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Global solvability of prey–predator models with indirect predator-taxis

Inkyung Ahn and Changwook Yoon

Abstract. This paper analyzes prey–predator models with indirect predator-taxis in such a way that chemical secreted by the predator triggers the repellent behavior of prey against the predator. Under the assumption of quadratic decay of predator, we prove the global existence and uniform boundedness of classical solutions up to two spatial dimensions. Moreover, via the linear stability analysis, we show that large chemosensitivity gives rise to the occurrence of pattern formations. We also obtain the global stability results for the nontrivial constant steady states by establishing proper Lyapunov functionals.

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1. Introduction

Many observations in nature demonstrate that the emergence of prey or predator can induce the directed movement of a species. Researchers have proposed and investigated several mathematical models with prey-taxis or predator-taxis to describe the prey–predator behavior involving the advection effect (see [5,10,12,21,32,33]). Instead of advection induced by direct contact between prey and predator, chemical effects such as specific odor, pheromone, and excrement may influence the mobility of species. Taking the effects of chemicals secreted by the predator on prey into consideration, we may come up with the chemotactic behavior of the indirect predator-taxis as a survival strategy of prey. For example, in nature, some whales avoid the underwater sounds of killer whales as a sign of potential danger or juvenile toads detect and avoid chemical cues from snake species that prey on them (see [6–8]).

In this paper, we study a diffusion–advection–reaction PDE model with indirect predator-taxis, i.e., we assume that the prey moves away from the gradient of a chemical emitted by the predator. From this perspective, we consider the following 3-component parabolic system:

$$\begin{cases} u_t = d_u \Delta u + \nabla \cdot (\chi u \nabla c) + f(u) - v g(u,v), & x \in \Omega, \quad t > 0, \\ c_t = d_c \Delta c + \alpha v - \beta c, & x \in \Omega, \quad t > 0, \\ v_t = d_v \Delta v + r v g(u,v) - k(v) v, & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad c(x,0) = c_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$, and ν denote the outer normal vector at the boundary. The unknowns u, c and v represent the population densities of prey, the concentration of the chemical and the population densities of predator, respectively. In addition, $\chi > 0$ is the chemotactic sensitivity of prey, $f(u)$ is the growth rate of prey, $g(u,v)$ is the functional response of predation, $k(v)$ is the mortality rate of predator, and constants d_u, d_c, d_v, α, β, r are positive.

We assume that $f(u)$ satisfies the following hypothesis:

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$(H1)$ The function $f : [0,\infty) \to [0,\infty)$ is continuously differentiable, and there exist two constants $N_1, N_2 > 0$ such that $f(u) \le N_1 u - N_2 u^2$ for any $u \ge 0$.

The hypothesis $(H1)$ indicates that the growth of prey follows the logistic law. On the other hand, we assume that $g(u,v)$ satisfies either $(H2)$ or $(H3)$ below.

$(H2)$ The function $g : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuously differentiable, $g(0,v) = 0$, $g_u(u,v) > 0$ and there exists a constant $N_3 > 0$ such that $g(u,v) \le N_3$ for any $u, v \ge 0$.

$(H3)$ The function $g : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuously differentiable, $g(0,v) = 0$, $g_u(u,v) > 0$ and there exists a constant $N_3 > 0$ such that $g(u,v) \le N_3 u$ for any $u, v \ge 0$.

The hypothesis $(H2)$ covers various type of functional responses such as:

$$\begin{aligned}
&g(u,v) = g(u) = \frac{u}{d_1 u + d_2} \ \text{ (Holling type } II);\\
&g(u,v) = \frac{u}{d_1 u + d_2 v} \ \text{ (Ratio-dependent type)};\\
&g(u,v) = \frac{u}{d_1 u + d_2 v + d_3} \ \text{ (Beddington-DeAngelis type)}
\end{aligned}$$

for some $d_1, d_2, d_3 > 0$. The typical example of $(H3)$ is the Lotka–Volterra type, i.e., $g(u,v) = u$. As for $k(v)$, we assume the following:

$(H4)$ The function $k : [0,\infty) \to (0,\infty)$ is continuously differentiable, and there exist three constants $k_1, k_2 > 0$, $k_3 \in \mathbb{R}$ such that $k'(v) \ge k_1$ and $k(v) \ge k_2 v + k_3$ for any $v \ge 0$.

The hypothesis $(H4)$ accounts for the intra-specific competition ($k_2 > 0$) or the logistic growth ($k_3 < 0$).

The indirect signal production mechanisms have been studied for various taxis-type models. When the predator detects the chemical emitted by prey and chases the trace, we can adopt the role of chemicals into the prey-taxis model. Recently, there have been several related studies on the indirect prey-taxis models (see [1,27,28]). Note that system (1.1) shares the same dynamics as the indirect prey-taxis models except for the taxis term. The indirect mechanisms can also be applied to a chemotaxis model related to two different states of a species. In this regard, Tao and Winkler [25] proposed a prototypical parabolic–elliptic ODE system describing the spread and aggregative behavior of the Mountain Pine Beetle (MPB) as the following:

$$\begin{cases}
u_t = \Delta u - \nabla \cdot (u \nabla c), & x \in \Omega, \quad t > 0,\\
c_t = \Delta c - \frac{1}{|\Omega|}\int_\Omega v + v, & x \in \Omega, \quad t > 0,\\
\tau v_t = u - \delta v, & x \in \Omega, \quad t > 0,
\end{cases}$$

where u and v denote the density of flying MPB and nesting MPB, respectively, and c is the concentration of beetle pheromone. They verified that the indirect mechanism plays a significant role in excluding the occurrence of blow-up in finite time. We refer the reader to [9,14,16] for more related results.

We state the first main result about the global existence of solutions to (1.1).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $f(u)$ and $k(v)$ satisfy $(H1)$ and $(H4)$, respectively. Assume that either $g(u,v)$ satisfies $(H2)$ if $n \ge 1$, or $(H3)$ if $n \in \{1,2\}$. Then, for any initial data $(u_0, c_0, v_0) \in [W^{1,p}(\Omega)]^3$ with $p > n$ satisfying $u_0(x)$, $c_0(x)$, $v_0(x) \ge 0 (\not\equiv 0)$ for $x \in \Omega$, system (1.1) possesses a unique global-in-time nonnegative classical solution (u,c,v) such that*

$$(u,c,v) \in \left[\mathcal{C}([0,\infty); W^{1,p}(\Omega)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0,\infty))\right]^3,$$

and there exists a constant $M_1 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le M_1 \quad \text{for all} \quad t > 0.$$

In the case of direct predator-taxis, Wu et al. [33] proved the global existence and uniform boundedness of classical solutions when $g(u,v)$ satisfies $(H2)$. If $g(u,v)$ is bounded above by some positive constant, then we can use the comparison principle so that we obtain the uniform boundedness of v. However, if $g(u,v)$ satisfies $(H3)$, i.e., the Lotka–Volterra type reaction term, due to the strong coupling of equations of u and v, the boundedness of v is not easily attained. We remark that a similar difficulty arises in urban crime models. A seminal urban crime model was proposed by Short et al. [20]:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(\frac{u}{v}\nabla v) - uv + B_1(x,t), & x\in\Omega, \quad t>0,\\ v_t = \Delta v + uv - v + B_2(x,t), & x\in\Omega, \quad t>0, \end{cases} \tag{1.2}$$

where u represents the density of criminal agents and v the attractiveness value. Similar to the logarithmic chemotaxis model, system (1.2) possesses an advection term $-\chi\nabla\cdot(\frac{u}{v}\nabla v)$, which is interpreted as a directed movement of criminals toward increasing attractiveness values. The Lotka–Volterra type $+uv$ term in the second equation of (1.2) is driven by the assumption that criminal activity increases attractiveness. To the best of our knowledge, the results on the existence of global classical solutions are only obtained in one-spatial dimension (see [19,29]). For the two-dimensional case, Winkler [31] showed the existence of global renormalized solution on a radially symmetric domain. Therefore, Theorem 1.1 implicates that up to two spatial dimensions, the indirect taxis is a key contribution in obtaining the well-posedness compared to direct taxis mechanisms, thus preventing the blow-up of solutions.

The second key contribution is the growth restriction of logistic-type specified by $(H1)$. As for classical chemotaxis models with logistic source, that is, for

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + au - bu^2, & x\in\Omega, \quad t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega, \quad t>0, \end{cases} \tag{1.3}$$

the existence theory has been well investigated by several studies. In the two-dimensional case, due to the pure existence of the quadratic degradation term, the global existence and uniform boundedness of classical solutions to (1.3) are obtained for any $b>0$. In contrast, in the higher-dimensional cases, the global boundedness is guaranteed for sufficiently large $b>0$ (see [26] for a parabolic–elliptic version $(\tau=0)$ and [17,30] for a parabolic–parabolic version $(\tau=1)$). In particular, for the parabolic–parabolic case in the higher dimensions, Winkler [30] verified the global boundedness by using a specific functional

$$\sum_{k=0}^{m} c_k \int_\Omega u^k |\nabla v|^{2m-2k},$$

with arbitrarily large $m\in\mathbb{N}$ and some proper constants c_k's. Due to the complexity, it is not easy to apply the approach in [30] to system (1.1), and thus, the existence problem of (1.1) in the higher dimensions seems to need more consideration.

