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On regularity criteria for the Navier–Stokes equations based on one directional derivative of the velocity or one diagonal entry of the velocity gradient

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Abstract. It is proved that if the solution of the Navier–Stokes system satisfies

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{22}{13}+\frac{3}{13q}, \quad 3<q<4,$$

or

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{3(\sqrt{65\alpha^2-78\alpha+49}+7-\alpha)}{16\alpha}, \quad \frac{3+\sqrt{17}}{4}\leq\alpha\leq\infty,$$

then the solution is smooth on $(0,T]$. These two improve many previous results.

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1. Introduction

In this paper, we continue our study [31,35,36] of the regularity criteria of the following Navier–Stokes equations (NSE):

$$\begin{cases} \partial_t \boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{0},\\ \nabla\cdot\boldsymbol{u}=0,\\ \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0, \end{cases} \tag{1}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the fluid velocity field, π is a scalar pressure, $\boldsymbol{u}_0$ is the prescribed initial velocity field satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0=0$, and

$$\partial_t\boldsymbol{u}=\frac{\partial\boldsymbol{u}}{\partial t}, \quad \partial_i=\frac{\partial}{\partial x_i}, \quad (\boldsymbol{u}\cdot\nabla)=\sum_{i=1}^3 u_i\partial_i, \quad \Delta=\partial_1\partial_1+\partial_2\partial_2+\partial_3\partial_3.$$

Leray [18] and Hopf [13] have established a global weak solution to (1); however, it remains an open problem of its regularity and uniqueness. Serrin [25] first showed that if

$$\boldsymbol{u}\in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3\leq q\leq\infty, \tag{2}$$

then the solution is regular on $(0,T]$. See also [8,22]. The so-called Serrin-type regularity criterion (2) was generalized by Beirão da Veiga [1] by adding integrability conditions on the velocity gradient,

$$\nabla\boldsymbol{u}\in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}\leq q\leq\infty. \tag{3}$$

In view of the divergence-free condition $\nabla \cdot \boldsymbol{u} = 0$, it is natural and important to consider components reduction improvements of (2) and (3), that is, whether or not integrability conditions on partial components of the velocity or velocity gradient could still ensure the smoothness of the solution. There are so many studies devoted to this refinement, and without no intention to be complete, we recommend [2,3,5,6,12,15,16,19–21,26,28,31,32,34,37–40].

In this paper, we would like to investigate the regularity criterion of (1) based on one directional derivative of the velocity field, say $\partial_3 \boldsymbol{u}$, or one diagonal entry of the velocity gradient, say $\partial_3 u_3$. Let us first review what have happened in the last decades. In [20, Theorem 4 (i)], Penel–Pokorný showed that if

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{3}{2}, \quad 2 \le q \le \infty, \tag{4}$$

then the solution is smooth. This is based on a regularity criterion in terms of u_3, $\partial_3 u_1$ and $\partial_3 u_2$ [20, Theorem 1 (a)]:

$$\begin{aligned} u_3 &\in L^{\frac{2s}{s-3}}(0,T;L^s(\mathbb{R}^3)), && 3 < s \le \infty; \\ \partial_3 u_1,\ \partial_3 u_2 &\in L^{\frac{2q}{2q-3}}(0,T;L^q(\mathbb{R}^3)), && \frac{3}{2} < q \le \infty. \end{aligned} \tag{5}$$

Then, Kukavica–Ziane [16] established a fine property of the horizontal convective terms (denoting by $\Delta_h = \partial_1\partial_1 + \partial_2\partial_2$ the horizontal Laplacian)

$$\begin{aligned} \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta_h u_j \, \mathrm{d}x = \frac{1}{2} \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_i u_j \partial_i u_j \partial_3 u_3 \, \mathrm{d}x \\ - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 \, \mathrm{d}x + \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 \, \mathrm{d}x, \end{aligned} \tag{6}$$

and refined (4) to be critical, but with limited range of space integrability indices,

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{9}{4} \le q \le 3. \tag{7}$$

Later on, Cao [2] employed multiplicative Sobolev inequalities

$$1 \le q < \infty \Rightarrow \|f\|_{L^{3q}} \le C \|\partial_1 f\|_{L^2}^{\frac{1}{3}} \|\partial_2 f\|_{L^2}^{\frac{1}{3}} \|\partial_3 f\|_{L^q}^{\frac{1}{3}} \tag{8}$$

and

$$1 \le q < \infty \Rightarrow \|f\|_{L^{5q}} \le C \left\|\partial_1 (f^2)\right\|_{L^2}^{\frac{1}{5}} \left\|\partial_2 (f^2)\right\|_{L^2}^{\frac{1}{5}} \|\partial_3 f\|_{L^q}^{\frac{1}{5}} \tag{9}$$

to get the following extended regularity condition

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{27}{16} \le q \le \frac{5}{2}. \tag{10}$$

It should be remarked that Cao [2] claimed the range of q in (10) is $q \ge \frac{27}{16}$, but it is indeed (10) which is actually proved. See the footnote of [31, p. 35] for more information.

