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Stability for a boundary contact problem in thermoelastic Timoshenko's beam

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Abstract. We demonstrate the existence of solutions to Signorini's problem for the Timoshenko's beam by using a hybrid disturbance. This disturbance enables the use of semigroup theory to show the existence and asymptotic stability. We show that stability is exponential, when the waves speed of propagation is equal. When the waves speed is different, we show that the solution decays polynomially. This result is new. We perform numerical experiments to visualize the asymptotic properties.

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1. Introduction

We study the mechanical and thermal evolution of a thermoelastic beam in unilateral contact. These contact problems arise naturally in many situations in industrial processes, when two or more materials can come into contact or lose contact as a result of thermoelastic expansion or contraction. Here, we consider the cross-contact problem with Timoshenko's beam model. The physical setting is represented in Fig. 1.

The equations of motion and energy balance are described by

$$\begin{aligned}&\rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=0,\\&\rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)+\sigma\theta_x=0,\\&\rho_3\theta_t-\tau\theta_{xx}+\sigma\psi_{xt}=0.\end{aligned}\tag{1.1}$$

The mathematical modelling can be found in [1,2]. Here, $\varphi(x,t)$, stands for the transversal displacement of the point x on the beam, ψ is the rotatory angle of the cross section and θ is the difference of temperature of the beam. Here, $\rho_1=\rho A,\;\;\rho_2=\rho I,\;k=\kappa GA,\;\;b=EI$ where E is Young's modulus, G is the modulus of rigidity and κ is the transversal shear factor. The terms $\rho,\;A$ and I are density of body, the area of the cross section and the moment of inertia, respectively. The constants $\rho_3,\;\tau,\;\sigma\;>0$ represent the physical parameters from thermoelasticity theory. The initial conditions of the model are given by

$$\begin{aligned}&\varphi(x,0)=\varphi_0(x),\quad\varphi_t(x,0)=\varphi_1(x),\;\theta(x,0)=\theta_0(x),\;\forall x\in(0,L)\\&\psi(x,0)=\psi_0(x),\quad\psi_t(x,0)=\psi_1(x),\;\forall x\in(0,L),\end{aligned}\tag{1.2}$$

and we use the following boundary conditions

$$\varphi(0,t)=\psi(0,t)=\theta(0,t)=\psi(L,t)=\theta(L,t)=0,\quad\text{in}\quad(0,T).\tag{1.3}$$

For the free end of the beam, where contact with the obstacle can occur, we consider Signorini's contact condition.

$$g_1\leq\varphi(L,t)\leq g_2,\quad 0\leq t\leq T.\tag{1.4}$$

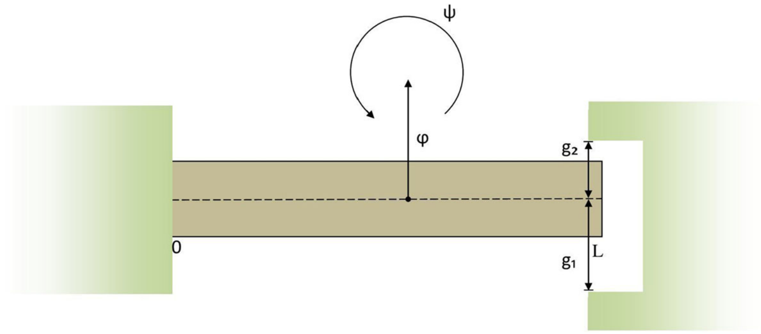

Fig. 1. Beam subject to a constraint at the free end L

This condition ensures that the transversal displacement at $x = L$ is restricted between stops g_1 and g_2. The mathematical boundary conditions for this physical setting are as follows

$$\begin{aligned} &S(L,t) \geq 0 \quad \text{if } \varphi(L,t) = g_1, \\ &S(L,t) = 0 \quad \text{if } g_1 < \varphi(L,t) < g_2, \\ &S(L,t) \leq 0 \quad \text{if } \varphi(L,t) = g_2, \end{aligned} \tag{1.5}$$

where $S = k(\varphi_x + \psi)$ and $M = b\psi_x$.