The next result is concerned with the linear stability of a positive constant steady state in the aspect of pattern formations. Indeed, the positive constant steady state (u_c,c_c,v_c) satisfies

$$f(u_c) = v_c g(u_c,v_c), \quad g(u_c,v_c) = \frac{k(v_c)}{r}, \quad c_c = \frac{\alpha}{\beta}v_c, \quad u_c,c_c,v_c>0. \tag{1.4}$$

Assuming the direct predator-taxis, Wu et al. [33] investigated the stability of a positive steady state, and it turns out that the presence of large predator-taxis may annihilate the spatial patterns. This result is similar to that of the direct prey-taxis model. Lee et al. [15] showed that prey-taxis stabilizes the system in the sense that, for large prey-taxis sensitivity, pattern formation does not occur. However, the indirect taxis provides different dynamics. For the indirect prey-taxis case, Ahn and Yoon [1] showed that large prey-taxis sensitivity results in generating pattern formations without diffusion-driven instability. Due to the similar structure to the prey-taxis case, we can find that indirect predator-taxis tends to generate pattern formations as the following proposition.

Proposition 1.2. *Let $f(u)$ satisfy $(H1)$ and $g(u,v)$ satisfy $(H2)$ or $(H3)$. Suppose that there exists a positive constant steady state (u_c,c_c,v_c) of (1.1) satisfying (1.4), and it is linearly stable for $\chi=0$. Then, there exists $\chi_c>0$ such that (u_c,c_c,v_c) is linearly unstable for any $\chi>\chi_c$.*

The last part of this paper is devoted to show the global behavior of solutions to (1.1). To do so, we impose specific conditions for f,g,k, and thus, we consider the following Lotka–Volterra system:

$$\begin{cases} \dfrac{\partial u}{\partial t}=d_u\Delta u+\nabla\cdot(\chi u\nabla c)+u(a_1-b_1u-c_1v), & x\in\Omega,\quad t>0,\\ \dfrac{\partial c}{\partial t}=d_c\Delta c+\alpha v-\beta c, & x\in\Omega,\quad t>0,\\ \dfrac{\partial v}{\partial t}=d_v\Delta v+v(a_2+b_2u-c_2v), & x\in\Omega,\quad t>0,\\ u(x,0)=u_0(x),\quad c(x,0)=c_0(x),\quad v(x,0)=v_0(x), & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}=\dfrac{\partial c}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0, & x\in\partial\Omega,\quad t>0, \end{cases} \tag{1.5}$$

where $\Omega\subset\mathbb{R}^2$ is a smooth bounded domain, $a_1>0$, $a_2\in\mathbb{R}$ and $b_i,c_i>0$, $i=1,2$. The functional response $g(u,v)=c_1u$ satisfies $(H3)$ which is not considered in [33]. Here, $a_2>0$ means that there exists another resource for predator so that the predator can survive even in the absence of the prey, whereas $a_2\le 0$ indicates that the reproduction of the predator entirely depends on the prey. Direct calculations on the kinetic terms of the equations of u and v yield that the existence of the positive constant steady state or the semi-trivial steady states depends on the parameters. More precisely, if $a_2>0$, then there exist nontrivial constant steady states (u^*,c^*,v^*) such that

$$(u^*,c^*,v^*)=\begin{cases}(\frac{a_1}{b_1},0,0)\quad\text{or}\quad(0,\frac{\alpha a_2}{\beta c_2},\frac{a_2}{c_2}) & \text{if}\quad \frac{a_1}{c_1}\le\frac{a_2}{c_2},\\ (\frac{a_1}{b_1},0,0)\quad\text{or}\quad(0,\frac{\alpha a_2}{\beta c_2},\frac{a_2}{c_2})\ \text{or}\ (u_c,c_c,v_c) & \text{if}\quad \frac{a_1}{c_1}>\frac{a_2}{c_2},\end{cases}$$

where (u_c,c_c,v_c) is given by

$$(u_c,c_c,v_c)=\left(\frac{a_1c_2-a_2c_1}{b_1c_2+b_2c_1},\frac{\alpha(a_2b_1+a_1b_2)}{\beta(b_1c_2+b_2c_1)},\frac{a_2b_1+a_1b_2}{b_1c_2+b_2c_1}\right). \tag{1.6}$$

On the contrary, if $a_2\le 0$, then we have two nontrivial constant steady states (u^*,c^*,v^*):

$$(u^*,c^*,v^*)=\begin{cases}(\frac{a_1}{b_1},0,0) & \text{if}\quad \frac{a_1}{b_1}\le-\frac{a_2}{b_2},\\ (\frac{a_1}{b_1},0,0)\quad\text{or}\quad(u_c,c_c,v_c) & \text{if}\quad \frac{a_1}{b_1}>-\frac{a_2}{b_2}.\end{cases}$$

The global existence and the uniform boundedness of classical solutions to (1.5) obtained in Theorem 1.1 enable us to have the following results on the global behavior of solutions to (1.5).

Theorem 1.3. *Let (u,c,v) be a global classical solution to system (1.5). Then, it holds that:*

(1) *If $a_2>0$ and $\frac{a_1}{c_1}\le\frac{a_2}{c_2}$, then the semi-trivial steady state $(u^*,c^*,v^*)=(0,\frac{\alpha a_2}{\beta c_2},\frac{a_2}{c_2})$ is globally asymptotically stable. Moreover, there exist positive constants η_1, C_{S_1}, T_1 such that for all $t>T_1$,*

$$\|u\|_{L^\infty(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^\infty(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^\infty(\Omega)}\le C_{S_1}e^{-\eta_1 t}\qquad \textit{if}\quad \frac{a_1}{b_1}<-\frac{a_2}{b_2},$$

$$\|u\|_{L^\infty(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^\infty(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^\infty(\Omega)}\le \frac{C_{S_1}}{(1+t)^{\eta_1}}\qquad \textit{if}\quad \frac{a_1}{b_1}=-\frac{a_2}{b_2}.$$

(2) *If $a_2<0$ and $\frac{a_1}{b_1}\le-\frac{a_2}{b_2}$, then the semi-trivial steady state $(u^*,c^*,v^*)=(\frac{a_1}{b_1},0,0)$ is globally asymptotically stable provided that $\chi>0$ satisfies*

$$\chi<\frac{4}{\alpha}\sqrt{\frac{d_ud_c\beta b_1c_1c_2}{a_1b_2}}.$$

Moreover, there exist positive constants η_2, C_{S_2}, T_2 *such that for all* $t > T_2$,

$$\|u - \frac{a_1}{b_1}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C_{S_2} e^{-\eta_2 t} \quad \text{if } \frac{a_1}{b_1} < -\frac{a_2}{b_2},$$

$$\|u - \frac{a_1}{b_1}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le \frac{C_{S_2}}{(1+t)^{\eta_2}} \quad \text{if } \frac{a_1}{b_1} = -\frac{a_2}{b_2}.$$

(3) *If* $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, *then there exists a positive constant steady state* $(u^*, c^*, v^*) = (u_c, c_c, v_c)$ *given as* (1.6). *Moreover, if* $\chi > 0$ *satisfies*

$$\chi < \frac{4}{\alpha}\sqrt{\frac{d_u d_c \beta c_1 c_2}{b_2}},$$

then (u_c, c_c, v_c) *is globally asymptotically stable, and there exist positive constants* η_3, C_{S_3}, T_3 *such that*

$$\|u - u_c\|_{L^\infty(\Omega)} + \|c - c_c\|_{L^\infty(\Omega)} + \|v - v_c\|_{L^\infty(\Omega)} \le C_{S_3} e^{-\eta_3 t} \quad \text{for all } t > T_3.$$

We note that if one obtains the uniform boundedness of solutions to (1.1), then the global behavior of solutions can be verified by the method similar to the proof of Theorem 1.3, even for higher dimensions than $n = 2$.

The rest of this paper is organized as follows: In Sect. 2, we provide several preliminaries. In Sect. 3, we prove the global existence and uniform boundedness of (1.1). A case study for global stability is given in Sect. 4. We investigate the global asymptotic behaviors of solutions to (1.5) and find the convergence rates. In the appendix, the proof of Proposition 1.2 will be provided.

Throughout this paper, C and C_i $(i = 1, 2, 3, \ldots)$ denote generic positive constants which change from line to line.

2. Preliminaries

First, we obtain the local existence in time of a classical solution to (1.1). Moreover, we assert that the total mass of u, c, and v are uniformly bounded.

