Z. Angew. Math. Phys. (2021) 72:19

0044-2275/21/010001-23
published online January 11, 2021
https://doi.org/10.1007/s00033-020-01453-y

Zeitschrift für angewandte Mathematik und Physik ZAMP

Global well-posedness of 3D magneto-micropolar fluid equations with mixed partial viscosity near an equilibrium

Yuzhu Wang and Weijia Li

Abstract. In this paper, we investigate the initial value problem for the 3D magneto-micropolar fluid equations with mixed partial viscosity. The main purpose of this paper is to establish global well-posedness of classical small solutions. More precisely, we prove that the global stability of perturbations near the steady solution is given by a background magnetic field. The proof is mainly based on the energy estimate and bootstrapping argument.

Mathematics Subject Classification. 35L30, 35B40.

Keywords. Magneto-micropolar fluid equations, Mixed partial viscosity, Global classical solutions.

1. Introduction

In this paper, we investigate three-dimensional (3D) magneto-micropolar fluid equations

$$\begin{cases} \partial_t u - (\mu+\chi)\Delta u + u\cdot\nabla u - B\cdot\nabla B + \nabla\left(p+\frac{1}{2}|B|^2\right) - 2\chi\nabla\times v = 0, \\ \partial_t v - \gamma\Delta v - \kappa\nabla\nabla\cdot v + 4\chi v + u\cdot\nabla v - 2\chi\nabla\times u = 0, \\ \partial_t B - \nu\Delta B + u\cdot\nabla B - B\cdot\nabla u = 0, \\ \nabla\cdot u = 0, \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $v = v(x,t)$, $B = B(x,t) \in \mathbb{R}^3$, $p = p(x,t) \in \mathbb{R}$ are the velocity of the fluid, the micro-rotational velocity, magnetic field and hydrostatic pressure field, respectively, and the μ, χ, γ and κ represent the kinematic viscosity, vortex viscosity and spin viscosity, respectively, and $\frac{1}{\nu}$ is the magnetic Reynold.

The incompressible magneto-micropolar fluid equations have made analytic studies a great challenge but offer new opportunities due to their distinctive mathematical features. The local existence and uniqueness of strong solution were proved by Galerkin method in [21]. Later, the global existence of strong solution with the small initial data was established in [18]. We refer to Rojas-Medar and Boldrini [22] for the existence and the uniqueness of weak solutions for 2D incompressible magneto-micropolar fluid equations. Global existence of smooth solutions and global regularity of weak solutions are important topic in the study field of the magneto-micropolar fluid equations. The blow-up criteria for smooth solutions and regularity criteria of weak solutions were obtained in different function spaces, such as Morrey–Campanato space, Besov space and homogeneous Besov space, we may refer to [8,32,38,39,41] and [43]. Very recently, based on Serrin's type non-blow up criterion established by [39], Wang and Gu [34] proved global existence of a class of smooth solutions, which ensure the L^3 norm is large. When partial viscosities disappear, the case becomes more complex, we may refer to [19,23,24] and [28]. Regmi and Wu [19] singled out three special partial dissipation cases and established the global regularity for each case. By fully exploiting the special structure of the system and using the maximal regularity property of the 1D heat operator, Shang

Birkhäuser

and Gu [23] established the global existence of classical solution for 2D magneto-micropolar equations with only velocity dissipation and partial magnetic diffusion. The blow-up criterion for two-dimensional magneto-micropolar fluid equations with partial viscosity has been proved in [28]. Mixed partial viscosity means the viscosity coefficients are different in different directions, even the viscosity coefficients disappear in some directions. Therefore, some nonlinear terms cannot be controlled by the energy functions and the dissipative parts, which causes difficulty in dealing with these nonlinear terms. The regularity of the 2D anisotropic magneto-micropolar fluid equations with vertical kinematic viscosity, horizontal magnetic diffusion and horizontal vortex viscosity is established in Cheng and Liu [3]. Wang and Wang [33] studied the global existence of smooth solutions for 3D magneto-micropolar fluid equations with mixed partial viscosity by energy method. The results in [33] imply that there are two directional viscosities in every equation.

If $v = 0$ and $\chi = 0$, then the magneto-micropolar fluid equations (1.1) reduce to MHD equations. In additional, if $\mu = \nu = 0$, then the MHD equations reduce to ideal MHD equations. The MHD equations govern the dynamics of the velocity and magnetic fields in electrically conducting fluids such as plasmas, liquid metals and salt water (see [12]). The global well-posedness of MHD equations has attracted the attention of many mathematicians, and lots of interesting results were established. On the one hand, we could refer to [6,9,15,17,27,29–31] for global well-posedness of MHD equations with full viscosity. On the other hand, the global well-posedness problem for the MHD equations with partial viscosity has been successfully investigated in lots of work(see, e.g., [1,2,4,5,7,10,11,13,14,16,20,26,35–37,40,42,44] and [45]). We only recall the global existence of classical solutions to 3D MHD equations with partial viscosity for our purpose. The globally well-posed is proved by Cai and Lei [1] under the assumption that the initial velocity field and the displacement of the initial magnetic field from a nonzero constant are sufficiently small in certain weighted Sobolev spaces. Lin and Zhang [14] proved the global well-posedness to 3D MHD-type equations by the energy method, which depends crucially on the divergence-free condition of the velocity field. Under the condition of small initial data, Wu and Zhu [37] proved that the MHD equations with mixed partial dissipation and magnetic diffusion have a unique global smooth solution. The global well-posedness of smooth solutions to the 3D MHD equations with mixed partial dissipation and magnetic diffusion was proved by Wang and Wang [26]. We also refer to [16] for global well-posedness of classical solutions for a family of special axisymmetric initial data whose swirl components of the velocity field and magnetic vorticity field are trivial.

In this paper, we are interested in the following stability problem of the 3D magneto-micropolar equations with mixed partial viscosity

$$\begin{cases} \partial_t u - \mu_i \partial_i^2 u - \mu_j \partial_j^2 u - \chi \Delta u + u \cdot \nabla u - B \cdot \nabla B + \nabla \left(p + \frac{1}{2}|B|^2 \right) - 2\chi \nabla \times v = 0, \\ \partial_t v - \gamma_m \partial_m^2 v - \gamma_n \partial_n^2 v - \kappa \nabla \nabla \cdot v + 4\chi v + u \cdot \nabla v - 2\chi \nabla \times u = 0, \\ \partial_t B - \nu_k \partial_k^2 B + u \cdot \nabla B - B \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \nabla \cdot B = 0, \end{cases} \tag{1.2}$$

where $\mu_i, \mu_j, \gamma_m, \gamma_n, \nu_k > 0 (i, j, k, m, n = 1, 2, 3, i \neq j \neq k,\ m \neq n)$ and $\chi, \kappa \geq 0$. It is easy to find a special solution of (1.2) which is given by the zero velocity field, zero micro-rotational velocity and the background magnetic field $B^0 = e_l (l = 1, 2, 3)$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. The perturbation (u, v, b) around this equilibrium with $b = B - e_l (l = i, j)$ satisfies

$$\begin{cases} \partial_t u - \mu_i \partial_i^2 u - \mu_j \partial_j^2 u - \chi \Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla \left(p + \frac{1}{2}|b|^2 \right) - 2\chi \nabla \times v - \partial_l b = 0, \\ \partial_t v - \gamma_m \partial_m^2 v - \gamma_n \partial_n^2 v - \kappa \nabla \nabla \cdot v + 4\chi v + u \cdot \nabla v - 2\chi \nabla \times u = 0, \\ \partial_t b - \nu_k \partial_k^2 b + u \cdot \nabla b - b \cdot \nabla u - \partial_l u = 0, \\ \nabla \cdot u = 0, \nabla \cdot b = 0. \end{cases} \tag{1.3}$$

Inspired by the recent works [33] for 3D magneto-micropolar equations with mixed partial viscosity and [26,37] for 3D incompressible MHD equations with mixed partial viscosity, the main aim of this paper is to investigate the stability problem on the perturbation (u, v, b). In other words, we shall prove the global small classical solutions to (1.3) with the initial data

$$t = 0: \quad u = u_0(x), \quad v = v_0(x), \quad b = b_0(x), \quad x \in \mathbb{R}^3. \tag{1.4}$$

Next, we start our main results as follows:

Theorem 1.1. *Let $\mu_i, \mu_j, \gamma_m, \gamma_n, \nu_k > 0(i,j,k,m,n = 1,2,3, i \neq j \neq k,\ m \neq n, l = i,j)$ and $\chi, \kappa \geq 0$. Assume that $(u_0, v_0, b_0) \in H^3(\mathbb{R}^3)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Put*

$$\mathscr{E}_0 = \|u_0\|_{H^3} + \|v_0\|_{H^3} + \|b_0\|_{H^3}.$$

There exists a constant $\delta > 0$ such that if $\mathscr{E}_0 \leq \delta$, then the problem (1.3), (1.4) has a unique global classical solution $(u, v, b) \in C([0,\infty); H^3(\mathbb{R}^3))$. Moreover, it holds that for any $t > 0$

$$\begin{aligned}\|u(t)\|_{H^3}^2 + \|v(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t (\|\partial_i u(\tau)\|_{H^3}^2 + \|\partial_j u(\tau)\|_{H^3}^2 + \|\partial_m v(\tau)\|_{H^3}^2 \\ + \|\partial_n v(\tau)\|_{H^3}^2 + \|\partial_k b(\tau)\|_{H^3}^2 + \|\partial_l b(\tau)\|_{H^2}^2 + \kappa \|\nabla \cdot v(\tau)\|_{H^3}) d\tau \leq C\mathscr{E}_0.\end{aligned}$$

Remark 1.2. The result in Theorem 1.1 implies that it only needs one directional viscosity in the equations for b, which reduces to the requirement for the viscosity in [33].

Theorem 1.1 contains eighteen cases; the proof is similar. We only prove Theorem 1.1 with case: $i = m = 2$, $j = n = 3$, $k = 1$ and $l = 3$. To obtain global solutions, we need to bound $\|u(t)\|_{H^3} + \|v(t)\|_{H^3} + \|b(t)\|_{H^3}$ via the energy estimates and bootstrapping argument. But there are only two directional velocities and micro-rotational velocity viscosity (in x_2 and x_3) and one directional magnetic diffusion (in x_1); some nonlinear terms cannot be controlled by $\|u(t)\|_{H^3} + \|v(t)\|_{H^3} + \|b(t)\|_{H^3}$ and the dissipative parts $\|\partial_2 u\|_{H^3}$, $\|\partial_3 u\|_{H^3}$, $\|\partial_2 v\|_{H^3}$, $\|\partial_3 v\|_{H^3}$, $\|\partial_1 b\|_{H^3}$. Consequently, we are not able to build a closed differential inequality for

$$\begin{aligned}E_0(t) = \sup_{0\leq\tau\leq t} \left(\|u(\tau)\|_{H^3}^2 + \|v(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\right) + 2\int_0^t (\mu_2 \|\partial_2 u(\tau)\|_{H^3}^2 \\ + \mu_3 \|\partial_3 u(\tau)\|_{H^3}^2 + \gamma_2 \|\partial_2 v(\tau)\|_{H^3}^2 + \gamma_3 \|\partial_3 v(\tau)\|_{H^3}^2 + \nu_1 \|\partial_1 b(\tau)\|_{H^3}^2 \\ + \kappa \|\nabla \cdot v(\tau)\|_{H^3}^2) d\tau,\end{aligned}$$

which forces us to introduce suitable extra terms in the energy estimate. To this end, the following term

$$E_1(t) = \int_0^t \|\partial_3 b(\tau)\|_{H^2}^2 d\tau$$

is introduced, which serves and achieves our purpose perfectly. Therefore, we could build a closed inequality for $E_0(t)$ and $E_1(t)$; then, global classical solutions follow from bootstrapping argument.