In a series of articles by Andrews et al. [3], Kuttler and Shillor [4], the authors studied the problem of one-dimensional semi-static thermoelastic contact. They demonstrated the existence of global weak solutions of their respective models. The numerical aspects of the problem were studied in [5,6]. In [7] was considered the Signorini's problem of the Euler–Bernoulli thermoelastic beams system, the authors showed the global existence of weak solutions of the model which decay exponentially to zero. Finally, in [8], the authors demonstrated the global existence of weak solutions of the Signorini's problem to Timoshenko's thermoelastic beam model and by introducing an additional friction mechanism, the authors were able to show that the solution of the models decay exponentially to zero. This additional dissipative mechanism makes the difference in the proof of the exponential stability of the problem. Here, we only consider the dissipation produced by the difference of temperature, for this reason we need that the waves speed of propagation be equal.

$$\chi_0 := \frac{k}{\rho_1} - \frac{b}{\rho_2} = 0. \tag{1.6}$$

In the general case (different propagation speeds), we prove that the decay rate is polynomial. Our method is different and follows the theory of semigroups.

The main contribution of this article is the use of semigroup theory to solve the Signorini's problem. We do this by taking dynamic boundary conditions and addressing the Signorini's problem using a Lipschitz disturbance, to obtain the normal compliance condition, then to arrive at the contact problem we use the observability inequalities. We believe that this method is stronger than the penalty method used in all the articles cited above. This is because we get more information about the asymptotic behaviour of the solution, under all boundary conditions. Unlike the articles [7–9] where special boundary conditions had to be used to show the exponential decay. Furthermore, with this method it is possible to prove the polynomial decay of the solutions of Timoshenko's contact problem. Finally, we believe that the polynomial decay rate that we obtain is optimal in the sense that it is the same rate as that obtained in [10,11] where optimality is demonstrated.

The remaining part of this manuscript is organized as follows: Sects. 2 and 3 deal with the global existence and the uniform stability of the hybrid system, respectively. In Sect. 4, we consider the normal compliance condition as a Lipschitz disturbance, then we take the limit $\epsilon \to 0$ to show the existence of global weak solutions to Signorini's problem. In addition, we show the exponential stability, provided the waves speed of propagation of the system is equal and the polynomial stability in the general case. Finally, in Sect. 5 we develop numerical experiments that verify the decay properties of solutions.

2. The semigroup setting

Our starting point is to consider the linear hybrid Timoshenko system that is given by

$$\begin{array}{l} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + \sigma\theta_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_3\theta_t - \tau\theta_{xx} + \sigma\psi_{xt} = 0, \quad \text{in } (0,L)\times(0,\infty). \end{array} \tag{2.1}$$

Verifying the initial data (1.2) and the boundary conditions (1.3) with $\varphi(L,t) = v(t)$, where

$$\epsilon v_{tt} + \epsilon v_t + \epsilon v + S(L,t) = 0. \tag{2.2}$$

Equation (2.2) is called the dynamic boundary condition. The equations describe the oscillations of the uniform cantilever curved beam with a load mass ϵ at its tip, with damping term proportional to the velocity. In a first moment, we omit the super index ϵ in system (2.1)–(2.2), we use this dependence later when we begin the limit process $\epsilon \to 0$. The objective of these boundary conditions is to apply the Lipschitz perturbation to obtain the normal compliance condition and then arrive to the Signorini's problem.