Lemma 2.1. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^n$, $n \ge 1$. *Suppose that* $f(u)$ *and* $k(v)$ *satisfy* $(H1)$ *and* $(H4)$, *respectively, and* $g(u, v)$ *satisfies* $(H2)$ *or* $(H3)$. *For any* $(u_0, c_0, v_0) \in [W^{1,p}(\Omega)]^3$ *where* $p > n$, *satisfying* $u_0(x)$, $c_0(x)$, $v_0(x) \ge 0 (\not\equiv 0)$ *for* $x \in \Omega$, *we have:*

(1) *There exists a maximal time of existence* $T_{\max} > 0$ *such that a unique nonnegative classical solution* (u, c, v) *satisfies*

$$(u, c, v) \in \left[\mathcal{C}([0, T_{\max}); W^{1,p}(\Omega)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right]^3.$$

(2) *There exists a constant* $M_2 > 0$ *such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} + \|c(\cdot,t)\|_{L^1(\Omega)} + \|v(\cdot,t)\|_{L^1(\Omega)} \le M_2 \quad \text{for } t \in (0, T_{\max}). \tag{2.1}$$

(3) *If* $T_{\max} < \infty$, *we have*

$$\lim_{t \nearrow T_{\max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\right) = \infty. \tag{2.2}$$

(4) *If* $g(u, v)$ *satisfies* $(H2)$, *there exists a constant* $M_3 > 0$ *such that*

$$0 \le v(x,t) \le M_3 \quad \text{for } (x,t) \in (\overline{\Omega} \times (0, T_{\max})). \tag{2.3}$$

Proof. The application of Amman's fixed point argument from Theorems 14.4 and 14.6 in [3] implies the local existence and uniqueness of solutions to (1.1) and blow-up criterion (2.2) (see e.g., [32,33]). In view of the hypotheses $(H1)$–$(H4)$, it follows from the maximum principle that the solutions are nonnegative. To show (2.1), we multiply the first equation of (1.1) by r and add it to the third equation of (1.1). Then integrating it over Ω implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r\int_\Omega u+\int_\Omega v\right)+\int_\Omega k(v)v=r\int_\Omega f(u)\le r\int_\Omega (N_1u-N_2u^2). \tag{2.4}$$

For given $r, N_1, N_2 > 0$, one can easily find a constant $C_1 > 0$ such that

$$rN_1u-rN_2u^2\le -ru+C_1 \quad \text{for all} \;\; u\ge 0. \tag{2.5}$$

From $(H4)$, we observe that there exists $C_2 > 0$ such that

$$-C_2+\int_\Omega v\le \int_\Omega k_2v^2+k_3v\le \int_\Omega k(v)v. \tag{2.6}$$

Using (2.5) and (2.6), (2.4) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r\int_\Omega u+\int_\Omega v\right)+\left(r\int_\Omega u+\int_\Omega v\right)\le C_3$$

for some positive constant C_3. Thus, we show the uniform boundedness of $\|u(\cdot,t)\|_{L^1(\Omega)}$ and $\|v(\cdot,t)\|_{L^1(\Omega)}$ for any $t > 0$. Integrating the second equation of (1.1) over Ω, the boundedness of $\|c(\cdot,t)\|_{L^1(\Omega)}$ directly follows from the boundedness of $\|v(\cdot,t)\|_{L^1(\Omega)}$. If $g(u,v)$ satisfies $(H2)$, due to Theorem 3.1 in [2] and comparison principle, we obtain (2.3). □

Next, we give the estimates introduced in [11] related to the fractional operator $(-\Delta+1)^\theta$, $\theta\in(0,1)$ in Ω with the Neumann boundary conditions. For any $1 < q < \infty$, the operator $-\Delta+1$ is sectorial in $L^q(\Omega)$ and the fractional operator $(-\Delta+1)^\theta$ is defined on a domain $D((-\Delta+1)^\theta)\subset L^q(\Omega)$ such that

$$\|w\|_{D((-\Delta+1)^\theta)}:=\|(-\Delta+1)^\theta w\|_{L^q(\Omega)}<\infty.$$

Lemma 2.2. *If $m\in\{0,1\}$, $1\le p\le\infty$, $1<q<\infty$ and $m-\frac{n}{p}<2\theta-\frac{n}{q}$, then there exists a constant $C>0$ that satisfies*

$$\|w\|_{W^{m,p}(\Omega)}\le C\|(-\Delta+1)^\theta w\|_{L^q(\Omega)} \tag{2.7}$$

for all $w\in D((-\Delta+1)^\theta)$. Moreover, there exist $C>0$ and $\zeta>0$ such that, for $q\ge p$,

$$\|(-\Delta+1)^\theta e^{t(\Delta-1)}w\|_{L^q(\Omega)}\le Ct^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\zeta t}\|w\|_{L^p(\Omega)} \tag{2.8}$$

for all $w\in L^p(\Omega)$.

3. Global existence

In this section, we show the uniform boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}$, which guarantees the global existence of solutions. To this end, we establish a boundedness criterion with respect to $\|v(\cdot,t)\|_{L^p(\Omega)}$ for $p>n$. As long as $\|v(\cdot,t)\|_{L^p(\Omega)}$ is bounded, one can achieve the boundedness of $\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ and $\|u(\cdot,t)\|_{L^\infty}$. Sequentially, the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ implies the boundedness of $\|v(\cdot,t)\|_{L^\infty(\Omega)}$. If $g(u,v)$ satisfies $(H2)$, by (2.3), $\|v(\cdot,t)\|_{L^\infty}$ is uniformly bounded, so we shall investigate the case for $n\in\{1,2\}$ and $(H3)$.

First, we state the decay estimate for u induced by the logistic growth of u.

Lemma 3.1. *Let $n = 1, 2$, and let $f(u)$, $g(u, v)$ and $k(v)$ satisfy (H1), (H3) and (H4), respectively. Suppose that (u, c, v) is a classical solution of system (1.1) in $\Omega\times(0, T_{\max})$. Then, there exist two constants $M_4, M_5 > 0$ independent of t such that*

$$\int_0^t \int_\Omega u^2 \le M_4(t+1) \quad \text{for} \quad t \in (0, T_{\max}), \tag{3.1}$$

$$\int_t^{t+\tau_0} \int_\Omega u^2 \le M_5 \quad \text{for} \quad t \in (0, T_{\max} - \tau_0), \tag{3.2}$$

where $\tau_0 := \min\{1, \frac{1}{6}T_{\max}\}$.

Proof. Integrating the first equation of (1.1) over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u \le N_1 \int_\Omega u - N_2 \int_\Omega u^2. \tag{3.3}$$

Using Young's inequality, the Cauchy–Schwarz inequality and (2.1), we have from (3.3) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u &\le -\frac{N_2}{2}\int_\Omega u^2 + C\\ &\le -\frac{N_2}{2|\Omega|}\left(\int_\Omega u\right)^2 + C \quad \text{for all} \quad t \in (0, T_{\max}).\end{aligned}$$

Thus, a standard ODE comparison implies the boundedness of $\int_\Omega u(\cdot, t)$ on $(0, T_{\max})$, which proves (3.1). Moreover, integration of (3.3) in time shows (3.2). □

We now introduce a boundedness criterion as below.

Lemma 3.2. *Let $n = 1, 2$, and let $f(u)$, $g(u, v)$ and $k(v)$ satisfy (H1), (H3) and (H4), respectively. Suppose that (u, c, v) is a classical solution of system (1.1) in $\Omega \times (0, T_{\max})$. If there exists $C > 0$ such that for some $p > n$,*

$$\|v(\cdot, t)\|_{L^p(\Omega)} \le C, \quad \text{for any} \quad t \in (0, T_{\max}), \tag{3.4}$$

then we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} < \infty \quad \text{for all} \quad t \in (0, T_{\max}). \tag{3.5}$$

Proof. Suppose that for some $p > n$,

$$\|v(\cdot, t)\|_{L^p(\Omega)} \le C \quad \text{for any} \quad t \in (0, T_{\max}). \tag{3.6}$$

Using (3.6), we claim that

$$\|\nabla c(\cdot, t)\|_{L^p(\Omega)} \le C \quad \text{for} \ \ t \in (0, T_{\max}). \tag{3.7}$$

For simplicity, we set $d_c = 1$ and $\beta = 1$. We consider the Duhamel formulation of the second equation of (1.1) as

$$c(x, t) = e^{(\Delta-1)t}c_0(x) + \alpha \int_0^t e^{(\Delta-1)(t-s)}v(x, s)\mathrm{d}s \quad \text{for} \quad (x, t) \in \Omega \times (0, T_{\max}). \tag{3.8}$$

Let $t_0 \in (0, T_{\max})$ be fixed. Applying (2.7) and (2.8) to (3.8), by (3.6), we find that for $p > n$ and $\theta \in (\frac{1}{2} + \frac{n}{2p}, 1)$, there exists $\zeta > 0$ such that

$$\begin{aligned}\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C\|(-\Delta+1)^{\theta} c\|_{L^p(\Omega)} \\ &\le Ct^{-\theta}e^{-\zeta t}\|c_0\|_{L^p(\Omega)} + C\int_0^t (t-s)^{-\theta} e^{-\zeta(t-s)}\|v(\cdot,s)\|_{L^p(\Omega)}\mathrm{d}s \\ &\le Ct^{-\theta} + C\int_0^t (t-s)^{-\theta}e^{-\zeta(t-s)}\mathrm{d}s \\ &\le Ct_0^{-\theta} + C\int_0^\infty s^{-\theta}e^{-\zeta s}\mathrm{d}s \\ &\le C(t_0^{-\theta}+1) \quad \text{for all} \quad t \in (t_0, T_{\max}).\end{aligned}$$

Thereofore, we infer that

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all} \quad t \in (0, T_{\max}),$$

which shows (3.7). Using the standard Moser iterative technique [2] with (3.7), we conclude that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded for all $t \in (0, T_{\max})$ (see also [13,22,32] for similar approaches). Due to the comparison principle, the boundedness of u guarantees the boundedness of v. □

3.1. One-dimensional case

For one-dimensional setting, via (3.4), it suffices to show the boundedness of $\|v(\cdot,t)\|_{L^2(\Omega)}$.