In a recent paper, Zhang [31] generalized (9) as

$$0 < \lambda < \infty, \quad 1 \le q < \infty \Rightarrow \|f\|_{L^{(2\lambda+1)q}} \le C \left\|\partial_1 (|f|^\lambda)\right\|_{L^2}^{\frac{1}{2\lambda+1}} \left\|\partial_2 (|f|^\lambda)\right\|_{L^2}^{\frac{1}{2\lambda+1}} \|\partial_3 f\|_{L^q}^{\frac{1}{2\lambda+1}}, \tag{11}$$

and employed general $L^{2\lambda}$ estimate (instead of L^4 estimate as in [2]) to improve (7) and (10) simultaneously. Precisely, he showed the following regularity criterion,

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad 1.56207 \approx \frac{3\sqrt{37}}{4} - 3 \le q \le 3. \tag{12}$$

Notice in establishing (12), Zhang have missed a condition (say, in [31, (31)], we need $1 \leq c \leq \infty$), which was noticed by Yuliya–Skalak [30]. Skalak [27] then covered all of the range $\left(\frac{3}{2}, 3\right]$,

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leq 3. \tag{13}$$

Finally, Zhang–Yuan–Zhou [36] showed two new refinements of (4),

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{8}{5}+\frac{3}{5q}, \quad 4\leq q\leq \infty, \tag{14}$$

and

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{14}{11}+\frac{9}{11q}, \quad \frac{5}{2}\leq q\leq \infty. \tag{15}$$

Whence, the state of the art is the following smoothness condition

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\begin{cases} 2, & \frac{3}{2}<q\leq 3,\\ \frac{14}{11}+\frac{9}{11q}, & 3<q<\frac{18}{5},\\ \frac{3}{2}, & \frac{18}{5}\leq q<4,\\ \frac{8}{5}+\frac{3}{5q}, & 4\leq q\leq\infty. \end{cases} \tag{16}$$

The first purpose of this paper is to improve (16) in the range $3<q<4$.

As far as regularity criterion $\partial_3 u_3$ is concerned, Zhou–Pokorný [39] first established a regularity condition based on u_3 and then showed that if

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}<\frac{4}{5}, \quad \frac{15}{4}<\alpha\leq\infty, \tag{17}$$

then the solution is smooth. The equality in (17) was verified by Jia–Zhou [14]:

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{4}{5}, \quad \frac{15}{4}\leq\alpha\leq\infty. \tag{18}$$

Later, Cao–Titi [4] established a bilateral multiplicative Sobolev inequality (see [32, Remark 8] for more information, and [35] for a more efficient form)

$$\begin{gathered}\left|\int_{\mathbb{R}^3}\phi f g\ \mathrm{d}x\right| \leq C\left\|\phi\right\|_{L^2}^{\frac{r-1}{r}}\left\|\partial_i\phi\right\|_{L^{\frac{2}{3-r}}}^{\frac{1}{r}}\left\|f\right\|_{L^2}^{\frac{r-2}{r}}\left\|\partial_j f\right\|_{L^2}^{\frac{1}{r}}\left\|\partial_k f\right\|_{L^2}^{\frac{1}{r}}\left\|g\right\|_{L^2}, \\ 2<r\leq 3,\ \{i,j,k\}=\{1,2,3\}. \end{gathered} \tag{19}$$

With (19) in hand, Cao–Titi showed the following two smoothness conditions,

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{3}{4}+\frac{3}{2\alpha}, \quad 2<\alpha<\infty, \tag{20}$$

and

$$\partial_1 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{1}{2}+\frac{3}{2\alpha}, \quad 3<\alpha<\infty. \tag{21}$$

Then Fang–Qian [9, Theorems 1.1 and 1.2] dominated u_3 by $\partial_1 u_3$, employed some tricks in [4] and improved (21) as (after rationalizing the denominator of [9, Equation (1.10)])

$$\partial_1 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{\sqrt{103\alpha^2-12\alpha+9}+3-9\alpha}{2\alpha}, \quad 3\leq\alpha<\infty. \tag{22}$$

Finally, Zhang–Zhong–Huang [35] found a more effective substitute of (19),

$$\begin{aligned}&\int_{\mathbb{R}^3} |f|^2|g|^2\ \mathrm{d}x_1\ \mathrm{d}x_2\ \mathrm{d}x_3\\&\quad\le C\,\|\partial_i f\|_{L^2(\mathbb{R}^3)}^{\frac{2(q-1)}{3q-2}}\|\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{2(q-1)}{3q-2}}\|\partial_k f\|_{L^q(\mathbb{R}^3)}^{\frac{2q}{3q-2}}\|g\|_{L^2(\mathbb{R}^3)}^{\frac{6q-8}{3q-2}}\|\partial_i g\|_{L^2(\mathbb{R}^3)}^{\frac{2}{3q-2}}\|\partial_j g\|_{L^2(\mathbb{R}^3)}^{\frac{2}{3q-2}},\\&\quad 2\le q<\infty,\ \{i,j,k\}=\{1,2,3\}.\end{aligned}\tag{23}$$

Invoking (23), Zhang–Zhong–Huang [35] were able to improve (21) and (22) as

$$\partial_1 u_3\in L^\beta(0,T;L^\alpha(\mathbb{R}^3)),\quad \frac{2}{\beta}+\frac{3}{\alpha}=\frac{3}{4}+\frac{5}{4\alpha},\quad \frac{7}{3}\le\alpha<\infty,\tag{24}$$

but could not refine (20).