Let us introduce the Hilbert space $\mathcal{H}$

$$\mathcal{H} = V_0 \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L) \times \mathbb{R}^2,$$

where

$$V_0 = \{u \in H^1(0,L); u(0) = 0\}$$

which is a Hilbert space with the norm

$$\|U\|_{\mathcal{H}}^2 = \int_0^L \Big[\rho_1|\Phi|^2 + \rho_2|\Psi|^2 + k|\varphi_x+\psi|^2 + b|\psi_x|^2 + \rho_3|\theta|^2\Big]\,\mathrm{d}x + \epsilon|V|^2 + \epsilon|v|^2.$$

Denoting by $\Phi = \varphi_t$, $\Psi = \psi_t$, $V = v_t$ system (2.2) can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}U, \quad U(0) = U_0,$$

where $(\varphi, \Phi, \psi, \Psi, \theta, v, V)$, $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, v_0, v_1)$ and $\mathcal{A}$ is the operator

$$\mathcal{A}U = \left(\Phi, \frac{1}{\rho_1}S_x, \Psi, \frac{1}{\rho_2}M_x - \frac{1}{\rho_2}S - \frac{\sigma}{\rho_2}\theta_x, \frac{\tau}{\rho_3}\theta_{xx} - \frac{\sigma}{\rho_3}\Psi_x, V, -[V + v + \frac{1}{\epsilon}S(L)]\right)^\top \tag{2.3}$$

with domain of $\mathcal{A}$ given by

$$D(\mathcal{A}) := \{U \in \mathbb{H}; \text{ and } \varphi(L) = v\}$$

where

$$\mathbb{H} := [H^2(0,L)\cap V_0] \times V_0 \times [H^2(0,L)\cap H_0^1(0,L)] \times H_0^1(0,L) \times [H^2(0,L)\cap H_0^1(0,L)] \times \mathbb{R}^2.$$

Moreover $\mathcal{A}$ is dissipative,

$$\mathrm{Re}(\mathcal{A}U, U) = -\tau \int_0^L |\theta_x|^2 \, \mathrm{d}x - \epsilon |V|^2 \leq 0. \tag{2.4}$$

To show the well-posedness of (2.1)–(2.2), we only need to prove that $\mathcal{A}$ is an infinitesimal generator of a C_0 semigroup. To do that it is enough to show that $0 \in \varrho(\mathcal{A})$, see [12] . That is for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^\top \in \mathcal{H}$, there exists only one $U \in D(\mathcal{A})$ such that $\mathcal{A}U = F$. In fact, recalling the definition of $\mathcal{A}$ we get the system

$$\Phi = f_1 \in H^1(0, L), \quad \Psi = f_3 \in H^1(0, L), \quad V = f_6,$$

and

$$S_x = \rho_1 f_2, \quad M_x - S - \sigma\theta_x = \rho_2 f_4, \quad \tau\theta_{xx} - \sigma\Psi_x = \rho_3 f_5, \quad V + v + \frac{1}{\epsilon} S(L) = -f_7.$$

Since θ verify Dirichlet boundary condition and Ψ is already given by f_3, using the Lax–Milgram Lemma we conclude that there exists only one $\theta \in H^2(0, L)$. It remains to show the existence of ψ and φ.

$$k(\varphi_x + \psi)_x = \rho_1 f_2, \quad b\psi_{xx} - k(\varphi_x + \psi) = \sigma\theta_x + \rho_2 f_4,$$

verifying the following boundary conditions

$$\varphi(0) = \psi(0) = \psi(L) = 0, \quad \varphi(L) + \frac{1}{\epsilon} S(L) = -f_7 - f_6.$$

Denoting by $U^i = (\varphi^i, \psi^i)$ the bilinear form

$$a(U^1, U^2) = \int_0^L k(\varphi_x^1 + \psi^1)(\varphi_x^2 + \psi^2) + b\psi_x^1 \psi_x^2 \, \mathrm{d}x$$

is symmetric, continuous and coercive over the convex set

$$\mathbb{K} = \left\{ (\varphi, \psi); \;\; \varphi \in V_0, \psi \in H_0^1(0, L), \;\; \varphi(L) + \frac{1}{\epsilon} S(L) = -f_7 - f_6 \right\}.$$

Thus, for any $(f_6, f_7) \in L^2(0, L) \times L^2(0, L)$ there exists only one weak solution to the above system (see Theorem 5.6 (Stampacchia) page 138 of [13]). Using the equations, we conclude that $(\varphi, \Phi, \psi, \Psi, \theta, v, V) \in D(\mathcal{A})$

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions.*

The above theorem implies the global existence of solution for the corresponding hybrid problem.