We introduce some notations which are used in this paper. $L^2 = L^2(\mathbb{R}^3)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^2}$. Sobolev space of order m is defined by $H^m = H^m(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) | \nabla^m u \in L^2\right\}$ with the norm $\|u\|_{H^m} = \left(\|u\|_{L^2}^2 + \|\nabla^m u\|_{L^2}^2\right)^{\frac{1}{2}}$.

The paper is organized as follows: In Sect. 2, we shall establish two a priori estimates and build the closed inequality for $E_0(t)$ and $E_1(t)$. More precisely, we derive the inequalities for $E_0(t)$ and $E_1(t)$ by the energy estimate and the tricky interpolation techniques in Sects. 2.1 and 2.2, respectively. In the last section (Sect. 3), we shall complete the proof of Theorem 1.1 by the bootstrapping argument.

2. Preliminaries

The main aim of this section is to establish the following two a priori estimates

$$E_0(t) \le C\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}} + E_0(t)^{\frac{3}{2}} + E_0(t)^2 + E_1(t)^2\right) \tag{2.1}$$

and

$$E_1(t) \le C\left(\mathscr{E}_0 + E_0(t) + E_0(t)^{\frac{3}{2}} + E_0(t)^2 + E_1(t)^2\right), \tag{2.2}$$

respectively.

The following lemma which will play very important roles in proving (2.1) and (2.2) has been obtained in [37].

Lemma 2.1. *Assume that the right-hand sides of the following estimates are all bounded, then*

$$\int_{\mathbb{R}^3} |fgh|\, dx \le C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_1 f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 h\|_{L^2}^{\frac{1}{2}}, \tag{2.3}$$

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh\phi|\, dx \le\ & C\|f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_3 f\|_{L^2}^{\frac{1}{4}}\|\partial_3\partial_2 f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{4}}\|\partial_2 g\|_{L^2}^{\frac{1}{4}}\\ &\cdot\|\partial_3 g\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3 g\|_{L^2}^{\frac{1}{4}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}\|\phi\|_{L^2}^{\frac{1}{2}}\|\partial_1\phi\|_{L^2}^{\frac{1}{2}}\end{aligned} \tag{2.4}$$

and

$$\int_{\mathbb{R}^3} |fgh|\, dx \le C\|f\|_{L^2}^{\frac{1}{4}}\|\partial_2 f\|_{L^2}^{\frac{1}{4}}\|\partial_3 f\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3 f\|_{L^2}^{\frac{1}{4}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_1 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}. \tag{2.5}$$

2.1. Proof of (2.1)

Owing to the equivalence of $\|(u,v,b)\|_{H^3}$ with $\|(u,v,b)\|_{L^2} + \|(u,v,b)\|_{\dot{H}^3}$, we estimate the L^2 and the homogeneous $\dot{H}^3$-norm of (u,v,b), respectively. By standard energy estimate, $\nabla\cdot u = \nabla\cdot b = 0$ and Cauchy inequality, we obtain

$$\begin{aligned}&(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + 2\int_0^t (\mu_2\|\partial_2 u\|_{L^2}^2 + \mu_3\|\partial_3 u\|_{L^2}^2 + \gamma_2\|\partial_2 v\|_{L^2}^2\\ &\quad + \gamma_3\|\partial_3 v\|_{L^2}^2 + \kappa\|\nabla\cdot v\|_{L^2}^2 + \nu_1\|\partial_1 b\|_{L^2}^2)d\tau\\ &\le \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.\end{aligned} \tag{2.6}$$

Next we investigate the $\dot{H}^3$-norm. Applying $\partial_i^3 (i = 1, 2, 3)$ to (1.3) and then taking the inner product with $(\partial_i^3 u, \partial_i^3 v, \partial_i^3 b)$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 [(\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 v\|_{L^2}^2 + \|\partial_i^3 b\|_{L^2}^2) + \mu_2\|\partial_i^3\partial_2 u\|_{L^2}^2 + \mu_3\|\partial_i^3\partial_3 u\|_{L^2}^2 + \chi\|\partial_i^3\nabla u\|_{L^2}^2\\ &+ \gamma_2\|\partial_i^3\partial_2 v\|_{L^2}^2 + \gamma_3\|\partial_i^3\partial_3 v\|_{L^2}^2 + \kappa\|\partial_i^3\nabla\cdot v\|_{L^2}^2 + 4\chi\|\partial_i^3 v\|_{L^2}^2 + \nu_1\|\partial_i^3\partial_1 b\|_{L^2}^2] = \sum_{j=1}^7 \mathcal{I}_j,\end{aligned} \tag{2.7}$$

where

$$\mathcal{I}_1 = \sum_{i=1}^3 \int_{\mathbb{R}^3} (\partial_i^3\partial_3 b\cdot\partial_i^3 u + \partial_i^3\partial_3 u\cdot\partial_i^3 b)\, dx,$$

$$\mathcal{I}_2 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \, dx,$$

$$\mathcal{I}_3 = \sum_{i=1}^{3} \int_{\mathbb{R}^3} [\partial_i^3 (b \cdot \nabla b) - b \cdot \nabla \partial_i^3 b] \cdot \partial_i^3 u \, dx,$$

$$\mathcal{I}_4 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \, dx,$$

$$\mathcal{I}_5 = \sum_{i=1}^{3} \int_{\mathbb{R}^3} [\partial_i^3 (b \cdot \nabla u) - b \cdot \nabla \partial_i^3 u] \cdot \partial_i^3 b \, dx,$$

$$\mathcal{I}_6 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla v) \cdot \partial_i^3 v \, dx,$$

$$\mathcal{I}_7 = -2\chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} [\partial_i^3 (\nabla \times v) \partial_i^3 u + \partial_i^3 (\nabla \times u) \partial_i^3 v] \, dx.$$

By integration by parts, we have

$$\mathcal{I}_1 = 0. \tag{2.8}$$

To bound $\mathcal{I}_2$, we decompose it into three pieces as

$$\mathcal{I}_2 = -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \, dx =: \mathcal{I}_{2,1} + \mathcal{I}_{2,2} + \mathcal{I}_{2,3}.$$

By Lemma 2.1 and Cauchy's inequality, we obtain

$$\begin{aligned}
\mathcal{I}_{2,1} &= -\int_{\mathbb{R}^3} \partial_1^3 (u \cdot \nabla u) \cdot \partial_1^3 u \, dx \\
&= -\int_{\mathbb{R}^3} [\partial_1^3 (u_2 \cdot \partial_2 u) \cdot \partial_1^3 u + \partial_1^3 (u_3 \cdot \partial_3 u) \cdot \partial_1^3 u + \partial_1^3 (u_1 \cdot \partial_1 u) \cdot \partial_1^3 u] \, dx \\
&= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} (\partial_1^k u_2 \cdot \partial_1^{3-k} \partial_2 u \cdot \partial_1^3 u + \partial_1^k u_3 \cdot \partial_1^{3-k} \partial_3 u \cdot \partial_1^3 u) \, dx \\
&\quad - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1^k u_1 \cdot \partial_1^{3-k} \partial_1 u \cdot \partial_1^3 u \, dx \\
&\le C \sum_{k=1}^{3} \|\partial_1^k u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^k u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \partial_1^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1^k u_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^k u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \partial_1^{3-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1^{k-1} (\partial_2 u_2 + \partial_3 u_3)\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1^{k-1} (\partial_2 u_2 + \partial_3 u_3)\|_{L^2}^{\frac{1}{2}} \|\partial_1^{4-k} u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^{4-k} u\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\le C \|\partial_2 u\|_{H^3}^{\frac{3}{2}} \|\partial_3 u\|_{H^3}^{\frac{1}{2}} \|u\|_{H^3} + \|\partial_3 u\|_{H^3}^{\frac{3}{2}} \|\partial_2 u\|_{H^3}^{\frac{1}{2}} \|u\|_{H^3} \\
&\le C \|u\|_{H^3} (\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2).
\end{aligned}$$

Hölder's inequality gives

$$\mathcal{I}_{2,2} \leq C\|\partial_2(u\cdot\nabla u)\|_{\dot{H}^2}\|\partial_2 u\|_{\dot{H}^2} \leq C\|u\|_{H^3}\|\partial_2 u\|_{H^3}^2.$$

Similarly, it holds that

$$\mathcal{I}_{2,3} \leq C\|\partial_3(u\cdot\nabla u)\|_{\dot{H}^2}\|\partial_3 u\|_{\dot{H}^2} \leq C\|u\|_{H^3}\|\partial_3 u\|_{H^3}^2.$$

Combining the above three estimates yields

$$\mathcal{I}_2 \leq C\|u\|_{H^3}(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2). \tag{2.9}$$

The next term is $\mathcal{I}_3$, which is estimated as

$$\begin{aligned}\mathcal{I}_3 &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}[\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b]\cdot\partial_i^3 u\,dx = \sum_{i=1}^{3}\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_i^k b\cdot\nabla_i^{3-k}b\cdot\partial_i^3 u\,dx\\ &=: \mathcal{I}_{3,1}+\mathcal{I}_{3,2}+\mathcal{I}_{3,3}.\end{aligned}$$

Thanks to Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{I}_{3,1} &= \sum_{k=1}^{2}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_1^k b\cdot\nabla\partial_1^{3-k}b\cdot\partial_1^3 u\,dx + \int_{\mathbb{R}^3}\partial_1^3 b\cdot\nabla b\cdot\partial_1^3 u\,dx\\ &\leq C\sum_{k=1}^{2}\|\partial_1^k b\|_{L^2}^{\frac12}\|\partial_3\partial_1^k b\|_{L^2}^{\frac12}\|\nabla\partial_1^{3-k}b\|_{L^2}^{\frac12}\|\partial_1\nabla\partial_1^{3-k}b\|_{L^2}^{\frac12}\|\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 u\|_{L^2}^{\frac12}\\ &\quad+\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_1^3 b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_3\nabla b\|_{L^2}^{\frac12}\|\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 u\|_{L^2}^{\frac12}\\ &\leq C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_1 b\|_{H^3}\\ &\leq C\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}(\|\partial_2 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).\end{aligned}$$

$\mathcal{I}_{3,2}$ could be treated along the same line; we arrive at

$$\mathcal{I}_{3,2} \leq C\|b\|_{H^3}(\|\partial_2 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).$$

It follows from Hölder's inequality and the embedding theorem that

$$\begin{aligned}\mathcal{I}_{3,3} &= \sum_{k=1}^{2}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k b\cdot\nabla\partial_3^{3-k}b\cdot\partial_3^3 u\,dx + \int_{\mathbb{R}^3}\partial_3^3 b\cdot\nabla b\cdot\partial_3^3 u\,dx\\ &\leq C\sum_{k=1}^{2}\|\partial_3^k b\|_{L^3}\|\nabla\partial_3^{3-k}b\|_{L^2}\|\partial_3^3 u\|_{L^6} + \|\partial_3^3 b\|_{L^2}\|\nabla b\|_{L^3}\|\partial_3^3 u\|_{L^6}\\ &\leq C\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2}\|b\|_{H^3}\\ &\leq C\|b\|_{H^3}(\|\partial_3 u\|_{H^3}^2 + \|\partial_3 b\|_{H^2}).\end{aligned}$$

Collecting the estimates for $\mathcal{I}_3$ and using Cauchy's inequality, we obtain

$$I_3 \leq C\left(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12} + \|b\|_{H^3})(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2\right). \tag{2.10}$$

We split the next term into three parts as

$$\mathcal{I}_4 = -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b dx =: \mathcal{I}_{4,1}+\mathcal{I}_{4,2}+\mathcal{I}_{4,3}.$$