As for (20), Fang–Qian [9] made a contribution by invoking a regularity criterion of Zhang [33]. Fang–Qian [10, Theorem 1.8] then used an integration by parts technique in estimating u_3 by $\partial_3 u_3$ and obtained the finest result up to now,

$$\begin{aligned}&\partial_3 u_3\in L^\beta(0,T;L^\alpha(\mathbb{R}^3)),\\&\frac{2}{\beta}+\frac{3}{\alpha}=\frac{\sqrt{289\alpha^2-264\alpha+144}+12-7\alpha}{8\alpha},\quad \frac{9}{5}<\alpha\le\infty.\end{aligned}\tag{25}$$

For later developments in anisotropic Lebesgue spaces, see [11,24]. The second aim of the present paper is to make (25) better.

Before stating the precise result, let us recall the weak formulation of (1), see [7,17,23,29] for instance.

Definition 1. Let $\boldsymbol{u}_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0=0$, $T>0$. A measurable $\mathbb{R}^3$-valued function $\boldsymbol{u}$ defined in $[0,T]\times\mathbb{R}^3$ is said to be a weak solution to (1) if

(1) $\boldsymbol{u}\in L^\infty(0,T;L^2(\mathbb{R}^3)\cap L^2(0,T;H^1(\mathbb{R}^3))$;
(2) $(1)_1$ and $(1)_2$ hold in the sense of distributions, i.e.,

$$\int_0^t\int_{\mathbb{R}^3}\boldsymbol{u}\cdot[\partial_t\boldsymbol{\phi}+(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{\phi}]\ \mathrm{d}x\ \mathrm{d}s+\int_{\mathbb{R}^3}\boldsymbol{u}_0\cdot\boldsymbol{\phi}(0)\ \mathrm{d}x=\int_0^T\int_{\mathbb{R}^3}\nabla\boldsymbol{u}:\nabla\boldsymbol{\phi}\ \mathrm{d}x\ \mathrm{d}t,$$

for each $\boldsymbol{\phi}\in C_c^\infty([0,T)\times\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{\phi}=0$, where $A:B=\sum_{i,j=1}^3 a_{ij}b_{ij}$ for 3×3 matrices $A=(a_{ij})$, $B=(b_{ij})$, and

$$\int_0^T\int_{\mathbb{R}^3}\boldsymbol{u}\cdot\nabla\psi\ \mathrm{d}x\ \mathrm{d}t=0,$$

for each $\psi\in C_c^\infty(\mathbb{R}^3\times[0,T))$;
(3) the strong energy inequality, that is,

$$\|\boldsymbol{u}(t)\|_{L^2}^2+2\int_s^t\|\nabla\boldsymbol{u}(s)\|_{L^2}^2\ \mathrm{d}s\le\|\boldsymbol{u}(s)\|_{L^2}^2,\quad s\le t\le T,$$

holds for $s=0$ and almost all times $s\in(0,T)$.

Now, our main result reads

Theorem 2. *Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \boldsymbol{u}_0 = 0$, $T > 0$. Assume that $\boldsymbol{u}$ is a weak solution to (1) on $[0,T]$ with initial data $\boldsymbol{u}_0$. If one of the following two conditions holds,*

$$\partial_3 \boldsymbol{u} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{22}{13}+\frac{3}{13q}, \quad 3<q<4, \tag{26}$$

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)),$$
$$\frac{2}{\beta}+\frac{3}{\alpha}=\frac{3\left(\sqrt{65\alpha^2-78\alpha+49}+7-\alpha\right)}{16\alpha}, \quad 1.78078 \approx \frac{3+\sqrt{17}}{4} \le \alpha \le \infty, \tag{27}$$

then the solution $\boldsymbol{u}$ is smooth in $(0,T]\times\mathbb{R}^3$.

Remark 3. (1) Our regularity criterion (26) is better than (16) in case $3<\alpha<4$. See Fig. 1, where "Skalak" refers to (13), "one Zhang-Zhou" (the upper one) means (14), "two Zhang-Zhou (the lower one)" demonstrates (15), "Penel-Pokorny" reveals (4), and "this" reflects (26).

(2) Our regularity criterion (27) is better than (17), (20) and (25). See Fig. 2, where "Zhou-Pokorný" refers to (17); "Cao–Titi" means (20); "Fang-Qian" demonstrates (25); and "this" reflects our result (27).

(3) It is not so hard to deduce that the scaling dimension $\frac{3\left(\sqrt{65\alpha^2-78\alpha+49}+7-\alpha\right)}{16\alpha}$ in (27) is strictly decreasing with respect to $\frac{3+\sqrt{17}}{4} \le \alpha \le \infty$. Notice that

$$\lim_{\alpha\to\frac{3+\sqrt{17}}{4}} \frac{3\left(\sqrt{65\alpha^2-78\alpha+49}+7-\alpha\right)}{16\alpha}=\frac{3(\sqrt{17}-3)}{2}\approx 1.68466,$$
$$\lim_{\alpha\to\infty} \frac{3\left(\sqrt{65\alpha^2-78\alpha+49}+7-\alpha\right)}{16\alpha}=\frac{3(\sqrt{65}-1)}{16}\approx 1.32417,$$

we have the following rough, but maybe more elegant regularity criterion in terms of $\partial_3 u_3$,

$$\partial_3 u_3 \in L^\beta(0,T;L^\alpha(\mathbb{R}^3)),$$
$$\frac{2}{\beta}+\frac{3}{\alpha}=\frac{3(\sqrt{65}-1)}{16}\approx 1.32417, \quad 1.78078\approx\frac{3+\sqrt{17}}{4}\le\alpha\le\infty. \tag{28}$$

2. Proof of Theorem 2

In this section, we shall prove Theorem 2.