3. Asymptotic behaviour of the hybrid system

The main tool we use in this section is the result due to Pruess [14] and Borichev and Tomilov [15].

Theorem 3.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions over a Hilbert space $\mathcal{H}$. Then, ([14]) $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathcal{A}) \quad \text{and} \quad \|(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \qquad \forall\, \lambda \in \mathbb{R}.$$

Moreover, if $i\mathbb{R} \subset \varrho(\mathcal{A})$ then we have ([15])

$$\|\, (i\lambda - \mathcal{A})^{-1} \,\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^\beta \;\; \Leftrightarrow \;\; \|\mathrm{e}^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{\beta}}} \|\mathcal{A}\Phi_0\|_{\mathcal{H}}.$$

Let us consider $U = (\varphi, \Phi, \psi, \Psi, \theta, v, V)^\top \in D(\mathcal{A})$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^\top \in \mathcal{H}$. The resolvent equation $i\lambda U - \mathcal{A}U = F$ in terms of its components can be written as

$$\begin{aligned} & i\lambda\varphi - \Phi = f_1, \\ & i\lambda\rho_1\Phi - k(\varphi_x + \psi)_x = \rho_1 f_2, \\ & i\lambda\psi - \Psi = f_3, \\ & i\lambda\rho_2\Psi - b\psi_{xx} + k(\varphi_x + \psi) + \sigma\theta_x = \rho_2 f_4, \\ & i\lambda\rho_3\theta - \tau\theta_{xx} + \sigma\Psi_x = \rho_3 f_5, \\ & i\lambda v - V = f_6, \\ & i\lambda\epsilon V + \epsilon v + \epsilon V + S(L,t) = \epsilon f_7. \end{aligned} \tag{3.1}$$

From (2.4) and the resolvent equation $i\lambda U - \mathcal{A}U = F$, we get

$$\epsilon|V|^2 + \int_0^L \tau|\theta_x|^2 \,\mathrm{d}x = \mathrm{Re}(F, U)_{\mathcal{H}}. \tag{3.2}$$

To get exponentially stability, we use condition (1.6). Let us introduce the functionals

$$\begin{aligned} \mathcal{I}(x) &= \underbrace{\rho_2 b|\Psi(x)|^2 + |M(x)|^2}_{:=\mathcal{I}_\psi(x)} + \underbrace{\rho_1 k|\Phi(x)|^2 + |S(x)|^2}_{:=\mathcal{I}_\varphi(x)} \\ \mathcal{L}(s) &= q'\left(\rho_2|\Psi|^2 + |M|^2 + \rho_1|\Phi|^2 + |S|^2\right) - q\rho_1 k\Phi\overline{\Psi} - qS\overline{M}, \end{aligned}$$

where

$$q(x) = \frac{e^{nx} - e^{n\xi_1}}{n}, \qquad q_0(x) = \frac{e^{-nx} - e^{-n\xi_2}}{n}. \tag{3.3}$$

Note that in this case $q'(x)$ is large in comparison with q for n large, therefore there exist positive constants such that

$$c_0 \int_{\xi_1}^{\xi_2} \mathcal{I}(x) \,\mathrm{d}x \le \int_{\xi_1}^{\xi_2} \mathcal{L}(s)\mathrm{d}s \le c_1 \int_{\xi_1}^{\xi_2} \mathcal{I}(x) \,\mathrm{d}x.$$

Lemma 3.1. *For any $[\xi_1, \xi_2] \subset [0, L]$, the solution of system (2.1) satisfies*

$$\left| q(\xi_1)\mathcal{I}_\psi(\xi_1) + q(\xi_2)\mathcal{I}_\psi(\xi_2) - \int_{\xi_1}^{\xi_2} \mathcal{I}_\psi \; ds \right| \le c\|U\|\|F\| + c\|S\|\|M\|,$$

$$\left| q(\xi_1)\mathcal{I}(\xi_1) + q(\xi_2)\mathcal{I}(\xi_2) - \int_{\xi_1}^{\xi_2} \mathcal{L}(s) \; ds \right| \le c\|U\|\|F\|.$$