Because $\mathcal{I}_{4,1}$ and $\mathcal{I}_{4,3}$ have partial derivatives in x_1 and x_3, respectively, we can handle them not difficultly. However, $\mathcal{I}_{4,2}$ involves the partial derivative in x_2 and the control of $\mathcal{I}_{4,2}$ is very complex. Lemma 2.1 and Cauchy's inequality entail that

$$
\begin{aligned}
\mathcal{I}_{4,1} &= -\int_{\mathbb{R}^3} \partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\,dx \\
&= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1^k u\cdot\nabla\partial_1^{3-k} b\cdot\partial_1^3 b\,dx \\
&\le C\|\partial_1^k u\|_{L^2}^{\frac12}\|\partial_2\partial_1^k u\|_{L^2}^{\frac12}\|\nabla\partial_1^{3-k} b\|_{L^2}^{\frac12}\|\partial_1\nabla\partial_1^{3-k} b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\
&\le C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^3}^{\frac32} \\
&\le C\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}(\|\partial_2 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2).
\end{aligned}
$$

The embedding theorem and Cauchy's inequality give

$$
\begin{aligned}
\mathcal{I}_{4,3} &= -\int_{\mathbb{R}^3} \partial_3^3(u\cdot\nabla b)\cdot\partial_3^3 b\,dx \\
&= -\sum_{k=2}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_3^k u\cdot\nabla\partial_3^{3-k} b\cdot\partial_3^3 b\,dx - 3\int_{\mathbb{R}^3} \partial_3 u\cdot\nabla\partial_3^2 b\cdot\partial_3^3 b\,dx \\
&\le C\|\partial_3^k u\|_{L^3}\|\nabla\partial_3^{3-k} b\|_{L^6}\|\partial_3^3 b\|_{L^2} + \|\partial_3 u\|_{L^\infty}\|\nabla\partial_3^2 b\|_{L^2}\|\partial_3^3 b\|_{L^2} \\
&\le C\|b\|_{H^3}(\|\partial_3 u\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

The difficult term is $\mathcal{I}_{4,2}$; we further decompose it into three parts as

$$
\begin{aligned}
\mathcal{I}_{4,2} &= -\int_{\mathbb{R}^3} \left[\partial_2^3(u_1\partial_1 b)\cdot\partial_2^3 b + \partial_2^3(u_2\partial_2 b)\cdot\partial_2^3 b + \partial_2^3(u_3\partial_3 b)\cdot\partial_2^3 b\right]\,dx \\
&=: \mathcal{I}_{4,2,1}+\mathcal{I}_{4,2,2}+\mathcal{I}_{4,2,3}.
\end{aligned}
$$

The same produce of treating $\mathcal{I}_{43}$ yields

$$
\begin{aligned}
\mathcal{I}_{4,2,1} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_1\cdot\partial_1\partial_2^{3-k} b\cdot\partial_2^3 b\,dx \\
&\le C\sum_{k=1}^{3} \|\partial_2^k u_1\|_{L^3}\|\partial_1\partial_2^{3-k} b\|_{L^6}\|\partial_2^3 b\|_{L^2} \\
&\le C\|\partial_2 u\|_{H^3}\|b\|_{H^3}\|\partial_1 b\|_{H^3} \\
&\le C\|b\|_{H^3}(\|\partial_2 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{4,2,3} &= -\sum_{k=2}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k u_3\cdot\partial_3\partial_2^{3-k} b\cdot\partial_2^3 b\,dx - 3\int_{\mathbb{R}^3} \partial_2 u_3\cdot\partial_3\partial_2^2 b\cdot\partial_2^3 b\,dx \\
&\le C\sum_{k=2}^{3} \|\partial_2^k u_3\|_{L^3}\|\partial_3\partial_2^{3-k} b\|_{L^6}\|\partial_2^3 b\|_{L^2} + \|\partial_2 u_3\|_{L^\infty}\|\partial_3\partial_2^2 b\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\le C\|b\|_{H^3}^2(\|\partial_2 u\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

We now turn to $\mathcal{I}_{4,2,2}$; we further break it down

$$\begin{aligned}\mathcal{I}_{4,2,2} &= -\sum_{k=2}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_2^k u_2\cdot\partial_2^{4-k}b\cdot\partial_2^3 b\,dx - 3\int_{\mathbb{R}^3}\partial_2 u_2\cdot\partial_2^3 b\cdot\partial_2^3 b\,dx\\ &= -\sum_{k=2}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_2^k u_2\cdot\partial_2^{4-k}b\cdot\partial_2^3 b\,dx + 3\int_{\mathbb{R}^3}(\partial_1 u_1+\partial_3 u_3)\cdot\partial_2^3 b\cdot\partial_2^3 b\,dx\\ &=: \mathcal{I}_{4,2,2,1}+\mathcal{I}_{4,2,2,2}+\mathcal{I}_{4,2,2,3}.\end{aligned}$$

We derive from Lemma 2.1 and Cauchy's inequality

$$\begin{aligned}\mathcal{I}_{4,2,2,1} &\le C\sum_{k=2}^{3}\|\partial_2^k u_2\|_{L^2}^{\frac12}\|\partial_2\partial_2^k u_2\|_{L^2}^{\frac12}\|\partial_2^{4-k}b\|_{L^2}^{\frac12}\|\partial_3\partial_2^{4-k}b\|_{L^2}^{\frac12}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\\ &\le C\|\partial_2 u\|_{H^3}\|b\|_{H^3}\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\\ &\le C\|b\|_{H^3}(\|\partial_2 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).\end{aligned}$$

From integration by parts and Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{I}_{4,2,2,2} &= -6\int_{\mathbb{R}^3}u_1\cdot\partial_2^3 b\cdot\partial_1\partial_2^3 b\,dx\\ &\le C\|u_1\|_{L^2}^{\frac14}\|\partial_2 u_1\|_{L^2}^{\frac14}\|\partial_3 u_1\|_{L^2}^{\frac14}\|\partial_2\partial_3\partial_2 u_1\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}\\ &\le C\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2).\end{aligned}$$

We cannot estimate $\mathcal{I}_{4,2,2,3}$ to yield a suitable bound directly. If we use Lemma 2.1 to estimate $\mathcal{I}_{4,2,2,3}$ directly,

$$\begin{aligned}\mathcal{I}_{4,2,2,3} &= 3\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_2^3 b\cdot\partial_2^3 b\,dx\\ &\le C\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_2\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_2^3 b\|_{L^2}^{\frac12},\end{aligned}$$

there will appear $\|\partial_3 b\|_{H^3}$ and the differential inequality would not be closed. We use the special structure of the equation for b in (1.3) to replace $\partial_3 u_3$ as follows

$$\partial_3 u_3 = \partial_t b_3 - \nu_1\partial_1^2 b_3 + u\cdot\nabla b_3 - b\cdot\nabla u_3.$$

We write $\mathcal{I}_{4,2,2,3}$ as

$$\begin{aligned}\mathcal{I}_{4,2,2,3} &= 3\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_2^3 b\cdot\partial_2^3 b\,dx\\ &= 3\int_{\mathbb{R}^3}[\partial_t b_3-\nu_1\partial_1^2 b_3+u\cdot\nabla b_3-b\cdot\nabla u_3]\cdot|\partial_2^3 b|^2\,dx\\ &=: \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4.\end{aligned}$$

In what follows, we can deal with $\mathcal{J}_2$, $\mathcal{J}_3$, $\mathcal{J}_4$. Hölder's inequality and Cauchy's inequality give

$$\begin{aligned}\mathcal{J}_2 &= 6\nu_1\int_{\mathbb{R}^3}\partial_1 b_3\cdot\partial_2^3 b\cdot\partial_1\partial_2^3 b\,dx\\ &\le C\|\partial_1 b_3\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_1\partial_2^3 b\|_{L^2}\end{aligned}$$

$$\leq C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^2.$$

From Lemma 2.1 and Cauchy inequality, we obtain

$$\begin{aligned}\mathcal{J}_3 &\leq C\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla b_3\|_{L^2}^{\frac14}\|\partial_2\nabla b_3\|_{L^2}^{\frac14}\|\partial_3\nabla b_3\|_{L^2}^{\frac14}\\ &\quad\times\|\partial_2\partial_3\nabla b_3\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}\|\partial_1\partial_2^3 b\|_{L^2}\\ &\leq C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac14}\|\partial_1 b\|_{H^3}\|b\|_{H^3}^{\frac32}\|\partial_3 b\|_{H^2}^{\frac12}\\ &\leq C\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_4 &\leq C\|b\|_{L^2}^{\frac14}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_3 b\|_{L^2}^{\frac14}\|\partial_2\partial_3 b\|_{L^2}^{\frac14}\|\nabla u_3\|_{L^2}^{\frac14}\|\partial_2\nabla u_3\|_{L^2}^{\frac14}\|\partial_3\nabla u_3\|_{L^2}^{\frac14}\\ &\quad\times\|\partial_2\partial_3\nabla u_3\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}\|\partial_1\partial_2^3 b\|_{L^2}\\ &\leq C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac14}\|\partial_1 b\|_{H^3}\|b\|_{H^3}^{\frac32}\|\partial_3 b\|_{H^2}^{\frac12}.\\ &\leq C\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).\end{aligned}$$

We write $\mathcal{J}_1$ as

$$\begin{aligned}\mathcal{J}_1 &= 3\frac{d}{dt}\int_{\mathbb{R}^3} b_3\cdot|\partial_2^3 b|^2dx - 3\int_{\mathbb{R}^3} b_3\cdot\partial_t|\partial_2^3 b|^2\,dx\\ &=: \mathcal{J}_{1,1}+\mathcal{J}_{1,2}.\end{aligned}$$

We use the equation of b in (1.3) to estimate $\mathcal{J}_{1,2}$

$$\begin{aligned}\mathcal{J}_{1,2} &= -6\int_{\mathbb{R}^3} b_3\cdot\partial_t\partial_2^3 b\cdot\partial_2^3 b\,dx\\ &= -6\int_{\mathbb{R}^3} b_3\cdot[\nu_1\partial_2^3\partial_1^2 b-\partial_2^3(u\cdot\nabla b)+\partial_2^3(b\cdot\nabla u)+\partial_2^3\partial_3 u)]\cdot\partial_2^3 b\,dx\\ &=: \mathcal{J}_{1,2,1}+\mathcal{J}_{1,2,2}+\mathcal{J}_{1,2,3}+\mathcal{J}_{1,2,4}.\end{aligned}$$

From integration by parts and Hölder's inequality, it follows that

$$\begin{aligned}\mathcal{J}_{1,2,1} &= 6\nu_1\int_{\mathbb{R}^3}\partial_1(b_3\cdot\partial_2^3 b)\cdot\partial_2^3\partial_1 b\,dx\\ &= 6\nu_1\int_{\mathbb{R}^3}\partial_1 b_3\cdot\partial_2^3 b\cdot\partial_2^3\partial_1 b\,dx + 6\nu_1\int_{\mathbb{R}^3} b_3\cdot|\partial_1\partial_2^3 b|^2\,dx\\ &\leq C\|\partial_1 b_3\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_1\partial_2^3 b\|_{L^2}+\|b_3\|_{L^\infty}\|\partial_1\partial_2^3 b\|_{L^2}^2\\ &\leq C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^2.\end{aligned}$$