Case I (26) holds. For any $\varepsilon \in (0,T)$, due to the fact that $\nabla\boldsymbol{u} \in L^2(0,T;L^2(\mathbb{R}^3))$, we may find a $\delta \in (0,\varepsilon)$, such that $\nabla\boldsymbol{u}(\delta)\in L^2(\mathbb{R}^3)$. Take this $\boldsymbol{u}(\delta)$ as initial data, there exists an $\tilde{\boldsymbol{u}} \in C([\delta,\Gamma^*),H^1(\mathbb{R}^3)) \cap L^2(\delta,\Gamma^*;H^2(\mathbb{R}^3))$, where $[\delta,\Gamma^*)$ is the life span of the unique strong solution, see [29]. Moreover, $\tilde{\boldsymbol{u}} \in C^\infty(\mathbb{R}^3\times(\delta,\Gamma^*))$. According to the uniqueness result, $\tilde{\boldsymbol{u}}=\boldsymbol{u}$ on $[\delta,\Gamma^*)$. If $\Gamma^*\ge T$, we have already that $\boldsymbol{u}\in C^\infty(\mathbb{R}^3\times(0,T))$, due to the arbitrariness of $\varepsilon\in(0,T)$. In case $\Gamma^*<T$, our strategy is to show that $u_3 \in L^3(\delta,\Gamma^*;L^9(\mathbb{R}^3))$. Then, by the fact that

$$(26) \Rightarrow \partial_3\boldsymbol{u} \in L^{\frac{13q}{11q-18}}(\delta,\Gamma^*;L^q(\mathbb{R}^3)) \subset L^{\frac{2q}{2q-3}}(\delta,\Gamma^*;L^q(\mathbb{R}^3)),$$

we may conclude the proof by invoking (5).

For this purpose, we multiply the equation of u_3 in (1) by $|u_3|u_3$ and integrate over $\mathbb{R}^3$,

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left\||u_3|^{\frac{3}{2}}\right\|_{L^2}^2+\frac{4}{9}\left\|\nabla|u_3|^{\frac{3}{2}}\right\|_{L^2}^2+\int_{\mathbb{R}^3}|u_3|\cdot|\nabla u_3|^2\,\mathrm{d}x=-\int_{\mathbb{R}^3}\partial_3\pi|u_3|u_3\,\mathrm{d}x\equiv I. \tag{29}$$

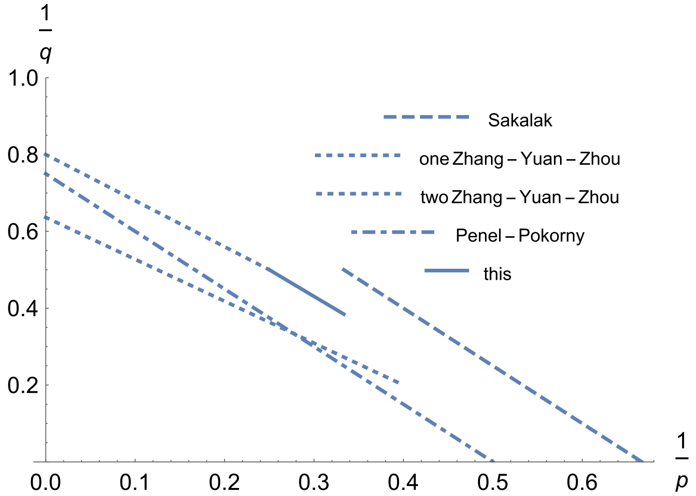

Fig. 1. Comparison of regularity criterion based on $\partial_3 \boldsymbol{u}$

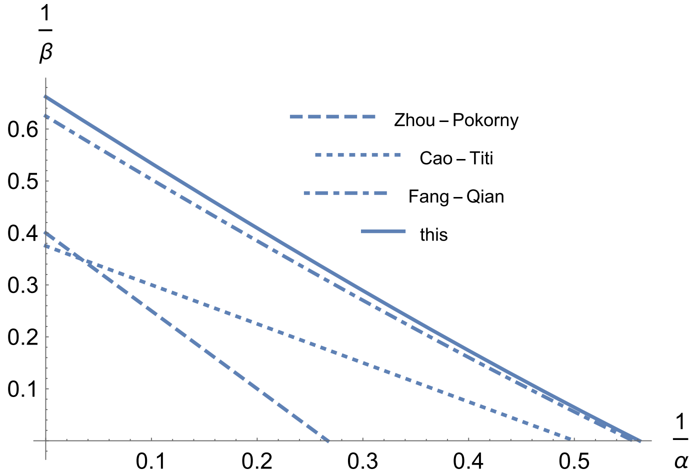

Fig. 2. Comparison of regularity criterion based on $\partial_3 u_3$,

By the Hölder inequality,

$$I \le \|\partial_3 \pi\|_{L^{\frac{3q}{4}}} \|u_3\|^2_{L^{\frac{6q}{3q-4}}}.$$

To dominate $\partial_3\pi$, we apply the divergence of $(1)_1$ to obtain

$$\begin{aligned}
&-\Delta\pi = \nabla\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} u_i\partial_i u_j\right) = \sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j)\\
&\Rightarrow \begin{cases} \pi = \displaystyle\sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j(u_iu_j) \\ -\Delta\partial_3\pi = \displaystyle\sum_{i,j=1}^{3}\partial_i\partial_j(\partial_3 u_i u_j + u_i\partial_3 u_j) \Rightarrow \partial_3\pi = \displaystyle\sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j(\partial_3 u_i u_j + u_i\partial_3 u_j)\end{cases} \qquad (30)\\
&\quad\left(\mathcal{R}_i = \partial_i(-\Delta)^{-\frac{1}{2}} \text{ is the Riesz transform}\right),
\end{aligned}$$