Proof. Multiplying Eq. $(3.1)_4$ by $q\overline{M}$, we get

$$-\frac{1}{2}\int_{\xi_1}^{\xi_2} q\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho_2 b|\Psi|^2 + |M|^2\right] = \rho_2 b\int_{\xi_1}^{\xi_2} q\overline{f_{3,x}}\Psi \,\mathrm{d}x - \int_{\xi_1}^{\xi_2} q\overline{M}S \,\mathrm{d}x + \rho_2\int_{\xi_1}^{\xi_2} q\overline{M}f_4 \,\mathrm{d}x. \tag{3.4}$$

Similarly, multiplying Eq. $(3.1)_2$ by $q\overline{S}$, we get

$$-\frac{1}{2}\int_{\xi_1}^{\xi_2} q\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho_1 k|\Phi|^2+|S|^2\right]=\rho_1\int_{\xi_1}^{\xi_2} qf_1\overline{S}\,\mathrm{d}x+\rho_1\int_{\xi_1}^{\xi_2} qk\Phi\overline{f_{1,x}}\,\mathrm{d}x+\rho_1 k\int_{\xi_1}^{\xi_2} q\Phi\overline{\Psi}\,\mathrm{d}x$$
$$-\rho_1 k\int_{\xi_1}^{\xi_2} q\Phi\overline{f_3}\,\mathrm{d}x. \tag{3.5}$$

So, our result follows. □

Let us denote by Q and R any functions satisfying

$$|Q|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{c}{|\lambda|}\|U\|^2_{\mathcal{H}},\quad |R|\leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\|F\|^2_{\mathcal{H}}.$$

Lemma 3.2. *Under the above conditions, we have*

$$\int_0^L|\psi_x|^2\,dx\leq c\int_0^L|\Psi|^2\,dx+\frac{c}{|\lambda|^2}\|\Phi\|^2_{L^2}+\frac{c}{|\lambda|^2}\|F\|^2_{\mathcal{H}}+\frac{c}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.6}$$

$$\int_0^L|\Phi|^2\,dx\leq c\int_0^L|S|^2\,dx+\frac{c}{|\lambda|^2}\|U\|^2_{\mathcal{H}}+\frac{c}{|\lambda|^2}\|F\|^2_{\mathcal{H}}+\frac{c}{|\lambda|^2}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.7}$$

Proof. Multiplying $(3.1)_4$ by $\overline{\psi}$ and using integration by parts, we get

$$\int_0^L b|\psi_x|^2\,\mathrm{d}x=\int_0^L\rho_2|\Psi|^2\,\mathrm{d}x+k\int_0^L\varphi\overline{\psi}_x\,\mathrm{d}x-k\int_0^L|\psi|^2\,\mathrm{d}x-\sigma\int_0^L\theta_x\overline{\psi}\,\mathrm{d}x+\int_0^L f_4\overline{\psi}\,\mathrm{d}x.$$

from where we get (3.6). Similarly, multiplying $(3.1)_2$ by $\overline{\varphi}$ we get the other inequality. □

Lemma 3.3. *Under the above conditions, we have that for any* $[\xi_1,\xi_2]\subset[0,L]$ *it follows*

$$\left|i\lambda\int_{\xi_1}^{\xi_2}\Psi\,ds\right|\leq c\|U\|^{1/2}_{\mathcal{H}}\|\Psi\|^{1/2}_{L^2}+\frac{c}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}}+c\|F\|_{\mathcal{H}}.$$