By using integration by parts and applying Lemma 2.1 and Cauchy's inequality, we arrive at

$$\begin{aligned}\mathcal{J}_{1,2,2} &= 6\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3} b_3\cdot\partial_2^3 b\cdot\partial_2^k u\cdot\nabla\partial_2^{3-k} b\,dx + 6\int_{\mathbb{R}^3} b_3\cdot\partial_2^3 b\cdot u\cdot\nabla\partial_2^3 b\,dx\\ &= 6\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3} b_3\cdot\partial_2^3 b\cdot\partial_2^k u\cdot\nabla\partial_2^{3-k} b\,dx - 3\int_{\mathbb{R}^3}\nabla b_3\cdot u\cdot|\partial_2^3 b|^2\,dx\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=2}^{3}\|b_3\|_{L^\infty}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_2^k u\|_{L^2}^{\frac12}\|\partial_2\partial_2^k u\|_{L^2}^{\frac12}\|\nabla\partial_2^{3-k} b\|_{L^2}^{\frac12}\|\partial_3\nabla\partial_2^{3-k} b\|_{L^2}^{\frac12}\\
&\quad+\|b_3\|_{L^2}^{\frac14}\|\partial_2 b_3\|_{L^2}^{\frac14}\|\partial_3 b_3\|_{L^2}^{\frac14}\|\partial_2\partial_3 b_3\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_2\partial_2 u\|_{L^2}^{\frac14}\|\partial_3\partial_2 u\|_{L^2}^{\frac14}\\
&\quad\times\|\partial_2\partial_3\partial_2 u\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\|\nabla\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_1\nabla\partial_2^2 b\|_{L^2}^{\frac12}+|J_3|\\
&\le C(\|b\|_{H^3}^2\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^3}\\
&\quad+\|b\|_{H^3}^{\frac32}\|\partial_1 b\|_{H^3}\|\partial_3 b\|_{H^2}^{\frac12}\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac14})\\
&\le C(\|b\|_{H^3}^2+\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32})(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

Similarly, it holds that

$$
\mathcal{J}_{1,2,3}\le C\left(\|b\|_{H^3}^2+\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}\right)\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2\right).
$$

Thanks to Lemma 2.1 and Cauchy's inequality, we deduce that

$$
\begin{aligned}
\mathcal{J}_{1,2,4}&\le C\|b_3\|_{L^2}^{\frac14}\|\partial_2 b_3\|_{L^2}^{\frac14}\|\partial_3 b_3\|_{L^2}^{\frac14}\|\partial_2\partial_3 b_3\|_{L^2}^{\frac14}\|\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_2^3 u\|_{L^2}\\
&\le C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^3}\\
&\le C\|b\|_{H^3}(\|\partial_2 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

It follows from the above estimates for $\mathcal{I}_4$ and Cauchy's inequality that

$$
\begin{aligned}
\mathcal{I}_4&\le C(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}+\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32}+\|b\|_{H^3}+\|b\|_{H^3}^2)(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2\\
&\quad+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2)+\mathcal{J}_{1,1}.
\end{aligned}\tag{2.11}
$$

Next we estimate $\mathcal{I}_5$. $\mathcal{I}_5$ is rewritten as

$$
\mathcal{I}_5=\sum_{i=1}^{3}\int_{\mathbb{R}^3}(\partial_i^3(b\cdot\nabla u)-b\cdot\partial_i^3\nabla u)\cdot\partial_i^3 b\,dx=:\mathcal{I}_{5,1}+\mathcal{I}_{5,2}+\mathcal{I}_{5,3}.
$$

Hölder's inequality and embedding theorem yield

$$
\begin{aligned}
\mathcal{I}_{5,1}&=\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_1^k b\cdot\nabla\partial_1^{3-k}u\cdot\partial_1^3 b\,dx\\
&\le C\|\partial_1^k b\|_{L^3}\|\nabla\partial_1^{3-k}u\|_{L^2}\|\partial_1^3 b\|_{L^6}\\
&\le C\|u\|_{H^3}\|\partial_1 b\|_{H^3}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{5,3}&=\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k b\cdot\nabla\partial_3^{3-k}u\cdot\partial_3^3 b\,dx\\
&\le C\sum_{k=1}^{2}\|\partial_3^k b\|_{L^3}\|\nabla\partial_3^{3-k}u\|_{L^6}\|\partial_3^3 b\|_{L^2}+\|\nabla u\|_{L^\infty}\|\partial_3^3 b\|_{L^2}^2\\
&\le C(\|b\|_{H^3}\|\partial_3 b\|_{H^2}\|\partial_3 u\|_{H^3}+\|\partial_3 b\|_{H^2}^2\|u\|_{H^3})\\
&\le C(\|u\|_{H^3}+\|b\|_{H^3})(\|\partial_3 u\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

To deal with $I_{5,2}$, we further decomposed it into

$$
\mathcal{I}_{5,2}=\int_{\mathbb{R}^3}\left\{[\partial_2^3(b_1\cdot\partial_1 u)+\partial_2^3(b_2\cdot\partial_2 u)+\partial_2^3(b_3\cdot\partial_3 u)]-b\cdot\partial_2^3\nabla u\right\}\cdot\partial_2^3 b\,dx
$$

$$
\begin{aligned}
&= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \cdot \partial_1 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx + \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_2 \cdot \partial_2^{4-k} u \cdot \partial_2^3 b \, dx \\
&\quad + \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u \cdot \partial_2^3 b \, dx \\
&=: \mathcal{I}_{5,2,1} + \mathcal{I}_{5,2,2} + \mathcal{I}_{5,2,3}.
\end{aligned}
$$

We can estimate $\mathcal{I}_{5,2,1}$ and $\mathcal{I}_{5,2,2}$ easily; using integration by parts and Lemma 2.1, we have

$$
\begin{aligned}
\mathcal{I}_{5,2,1} &= -\sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_1 \partial_2^k b_1 \cdot \partial_2^{3-k} u \cdot \partial_2^3 b \, dx - \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_1 \cdot \partial_2^{3-k} u \cdot \partial_1 \partial_2^3 b \, dx \\
&\le C \sum_{k=1}^{3} \|\partial_1 \partial_2^k b_1\|_{L^2} \|\partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_2 \partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_3 \partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_2 \partial_3 \partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_2^3 b\|_{L^2}^{\frac12} \|\partial_1 \partial_2^3 b\|_{L^2}^{\frac12} \\
&\quad + C \sum_{k=1}^{3} \|\partial_1 \partial_2^3 b\|_{L^2} \|\partial_2^k b_1\|_{L^2}^{\frac12} \|\partial_1 \partial_2^k b_1\|_{L^2}^{\frac12} \|\partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_2 \partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_3 \partial_2^{3-k} u\|_{L^2}^{\frac14} \|\partial_2 \partial_3 \partial_2^{3-k} u\|_{L^2}^{\frac14} \\
&\le C \|b\|_{H^3}^{\frac12} \|\partial_1 b\|_{H^3}^{\frac32} \|u\|_{H^3}^{\frac12} \|\partial_2 u\|_{H^3}^{\frac12} \\
&\le C \|b\|_{H^3}^{\frac12} \|u\|_{H^3}^{\frac12} (\|\partial_2 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2).
\end{aligned}
$$

Thanks to $\nabla \cdot b = 0$ and Hölder's inequality, it yields

$$
\begin{aligned}
\mathcal{I}_{5,2,2} &= -\sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^{k-1} (\partial_1 b_1 + \partial_3 b_3) \cdot \partial_2^{3-k} \partial_2 u \cdot \partial_2^3 b \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_2 \cdot \partial_2 u \cdot \partial_2^3 b \, dx \\
&\le C \sum_{k=1}^{2} \|\partial_2^{k-1} (\partial_1 b_1 + \partial_3 b_3)\|_{L^3} \|\partial_2^{3-k} \partial_2 u\|_{L^6} \|\partial_2^3 b\|_{L^2} \\
&\quad + \|\partial_2^2 (\partial_1 b_1 + \partial_3 b_3)\|_{L^2} \|\partial_2 u\|_{L^\infty} \|\partial_2^3 b\|_{L^2} \\
&\le C \|b\|_{H^3} (\|\partial_2 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

$\mathcal{I}_{5,2,3}$ cannot be estimate directly; we decomposed it into

$$
\begin{aligned}
\mathcal{I}_{5,2,3} &= \sum_{k=1}^{3} \mathcal{C}_3^k \int_{\mathbb{R}^3} (\partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_1 \cdot \partial_2^3 b_1 + \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_2 \cdot \partial_2^3 b_2 + \\
&\qquad \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_3 \cdot \partial_2^3 b_3) \, dx \\
&=: \mathcal{I}_{5,2,3,1} + \mathcal{I}_{5,2,3,2} + \mathcal{I}_{5,2,3,3}.
\end{aligned}
$$

We could bound $\mathcal{I}_{5,2,3,2}$ by Hölder's inequality and the embedding theorem

$$
\begin{aligned}
\mathcal{I}_{5,2,3,2} &= \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_2 \cdot \partial_2^3 b_2 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_3 \cdot \partial_3 u_2 \cdot \partial_2^3 b_2 \, dx \\
&\le C \sum_{k=1}^{2} \|\partial_2^k b_3\|_{L^3} \|\partial_3 \partial_2^{3-k} u_2\|_{L^6} \|\partial_2^3 b_2\|_{L^2} + \|\partial_2^3 b_3\|_{L^2} \|\partial_3 u_2\|_{L^\infty} \|\partial_2^3 b_2\|_{L^2} \\
&\le C (\|\partial_1 b\|_{H^3} + \|\partial_3 b\|_{H^2}) \|\partial_3 u\|_{H^3} \|b\|_{H^3} \\
&\le C \|b\|_{H^3}^2 (\|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

$\mathcal{I}_{5,2,3,3}$ is estimated as by Lemma 2.1

$$\begin{aligned}\mathcal{I}_{5,2,3,3} &= \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_3 \cdot \partial_2^3 b_3 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_3 \cdot \partial_3 u_3 \cdot \partial_2^3 b_3 \, dx \\ &\leq C \sum_{k=1}^{2} \|\partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^{3-k} u_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 \partial_2^{3-k} u_3\|_{L^2}^{\frac{1}{2}} \\ &\quad \times \|\partial_2^3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 b_3\|_{L^2}^{\frac{1}{2}} + |\mathcal{I}_{4,2,2,2}| \\ &\leq C \|\partial_1 b\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 u\|_{H^3} \|b\|_{H^3} + |\mathcal{I}_{4,2,2,2}| \\ &\leq C \|b\|_{H^3} (\|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2) + |\mathcal{I}_{4,2,2,2}|.\end{aligned}$$

The last term $\mathcal{I}_{5,2,3,1}$ contains a part, which cannot be directly handled

$$\begin{aligned}\mathcal{I}_{5,2,3,1} &= \sum_{k=1}^{2} \mathcal{C}_3^k \int_{\mathbb{R}^3} \partial_2^k b_3 \cdot \partial_3 \partial_2^{3-k} u_1 \cdot \partial_2^3 b_1 \, dx + \int_{\mathbb{R}^3} \partial_2^3 b_3 \cdot \partial_3 u_1 \cdot \partial_2^3 b_1 \, dx \\ &\leq C \sum_{k=1}^{2} \|\partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^k b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2^{3-k} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 \partial_2^{3-k} u_1\|_{L^2}^{\frac{1}{2}} \\ &\quad \times \|\partial_2^3 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 b_1\|_{L^2}^{\frac{1}{2}} + K_1 \\ &\leq C \|\partial_1 b\|_{H^3}^{\frac{1}{2}} \|\partial_3 b\|_{H^2}^{\frac{1}{2}} \|\partial_3 u\|_{H^3} \|b\|_{H^3} + K_1 \\ &\leq C \|b\|_{H^3} (\|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2) + K_1.\end{aligned}$$

To estimate $\mathcal{K}_1$, we shall use the special structure of the equation for b in (1.3) again

$$\partial_3 u_1 = \partial_t b_1 - \nu_1 \partial_1^2 b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1.$$

