{\Omega}_N$ [10]. For $N = 200$ and $\beta \geq 1.0004$, $\varepsilon_2 \leq 0.003\,07$ and $\varepsilon_3 \leq 0.00022$.

6. Stress intensity factor for a large crack.

The norms of the operator $\mathbf{R}$ in $C[\beta, \infty)$ and in $L^1(\beta, \infty)$ are given by the Laplace integral:

$$\|\mathbf{R}\|_C = \|\mathbf{R}\|_{L^1} = \int_\beta^\infty R(\beta + x)\, \mathrm{d}x = q(\beta),$$

$$q(\beta) = \frac{1}{2} \int_0^\infty \frac{I_1(\xi)\, e^{-2\xi\beta}}{\xi K_1(\xi)} \mathrm{d}\xi,$$

where $q(\beta)$ is a monotonically decreasing function of the parameter β, $q(1.0004) = 0.985$. Then, the Neumann series

$$\omega(x) = \sum_{k=0}^{\infty} (-\mathbf{R})^k R(x+\beta) \tag{67}$$

converges absolutely and uniformly for $\beta \geq 1.0004$.

The inequality

$$0 \leq \mathbf{R}R(x+\beta) \leq R(x+\beta)\, q(\beta)$$

involves that for $\beta \geq 1.0004$ the terms of the Neumann series are alternating and monotonically decreasing in magnitude. For large β:

$$\left\| (-\mathbf{R})^k R(x+\beta) \right\|_C \leq q^{k+1}(\beta), \tag{68}$$

$$\left\| (-\mathbf{R})^k R(x+\beta) \right\|_{L^1} \leq q^{k+1}(\beta), \tag{69}$$

$$q(\beta) = \frac{1}{4\beta^2} + O\left(\frac{1}{\beta^3}\right). \tag{70}$$

The above estimates manifest that terms of the Neumann series constitute an asymptotic scale as $(2\beta)^2$ is large and the Neumann series therefore is an asymptotic expansion. Hence, we find

$$\int_\beta^\infty \frac{x}{\beta}\omega(x)\,\mathrm{d}x = C(\beta) - \int_\beta^\infty C(x,\beta)\left(\sum_{k=0}^{N-1}(-\mathbf{R})^k R(x+\beta)\right)\mathrm{d}x + O\left(\frac{1}{(2\beta)^{2N+4}}\right),$$

where the asymptotic order of error is derived by using the estimates

$$C(x,\beta) \leq C(\beta) = \frac{3}{16\beta^2} + O\left(\frac{\ln(2\beta)}{(2\beta)^4}\right),$$

(69) and (70):

$$\int_\beta^\infty C(x,\beta)\left(\mathbf{R}^N R(x+\beta)\right)\mathrm{d}x \leq C(\beta)\max\int_\beta^\infty \left(\mathbf{R}^N R(x+\beta)\right)\mathrm{d}x$$

$$\leq C(\beta)\, q^{N+1}(\beta) = O\left(\frac{1}{(2\beta)^{2N+4}}\right).$$

Taking $N = 2$ and expanding Laplace integrals $C(x,\beta)$ and $R(s+x)$ into asymptotic series, we finally obtain

$$\int_\beta^\infty x\omega(x)\,\mathrm{d}t = \beta\left(H(\beta) + r(\beta)\right),$$

$$H(\beta) = \frac{3}{16\beta^2} - \frac{d_1 - d_2\ln(2\beta)}{(2\beta)^4} - \frac{d_3 + d_4\ln(2\beta) - 7d_2\ln^2(2\beta)}{(2\beta)^6},$$

$$d_1 = 1.782\,28,\ d_2 = \frac{3}{2},\ d_3 = 17.261\,73,\quad d_4 = 25.445\,50,$$

$$r(\beta) = \frac{d_5}{\beta^6} + o\left(\frac{1}{(2\beta)^6}\right),\quad 0 < d_5 < \frac{11}{98\,304}. \tag{71}$$

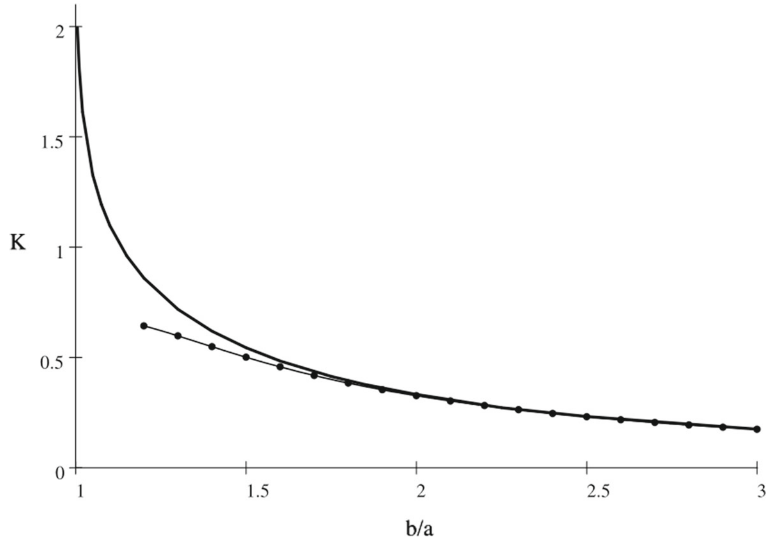

FIG. 1. Normalized stress intensity factor $K = \frac{K_{III}}{\sqrt{a}\gamma G}$ versus $\frac{b}{a}$