Proof. Integrating over $]0,L[$ $(3.1)_4$, we get

$$\left|i\lambda\rho_2\int_{\xi_1}^{\xi_2}\Psi\,\mathrm{d}s\right|=\left|b[\psi_x(\xi_2)-\psi_x(\xi_1)]-k[\varphi(\xi_2)-\varphi(\xi_1)]-\int_{\xi_1}^{\xi_2}\left(k\psi+\sigma\theta_x\right)\mathrm{d}x+\rho_2\int_{\xi_1}^{\xi_2}f_4\,\mathrm{d}x\right|,$$
$$\leq c[\mathcal{I}_\psi(\xi_1)+\mathcal{I}_\psi(\xi_2)]^{1/2}+\frac{c}{|\lambda|}[\mathcal{I}(\xi_1)+\mathcal{I}(\xi_2)]^{1/2}+\left|\frac{k}{i\lambda}\int_{\xi_1}^{\xi_2}\Psi\,\mathrm{d}s\right|+c\|\theta_x\|_{L^2}+c\|F\|_{\mathcal{H}}.$$

So, using Lemma 3.2 and (3.4) we get for λ large enough that

$$\mathcal{I}_\psi(\xi_1)+\mathcal{I}_\psi(\xi_2)\leq c\|S\|_{L^2}\|\psi_x\|_{L^2}+c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

From Lemma 3.2, we get

$$\mathcal{I}_\psi(\xi_1)+\mathcal{I}_\psi(\xi_2)\leq c\|S\|_{L^2}\|\Psi\|_{L^2}+\frac{c}{|\lambda|}\|S\|^2_{L^2}+c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{3.8}$$

from where our conclusion follows. □

Theorem 3.2. *The semigroup $e^{\mathcal{A}t}$ associated with system (2.1) verifying (1.3) is exponentially stable provided (1.6) holds. If $\chi_0 \neq 0$ the semigroup decays polynomially to zero, that is*

$$\|T(t)U_0\| \leq \frac{c}{t^{\frac{1}{2}}}\|U_0\|_{D(\mathcal{A})},$$

where c is independent of ϵ.

Proof. Multiplying $(3.1)_5$ by $\int_x^L \overline{\Psi}\,\mathrm{d}s$, we get

$$\underbrace{-\rho_3\int_0^L \theta\int_x^L \overline{i\lambda\Psi}\,\mathrm{d}s\,\mathrm{d}x}_{:=J_1}+\underbrace{\tau\theta_x(0)\int_0^L \overline{\Psi}\,\mathrm{d}s}_{:=J_2}-\tau\int_0^L \theta_x\overline{\Psi}\,\mathrm{d}x+\sigma\int_0^L|\Psi|^2\,\mathrm{d}x=\int_0^L f_4\int_x^L\overline{\Psi}\,\mathrm{d}s\,\mathrm{d}x.$$

Therefore, we have

$$\sigma\int_0^L|\Psi|^2\,\mathrm{d}x = J_1-J_2+\tau\int_0^L\theta_x\overline{\Psi}\,\mathrm{d}x+\int_0^L f_4\int_x^L\overline{\Psi}\,\mathrm{d}s\,\mathrm{d}x,$$

from where we get

$$\int_0^L|\Psi|^2\,\mathrm{d}x \leq c|J_1|+c|J_2|+c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.9}$$

From Lemma 3.3,

$$\left|\rho_2\int_0^x \overline{i\lambda\Psi}\,\mathrm{d}s\right| \leq c\|U\|_{\mathcal{H}}^{1/2}\|\Psi\|_{L^2}^{1/2}+\frac{c}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}}+c\|F\|_{\mathcal{H}}. \tag{3.10}$$

Using Gagliardo–Nirenberg's inequality and relation (3.1), we get

$$\begin{aligned}|\theta_x(0)| &\leq c\|\theta_x\|^{1/2}\|\theta_{xx}\|^{1/2}\leq c\|\theta_x\|_{L^2}^{1/2}|\lambda|^{1/2}\left[\|\theta\|_{L^2}^{1/2}+\|\psi_x\|_{L^2}^{1/2}+\frac{1}{|\lambda|^{1/2}}\|F\|_{\mathcal{H}}^{1/2}\right].\\ &\leq c|\lambda|^{1/2}\left[\sqrt{R}+\|\theta_x\|_{L^2}^{1/2}\|\psi_x\|_{L^2}^{1/2}\right].\end{aligned}$$