Lemma 3.3. *Let $n = 1$, and let $f(u)$, $g(u,v)$ and $k(v)$ satisfy $(H1)$, $(H3)$ and $(H4)$, respectively. Suppose that (u,c,v) is a classical solution of system (1.1) in $\Omega \times (0, T_{\max})$. Then, there exists $C > 0$ independent of t such that*

$$\|v(\cdot,t)\|_{L^2(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{\max}). \tag{3.9}$$

Proof. Multiplying the third equation of (1.1) by v, we deduce from Young's inequality that there exists $C_1 > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2 + d_v\int_\Omega v_x^2 &= \int_\Omega rg(u,v)v^2 - \int_\Omega k(v)v^2 \\ &\le rN_3\int_\Omega uv^2 - k_2\int_\Omega v^3 - k_3\int_\Omega v^2 \\ &\le rN_3\int_\Omega uv^2 - \frac{1}{2}\int_\Omega v^2 + C_1.\end{aligned}$$

Owing to the Gagliardo–Nirenberg inequality, we find that there exists $C_2 > 0$ such that

$$\|v\|_{L^4(\Omega)}^4 \le C_2\left(\|v_x\|_{L^2(\Omega)}^2\|v\|_{L^1(\Omega)}^2 + \|v\|_{L^1(\Omega)}^4\right),$$

which, along with Hölder's inequality, Young's inequality and the hypothesis $(H3)$, helps to infer that for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2 + d_v\int_\Omega v_x^2 &\le -\frac{1}{2}\int_\Omega v^2 + C_1 + rN_3\|u\|_{L^2(\Omega)}\|v\|_{L^4(\Omega)}^2 \\ &\le -\frac{1}{2}\int_\Omega v^2 + C_1 + \frac{r^2N_3^2}{4\varepsilon}\int_\Omega u^2 + \varepsilon C_2\left(\|v_x\|_{L^2(\Omega)}^2\|v\|_{L^1(\Omega)}^2 + \|v\|_{L^1(\Omega)}^4\right).\end{aligned}$$

Choosing $\varepsilon = \frac{d_v}{2C_2M_2^2}$, we see that there exists $C_3 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2 + d_v \int_\Omega v_x^2 + \int_\Omega v^2 \le C_3\left(1+\int_\Omega u^2\right). \tag{3.10}$$

Let $y(t) := \int_\Omega v^2(\cdot,t)$. Neglecting the second summand on the left-hand side of (3.10), we obtain

$$y'(t) + y(t) \le C_3\left(1+\int_\Omega u^2\right).$$

Employing Lemma 3.2 in [24] with (3.2), we have for $\tau_0 \le 1$ that

$$y(t) \le \max\left\{\int_\Omega u_0^2 + C_3(C_1+1), \frac{C_3(C_1+1)}{\tau_0} + 2C_3(C_1+1)\right\},$$

which completes the proof of (3.9). □

3.2. Two-dimensional case

In the two-dimensional case, the uniform boundedness of $\|v(\cdot,t)\|_{L^p(\Omega)}$, $p > 2$ is not directly obtained. Thus, we first proceed to show the global solvability of (1.1).

Lemma 3.4. *Let $n = 2$, $0 < T < T_{\max}$, and let $g(u,v)$ satisfy $(H3)$. Suppose that (u,c,v) is a classical solution of system (1.1) in $\Omega\times(0,T_{\max})$. Then, for any $p > 1$, there exists $C = C(p,T) > 0$ such that*

$$\|v(\cdot,t)\|_{L^p(\Omega)} \le C \quad \textit{for all} \quad t\in(0,T). \tag{3.11}$$

Proof. Let $0 < T < T_{\max}$. Multiplying the third equation of (1.1) by pv^{p-1} and integrating over Ω imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \frac{4d_v}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 &= rp\int_\Omega g(u,v)v^p - p\int_\Omega k(v)v^p\\ &\le rN_3p\int_\Omega uv^p - pk_2\int_\Omega v^{p+1} - pk_3\int_\Omega v^p\\ &\le C_1\|u\|_{L^2(\Omega)}\|v^{\frac{p}{2}}\|^2_{L^4(\Omega)} - pk_2\int_\Omega v^{p+1} - pk_3\int_\Omega v^p.\end{aligned} \tag{3.12}$$

We observe that by the Gagliardo–Nirenberg inequality, there exists $C_2 > 0$ such that

$$\|v^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le C_2\left(\|\nabla v^{\frac{p}{2}}\|_{L^2(\Omega)}\|v^{\frac{p}{2}}\|_{L^2(\Omega)} + \|v^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right). \tag{3.13}$$

Moreover, by Young's inequality, we have from $(H4)$ that for any $C_3 > 0$, there exists $C_4 > 0$ such that

$$-pk_2\int_\Omega v^{p+1} - pk_3\int_\Omega v^p \le -C_3\int_\Omega v^p + C_4. \tag{3.14}$$

Plugging (3.13) and (3.14) into (3.12), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p + \frac{4d_v}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2\\ &\le C_1C_2\|u\|_{L^2(\Omega)}\left(\|\nabla v^{\frac{p}{2}}\|_{L^2(\Omega)}\|v^{\frac{p}{2}}\|_{L^2(\Omega)} + \|v^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right) - C_3\int_\Omega v^p + C_4\\ &\le \frac{2d_v}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + C_5\int_\Omega u^2\int_\Omega v^p + C_6\left(\int_\Omega u^2\right)^{\frac{1}{2}} - C_3\int_\Omega v^p + C_4\\ &\le \frac{2d_v}{p}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + C_7\int_\Omega u^2\left(1+\int_\Omega v^p\right) - C_3\left(1+\int_\Omega v^p\right) + C_8,\end{aligned} \tag{3.15}$$

which turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1+\int_\Omega v^p\right)+\frac{2d_v}{p}\int_\Omega|\nabla v^{\frac{p}{2}}|^2\le C_7\int_\Omega u^2\left(1+\int_\Omega v^p\right)+C_8\quad\text{for all}\quad t\in(0,T).$$

Then, along with (3.1), Grönwall's inequality leads to

$$\begin{aligned}1+\int_\Omega v^p&\le\left(1+\int_\Omega v_0^p\right)e^{C_7\int_0^t\int_\Omega u^2(\cdot,s)ds}+C_8\int_0^t e^{C_7\int_s^t\int_\Omega u^2(\cdot,\sigma)d\sigma}\mathrm{d}s\\&\le\left(1+\int_\Omega v_0^p\right)e^{C_7M_4(T+1)}+C_8Te^{C_7M_4(T+1)}\quad\text{for all}\quad t\in(0,T).\end{aligned}$$

This completes the proof of (3.11). □

Invoking Lemmas 2.1 and 3.2, we obtain the global solvability of (1.1), i.e., $T_{\max}=\infty$ and the solution of (1.1) exists for any finite time interval.

Lemma 3.5. *Let $n=2$, $T>0$, and let $f(u)$, $g(u,v)$ and $k(v)$ satisfy $(H1)$, $(H3)$ and $(H4)$, respectively. Then, for any initial data $(u_0,c_0,v_0)\in[W^{1,p}(\Omega)]^3$ with $p>2$ satisfying $u_0(x)$, $c_0(x)$, $v_0(x)\ge 0(\not\equiv 0)$ for $x\in\Omega$, system (1.1) possesses a unique global-in-time nonnegative classical solution (u,c,v) such that*

$$(u,c,v)\in\left[C([0,T);W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T))\right]^3,$$

and there exists a constant $C>0$ depending on T such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}\le C\quad\textit{for all}\quad t\in(0,T).$$

Now, we are in position to show the uniform boundedness of $\|v(\cdot,t)\|_{L^p}$ with $p>2$.