We can write $\mathcal{K}_1$ as

$$\begin{aligned}\mathcal{K}_1 &= \int_{\mathbb{R}^3} \partial_2^3 b_3 \cdot (\partial_t b_1 - \nu_1 \partial_1^2 b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1) \cdot \partial_2^3 b_1 \, dx \\ &=: \mathcal{K}_{1,1} + \mathcal{K}_{1,2} + \mathcal{K}_{1,3} + \mathcal{K}_{1,4}.\end{aligned}$$

We can handle $\mathcal{K}_{1,2}$ $\mathcal{K}_{1,3}$ and $\mathcal{K}_{1,4}$ as $\mathcal{J}_2$, $\mathcal{J}_3$ and $\mathcal{J}_4$

$$\begin{aligned}\mathcal{K}_{1,2} &\leq C \|b\|_{H^3} \|\partial_1 b\|_{H^3}^2, \\ \mathcal{K}_{1,3} &\leq C \|u\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}^{\frac{1}{4}} \|\partial_3 u\|_{H^3}^{\frac{1}{4}} \|b\|_{H^3}^{\frac{3}{2}} \|\partial_1 b\|_{H^3} \|\partial_3 b\|_{H^2}^{\frac{1}{2}} \\ &\leq C \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} (\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2), \\ \mathcal{K}_{1,4} &\leq C \|u\|_{H^3}^{\frac{1}{2}} \|\partial_2 u\|_{H^3}^{\frac{1}{4}} \|\partial_3 u\|_{H^3}^{\frac{1}{4}} \|b\|_{H^3}^{\frac{3}{2}} \|\partial_1 b\|_{H^3} \|\partial_3 b\|_{H^2}^{\frac{1}{2}} \\ &\leq C \|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{3}{2}} (\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).\end{aligned}$$

$\mathcal{K}_{1,1}$ could be dealt by integration by parts

$$\mathcal{K}_{1,1} = \frac{d}{dt} \int_{\mathbb{R}^3} b_1 \cdot \partial_2^3 b_3 \cdot \partial_2^3 b_1 \, dx - \int_{\mathbb{R}^3} b_1 \cdot \partial_t (\partial_2^3 b_3 \cdot \partial_2^3 b_1) \, dx =: \mathcal{K}_{1,1,1} + \mathcal{K}_{1,1,2}.$$

According to the following equations

$$\begin{cases} \partial_t b_1 = \nu_1 \partial_1^2 b_1 - u \cdot \nabla b_1 + b \cdot \nabla u_1 + \partial_3 u_1, \\ \partial_t b_3 = \nu_1 \partial_1^2 b_3 - u \cdot \nabla b_3 + b \cdot \nabla u_3 + \partial_3 u_3, \end{cases}$$

we rewrite $\mathcal{K}_{1,1,2}$ as

$$\begin{aligned}\mathcal{K}_{1,1,2} &= -\int_{\mathbb{R}^3} b_1 \cdot \partial_2^3 b_3 \cdot \partial_2^3(\nu_1 \partial_1^2 b_1 - u\cdot\nabla b_1 + b\cdot\nabla u_1 + \partial_3 u_1)\, dx\\ &\quad -\int_{\mathbb{R}^3} b_1 \cdot \partial_2^3 b_1 \cdot \partial_2^3(\nu_1 \partial_1^2 b_3 - u\cdot\nabla b_3 + b\cdot\nabla u_3 + \partial_3 u_3)\, dx\\ &= \int_{\mathbb{R}^3} \{b_1[\partial_2^3(u\cdot\nabla b_3)\partial_2^3 b_1 + \partial_2^3(u\cdot\nabla b_1)\partial_2^3 b_3] - \nu_1 b_1[\partial_2^3\partial_1^2 b_3 \partial_2^3 b_1 + \partial_2^3\partial_1^2 b_1 \partial_2^3 b_3]\\ &\quad - b_1[\partial_2^3(b\cdot\nabla u_3)\partial_2^3 b_1 + \partial_2^3(b\cdot\nabla u_1)\partial_2^3 b_3] - [\partial_2^3\partial_3 u_3 \partial_2^3 b_1 + \partial_2^3\partial_3 u_1\partial_2^3 b_3]\}\, dx\\ &=: \mathcal{K}_{1,1,2,1} + \mathcal{K}_{1,1,2,2} + \mathcal{K}_{1,1,2,3} + \mathcal{K}_{1,1,2,4}.\end{aligned}$$

Similar to the terms $\mathcal{J}_{1,2,1}$, $\mathcal{J}_{1,2,2}$, $\mathcal{J}_{1,2,3}$, $\mathcal{J}_{1,2,4}$, we have

$$\begin{aligned}\mathcal{K}_{1,1,2,1} &\le C\|b\|_{H^3}^2\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^3} + \|b\|_{H^3}^{\frac32}\|\partial_1 b\|_{H^3}\|\partial_3 b\|_{H^2}^{\frac12}\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac14}\\ &\le C(\|b\|_{H^3}^2 + \|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32})(\|\partial_2 u\|_{H^3}^2) + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2),\\ K_{1,1,2,2} &\le C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^2,\\ K_{1,1,2,3} &\le C\|b\|_{H^3}^2\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_2 u\|_{H^3} + \|b\|_{H^3}^{\frac32}\|\partial_1 b\|_{H^3}\|\partial_3 b\|_{H^2}^{\frac12}\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac14}\\ &\le C(\|b\|_{H^3}^2 + \|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32})(\|\partial_2 u\|_{H^3}^2) + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2),\\ K_{1,1,2,4} &\le C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^{\frac12}\|\partial_3 b\|_{H^2}^{\frac12}\|\partial_3 u\|_{H^3},\\ &\le C\|b\|_{H^3}(\|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2).\end{aligned}$$

Thanks to the above estimates for $\mathcal{I}_5$ and Cauchy's inequality, we deduce that

$$\mathcal{I}_5 \le C\left(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12} + \|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac32} + \|b\|_{H^3} + \|b\|_{H^3}^2)(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2\right). \tag{2.12}$$

It remains to estimate $\mathcal{I}_6$. We rewrite $\mathcal{I}_6$ as

$$\mathcal{I}_6 = -\sum_{i=1}^3\int_{\mathbb{R}^3} \partial_i^3(u\cdot\nabla v)\cdot\partial_i^3 v\, dx =: \mathcal{I}_{6,1} + \mathcal{I}_{6,2} + \mathcal{I}_{6,3}.$$

Lemma 2.1 and Cauchy's inequality entail that

$$\begin{aligned}\mathcal{I}_{6,1} &= -\int_{\mathbb{R}^3} \partial_1^3(u_1\cdot\partial_1 v + u_2\cdot\partial_2 v + u_3\cdot\partial_3 v)\cdot\partial_1^3 v\, dx\\ &= -\sum_{k=1}^3\int_{\mathbb{R}^3} \partial_1^k u_1\cdot\partial_1^{4-k} v\cdot\partial_1^3 v\, dx\\ &\quad -\int_{\mathbb{R}^3} (\partial_1^k u_2\cdot\partial_1^{3-k}\partial_2 v + \partial_1^k u_3\cdot\partial_1^{3-k}\partial_3 v)\cdot\partial_1^3 v\, dx\\ &\le C\|\partial_1^{k-1}(\partial_2 u_2 + \partial_3 u_3)\|_{L^2}^{\frac12}\|\partial_1\partial_1^{k-1}(\partial_2 u_2 + \partial_3 u_3)\|_{L^2}^{\frac12}\\ &\quad \times\|\partial_1^{4-k} v\|_{L^2}^{\frac12}\|\partial_2\partial_1^{4-k} v\|_{L^2}^{\frac12}\|\partial_1^3 v\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 v\|_{L^2}^{\frac12}\\ &\quad + \|\partial_1^k u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^k u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^{3-k} v_2\|_{L^2}^{\frac12}\|\partial_1\partial_2\partial_1^{3-k} v_2\|_{L^2}^{\frac12}\|\partial_1^3 v\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 v\|_{L^2}^{\frac12}\\ &\quad + \|\partial_1^k u_3\|_{L^2}^{\frac12}\|\partial_2\partial_1^k u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^{3-k} v_2\|_{L^2}^{\frac12}\|\partial_1\partial_3\partial_1^{3-k} v_2\|_{L^2}^{\frac12}\|\partial_1^3 v\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 v\|_{L^2}^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&\le C(\|\partial_2 u\|_{H^3}+\|\partial_3 u\|_{H^3})\|v\|_{H^3}\|\partial_2 v\|_{H^3}^{\frac{1}{2}}\|\partial_3 v\|_{H^3}^{\frac{1}{2}}\\
&\quad+\|u\|_{H^3}^{\frac{1}{2}}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|v\|_{H^3}^{\frac{1}{2}}\|\partial_2 v\|_{H^3}\|\partial_3 v\|_{H^3}^{\frac{1}{2}}+\|u\|_{H^3}^{\frac{1}{2}}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|v\|_{H^3}^{\frac{1}{2}}\|\partial_3 v\|_{H^3}^{\frac{3}{2}}\\
&\le C\left(\|v\|_{H^3}+\|u\|_{H^3}^{\frac{1}{2}}\|v\|_{H^3}^{\frac{1}{2}}\right)\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 v\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2\right).
\end{aligned}
$$

By Hölder inequality and the embedding theorem, we have

$$
\begin{aligned}
\mathcal{I}_{6,2}&=-\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_2^k u\cdot\nabla\partial_2^{3-k}v\cdot\partial_2^3 v\,dx\\
&\le C\|\partial_2^k u\|_{L^3}\|\nabla\partial_2^{3-k}v\|_{L^2}\|\partial_2^3 v\|_{L^6}\\
&\le C\|\partial_2 u\|_{H^3}\|v\|_{H^3}\|\partial_2 v\|_{H^3}\\
&\le C\|v\|_{H^3}(\|\partial_2 u\|_{H^3}^2+\|\partial_2 v\|_{H^3}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{6,3}&=-\sum_{k=1}^{3}\mathcal{C}_3^k\int_{\mathbb{R}^3}\partial_3^k u\cdot\nabla\partial_3^{3-k}v\cdot\partial_3^3 v\,dx\\
&\le C\|\partial_3^k u\|_{L^3}\|\nabla\partial_3^{3-k}v\|_{L^2}\|\partial_3^3 v\|_{L^6}\\
&\le C\|\partial_3 u\|_{H^3}\|v\|_{H^3}\|\partial_3 v\|_{H^3}\\
&\le C\|v\|_{H^3}(\|\partial_3 u\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2).
\end{aligned}
$$

The above estimates for $\mathcal{I}_6$ and Cauchy's inequality give

$$
\mathcal{I}_6\le C\left(\|u\|_{H^3}^{\frac{1}{2}}\|v\|_{H^3}^{\frac{1}{2}}+\|v\|_{H^3}\right)\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 v\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2\right). \tag{2.13}
$$

By integration by part and Cauchy's inequality, we arrive at

$$
\begin{aligned}
\mathcal{I}_7&=4\chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3(\nabla\times u)\partial_i^3 v\,dx\\
&\le\chi(\|\nabla\times\partial_i^3 u\|_{L^2}^2+4\|\partial_i^3 v\|_{L^2}^2)\\
&=\chi(\|\nabla\partial_i^3 u\|_{L^2}^2+4\|\partial_i^3 v\|_{L^2}^2).
\end{aligned} \tag{2.14}
$$

Integrating (2.7) with respect to time and combining (2.6), (2.8)–(2.14) yield

$$
\begin{aligned}
E_0(t)&\le C\mathscr{E}_0+\int_0^t(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7)d\tau\\
&\le C\mathscr{E}_0+C\int_0^t\Big(\|u\|_{H^3}+\|v\|_{H^3}+\|b\|_{H^3}+\|b\|_{H^3}^2+\|u\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}+\|u\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{3}{2}}\\
&\quad+\|u\|_{H^3}^{\frac{1}{2}}\|v\|_{H^3}^{\frac{1}{2}}\Big)\cdot\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 v\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2\right)d\tau\\
&\quad+\int_0^t\mathcal{J}_{1,1}d\tau\\
&\le C\mathscr{E}_0+C\left(E_0^{\frac{3}{2}}(t)+E_0^2(t)+E_0^{\frac{1}{2}}(t)E_1(t)+E_0(t)E_1(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+3\int_{R^3} b_3\cdot|\partial_2^3 b|^2\,dx-3\int_{\mathbb{R}^3} b_3(x,0)\cdot|\partial_2^3 b|^2(x,0)\,dx\\
&\le C\mathscr{E}_0+C\left(E_0^{\frac{3}{2}}(t)+E_0^2(t)+E_1^2(t)\right)\\
&\quad+C\left(\|b_3(0)\|_{L^\infty}\|\partial_2^3 b(0)\|_{L^2}^2+\|b_3(t)\|_{L^\infty}\|\partial_2^3 b(t)\|_{L^2}^2\right)\\
&\le C\left(\mathscr{E}_0+\mathscr{E}_0^{\frac{3}{2}}+E_0^{\frac{3}{2}}(t)+E_0^2(t)+E_1^2(t)\right).
\end{aligned}\tag{2.15}
$$

This completes the proof of (2.1).