and thus by the interpolation inequality,

$$I \le C\|\boldsymbol{u}\|_{L^{3q}}\|\partial_3\boldsymbol{u}\|_{L^q}\cdot\left[\|u_3\|_{L^3}^{\frac{6q-11}{4q-3}}\|u_3\|_{L^{4q}}^{\frac{2(4-q)}{4q-3}}\right]^2.$$

Employing the multiplicative Sobolev inequalities (8) and (11) yields

$$\begin{aligned}
I &\le C\|\partial_1\boldsymbol{u}\|_{L^2}^{\frac{1}{3}}\|\partial_2\boldsymbol{u}\|_{L^2}^{\frac{1}{3}}\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{1}{3}}\cdot\|\partial_3\boldsymbol{u}\|_{L^q}\\
&\quad\cdot\left\{\|u_3\|_{L^3}^{\frac{6q-11}{4q-3}}\cdot\left[\left\|\partial_1\left(|u_3|^{\frac{3}{2}}\right)\right\|_{L^2}^{\frac{1}{4}}\left\|\partial_2\left(|u_3|^{\frac{3}{2}}\right)\right\|_{L^2}^{\frac{1}{4}}\|\partial_3 u_3\|_{L^q}^{\frac{1}{4}}\right]^{\frac{2(4-q)}{4q-3}}\right\}^2.
\end{aligned}$$

After collection, we deduce by the Young inequality,

$$\begin{aligned}
I &\le C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{2}{3}}\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{13q}{3(4q-3)}}\|u_3\|_{L^3}^{\frac{2(6q-11)}{4q-3}}\left\|\nabla(|u_3|^{\frac{3}{2}})\right\|_{L^2}^{\frac{2(4-q)}{4q-3}}\\
&\le C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{2(4q-3)}{3(5q-7)}}\|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{13q}{3(5q-7)}}\|u_3\|_{L^3}^{\frac{2(6q-11)}{5q-7}} + \frac{2}{9}\left\|\nabla\left(|u_3|^{\frac{3}{2}}\right)\right\|_{L^2}^2\\
&\le C\left(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\partial_3\boldsymbol{u}\|_{L^q}^{\frac{13q}{11q-18}}\right)\left(1+\|u_3\|_{L^3}^3\right) + \frac{2}{9}\left\|\nabla\left(|u_3|^{\frac{3}{2}}\right)\right\|_{L^2}^2.
\end{aligned}$$

Putting this above inequality into (29) and applying the Gronwall inequality give

$$\|u_3\|_{L^3(\delta,\Gamma^*;L^9(\mathbb{R}^3))} = \left\||u_3|^{\frac{3}{2}}\right\|_{L^2(\delta,\Gamma^*;L^6(\mathbb{R}^3))} \le C\left\|\nabla|u_3|^{\frac{3}{2}}\right\|_{L^2(\delta,\Gamma^*;L^2(\mathbb{R}^3))} \le C,$$

as desired.

Case II (27) **holds.** Argue as in Case I, it suffices to show that $\|\nabla\boldsymbol{u}(t)\|_{L^2}$ is uniformly bounded as $t \nearrow \Gamma^*$. By the absolute continuity property of the Lebesgue integrable function, for $\delta_2 \in (0,1)$ to be determined, we can choose a $\delta_1 \in [\delta, \Gamma^*)$ such that

$$\nabla\boldsymbol{u}(\delta_1) \in L^2(\mathbb{R}^3), \quad \int_{\delta_1}^{\Gamma^*}\|\nabla\nabla_h\boldsymbol{u}\|_{L^2}^2\ \mathrm{d}t < \delta_2. \qquad (31)$$

We first establish the L^q bound of u_3 in terms of $\partial_3 u_3$, which have been used in [9,10]. Multiplying the third component of $(1)_1$:

$$\partial_t u_3 + (\boldsymbol{u}\cdot\nabla)u_3 - \Delta u_3 + \partial_3\pi = 0$$

by $|u_3|^{q-2}u_3$ with

$$2 < q \le 6, \tag{32}$$

integrating over $\mathbb{R}^3$ and applying integration by parts, we obtain

$$\begin{aligned}\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_{L^q}^q + c(q)\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^2 &= -\int\limits_{\mathbb{R}^3}\partial_3\pi\cdot|u_3|^{q-2}u_3\,\mathrm{d}x\\ &\le C\int\limits_{\mathbb{R}^3}|\pi|\cdot|u_3|^{q-2}|\partial_3 u_3|\,\mathrm{d}x\\ &\equiv J.\end{aligned} \tag{33}$$

By the Hölder inequality with

$$\frac{1}{a}+\frac{q-2}{(q+1)\alpha}+\frac{1}{\alpha}=1,\quad 1\le a\le\infty,\quad 1\le\alpha\le\infty, \tag{34}$$

we have

$$J \le C\|\pi\|_{L^a}\|u_3\|_{L^{(q+1)\alpha}}^{q-2}\|\partial_3 u_3\|_{L^\alpha}.$$