Lemma 3.6. *Let $n=2$, $T>0$, and let $f(u)$, $g(u,v)$ and $k(v)$ satisfy $(H1)$, $(H3)$ and $(H4)$, respectively. Suppose that (u,c,v) is a classical solution of system (1.1) in $\Omega\times(0,T)$. Then, for any $p>1$, there exists $C>0$ independent of t such that*

$$\|v(\cdot,t)\|_{L^p(\Omega)}\le C\quad\textit{for all}\quad t\in(0,T).\tag{3.16}$$

Proof. We recall from (3.15) that for any $C_1>0$, there exist $C_2,C_3>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\le C_2\int_\Omega u^2\left(1+\int_\Omega v^p\right)-C_1\left(1+\int_\Omega v^p\right)+C_3,\tag{3.17}$$

where C_2 is independent of C_1 whereas C_3 depends on C_1. On the other hand, we infer from (3.3) that for any $t>0$, there exists $C_4>0$ independent of t such that

$$\int_s^t\int_\Omega u^2\le C_4(t-s+1)\quad\text{for any}\quad s\in[0,t).\tag{3.18}$$

We choose $C_1>C_2(C_4+M_4)$. Then, in view of (3.1), there exists $t^*\in(0,\infty)$ such that

$$C_2\int_0^t\int_\Omega u^2-C_1t\le C_2M_4(t+1)-C_1t<0\quad\text{for any}\quad t>t^*.\tag{3.19}$$

Letting $y(t):=1+\int_\Omega v^p(\cdot,t)$, it follows from (3.17) that

$$y'(t)\le\left(C_2\int_\Omega u^2(\cdot,t)-C_1\right)y(t)+C_3\quad\text{for any}\quad t>0.\tag{3.20}$$

Then, using (3.18) and (3.19), we solve (3.20) such that for any $t > t^*$,

$$
\begin{aligned}
y(t) &\le y(0)\exp\left\{C_2\int_0^t\int_\Omega u^2(\cdot,s)\mathrm{d}s - C_1 t\right\} + C_3\int_0^t \exp\left\{C_2\int_s^t\int_\Omega u^2(\cdot,\sigma)d\sigma - C_1(t-s)\right\}\mathrm{d}s \\
&\le y(0) + C_3\int_0^t \exp\left\{C_2C_4(t-s+1) - C_1(t-s)\right\}\mathrm{d}s \\
&\le y(0) + C_5\int_0^t e^{-C_6(t-s)}\mathrm{d}s \\
&\le y(0) + C_7,
\end{aligned}
\tag{3.21}
$$

where C_5, C_6 and C_7 are positive constants independent of t. For such $t^* > 0$, Lemma 3.5 implies that there exists $C(t^*) > 0$ such that

$$
\int_\Omega v^p(\cdot,t) \le C(t^*) \quad \text{for any} \quad t \in (0, t^*]. \tag{3.22}
$$

Combining (3.21) and (3.22), we complete the proof of (3.16). □

Proof of Theorem 1.1. Suppose that $T_{\max} < \infty$, where $T_{\max}$ is the maximal time of existence given in Lemma 2.1 By the application of Lemmas 3.3 and 3.6 with some $p > 2$ to Lemma 3.2, we obtain (3.5). This is contrary to (2.2), which implies $T_{\max} = \infty$. □

4. Global stability

In this section, we study the global asymptotic behavior of solutions to (1.5). For reader's convenience, we write system (1.5) again.

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_u\Delta u + \nabla\cdot(\chi u\nabla c) + u(a_1 - b_1 u - c_1 v), & x\in\Omega, \quad t>0,\\
\dfrac{\partial c}{\partial t} = d_c\Delta c + \alpha v - \beta c, & x\in\Omega, \quad t>0,\\
\dfrac{\partial v}{\partial t} = d_v\Delta v + v(a_2 + b_2 u - c_2 v), & x\in\Omega, \quad t>0,\\
u(x,0) = u_0(x), \quad c(x,0) = c_0(x), \quad v(x,0) = v_0(x), & x\in\Omega,\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial c}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0,
\end{cases}
\tag{4.1}
$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $a_1 > 0$, $a_2 \in \mathbb{R}$ and $b_i, c_i > 0$, $i = 1, 2$.

Since the existence of the nontrivial constant steady states depends on the parameters as mentioned above, we shall show the global stability case by case. To this end, we introduce the following tool used in [4, Lemma 3.1].

Lemma 4.1. *Suppose that $h : (1,\infty)$ is a uniformly continuous nonnegative function such that*

$$
\int_1^\infty h(t)\mathrm{d}t < \infty.
$$

Then, $h(t) \to 0$ as $t \to \infty$.

If $a_2 > 0$ and $\frac{a_1}{c_1} \le \frac{a_2}{c_2}$, there is no positive constant steady state and the semi-trivial steady state $(u^*, c^*, v^*) = (0, \frac{\alpha a_2}{\beta c_2}, \frac{a_2}{c_2})$ is globally asymptotically stable.

Lemma 4.2. *Let* (u,c,v) *be a global classical solution to system* (4.1). *If* $a_2>0$ *and* $\frac{a_1}{c_1}\le\frac{a_2}{c_2}$, *then the semi-trivial steady state* $(u^*,c^*,v^*)=(0,\frac{\alpha a_2}{\beta c_2},\frac{a_2}{c_2})$ *is globally asymptotically stable. Moreover, there exist positive constants* η_1, C_{S_1}, T_1 *such that for all* $t>T_1$,

$$\|u\|_{L^\infty(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^\infty(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^\infty(\Omega)}\le C_{S_1}e^{-\eta_1 t}\quad \textit{if}\quad \frac{a_1}{b_1}<-\frac{a_2}{b_2}, \tag{4.2}$$

$$\|u\|_{L^\infty(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^\infty(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^\infty(\Omega)}\le \frac{C_{S_1}}{(1+t)^{\eta_1}}\quad \textit{if}\quad \frac{a_1}{b_1}=-\frac{a_2}{b_2}. \tag{4.3}$$

Proof. We shall use the following Lyapunov functional:

$$W_1(u(t),c(t),v(t)):=\frac{b_2}{c_1}\int_\Omega u+\frac{C_{W_1}}{2}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}-\frac{a_2}{c_2}\log\frac{v}{a_2/c_2}\right), \tag{4.4}$$

where a positive constant C_{W_1} satisfies

$$C_{W_1}<\frac{4\beta c_2}{\alpha^2}. \tag{4.5}$$

Using the Taylor's expansion of $v-\frac{a_2}{c_2}-\frac{a_2}{c_2}\log\frac{v}{a_2/c_2}$ about a_2/c_2, we infer that there exists ξ lying between v and a_2/c_2 such that

$$v-\frac{a_2}{c_2}-\frac{a_2}{c_2}\log\frac{v}{a_2/c_2}=\frac{a_2/c_2}{2\xi^2}\left(v-\frac{a_2}{c_2}\right)^2\ge 0,$$

which implies the nonnegativity of W_1. We differentiate $W_1(t)$ in time to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}W_1(t)=\frac{b_2}{c_1}\int_\Omega u(a_1-b_1u-c_1v)+C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)c_t+\int_\Omega\left(1-\frac{a_2/c_2}{v}\right)v_t.$$

Let

$$I_1:=\frac{b_2}{c_1}\int_\Omega u(a_1-b_1u-c_1v)+\int_\Omega\left(1-\frac{a_2/c_2}{v}\right)v_t,$$
$$I_2:=C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)c_t.$$

By integration by parts, we observe that

$$\begin{aligned}\int_\Omega\left(1-\frac{a_2/c_2}{v}\right)v_t&=\int_\Omega\left(1-\frac{a_2/c_2}{v}\right)(d_v\Delta v+v(a_2+b_2u-c_2v))\\&=-\frac{d_va_2}{c_2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2+\int_\Omega\left(1-\frac{a_2/c_2}{v}\right)v(a_2+b_2u-c_2v)\\&=-\frac{d_va_2}{c_2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2-c_2\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}\right)b_2u,\end{aligned}$$

which results in

$$\begin{aligned}I_1&=-\frac{d_va_2}{c_2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2-c_2\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2+\frac{b_2}{c_1}\int_\Omega\left(-b_1u^2+\left(a_1-\frac{a_2c_1}{c_2}\right)u\right)\\&\le-\frac{d_va_2}{c_2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2-c_2\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2-\frac{b_1b_2}{c_1}\int_\Omega u^2+b_2\int_\Omega\left(\frac{a_1}{c_1}-\frac{a_2}{c_2}\right)u.\end{aligned} \tag{4.6}$$