2.2. Proof of (2.2)

The main purpose of this section is to prove (2.2), namely

$$E_1(t)\le C\left(\mathscr{E}_0+E_0(t)+E_0(t)^{\frac{3}{2}}+E_0(t)^2+E_1(t)^2\right).$$

To bound the norm $\|\partial_3 b\|_{H^2}$, we need to bound the norm $\|\partial_3 b\|_{L^2}+\|\partial_3 b\|_{\dot H^2}$. We first estimate the L^2-norm of $\partial_3 b$. To this end, we write the equation of u in (1.3) as

$$\partial_3 b=\partial_t u-\mu_2\partial_2^2 u-\mu_3\partial_3^2 u-\chi\Delta u+u\cdot\nabla u-b\cdot\nabla b+\nabla p-2\chi\nabla\times v.$$

Taking the inner product of the above equation and $\partial_3 b$, we obtain

$$
\begin{aligned}
\|\partial_3 b\|_{L^2}^2 &= \int_{\mathbb{R}^3}\partial_t u\cdot\partial_3 b\,dx-\mu_2\int_{\mathbb{R}^3}\partial_2^2 u\cdot\partial_3 b\,dx\\
&\quad-\mu_3\int_{\mathbb{R}^3}\partial_3^2 u\cdot\partial_3 b\,dx+\int_{\mathbb{R}^3}u\cdot\nabla u\cdot\partial_3 b\,dx-\int_{\mathbb{R}^3}b\cdot\nabla b\cdot\partial_3 b\,dx\\
&\quad-\chi\int_{\mathbb{R}^3}\Delta u\cdot\partial_3 b\,dx-2\chi\int_{\mathbb{R}^3}(\nabla\times v)\cdot\partial_3 b\,dx\\
&=:\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5+\mathcal{L}_6+\mathcal{L}_7,
\end{aligned}\tag{2.16}
$$

where we have eliminated the pressure term by using $\nabla\cdot b=0$. We handle $\mathcal{L}_1$ by integrating by parts and the equation of b in (1.3)

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{d}{dt}\int_{\mathbb{R}^3}u\cdot\partial_3 b\,dx-\int_{\mathbb{R}^3}u\cdot\partial_3(\nu_1\partial_1^2 b-u\cdot\nabla b+b\cdot\nabla u+\partial_3 u)\,dx\\
&=:\mathcal{L}_{1,0}+\mathcal{L}_{1,1}+\mathcal{L}_{1,2}+\mathcal{L}_{1,3}+\mathcal{L}_{1,4}.
\end{aligned}
$$

It follows from Hölder's inequality and Cauchy's inequality that

$$\mathcal{L}_{1,1}=-\nu_1\int_{\mathbb{R}^3}u\cdot\partial_3\partial_1^2 b dx\le C\|\partial_3 u\|_{H^3}\|\partial_1 b\|_{H^3}\le C(\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2).$$

Thanks to Lemma 2.1 and Cauchy's inequality, we arrive at

$$
\begin{aligned}
\mathcal{L}_{1,2}&=\int_{\mathbb{R}^3}u\cdot\partial_3 u\cdot\nabla b+u\cdot u\cdot\nabla\partial_3 b\,dx\\
&\le C\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}^{\frac12}\|\nabla\partial_1 b\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}\\
&\quad+C\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\nabla\partial_3 b\|_{L^2}^{\frac12}\|\partial_1\nabla\partial_3 b\|_{L^2}^{\frac12}\\
&\le C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^3}^{\frac14}\|\partial_3 u\|_{H^3}^{\frac54}\|b\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^3}^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|u\|_{H^3}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 b\|_{H^2}\\
\le\ & C\|u\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2)\\
&+\|u\|_{H^3}^2(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2)+\epsilon\|\partial_3 b\|_{H^2}^2.
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{1,3} &= -\int_{\mathbb{R}^3} u\cdot\partial_3 b\cdot\nabla u + u\cdot b\cdot\nabla\partial_3 u\,dx\\
&\le C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\partial_2 u\|_{L^2}^{\frac{1}{2}}\\
&\le C\|u\|_{H^3}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 b\|_{H^2}\\
&\quad + C\|u\|_{H^3}^{\frac{1}{2}}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 u\|_{H^3}\|b\|_{H^3}^{\frac{1}{2}}\|\partial_1 b\|_{H^3}^{\frac{1}{2}}\\
&\le C\|u\|_{H^3}^{\frac{1}{2}}\|b\|_{H^3}^{\frac{1}{2}}(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2)\\
&\quad +\|u\|_{H^3}^2(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2)+\epsilon\|\partial_3 b\|_{H^2}^2.
\end{aligned}
$$

Integration by parts yields

$$
\mathcal{L}_{1,4} = -\int_{\mathbb{R}^3} u\cdot\partial_3^2 u\,dx = \int_{\mathbb{R}^3}\partial_3 u\cdot\partial_3 u\,dx \le C\|\partial_3 u\|_{H^3}^2.
$$

Hölder's inequality and Cauchy's inequality entail that

$$
\mathcal{L}_2 = -\mu_2\int_{\mathbb{R}^3}\partial_2^2 u\cdot\partial_3 b\,dx \le C\|\partial_2 u\|_{H^3}\|\partial_3 b\|_{H^2} \le C\|\partial_2 u\|_{H^3}^2+\epsilon\|\partial_3 b\|_{H^2}^2
$$

and

$$
\mathcal{L}_3 = -\mu_3\int_{\mathbb{R}^3}\partial_3^2 u\cdot\partial_3 b\,dx \le C\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2} \le C\|\partial_3 u\|_{H^3}^2+\epsilon\|\partial_3 b\|_{H^2}^2.
$$

We could deal with $\mathcal{L}_4$ and $\mathcal{L}_5$ by using Lemma 2.1 and Cauchy's inequality

$$
\begin{aligned}
\mathcal{L}_4 &= \int_{\mathbb{R}^3} u\cdot\nabla u\cdot\partial_3 b\,dx\\
&\le C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3 b\|_{L^2}^{\frac{1}{2}}\\
&\le C\|u\|_{H^3}\|\partial_2 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 u\|_{H^3}^{\frac{1}{2}}\|\partial_3 b\|_{H^2}\\
&\le C\|u\|_{H^3}^2(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2)+\epsilon\|\partial_3 b\|_{H^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_5 &= \int_{\mathbb{R}^3} b\cdot\nabla b\cdot\partial_3 b\,dx\\
&\le C\|b\|_{L^2}^{\frac{1}{2}}\|\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_3 b\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_3 b\|_{L^2}^{\frac{1}{2}}\\
&\le C\|b\|_{H^3}\|\partial_1 b\|_{H^3}^{\frac{1}{2}}\|\partial_3 b\|_{H^2}^{\frac{3}{2}}\\
&\le C\|b\|_{H^3}(\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}
$$

By integration by parts, Hölder's inequality and Cauchy's inequality, it holds that

$$\begin{aligned}
\mathcal{L}_6 &= -\chi \int_{\mathbb{R}^3} (\partial_1^2 u \cdot \partial_3 b + \partial_2^2 u \cdot \partial_3 b + \partial_3^2 u \cdot \partial_3 b)\, dx \\
&= -\chi \int_{\mathbb{R}^3} (\partial_1 \partial_3 u \cdot \partial_1 b + \partial_2^2 u \cdot \partial_3 b + \partial_3^2 u \cdot \partial_3 b)\, dx \\
&\le C(\|\partial_3 u\|_{H^3} \|\partial_1 b\|_{H^3} + \|\partial_2 u\|_{H^3} \|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3} \|\partial_3 b\|_{H^2}) \\
&\le C(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2) + \epsilon \|\partial_3 b\|_{H^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_7 &= -2\chi \int_{\mathbb{R}^3} [(\partial_2 v_3 - \partial_3 v_2) \cdot \partial_3 b_1 + (\partial_1 v_3 - \partial_3 v_1) \cdot \partial_3 b_2 + (\partial_1 v_2 - \partial_2 v_1) \cdot \partial_3 b_3]\, dx \\
&= -2\chi \int_{\mathbb{R}^3} [(\partial_3 v_2 \partial_1 b_3 - \partial_3 v_3 \partial_1 b_2) + (\partial_2 v_3 \partial_3 b_1 - \partial_2 v_1 \partial_2 b_3) + (\partial_3 v_1 \partial_3 b_2 - \partial_3 v_2 \partial_3 b_1)]\, dx \\
&\le C(\|\partial_3 v\|_{H^3} \|\partial_1 b\|_{H^3} + \|\partial_2 v\|_{H^3} \|\partial_3 b\|_{H^2} + \|\partial_3 v\|_{H^3} \|\partial_3 v\|_{H^2}) \\
&\le C(\|\partial_2 v\|_{H^3}^2 + \|\partial_3 v\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2) + \epsilon \|\partial_3 b\|_{H^2}^2.
\end{aligned}$$

Collecting the estimates for $\mathcal{L}_i (i = 1, \ldots, 7)$ and inserting them into (2.16) yield

$$\begin{aligned}
\|\partial_3 b\|_{L^2}^2 \le{}& C(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_2 v\|_{H^3}^2 + \|\partial_3 v\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2) \\
&+ C\left(\|u\|_{H^3}^{\frac{1}{2}} \|b\|_{H^3}^{\frac{1}{2}} + \|u\|_{H^3}^2 + \|b\|_{H^3}\right) \left(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2 + \|\partial_3 b\|_{H^2}^2\right) \\
&+ 7\epsilon \|\partial_3 b\|_{H^2}^2 + \mathcal{L}_{1,0}.
\end{aligned} \tag{2.17}$$

We have finished the L^2 estimate of $\partial_3 b$. Now we turn to the $\dot{H}^2$ estimate of $\partial_3 b$

$$\begin{aligned}
\sum_{i=1}^{3} \|\partial_i^2 \partial_3 b\|_{L^2}^2 ={}& \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 \partial_t u \cdot \partial_i^2 \partial_3 b\, dx - \mu_2 \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 \partial_2^2 u \cdot \partial_i^2 \partial_3 b dx \\
&- \mu_3 \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 \partial_3^2 u \cdot \partial_i^2 \partial_3 b\, dx - \chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 \Delta u \cdot \partial_i^2 \partial_3 b\, dx \\
&+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 \partial_3 b\, dx - \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 (b \cdot \nabla b) \cdot \partial_i^2 \partial_3 b\, dx \\
&- 2\chi \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 (\nabla \times v) \cdot \partial_i^2 \partial_3 b\, dx \\
=:{}& \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}_6 + \mathcal{M}_7.
\end{aligned} \tag{2.18}$$