Thanks to (30) and (11), it follows that

$$J \le C\|\boldsymbol{u}\|_{L^{2a}}^2\left(\left\|\nabla_h\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^{\frac{2}{q+1}}\|\partial_3 u_3\|_{L^\alpha}^{\frac{1}{q+1}}\right)^{q-2}\|\partial_3 u_3\|_{L^\alpha},$$

provided that

$$1 < a < \infty,\quad 1 \le \alpha < \infty. \tag{35}$$

Employing the Gagliardo–Nirenberg inequality with

$$\frac{3}{2a} = (1-\vartheta)\frac{3}{2}+\vartheta\left(-1+\frac{3}{2}\right),\quad 2 < 2a < 6 \tag{36}$$

gives

$$J \le C\left(\|\boldsymbol{u}\|_{L^2}^{1-\vartheta}\|\nabla\boldsymbol{u}\|_{L^2}^{\vartheta}\right)^2\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^{\frac{2(q-2)}{q+1}}\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{q+1}}.$$

By the fact that $\boldsymbol{u}\in L^\infty(0,T;L^2(\mathbb{R}^3))$ from Definition 1 and the Young inequality, we deduce

$$\begin{aligned}J &\le C\|\nabla\boldsymbol{u}\|_{L^2}^{2\vartheta}\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^{\frac{2(q-2)}{q+1}}\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{q+1}}\\ &\le \begin{cases} C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac{2(q+1)\vartheta}{3}}\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{3}}+\dfrac{c(q)}{2}\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^2, & \text{if } \dfrac{2(q+1)\vartheta}{3}<2\\ C\|\nabla\boldsymbol{u}\|_{L^2}^{2}\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{3}}+\dfrac{c(q)}{2}\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^2, & \text{if } \dfrac{2(q+1)\vartheta}{3}=2\end{cases}\\ &\le \begin{cases} C\|\nabla\boldsymbol{u}\|_{L^2}^{2}+C\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{3-(q+1)\vartheta}}+\dfrac{c(q)}{2}\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^2, & \text{if } \dfrac{2(q+1)\vartheta}{3}<2\\ C\|\nabla\boldsymbol{u}\|_{L^2}^{2}\|\partial_3 u_3\|_{L^\alpha}^{\frac{2q-1}{3}}+\dfrac{c(q)}{2}\left\|\nabla\left(|u_3|^{\frac{q}{2}}\right)\right\|_{L^2}^2, & \text{if } \dfrac{2(q+1)\vartheta}{3}=2\end{cases}.\end{aligned} \tag{37}$$

if

$$\frac{2(q+1)\vartheta}{3}\le 2,\quad \beta=\begin{cases}\dfrac{2q-1}{3-(q+1)\vartheta}, & \text{if } \dfrac{2(q+1)}{3}<2\\ \infty, & \text{if } \dfrac{2(q+1)\vartheta}{3}=2\end{cases}. \tag{38}$$

Plugging (37) into (33), absorbing the last term into the left-hand side and integrating with respect to the time, we find

$$u_3\in L^\infty(\delta_1,\Gamma^*;L^q(\mathbb{R}^3)). \tag{39}$$

Then, we establish the bound of $\|\nabla_h \boldsymbol{u}\|_{L^2}^2$ with $\nabla_h = (\partial_1, \partial_2)$ the horizontal gradient operator. Testing $(1)_1$ by $-\Delta_h \boldsymbol{u}$, it follows from [3,4,39] that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h \boldsymbol{u}\|_{L^2}^2 + \|\nabla\nabla_h \boldsymbol{u}\|_{L^2}^2 &= \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta_h \boldsymbol{u}\,\mathrm{d}x \\ &\le C\int_{\mathbb{R}^3} |u_3||\nabla \boldsymbol{u}||\nabla\nabla_h \boldsymbol{u}|\,\mathrm{d}x \\ &\equiv K.\end{aligned}\tag{40}$$

By the Hölder inequality, the Minkowski inequality and the Gagliardo–Nirenberg inequality,

$$\begin{aligned}K &\le C\int_{\mathbb{R}^2} \max_{x_3}|u_3| \left(\int_{\mathbb{R}} |\nabla \boldsymbol{u}|^2\,\mathrm{d}x_3\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\nabla\nabla_h \boldsymbol{u}|^2\,\mathrm{d}x_3\right)^{\frac{1}{2}} \mathrm{d}x_1\,\mathrm{d}x_2 \\ &\le C\left[\int_{\mathbb{R}^2}\left(\max_{x_3}|u_3|\right)^r \mathrm{d}x_1\,\mathrm{d}x_2\right]^{\frac{1}{r}} \left[\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} |\nabla \boldsymbol{u}|^2\,\mathrm{d}x_3\right)^{\frac{r}{r-2}} \mathrm{d}x_1\,\mathrm{d}x_2\right]^{\frac{r-2}{2r}} \\ &\quad\cdot\left[\int_{\mathbb{R}^3} |\nabla\nabla_h \boldsymbol{u}|^2\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\right]^{\frac{1}{2}} \\ &\le C\left[\int_{\mathbb{R}^3} |u_3|^{r-1}|\partial_3 u_3|\,\mathrm{d}x\right]^{\frac{1}{r}} \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2} |\nabla \boldsymbol{u}|^{\frac{2r}{r-2}}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{\frac{r-2}{r}} \mathrm{d}x_3\right]^{\frac{1}{2}} \|\nabla\nabla_h \boldsymbol{u}\|_{L^2} \\ &\le C\|u_3\|_{L^q}^{\frac{r-1}{r}} \|\partial_3 u_3\|_{L^\alpha}^{\frac{1}{r}} \cdot \|\nabla \boldsymbol{u}\|_{L^2}^{\frac{r-2}{r}} \|\nabla\nabla_h \boldsymbol{u}\|_{L^2}^{\frac{2}{r}} \cdot \|\nabla\nabla_h \boldsymbol{u}\|_{L^2} \\ &\le C\|u_3\|_{L^q}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_{L^\alpha}^{\frac{2}{r-2}} \|\nabla \boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla\nabla_h \boldsymbol{u}\|_{L^2}^2.\end{aligned}\tag{41}$$