On the other hand, Young's inequality implies

$$
\begin{aligned}
I_2 &= C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)(d_c\Delta c+\alpha v-\beta c)\\
&= -d_cC_{W_1}\int_\Omega|\nabla c|^2-\beta C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\alpha C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)\left(v-\frac{a_2}{c_2}\right)\\
&\le -d_cC_{W_1}\int_\Omega|\nabla c|^2-(\beta-\beta_1)C_{W_1}\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\frac{\alpha^2C_{W_1}}{4\beta_1}\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2 \qquad (4.7)
\end{aligned}
$$

for any $\beta_1\in(0,\beta)$. By (4.5), we can choose $\beta_1\in(\frac{\alpha^2C_{W_1}}{4c_2},\beta)$, and thus, plugging (4.6) and (4.7) into (4.4), there exist two constants $C_1,C_2>0$ such that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W_1(t) \le& -\frac{d_va_2}{c_2}\int_\Omega\left|\frac{\nabla v}{v}\right|^2-d_cC_{W_1}\int_\Omega|\nabla c|^2-\frac{b_1b_2}{c_1}\int_\Omega u^2+b_2\int_\Omega\left(\frac{a_1}{c_1}-\frac{a_2}{c_2}\right)u\\
&-C_1\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2-C_2\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2. \qquad (4.8)
\end{aligned}
$$

For the case of $\frac{a_1}{c_1}<\frac{a_2}{c_2}$, we employ the following functional:

$$
\mathcal{W}_1(t):=\int_\Omega u+\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2. \qquad (4.9)
$$

We see from (4.8) and (4.9) that there exists $C_3>0$ satisfying

$$
\frac{\mathrm{d}}{\mathrm{d}t}W_1(t)\le -C_3\mathcal{W}_1(t)\quad\text{for all}\quad t>0.
$$

Since W_1 is nonnegative, we have

$$
\int_1^\infty\mathcal{W}(t)\mathrm{d}t\le\frac{1}{C_3}\left(W_1(1)-W_1(t)\right)\le\frac{W_1(1)}{C_3}<\infty,
$$

which, by Lemma 4.1, yields

$$
\int_\Omega u+\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2\to 0\quad\text{as}\quad t\to\infty.
$$

Further, in view of Lemma 4.1 in [12], using the convexity of a function $v\mapsto v-\frac{a_2}{c_2}-\frac{a_2}{c_2}\log\frac{v}{a_2/c_2}$ and the asymptotic convergence of v to $\frac{a_2}{c_2}$, we find that there exists a constant $T_1>0$ such that

$$
C_4\left(v-\frac{a_2}{c_2}\right)^2\le\left(v-\frac{a_2}{c_2}-\frac{a_2}{c_2}\log\frac{v}{a_2/c_2}\right)\le C_5\left(v-\frac{a_2}{c_2}\right)^2\quad\text{for all}\quad t>T_1, \qquad (4.10)
$$

where C_4,C_5 are positive constants. We see from the computations in (4.6)–(4.10) that for all $t\ge T_1$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}W_1(t)\le -C_6\mathcal{W}_1(t)\le -C_7W_1(t),
$$

which yields

$$
W_1(t)\le C_8e^{-C_9t}\quad\text{for all}\quad t\ge T_1.
$$

Thus, by (4.10) and (4.9), we obtain

$$
\|u\|_{L^1(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^2(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^2(\Omega)}\le C_{10}e^{-C_{11}t}\quad\text{for all}\quad t>T_1. \qquad (4.11)
$$

From Theorem 1.1, we see that $\chi u\nabla c$ and $u(a_1-b_1u-c_1v)$ are uniformly bounded in $L^\infty(\Omega\times(0,\infty))$. Using the standard parabolic regularity theory (see [18, Theorem 1.3]), one can find a constant $\theta_0\in(0,1)$ such that

$$\|u\|_{\mathcal{C}^{\theta_0,\frac{\theta_0}{2}}(\Omega\times[t,t+1])}\le C_{12}\quad\text{for all}\quad t>1.$$

With the aid of the standard parabolic Schauder's theory, we have from the second and third equations of (1.1) that

$$\|c\|_{\mathcal{C}^{2+\theta_0,1+\frac{\theta_0}{2}}(\Omega\times[t,t+1])}+\|v\|_{\mathcal{C}^{2+\theta_0,1+\frac{\theta_0}{2}}(\Omega\times[t,t+1])}\le C_{13}\quad\text{for all}\quad t>1.$$

By a similar method in [23, Theorem 3.14], we find a constant $C_{14}>0$ such that

$$\|u\|_{W^{1,\infty}(\Omega)}\le C_{14}\quad\text{for all}\quad t>1. \tag{4.12}$$

Then, the Gagliardo–Nirenberg inequality with (4.11) and (4.12) entails that

$$\|u\|_{L^\infty(\Omega)}\le C_{15}\left(\|\nabla u\|^{\frac{2}{3}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{3}}_{L^1(\Omega)}+\|u\|_{L^1(\Omega)}\right)\le C_{16}e^{-C_{17}t}\quad\text{for all}\quad t>T_1. \tag{4.13}$$

Moreover, by Theorem 1.1, we have

$$c-\frac{\alpha a_2}{\beta c_2},\quad v-\frac{a_2}{c_2}\in W^{1,\infty}(\Omega),$$

which, together with the Gagliardo–Nirenberg inequality, implies that for all $t>T_1$,

$$\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^\infty(\Omega)}\le C_{18}\left(\|\nabla(c-\frac{\alpha a_2}{\beta c_2})\|^{\frac{2}{3}}_{L^\infty(\Omega)}\|c-\frac{\alpha a_2}{\beta c_2}\|^{\frac{1}{3}}_{L^2(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^2(\Omega)}\right)\le C_{19}e^{-C_{20}t}, \tag{4.14}$$

$$\|v-\frac{a_2}{c_2}\|_{L^\infty(\Omega)}\le C_{21}\left(\|\nabla(v-\frac{a_2}{c_2})\|^{\frac{2}{3}}_{L^\infty(\Omega)}\|v-\frac{a_2}{c_2}\|^{\frac{1}{3}}_{L^2(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^2(\Omega)}\right)\le C_{22}e^{-C_{23}t}. \tag{4.15}$$

Collecting (4.13)–(4.15), we complete the proof of (4.2). For the case of $\frac{a_1}{c_1}=\frac{a_2}{c_2}$, we use the following functional:

$$\mathcal{W}_2(t):=\int_\Omega u^2+\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2. \tag{4.16}$$

From (4.8), (4.10) and (4.16), we infer that

$$\frac{d}{dt}W_1(t)\le -C_{24}\mathcal{W}_2(t)\le -C_{25}W_1^2(t)\quad\text{for all}\quad t>T_1,$$

where we used

$$\begin{aligned}W_1(t)&\le C_{26}\left(\int_\Omega u+\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2+\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2\right)\\&\le C_{27}\left(\int_\Omega u^2\right)^{\frac12}+C_{28}\left(\int_\Omega\left(c-\frac{\alpha a_2}{\beta c_2}\right)^2\right)^{\frac12}+C_{29}\left(\int_\Omega\left(v-\frac{a_2}{c_2}\right)^2\right)^{\frac12}\\&\le C_{30}\mathcal{W}^{\frac12}(t).\end{aligned}$$

This gives $W_1(t)\le C_{31}(1+t)^{-1}$ for all $t>T_1$, and thus, via (4.16), we have

$$\|u\|_{L^1(\Omega)}+\|c-\frac{\alpha a_2}{\beta c_2}\|_{L^2(\Omega)}+\|v-\frac{a_2}{c_2}\|_{L^2(\Omega)}\le\frac{C_{32}}{(1+t)^{C_{33}}}. \tag{4.17}$$

Following a similar manner to the derivation of (4.2), by the Gagliado–Nirengerg inequality and (4.17), we can improve the decay estimates up to L^∞-norm. This completes the proof of (4.3). □

If $a_2 < 0$ and $\frac{a_1}{b_1} \le -\frac{a_2}{b_2}$, the semi-trivial steady state $(u^*, c^*, v^*) = (\frac{a_1}{b_1}, 0, 0)$ is the only nontrivial one, and due to the chemotactic term, its global asymptotical stability is obtained for sufficiently small $\chi > 0$. When $u^* \neq 0$, due to the advection term, we need to construct an energy functional related to a positive semidefinite matrix (see e.g., [4,12]).