Using integration by parts and the equation of b in (1.3) yields

$$\begin{aligned}
\mathcal{M}_1 ={}& \frac{d}{dt} \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \partial_3 \partial_i^2 b\, dx \\
&+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^2 u \cdot \partial_3 \partial_i^2 (u \cdot \nabla b - b \cdot \nabla u - \nu_1 \partial_1^2 b - \partial_3 u)\, dx \\
=:{}& \mathcal{M}_{1,0} + \mathcal{M}_{1,1} + \mathcal{M}_{1,2} + \mathcal{M}_{1,3} + \mathcal{M}_{1,4}.
\end{aligned}$$

Making use of Lemma 2.1, Hölder's inequality and Cauchy's inequality, we derive that

$$\begin{aligned}\mathcal{M}_{1,1} &= \int_{\mathbb{R}^3} \partial_3\partial_1^3 u\cdot(\partial_1 u\cdot\nabla b + u\cdot\partial_1\nabla b) + \partial_2^2 u\cdot\partial_2^2\partial_3(u\cdot\nabla b) + \partial_3^2 u\cdot\partial_3^2\partial_3(u\cdot\nabla b)\,dx\\
&\le C\|\partial_3\partial_1^3 u\|_{L^2}(\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2\partial_1 u\|_{L^2}^{\frac14}\|\partial_3\partial_1 u\|_{L^2}^{\frac14}\|\partial_2\partial_3\partial_1 u\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\\
&\quad + \|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\nabla b\|_{L^2}^{\frac12}\|\partial_1\partial_1\nabla b\|_{L^2}^{\frac12})\\
&\quad + C\|\partial_2 u\|_{H^3}(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\quad + C\|\partial_3 u\|_{H^3}(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\le C\|\partial_3 u\|_{H^3}^{\frac54}\|\partial_2 u\|_{H^3}^{\frac14}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^3}^{\frac12}\\
&\quad + C(\|\partial_2 u\|_{H^3} + \|\partial_3 u\|_{H^3})(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\le C(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12} + \|u\|_{H^3}^2 + \|b\|_{H^3})(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2) + \epsilon\|\partial_3 b\|_{H^2}^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{1,2} &= -\int_{\mathbb{R}^3} \partial_3\partial_1^3 u\cdot(\partial_1 b\cdot\nabla u + b\cdot\partial_1\nabla u) + \partial_2^2 u\cdot\partial_2^2\partial_3(b\cdot\nabla u) + \partial_3^2 u\cdot\partial_3^2\partial_3(b\cdot\nabla u)\,dx\\
&\le C\|\partial_3\partial_1^3 u\|_{L^2}(\|\partial_1 b\|_{L^2}^{\frac14}\|\partial_2\partial_1 b\|_{L^2}^{\frac14}\|\partial_3\partial_1 b\|_{L^2}^{\frac14}\|\partial_2\partial_3\partial_1 b\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1\nabla u\|_{L^2}^{\frac12}\\
&\quad + \|\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_3\partial_1\nabla u\|_{L^2}^{\frac14}\|\partial_2\partial_3\partial_1\nabla u\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12})\\
&\quad + C\|\partial_2 u\|_{H^3}(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\quad + C\|\partial_3 u\|_{H^3}(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\le C\|\partial_3 u\|_{H^3}^{\frac54}\|\partial_2 u\|_{H^3}^{\frac14}\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}\|\partial_1 b\|_{H^3}^{\frac12}\\
&\quad + C(\|\partial_2 u\|_{H^3} + \|\partial_3 u\|_{H^3})(\|u\|_{H^3}\|\partial_3 b\|_{H^2} + \|\partial_3 u\|_{H^3}\|b\|_{H^3})\\
&\le C(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12} + \|u\|_{H^3}^2 + \|b\|_{H^3})(\|\partial_2 u\|_{H^3}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2) + \epsilon\|\partial_3 b\|_{H^2}^2.\end{aligned}$$

It is not difficult to prove that

$$\begin{aligned}\mathcal{M}_{1,3} &= -\nu_1\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_3\partial_i^3 u\cdot\partial_i\partial_1^2 b\,dx\\
&\le C\|\partial_3 u\|_{H^3}\|\partial_1 b\|_{H^3}\\
&\le C\left(\|\partial_3 u\|_{H^3}^2 + \|\partial_1 b\|_{H^3}^2\right),\\
\mathcal{M}_{1,4} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}|\partial_3\partial_i^2 u|^2\,dx \le C\|\partial_3 u\|_{H^3}^2,\\
\mathcal{M}_2 &\le C\|\partial_2 u\|_{H^3}\|\partial_3 b\|_{H^2}\\
&\le C\|\partial_2 u\|_{H^3}^2 + \epsilon\|\partial_3 b\|_{H^2}^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_3 &\le C\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2}\\
&\le C\|\partial_3 u\|_{H^3}^2 + \epsilon\|\partial_3 b\|_{H^2}^2.\end{aligned}$$

Next, we estimate $\mathcal{M}_4$ and $\mathcal{M}_7$ by using integration by parts, Hölder's inequality and Cauchy's inequality

$$\mathcal{M}_4 = -\chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2\partial_1\partial_3 u\cdot\partial_i^2\partial_1 b\,dx - \chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2\partial_2^2 u\cdot\partial_i^2\partial_3 b\,dx - \chi\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2\partial_3^2 u\cdot\partial_i^2\partial_3 b\,dx$$

$$\begin{aligned}
&\leq C\left(\|\partial_3 u\|_{H^3}\|\partial_1 b\|_{H^3}+\|\partial_2 u\|_{H^3}\|\partial_3 b\|_{H^2}+\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2}\right)\\
&\leq C\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2\right)+\epsilon\|\partial_3 b\|_{H^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_7=&-2\chi\sum_{i=1}^{3}\int\limits_{\mathbb{R}^3}\partial_i^2(\partial_2 v_3-\partial_3 v_2)\cdot\partial_i^2\partial_3 b_1+\partial_i^2(\partial_3 v_1-\partial_1 v_3)\cdot\partial_i^2\partial_3 b_2\\
&+\partial_i^2(\partial_1 v_2-\partial_2 v_1)\cdot\partial_i^2\partial_3 b_3\,dx\\
\leq&\, C\|\partial_2 v\|_{H^3}\|\partial_3 b\|_{H^2}+C\|\partial_3 v\|_{H^3}\|\partial_1 b\|_{H^3}+C\|\partial_3 v\|_{H^3}\|\partial_3 b\|_{H^2}\\
\leq&\, C(\|\partial_2 v\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2)+\epsilon\|\partial_3 b\|_{H^2}^2.
\end{aligned}$$

To bound $\mathcal{M}_5$, we use Lemma 2.1 and Cauchy's inequality and obtain

$$\begin{aligned}
\mathcal{M}_5=&\int\limits_{\mathbb{R}^3}-\partial_3(u\cdot\nabla u)\cdot\partial_1^4 b+\partial_2^2(u\cdot\nabla u)\cdot\partial_2^2\partial_3 b+\partial_3^2(u\cdot\nabla u)\cdot\partial_3^2\partial_3 b\,dx\\
=&\int\limits_{\mathbb{R}^3}-(\partial_3 u\cdot\nabla u+u\cdot\partial_3\nabla u))\cdot\partial_1^4 b+\partial_2^2(u\cdot\nabla u)\cdot\partial_2^2\partial_3 b+\partial_3^2(u\cdot\nabla u)\cdot\partial_3^2\partial_3 b\,dx\\
\leq&\, C\|\partial_1^4 b\|_{L^2}\Big(\|\nabla u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\|\partial_3\nabla u\|_{L^2}^{\frac14}\|\partial_2\partial_3\nabla u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3 u\|_{L^2}^{\frac12}\\
&+\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_3\nabla u\|_{L^2}^{\frac12}\|\partial_1\partial_3\nabla u\|_{L^2}^{\frac12}\Big)\\
&+C\|u\|_{H^3}\|\partial_2 u\|_{H^3}\|\partial_3 b\|_{H^2}+C\|u\|_{H^3}\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2}\\
\leq&\, C\|\partial_1 b\|_{H^3}\|u\|_{H^3}\|\partial_2 u\|_{H^3}^{\frac12}\|\partial_3 u\|_{H^3}^{\frac12}+C\|u\|_{H^3}\|\partial_2 u\|_{H^3}\|\partial_3 b\|_{H^2}\\
&+C\|u\|_{H^3}\|\partial_3 u\|_{H^3}\|\partial_3 b\|_{H^2}\\
\leq&\, C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2\right)\\
&+C\|u\|_{H^3}^2\left(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2\right)+\epsilon\|\partial_3 b\|_{H^3}^2.
\end{aligned}$$

Similarly, it holds that

$$\begin{aligned}
\mathcal{M}_6=&-\int\limits_{\mathbb{R}^3}[\partial_1^2(b\cdot\nabla b)\cdot\partial_1^2\partial_3 b+\partial_3^2(b\cdot\nabla b)\cdot\partial_3^2\partial_3 b\\
&+\partial_2^2 b\cdot\nabla b\cdot\partial_3\partial_2^2 b+\partial_2 b\cdot\partial_2\nabla b\cdot\partial_3\partial_2^2 b+b\cdot\partial_2^2\nabla b\cdot\partial_3\partial_2^2 b]\,dx\\
\leq&\, C\|\partial_1 b\|_{H^3}\|b\|_{H^3}\|\partial_3 b\|_{H^2}+C\|\partial_3 b\|_{H^2}^2\|b\|_{H^3}\\
&+C\|\partial_3\partial_2^2 b\|_{L^2}(\|\nabla b\|_{L^2}^{\frac14}\|\partial_2\nabla b\|_{L^2}^{\frac14}\|\partial_3\nabla b\|_{L^2}^{\frac14}\|\partial_2\partial_3\nabla b\|_{L^2}^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 b\|_{L^2}^{\frac12}\\
&+\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_2\partial_2 b\|_{L^2}^{\frac14}\|\partial_3\partial_2 b\|_{L^2}^{\frac14}\|\partial_2\partial_3\partial_2 b\|_{L^2}^{\frac14}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla b\|_{L^2}^{\frac12}\\
&+\|b\|_{L^2}^{\frac14}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_3 b\|_{L^2}^{\frac14}\|\partial_2\partial_3 b\|_{L^2}^{\frac14}\|\partial_2^2\nabla b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2\nabla b\|_{L^2}^{\frac12})\\
\leq&\, C(\|\partial_1 b\|_{H^3}\|b\|_{H^3}\|\partial_3 b\|_{H^2}+\|\partial_3 b\|_{H^2}^2\|b\|_{H^3}+C\|\partial_1 b\|_{H^3}^{\frac12}\|b\|_{H^3}\|\partial_3 b\|_{H^2}^{\frac32})\\
\leq&\, C\|b\|_{H^3}(\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2).
\end{aligned}$$

We institute the estimates for $\mathcal{M}_i(i=1,\ldots,7)$ into (2.18) and arrive at

$$\begin{aligned}
\|\partial_i^2\partial_3 b\|_{L^2}^2\leq&\, C(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 v\|_{H^3}^2+\|\partial_3 v\|_{H^3}^2+\|\partial_1 b\|_{H^3}^2)\\
&+C(\|u\|_{H^3}^{\frac12}\|b\|_{H^3}^{\frac12}+\|u\|_{H^3}^2+\|b\|_{H^3}^2+\|b\|_{H^3})(\|\partial_2 u\|_{H^3}^2+\|\partial_3 u\|_{H^3}^2\\
&+\|\partial_1 b\|_{H^3}^2+\|\partial_3 b\|_{H^2}^2)+7\epsilon\|\partial_3 b\|_{H^2}^2+\mathcal{M}_{1,0}.
\end{aligned}\tag{2.19}$$