Here, the exponents appeared above should satisfy

$$2 < r < \infty, \quad \frac{r-1}{q} + \frac{1}{\alpha} = 1.\tag{42}$$

Putting (41) into (40) and integrating with respect to the time give

$$\begin{aligned}&\sup_{\delta_1\le t<\Gamma^*}\|\nabla_h \boldsymbol{u}(t)\|_{L^2}^2 + \int_{\delta_1}^{\Gamma^*} \|\nabla\nabla_h \boldsymbol{u}\|_{L^2}^2\,\mathrm{d}t \\ &\quad\le \|\nabla_h \boldsymbol{u}(\delta_1)\|_{L^2}^2 + C\int_{\delta_1}^{\Gamma^*} \|u_3\|_{L^q}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_{L^\alpha}^{\frac{2}{r-2}} \|\nabla \boldsymbol{u}\|_{L^2}^2\,\mathrm{d}t \\ &\quad\le C + C\int_{\delta_1}^{\Gamma^*} \|u_3\|_{L^q}^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_{L^\alpha}^{\frac{2}{r-2}} \|\nabla \boldsymbol{u}\|_{L^2}^2\,\mathrm{d}t.\end{aligned}\tag{43}$$

Finally, we obtain the H^1 estimate of the solution. Taking the inner product of (1)$_1$ by $-\Delta \boldsymbol{u}$ in $L^2(\mathbb{R}^3)$, it follows from [39] that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla \boldsymbol{u}\right\|_{L^2}^2 + \left\|\Delta \boldsymbol{u}\right\|_{L^2}^2 &= \int\limits_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\Delta\boldsymbol{u}\,\mathrm{d}x \\ &\le C\int\limits_{\mathbb{R}^3} |\nabla_h \boldsymbol{u}||\nabla\boldsymbol{u}|^2\,\mathrm{d}x \\ &\equiv L. \end{aligned} \tag{44}$$

Invoking the Hölder inequality (8) and the Young inequality, we get

$$\begin{aligned} L &\le C\left\|\nabla_h\boldsymbol{u}\right\|_{L^2}\left\|\nabla\boldsymbol{u}\right\|_{L^4}^2 \\ &\le C\left\|\nabla_h\boldsymbol{u}\right\|_{L^2}\cdot\left(\left\|\nabla\boldsymbol{u}\right\|_{L^2}^{\frac14}\left\|\nabla\boldsymbol{u}\right\|_{L^6}^{\frac34}\right)^2 \\ &\le C\left\|\nabla_h\boldsymbol{u}\right\|_{L^2}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^{\frac12}\left\|\nabla\nabla_h\boldsymbol{u}\right\|_{L^2}\left\|\Delta\boldsymbol{u}\right\|_{L^2}^{\frac12} \\ &\le C\left\|\nabla_h\boldsymbol{u}\right\|_{L^2}^{\frac43}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^{\frac23}\left\|\nabla\nabla_h\boldsymbol{u}\right\|_{L^2}^{\frac43} + \frac12\left\|\Delta\boldsymbol{u}\right\|_{L^2}^2. \end{aligned} \tag{45}$$

Gathering (45) into (44) and integrating with respect to the time provide

$$\begin{aligned} &\sup_{\delta_1\le t<\Gamma^*}\left\|\nabla\boldsymbol{u}(t)\right\|_{L^2}^2 + \int\limits_{\delta_1}^{\Gamma^*}\left\|\Delta\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t \\ &\quad\le \left\|\nabla\boldsymbol{u}(\delta)\right\|_{L^2}^2 + C\int\limits_{\delta_1}^{\Gamma^*}\left\|\nabla_h\boldsymbol{u}\right\|_{L^2}^{\frac43}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^{\frac23}\left\|\nabla\nabla_h\boldsymbol{u}\right\|_{L^2}^{\frac43}\,\mathrm{d}t \\ &\quad\le C + C\sup_{\delta_1\le t<\Gamma^*}\left\|\nabla_h\boldsymbol{u}(t)\right\|_{L^2}^{\frac43}\left(\int\limits_{\delta_1}^{\Gamma^*}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t\right)^{\frac13}\left(\int\limits_{\delta_1}^{\Gamma^*}\left\|\nabla\nabla_h\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t\right)^{\frac23}. \end{aligned} \tag{46}$$