Lemma 4.3. *Let (u, c, v) be a global classical solution to system (4.1). Suppose that $\chi > 0$ satisfies*

$$\chi < \frac{4}{\alpha}\sqrt{\frac{d_u d_c \beta b_1 c_1 c_2}{a_1 b_2}}. \tag{4.18}$$

If $a_2 < 0$ and $\frac{a_1}{b_1} \le -\frac{a_2}{b_2}$, then the semi-trivial steady state $(u^*, c^*, v^*) = (\frac{a_1}{b_1}, 0, 0)$ is globally asymptotically stable. Moreover, there exist positive constants η_2, C_{S_2}, T_2 such that for all $t > T_2$,

$$\|u - \frac{a_1}{b_1}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C_{S_2} e^{-\eta_2 t} \quad \text{if} \quad \frac{a_1}{b_1} < -\frac{a_2}{b_2}, \tag{4.19}$$

$$\|u - \frac{a_1}{b_1}\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le \frac{C_{S_2}}{(1+t)^{\eta_2}} \quad \text{if} \quad \frac{a_1}{b_1} = -\frac{a_2}{b_2}. \tag{4.20}$$

Proof. From (4.18), we choose a positive constant C_{W_2} satisfying

$$\frac{\chi^2 a_1 b_2}{4 d_u d_c b_1 c_1} < C_{W_2} < \frac{4\beta c_2}{\alpha^2}. \tag{4.21}$$

By making use of such C_{W_2}, we set the following Lyapunov functional:

$$W_2(u(t), c(t), v(t)) := \frac{b_2}{c_1}\int_\Omega \left(u - \frac{a_1}{b_1} - \frac{a_1}{b_1}\log\frac{u}{a_1/b_1}\right) + \frac{C_{W_2}}{2}\int_\Omega c^2 + \int_\Omega v.$$

Differentiating $W_2(t)$ in time, integration by parts implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W_2(t) &= \frac{b_2}{c_1}\int_\Omega \left(1 - \frac{a_1/b_1}{u}\right) u_t + C_{W_2}\int_\Omega c c_t + \int_\Omega v(a_2 + b_2 u - c_2 v) \\
&= \frac{b_2}{c_1}\int_\Omega \left(1 - \frac{a_1/b_1}{u}\right)(d_u \Delta u + \nabla\cdot(\chi u \nabla c) + u(a_1 - b_1 u - c_1 v)) \\
&\quad - d_c C_{W_2}\int_\Omega |\nabla c|^2 + \alpha C_{W_2}\int_\Omega cv - \beta C_{W_2}\int_\Omega c^2 + \int_\Omega v(a_2 + b_2 u - c_2 v) \\
&= -\frac{d_u a_1 b_2}{b_1 c_1}\int_\Omega \left|\frac{\nabla u}{u}\right|^2 - \frac{\chi a_1 b_2}{b_1 c_1}\int_\Omega \frac{\nabla u\cdot\nabla c}{u} - d_c C_{W_2}\int_\Omega |\nabla c|^2 \\
&\quad - \frac{b_1 b_2}{c_1}\int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 + \int_\Omega \left(-c_2 v^2 + \left(a_2 + \frac{a_1 b_2}{b_1}\right) v\right) \\
&\quad + \alpha C_{W_2}\int_\Omega cv - \beta C_{W_2}\int_\Omega c^2.
\end{aligned} \tag{4.22}$$

The derivative terms of (4.22) are expressed as

$$-\frac{d_u a_1 b_2}{b_1 c_1}\int_\Omega \left|\frac{\nabla u}{u}\right|^2 - \frac{\chi a_1 b_2}{b_1 c_1}\int_\Omega \frac{\nabla u\cdot\nabla c}{u} - d_c C_{W_2}\int_\Omega |\nabla c|^2 = -\int_\Omega \begin{bmatrix}\nabla u\\ \nabla c\end{bmatrix}^T M \begin{bmatrix}\nabla u\\ \nabla c\end{bmatrix},$$

where

$$M = \begin{bmatrix} \frac{d_u a_1 b_2}{b_1 c_1 u^2} & -\frac{\chi a_1 b_2}{2 b_1 c_1 u} \\ -\frac{\chi a_1 b_2}{2 b_1 c_1 u} & d_c C_{W_2}\end{bmatrix}.$$

By (4.18) and (4.21), the matrix M is positive semidefinite, thus we have

$$-\int_\Omega \begin{bmatrix}\nabla u\\ \nabla c\end{bmatrix}^T M \begin{bmatrix}\nabla u\\ \nabla c\end{bmatrix} \le 0. \tag{4.23}$$

On the other hand, owing to (4.21), we take a constant $\beta_2 \in (\frac{\alpha^2 C_{W_2}}{4c_2}, \beta)$. Then, Young's inequality gives

$$\begin{aligned}&\int_\Omega \left(-c_2v^2 + \left(a_2 + \frac{a_1b_2}{b_1}\right)v\right) + \alpha C_{W_2}\int_\Omega cv - \beta C_{W_2}\int_\Omega c^2\\ &\le \int_\Omega \left(-c_2v^2 + \left(a_2 + \frac{a_1b_2}{b_1}\right)v\right) + \frac{\alpha^2 C_{W_2}}{4\beta_2}\int_\Omega v^2 - (\beta-\beta_2)C_{W_2}\int_\Omega c^2\\ &\le -C_1\int_\Omega v^2 - C_2\int_\Omega c^2 + \int_\Omega \left(a_2 + \frac{a_1b_2}{b_1}\right)v \end{aligned} \tag{4.24}$$

for some positive constants $C_1, C_2 > 0$. By (4.23) and (4.24), it follows from (4.22) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}W_2(t) \le -C_3\left(\int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 + \int_\Omega c^2 + \int_\Omega v\right) \quad \text{if} \quad \frac{a_1}{b_1} < -\frac{a_2}{b_2},\\ &\frac{\mathrm{d}}{\mathrm{d}t}W_2(t) \le -C_4\left(\int_\Omega \left(u - \frac{a_1}{b_1}\right)^2 + \int_\Omega c^2 + \int_\Omega v^2\right) \quad \text{if} \quad \frac{a_1}{b_1} = -\frac{a_2}{b_2},\end{aligned}$$

where $C_3, C_4 > 0$ are positive constants. Following a similar argument as in the proof of Lemma 4.2, we prove (4.19) and (4.20). □

The global asymptotic stability of the positive constant steady state (u_c, c_c, v_c) is also obtained for sufficiently small $\chi > 0$.

Lemma 4.4. *Let (u, c, v) be a global classical solution to system (4.1). If $-\frac{a_1b_2}{b_1c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, then there exists a positive constant steady state $(u^*, c^*, v^*) = (u_c, c_c, v_c)$, $u_c, c_c, v_c > 0$ given as in (1.6). Moreover, if $\chi > 0$ satisfies*

$$\chi < \frac{4}{\alpha}\sqrt{\frac{d_u d_c \beta c_1 c_2}{b_2}}, \tag{4.25}$$

then (u_c, c_c, v_c) is globally asymptotically stable and there exist positive constants η_3, C_{S_3}, T_3 such that

$$\|u - u_c\|_{L^\infty(\Omega)} + \|c - c_c\|_{L^\infty(\Omega)} + \|v - v_c\|_{L^\infty(\Omega)} \le C_{S_3}e^{-\eta_3 t} \quad \textit{for all} \quad t > T_3, \tag{4.26}$$

Proof. The positivity of u_c, c_c, v_c is a direct consequence of (1.6) under the assumption $-\frac{a_1b_2}{b_1c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$. To show the convergence, we construct a Lyapunov functional as

$$W_3(u(t), c(t), v(t)) := \frac{b_2}{c_1}\int_\Omega \left(u - u_c - u_c\log\frac{u}{u_c}\right) + \frac{C_{W_3}}{2}\int_\Omega (c - c_c)^2 + \int_\Omega \left(v - v_c - v_c\log\frac{v}{v_c}\right), \tag{4.27}$$

where $u_c, c_c, v_c > 0$ satisfy (1.6) and C_{W_3} is chosen such that

$$\frac{\chi^2 b_2}{4d_u d_c c_1} < C_{W_3} < \frac{4\beta c_2}{\alpha^2}. \tag{4.28}$$

By differentiation of (4.27), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W_3(t) &= \frac{b_2}{c_1}\int_\Omega \left(1-\frac{u_c}{u}\right)u_t + C_{W_3}\int_\Omega (c-c_c)c_t + \int_\Omega \left(1-\frac{v_c}{v}\right)v_t \\
&= -\frac{d_u u_c b_2}{c_1}\int_\Omega \left|\frac{\nabla u}{u}\right|^2 - \frac{\chi u_c b_2}{c_1}\int_\Omega \frac{\nabla u\cdot\nabla c}{u} + \frac{b_2}{c_1}\int_\Omega\left(1-\frac{u_c}{u}\right)u(a_1-b_1u-c_1v) \\
&- d_c C_{W_3}\int_\Omega |\nabla c|^2 + \alpha C_{W_3}\int_\Omega (c-c_c)v - \beta C_{W_3}\int_\Omega (c-c_c)c \\
&- d_v v_c\int_\Omega \left|\frac{\nabla v}{v}\right|^2 + \int_\Omega\left(1-\frac{v_c}{v}\right)v(a_2+b_2u-c_2v). \qquad (4.29)
\end{aligned}
$$