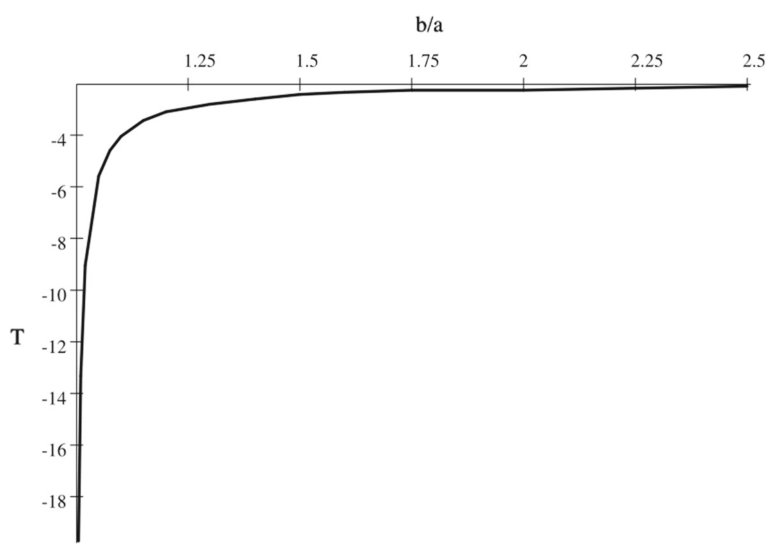

FIG. 2. Normalized contact stress $T = \frac{\tau_{\theta r}(a,0+)}{\gamma G}$ versus $\frac{b}{a}$

The approximate formula for the stress intensity factor

$$K_{III} \approx \frac{\sqrt{\pi a}\gamma G}{2\beta^{3/2}\left(1 - H\left(\beta\right)\right)} \tag{72}$$

is asymptotically exact for large $4\beta^2$, and its relative error is a decreasing function of the parameter β. For $\beta = 1.75$, the relative error is 3.4%, and for $\beta = 2.5$ the relative error is 0.5%.

7. Results

The results of calculations of the stress intensity factor are represented in Fig. 1, where the dotted line gives approximate values evaluated by the asymptotic formula (72). We observe a very rapid decrease in the stress intensity factor as the parameter β is increasing. This means that one can expect a state of the crack arrest for a sufficiently large $\beta = b/a$. If the tip of the crack is very close to the end of the rigid cylinder, then the crack can be unstable.

The normalized contact stress $T = \tau_{\theta r}(a, 0+)/\gamma G$ is presented in Fig. 2. Despite the fact that the contact stress is finite for any $\beta > 1$, the contact stress is very large as the tip of the crack is very close to the end of the rigid cylinder. In the latter case, a debonding or damage zone can be developed at the end of the cylinder along the lateral surface.

8. Conclusions

We study torsion of an elastic space by an embedded semi-infinite rigid cylinder which has caused a coaxial penny-shaped crack in the plane of the cylinder end. The problem is reformulated as a system of dual integral equations containing inverse Bessel and Weber–Orr integral transforms. By using special ansatzs, the system of dual integral equations is reduced to solving an integral equation of the second kind with Hankel integral operator given on a semi-infinite interval. The stress intensity factor and contact stress are expressed in terms of a solution to the integral equation. A detailed investigation of the integral equation permits us to suggest accurate methods for its efficient solving for any size of the crack. In particular, approximate formulas for the contact stress at the end of the cylinder and stress intensity factor are derived. When the crack is large, an asymptotic formula for the stress intensity factor is obtained as well. Analytical estimations and calculations manifest a strong increase in the stress intensity factor and the contact stress at the end of the cylinder as the crack tip is very close to the cylinder surface.

9. Appendix: Some estimates for $\omega(x)$ and $\psi(b)$

Let us rewrite (28) in the form

$$\omega(x) = ((1-\mu)\mathbf{I} - \mu\mathbf{R})\,\omega(x) + \mu R(x+\beta),$$
$$0 < \mu < \frac{1}{\|\mathbf{I}+\mathbf{R}\|_{L^2}},$$

where the self-adjoint operator $(1-\mu)\mathbf{I} - \mu\mathbf{R}$ is positive. According to [7], there exists a L^2-solution of the above equation that is a limit of the successive approximations

$$\omega_{n+1}(x) = ((1-\mu)\mathbf{I} - \mu\mathbf{R})\,\omega_n(x) + \mu R(x+\beta),$$
$$\omega_0(x) = \mu R(x+\beta),$$

and consequently this solution is positive and, as it shown in Sect. 2, unique,

$$\omega(x) > 0. \tag{73}$$

We have from (28)

$$x\omega(x) + \frac{xR(x+\beta)}{\beta}\int_{\beta}^{\infty} s\omega(s)\,\mathrm{d}s \geq xR(x+\beta).$$

Then, on integrating

$$(1 + C(\beta)) \int_{\beta}^{\infty} s\omega(s)\,\mathrm{d}s \geq \int_{\beta}^{\infty} xR(x+s)\,\mathrm{d}x = \beta C(\beta),$$

$$\int_{\beta}^{\infty} s\omega(s)\,\mathrm{d}s \geq \frac{\beta C(\beta)}{1 + C(\beta)}, \tag{74}$$

where $C(\beta)$ is defined by (31).

Now, (29) involves

$$\psi(\beta) \geq \sqrt{\frac{\pi}{8\beta^3}}\,(1 + C(\beta)) > 0.$$

Note that $C(\beta)$ is a monotonically decreasing function of the parameter β and $C(1) = \infty$.

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