Thanks to (31) and the obtained estimates (43) and (39), we have

$$\begin{aligned} &\sup_{\delta_1\le t<\Gamma^*}\left\|\nabla\boldsymbol{u}(t)\right\|_{L^2}^2 + \int\limits_{\delta_1}^{\Gamma^*}\left\|\Delta\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t \\ &\quad\le C + C\delta_2^{\frac13}\left[C + C\int\limits_{\delta_1}^{\Gamma^*}\left\|u_3\right\|_{L^q}^{\frac{2(r-1)}{r-2}}\left\|\partial_3 u_3\right\|_{L^\alpha}^{\frac{2}{r-2}}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t\right]^{\frac43} \\ &\quad\le C + C\delta_2^{\frac13}\sup_{\delta_1\le t<\Gamma^*}\left\|u_3(t)\right\|_{L^q}^{\frac{8(r-1)}{3(r-2)}}\cdot\sup_{\delta_1\le t<\Gamma^*}\left\|\nabla\boldsymbol{u}(t)\right\|_{L^2}^2\cdot\left(\int\limits_{\delta_1}^{\Gamma^*}\left\|\partial_3 u_3\right\|_{L^\alpha}^{\frac{2}{r-2}}\left\|\nabla\boldsymbol{u}\right\|_{L^2}^{\frac12}\,\mathrm{d}t\right)^{\frac43} \\ &\quad\le C + C\delta_2^{\frac13}\sup_{\delta_1\le t<\Gamma^*}\left\|\nabla\boldsymbol{u}(t)\right\|_{L^2}^2\left(\int\limits_{\delta_1}^{\Gamma^*}\left\|\partial_3 u_3\right\|_{L^\alpha}^{\frac{8}{3(r-2)}} + \left\|\nabla\boldsymbol{u}\right\|_{L^2}^2\,\mathrm{d}t\right)^{\frac43}. \end{aligned} \tag{47}$$

Now, if

$$\beta = \frac{8}{3(r-2)}, \tag{48}$$

then the last integral in (47) is finite, and once we choose δ_2 sufficiently small, then we can absorb the last term in (47) into the left-hand side to deduce

$$\sup_{\delta_1 \le t < \Gamma^*} \|\nabla \boldsymbol{u}(t)\|_{L^2}^2 \le C,$$

as desired. The proof of Theorem 2 is thus completed.

Now, we calculate all the parameters above. Denote by $\tilde{\beta} = \frac{1}{\beta}$, $\tilde{\alpha} = \frac{1}{\alpha}$, then

$$\begin{aligned} \frac{1}{q} &= \frac{1}{r-1}\left(1 - \frac{1}{\alpha}\right) = \frac{1}{(r-2)+1}(1-\tilde{\alpha}) \quad (\text{by } (42)) \\ &= \frac{1}{\frac{8}{3\beta}+1}(1-\tilde{\alpha}) = \frac{3(1-\tilde{\alpha})}{8\tilde{\beta}+3} \quad (\text{by } (48)). \end{aligned} \tag{49}$$

On the other hand,

$$\begin{aligned} \tilde{\beta} &= \frac{1}{\beta} = \frac{3-(q+1)\vartheta}{2q-1} \quad (\text{by } (38)) \\ &= \frac{3}{2q-1} - \frac{q+1}{2q-1} \cdot \frac{3}{2}\left(1 - \frac{1}{a}\right) \quad (\text{by } (36) \Rightarrow \vartheta = \frac{3}{2}\left(1-\frac{1}{a}\right)) \\ &= \frac{3}{2q-1} - \frac{q+1}{2q-1} \cdot \frac{3}{2} \cdot \frac{2q-1}{(q+1)\alpha} = \frac{3}{2q-1} - \frac{3}{2\alpha} \quad (\text{by } (34) \Rightarrow 1 - \frac{1}{a} = \frac{2q-1}{(q+1)\alpha}). \end{aligned} \tag{50}$$

Putting (49) into (50) yields

$$\tilde{\beta} = \frac{3}{2q-1} - \frac{3}{2}\tilde{\alpha} = \frac{3}{2\frac{8\tilde{\beta}+3}{3(1-\tilde{\alpha})}-1} - \frac{3}{2}\tilde{\alpha} \Rightarrow 32\tilde{\beta}^2 + (54\tilde{\alpha}+6)\tilde{\beta} + (9\tilde{\alpha}^2 + 27\tilde{\alpha} - 18) = 0.$$

Solving this quadratic equation gives

$$\tilde{\beta} = \frac{3(\sqrt{65 - 78\tilde{\alpha} + 49\tilde{\alpha}^2} - 9\tilde{\alpha} - 1)}{32}.$$

Hence,

$$\frac{2}{\beta} + \frac{3}{\alpha} = 2\tilde{\beta} + 3\tilde{\alpha} = \frac{3(\sqrt{65 - 78\tilde{\alpha} + 49\tilde{\alpha}^2} + 7\tilde{\alpha} - 1)}{16} = \frac{3\left(\sqrt{65\alpha^2 - 78\alpha + 49} + 7 - \alpha\right)}{16\alpha}.$$

Now, the main restriction of α comes from (32) and (38). After some calculations, we find (38) reduces to $\frac{3+\sqrt{17}}{4} \le \alpha \le \infty$ (in (38), $\beta = \infty$ corresponds to $\alpha = \frac{3+\sqrt{17}}{4}$), and all the assumptions, say (32), (34)-(36), (38), (42), (48), are all valid.

Remark 4. If we apply the same method in the proof of Theorem 2 to [9, Theorem 1.2], that is, in showing [9, Lemma 2.1], we use the generalized multiplicative Sobolev inequality (11), we get better result than (22), but no better result than (24).

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