Now, adding (2.17) and (2.19), then integrating in time and using the definition of $E_0(t)$ and $E_1(t)$ and choosing $\epsilon \leq \frac{1}{28}$, we obtain

$$E_1(t) \leq C(E_0(t) + E_0^{\frac{3}{2}}(t) + E_0^2(t) + E_0^{\frac{1}{2}}(t)E_1(t) + E_0(t)E_1(t)) \\ + \frac{1}{2}E_1(t) + \int_0^t \mathcal{L}_{1,0} d\tau + \int_0^t \mathcal{M}_{1,0} d\tau. \tag{2.20}$$

Noting that

$$\int_0^t \mathcal{L}_{1,0} d\tau = \int_{\mathbb{R}^3} u(x,t) \cdot \partial_3 b(x,t) dx - \int_{\mathbb{R}^3} u(x,0) \cdot \partial_3 b(x,0)\, dx \\ \leq C(E_0(t) + \mathscr{E}_0)$$

and

$$\int_0^t \mathcal{M}_{1,0} d\tau = \int_{\mathbb{R}^3} \partial_i^2 u(x,t) \cdot \partial_3 \partial_i^2 b(x,t) dx - \int_{\mathbb{R}^3} u(x,0) \cdot \partial_3 b(x,0)\, dx \\ \leq C(E_0(t) + \mathscr{E}_0).$$

Therefore, we have by Cauchy's inequality

$$E_1(t) \leq C(\mathscr{E}_0 + E_0(t) + E_0(t)^{\frac{3}{2}} + E_0^2(t) + E_1^2(t)).$$

This complete the proof of (2.2).

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We employ the bootstrapping argument(see, e.g.,[25]). It follows from (2.1) and (2.2) that

$$E_0(t) + E_1(t) \leq C(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}} + E_0(t)^{\frac{3}{2}} + E_0(t)^2 + E_1(t)^2),$$

or, for pure constants C_0, C_1, C_2,

$$E_0(t) + E_1(t) \leq C_0 \left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}}) + C_1(E_0(t)^{\frac{3}{2}} + E_1(t)^{\frac{3}{2}}\right) \\ + C_2(E_0(t)^2 + E_1(t)^2). \tag{3.1}$$

To initiate the bootstrapping argument, we make the ansatz

$$E_0(t) + E_1(t) \leq \min\left\{\frac{1}{16C_1^2}, \frac{1}{4C_2}.\right\} \tag{3.2}$$

We then show that (3.1) allows us to conclude that $E_0(t) + E_1(t)$ actually admits an even smaller bound by taking the initial H^3-norm $\mathscr{E}_0$ sufficiently small. In fact, when (3.2) holds, (3.1) implies

$$E_0(t) + E_1(t) \leq CC_0(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}}) + C_1\sqrt{E_0(t) + E_1(t)}(E_0(t) + E_1(t)) \\ + C_2(E_0(t) + E_1(t))(E_0(t) + E_1(t)) \\ \leq C_0(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}}) + \frac{1}{2}(E_0(t) + E_1(t)), \tag{3.3}$$

or

$$E_0(t) + E_1(t) \leq 2C_0\left(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}}\right). \tag{3.4}$$

Therefore, if we take $\mathscr{E}_0$ sufficiently small such that

$$2C_0(\mathscr{E}_0 + \mathscr{E}_0^{\frac{3}{2}}) \le \min\left\{\frac{1}{16C_1^2}, \frac{1}{4C_2},\right\} \tag{3.5}$$

then $E_0(t) + E_1(t)$ actually admits an smaller bound in (3.4) than the one in the ansatz (3.2). Then, the bootstrapping argument assesses that (3.4) holds for all time, provided that (3.5) holds. This completes the proof of Theorem 1.1.

Acknowledgements

This work was supported in part by the NNSF of China (Grant No. 11871212) and the Basic Research Project of Key Scientific Research Project Plan of Universities in Henan Province (Grant No. 20ZX002).

References

[1] Cai, Y., Lei, Z.: Global well-posedness of the incompressible Magnetohydrodynamics. Arch. Ration. Mech. Anal. **228**, 969–993 (2018)
[2] Cao, C., Wu, J.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803–1822 (2011)
[3] Cheng, J., Liu, Y.: Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity. Comput. Math. Appl. **70**, 66–72 (2015)
[4] Deng, W., Zhang, P.: Large time behavior of solutions to 3-D MHD system with initial data near equilibrium. Arch. Ration. Mech. Anal. **230**, 1017–1102 (2018)
[5] Dong, B., Li, J., Wu, J.: Global regularity for the 2D MHD equations with partial hyper-resistivity. Int. Math. Res. Not. IMRN **14**, 4261–4280 (2019)
[6] Fan, J., Malaikah, H., Monaquel, S., Nakamura, G., Zhou, Y.: Global Cauchy problem of 2D generalized MHD equations. Monatsh. Math. **175**, 127–131 (2014)
[7] Fan, J., Zhao, K.: Global Cauchy problem of 2D generalized magnetohydrodynamic equations. J. Math. Anal. Appl. **420**, 1024–1032 (2014)
[8] Gala, S.: Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space. Nonlinear Differ. Equ. Appl. **17**, 181–194 (2010)
[9] He, C., Huang, X., Wang, Y.: On some new global existence results for 3D magnetohydrodynamic equations. Nonlinearity **27**, 343–352 (2014)
[10] He, L., Xu, L., Yu, P.: On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves. Ann. PDE **4**, 5 (2018)
[11] Hu, X., Lin, F.: Global existence for two dimensional incompressible Magnetohydrodynamic flows with zero magnetic diffusivity. arXiv: 1405.0082v1
[12] Lifschitz, A.: Magnetohydrodynamics and spectral theory. In: Developments in Electromagnetic Theory and Applications, Vol. 4. Kluwer Academic Publishers Group, Dordrecht (1989)
[13] Lin, F., Xu, L., Zhang, P.: Global small solutions to 2-D incompressible MHD system. J. Differ. Equ. **259**, 5440–5485 (2015)
[14] Lin, F., Zhang, P.: Global small solutions to an MHD-type system: the three-dimensional case. Commun. Pure Appl. Math. **67**, 531–580 (2014)
[15] Lin, Y., Zhang, H., Zhou, Y.: Global smooth solutions of MHD equations with large data. J. Differ. Equ. **261**, 102–112 (2016)
[16] Lei, Z.: On axially symmetric incompressible Magnetohydrodynamics in three dimensions. J. Differ. Equ. **259**, 3202–3215 (2015)
[17] Miao, C., Yuan, B., Zhang, B.: Well-posedness for the incompressible magnetohydrodynamics system. Math. Methods Appl. Sci. **30**, 961–976 (2007)
[18] Ortega-Torres, E., Rojas-Medar, M.: Magneto-micropolar fluid motion: global existence of strong solutions. Abstr. Appl. Anal. **4**, 109–125 (1999)

[19] Regmi, D., Wu, J.: Global regularity for the 2D magneto-micropolar equations with partial dissipation. J. Math. Study **49**, 169–194 (2016)
[20] Ren, X., Wu, J., Xiang, Z., Zhang, Z.: Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion. J. Funct. Anal. **267**, 503–541 (2014)
[21] Rojas-Medar, M.: Magneto-micropolar fluid motion: existence and uniqueness of strong solutions. Math. Nachr. **188**, 307–319 (1997)
[22] Rojas-Medar, M., Boldrini, J.: Magneto-micropolar fluid motion: existence of weak solutions. Rev. Mat. Complut. **11**, 443–460 (1998)
[23] Shang, H., Gu, C.: Global regularity and decay estimates for 2D magnetomicropolar equations with partial dissipation. Z. Angew. Math. Phys. **70**, 85 (2019)
[24] Shang, H., Zhao, J.: Global regularity for 2D magneto-micropolar equations with only micro-rotational velocity dissipation and magnetic diffusion. Nonlinear Anal. **150**, 194–209 (2017)
[25] Tao, T.: Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence (2006)
[26] Wang, F., Wang, K.: Global existence of 3D MHD equations with mixed partial dissipation and magnetic diffusion. Nonlinear Anal. Real World Appl. **14**, 526–535 (2013)
[27] Wang, F.: On global regularity of incompressile MHD equations in $\mathbb{R}^3$. J. Math. Anal. Appl. **454**, 936–941 (2017)
[28] Wang, Y.-Z., Wang, Y.: Blow-up criterion for two-dimensional magneto-micropolar fluid equations with partial viscosity. Math. Methods Appl. Sci. **34**, 2125–2135 (2011)
[29] Wang, Y.-Z., Wang, K.: Global well-posedness of the three dimensional magnetohydrodynamics equations. Nonlinear Anal. Real World Appl. **17**, 245–251 (2014)
[30] Wang, Y.-Z., Li, P.: Global existence of three dimensional incompressible MHD flows. Math. Methods Appl. Sci. **39**, 4246–4256 (2016)
[31] Wang, Y.: Asymptotic decay of solutions to 3D MHD equations. Nonlinear Anal. **132**, 115–125 (2016)
[32] Wang, Y.X.: Blow-up criteria of smooth solutions to the three-dimensional magneto-micropolar fluid equations. Bound. Value Probl. **118**, 10 (2015)
[33] Wang, Y., Wang, K.: Global well-posedness of 3D magneto-micropolar fluid equations with mixed partial viscosity. Nonlinear Anal. Real World Appl. **33**, 348–362 (2017)
[34] Wang, Y., Gu, L.: Global regularity of 3D magneto-micropolar fluid equations. Appl. Math. Lett. **99**, 105980 (2020)
[35] Wei, D., Zhang, Z.: Global well-posedness of the MHD equations in a homogeneous magnetic field. Anal. PDE **10**, 1361–1406 (2017)
[36] Wu, J., Wu, Y.: Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion. Adv. Math. **310**, 759–888 (2017)
[37] Wu, J., Zhu, Y.: Global solution of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium. arXiv:1906.05054v1
[38] Xu, F.: Regularity criterion of weak solution for the 3D magneto-micropolar fluid equations in Besov spaces. Commun. Nonlinear Sci. **17**, 2426–2433 (2012)
[39] Yuan, B.: Regularity of weak solutions to magneto-micropolar fluid equations. Acta Math. Sci. **30**, 1469–1480 (2010)
[40] Yuan, B., Zhao, J.: Global regularity of 2D almost resistive MHD equations. Nonlinear Anal. Real World Appl. **41**, 53–65 (2018)
[41] Yuan, J.: Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations. Math. Methods Appl. Sci. **31**, 1113–1130 (2008)
[42] Zhang, T.: An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system. arXiv:1404.5681v1
[43] Zhang, Z., Yao, Z., Wang, X.: A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces. Nonlinear Anal. **74**, 2220–2225 (2011)
[44] Zhang, Z., Dong, B., Jia, Y.: Remarks on the global regularity and time decay of the 2D MHD equations with partial dissipation. Math. Methods Appl. Sci. **42**, 3388–3399 (2019)
[45] Zhou, Y., Fan, J.: Global Cauchy problem for a 2D Leray-α-MHD model with zero viscosity. Nonlinear Anal. **74**, 1331–1335 (2011)

Yuzhu Wang and Weijia Li
School of Mathematics and Statistics
North China University of Water Resources and Electric Power
Zhengzhou 450011
China
e-mail: yuzhu108@163.com

(Received: May 1, 2020; revised: November 28, 2020; accepted: December 14, 2020)