Similarly to the proof in Lemma 4.3, by (4.25) and (4.28), we have

$$
-\frac{d_u u_c b_2}{c_1}\int_\Omega \left|\frac{\nabla u}{u}\right|^2 - \frac{\chi u_c b_2}{c_1}\int_\Omega \frac{\nabla u\cdot\nabla c}{u} - d_c C_{W_3}\int_\Omega |\nabla c|^2 \le 0. \qquad (4.30)
$$

As to the reaction terms of u, we obtain

$$
\begin{aligned}
&\frac{b_2}{c_1}\int_\Omega\left(1-\frac{u_c}{u}\right)u(a_1-b_1u-c_1v) \\
&\quad= \frac{b_2}{c_1}\int_\Omega\left(1-\frac{u_c}{u}\right)u(a_1-b_1u-c_1v) \\
&\quad= \frac{b_2}{c_1}\int_\Omega (u-u_c)(a_1-b_1u_c-c_1v_c-b_1u-c_1v+b_1u_c+c_1v_c) \\
&\quad= -\frac{b_1b_2}{c_1}\int_\Omega (u-u_c)^2 + b_2\int_\Omega (u-u_c)(v_c-v), \qquad (4.31)
\end{aligned}
$$

where we used $a_1-b_1u_c-c_1v_c=0$. Similarly, we have for v that

$$
\begin{aligned}
&\int_\Omega\left(1-\frac{v_c}{v}\right)v(a_2+b_2u-c_2v) \\
&\quad= \int_\Omega (v-v_c)(a_2+b_2u_c-c_2v_c+b_2u-c_2v-b_2u_c+c_2v_c) \\
&\quad= -c_2\int_\Omega (v-v_c)^2 - b_2\int_\Omega (u-u_c)(v_c-v). \qquad (4.32)
\end{aligned}
$$

Lastly for c, we see that

$$
\begin{aligned}
&\alpha C_{W_3}\int_\Omega (c-c_c)v - \beta C_{W_3}\int_\Omega (c-c_c)c \\
&\quad= \alpha C_{W_3}\int_\Omega (c-c_c)v - \beta C_{W_3}\int_\Omega (c-c_c)^2 - \beta C_{W_3}\int_\Omega (c-c_c)c_c \\
&\quad= \alpha C_{W_3}\int_\Omega (c-c_c)(v-v_c) - \beta C_{W_3}\int_\Omega (c-c_c)^2. \qquad (4.33)
\end{aligned}
$$

Using Young's inequality, we infer from (4.28) and the choice of $\beta_3\in(\frac{\alpha^2C_{W_3}}{4c_2},\beta)$ that

$$
\begin{aligned}
&-c_2\int_\Omega (v-v_c)^2 + \alpha C_{W_3}\int_\Omega (c-c_c)(v-v_c) - \beta C_{W_3}\int_\Omega (c-c_c)^2 \\
&\quad\le -c_2\int_\Omega (v-v_c)^2 + \frac{\alpha^2C_{W_3}}{4\beta_3}\int_\Omega (v-v_c)^2 - (\beta-\beta_3)C_{W_3}\int_\Omega (c-c_c)^2 \\
&\quad\le -C_1\int_\Omega (v-v_c)^2 - C_2\int_\Omega (c-c_c)^2 \qquad (4.34)
\end{aligned}
$$

for some positive constants $C_1, C_2 > 0$. Substituting (4.30)–(4.34) into (4.29), we end up with

$$\frac{d}{dt}W_3(t) \leq -C_3\left(\int_\Omega (u-u_c)^2 + \int_\Omega (c-c_c)^2 + \int_\Omega (v-v_c)^2\right),$$

where C_3 is a positive constant. Thus, the decay rate (4.26) follows from a similar argument in the proof of Lemma 4.2. □

Proof of Theorem 1.3. Collecting the results of Lemmas 4.2–4.4, we complete the proof. □

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5. Appendix

Using standard linearization, we prove Proposition 1.2 related to the instability of a positive constant steady state for sufficiently large $\chi > 0$. From the threshold χ_c defined by (5.5), the positive constant steady state can become unstable for various eigenmodes depending on all possible parameters.

Proof of Proposition 1.2. To linearize system (1.1) at (u_c, c_c, v_c), we let $u_L := u - u_c$, $c_L := c - c_c$, $v_L := v - v_c$ and rewrite (1.1) as

$$\frac{\partial}{\partial t}\begin{pmatrix} u_L \\ c_L \\ v_L \end{pmatrix} = \begin{pmatrix} d_u\Delta + A_1 & \chi u_c \Delta & A_2 \\ 0 & d_c\Delta - \beta & \alpha \\ A_3 & 0 & d_v\Delta + A_4 \end{pmatrix}\begin{pmatrix} u_L \\ c_L \\ v_L \end{pmatrix}, \tag{5.1}$$

where

$$\begin{aligned} A_1 &= f'(u_c) - v_c g_u(u_c, v_c), \quad & A_2 &= -g(u_c, v_c) - v_c g_v(u_c, v_c), \\ A_3 &= r v_c g_u(u_c, v_c), & A_4 &= r g(u_c, v_c) + r v_c g_v(u_c, v_c) - k. \end{aligned} \tag{5.2}$$

Let $\{\mu_j\}_{j=1}^\infty$ and $\{\phi_j\}_{j=1}^\infty$ be positive eigenvalues and corresponding eigenfunctions of the Laplace operator $-\Delta$ on Ω with homogeneous Neumann boundary conditions. Then, we express u_L, c_L, v_L of (5.1) as

$$u_L = \sum_{j=1}^\infty u_j \phi_j(x) e^{\lambda_j t}, \qquad c_L = \sum_{j=1}^\infty c_j \phi_j(x) e^{\lambda_j t}, \qquad v_L = \sum_{j=1}^\infty v_j \phi_j(x) e^{\lambda_j t}$$

for some constants u_j, c_j and v_j's. For each $j = 1, 2, 3, \ldots$, there exists a 3×3 matrix M_j given by

$$M_j = \begin{pmatrix} -d_u\mu_j + A_1 & -\chi u_c \mu_j & A_2 \\ 0 & -d_c\mu_j - \beta & \alpha \\ A_3 & 0 & -d_v\mu_j + A_4 \end{pmatrix},$$

where $\lambda_j(u_L, c_L, v_L)^\mathsf{T} = M_j(u_L, c_L, v_L)^\mathsf{T}$ holds. Let λ_j^1, λ_j^2, λ_j^3 be eigenvalues of M_j. Then, by the Routh–Hurwitz criterion, (u_c, c_c, v_c) is linearly stable if and only if for all $j \in \mathbb{N}$,

$$P_j^1 > 0, \quad P_j^1 P_j^2 - P_j^3 > 0, \quad P_j^3 > 0, \tag{5.3}$$

where

$$
\begin{aligned}
P_j^1 &= -tr(M_j) = (d_u + d_c + d_v)\mu_j - A_1 + \beta - A_4,\\
P_j^2 &= det\begin{pmatrix} -d_u\mu_j + A_1 & -\chi u_c\mu_j \\ 0 & -d_c\mu_j - \beta \end{pmatrix} + det\begin{pmatrix} -d_c\mu_j - \beta & \alpha \\ 0 & -d_v\mu_j + A_4 \end{pmatrix}\\
&\quad + det\begin{pmatrix} -d_u\mu_j + A_1 & A_2 \\ A_3 & -d_v\mu_j + A_4 \end{pmatrix},\\
&= (d_u\mu_j - A_1)(d_c\mu_j + \beta) + (d_c\mu_j + \beta)(d_v\mu_j - A_4) + (d_u\mu_j - A_1)(d_v\mu_j - A_4) - A_2A_3,\\
P_j^3 &= -det(M_j)\\
&= (d_u\mu_j - A_1)(d_c\mu_j + \beta)(d_v\mu_j - A_4) - A_2A_3(d_c\mu_j + \beta) + A_3\alpha u_c\mu_j\chi\\
&=: P_j^{3,1} + P_j^{3,2}\chi.
\end{aligned}
$$

Since we assume that (u_c, c_c, v_c) is linearly stable for $\chi = 0$, via (5.3), we have

$$P_j^1 > 0, \quad P_j^1P_j^2 - P_j^{3,1} > 0, \quad P_j^{3,1} > 0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{5.4}$$

From (5.2), the hypotheses $(H2)$ and $(H3)$ imply $A_3 > 0$, and thus, $P_j^{3,2} > 0$ for all $j \in \mathbb{N}$. Therefore, combining (5.3) and (5.4), we can find $\chi_c > 0$ such that

$$\chi_c := \min_{j\in\mathbb{N}}\left\{\frac{P_j^1P_j^2 - P_j^{3,1}}{P_j^{3,2}}\right\}. \tag{5.5}$$

This completes the proof. □

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Inkyung Ahn and Changwook Yoon
Department of Mathematics
Korea University
Sejong 30019
Republic of Korea
e-mail: chwyoon@gmail.com

Inkyung Ahn
e-mail: ahnik@korea.ac.kr

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