Using Lemma 3.3, the above inequality and recalling the definition of J_2 we get

$$\begin{aligned}|J_2| &\leq \frac{c}{|\lambda|^{1/2}}\left[\sqrt{R}+\|\theta_x\|_{L^2}^{1/2}\|\psi_x\|_{L^2}^{1/2}\right](\|U\|_{\mathcal{H}}^{1/2}\|\Psi\|_{L^2}^{1/2}+\frac{1}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}),\\ &\leq \frac{c}{|\lambda|^{1/2}}\sqrt{R}\|U\|_{\mathcal{H}}^{1/2}\|\Psi\|_{L^2}^{1/2}+\underbrace{\frac{c}{|\lambda|^{1/2}}\|\theta_x\|_{L^2}^{1/2}(\|\psi_x\|_{L^2}^{1/2}\|\Psi\|_{L^2}^{1/2})\|U\|_{\mathcal{H}}^{1/2}}_{:=J_3}+\frac{c}{|\lambda|}\|U\|_{\mathcal{H}}\sqrt{R}\\ &\quad+\frac{c}{|\lambda|}\|U\|_{\mathcal{H}}\|\theta_x\|_{L^2}^{1/2}\|\psi_x\|_{L^2}^{1/2}+\frac{c}{|\lambda|^{1/2}}R.\end{aligned} \tag{3.11}$$

To estimate J_3, we use (3.6)

$$\begin{aligned}J_3 &\leq \frac{c}{|\lambda|^{1/2}}\|\theta_x\|_{L^2}^{1/2}\|\Psi\|_{L^2}\|U\|_{\mathcal{H}}^{1/2}+\frac{c}{|\lambda|}\|\theta_x\|_{L^2}^{1/2}\|\Psi\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}+\frac{c}{|\lambda|}R^{3/4}\|U\|_{\mathcal{H}}^{1/2},\\ &\leq \frac{\epsilon_0}{|\lambda|^2}\|U\|_{\mathcal{H}}^2+\epsilon_0\|\Psi\|_{L^2}^2+R.\end{aligned}$$

Using the same above procedure in (3.11), we get

$$J_2 \leq \frac{\epsilon_0}{|\lambda|^2}\|U\|_{\mathcal{H}}^2 + \epsilon_0\|\Psi\|_{L^2}^2 + R.$$

On the other hand, using interpolation we get

$$\begin{aligned}\|\theta\|_{L^2} \leq c\|\theta\|_{-1}^{1/2}\|\theta_x\|_{L^2}^{1/2} &\leq \frac{c}{|\lambda|^{1/2}}\left[\|\theta_x\|_{L^2}^{1/2} + \|\Psi\|_{L^2}^{1/2} + \|F\|_{\mathcal{H}}^{1/2}\right]\|\theta_x\|_{L^2}^{1/2},\\ &\leq \frac{c}{|\lambda|^{1/2}}\left[\|\theta_x\|_{L^2} + \|\Psi\|_{L^2}^{1/2}\|\theta_x\|_{L^2}^{1/2} + \|F\|_{\mathcal{H}}^{1/2}\|\theta_x\|_{L^2}^{1/2}\right],\\ &\leq \frac{c}{|\lambda|^{1/2}}\left[c_\epsilon\sqrt{R} + \|\Psi\|_{L^2}^{1/2}\|\theta_x\|_{L^2}^{1/2}\right]. \end{aligned} \tag{3.12}$$

Recalling the definition of J_1 and using (3.10) and (3.12), we get

$$\begin{aligned}|J_1| &\leq \frac{c}{|\lambda|^{1/2}}\left[c_\epsilon\sqrt{R} + \|\Psi\|_{L^2}^{1/2}\|\theta_x\|_{L^2}^{1/2}\right]\left[\|U\|_{\mathcal{H}}^{1/2}\|\Psi\|_{L^2}^{1/2} + \frac{1}{|\lambda|^{1/2}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\right],\\ &\leq \frac{\epsilon}{|\lambda|^2}\|U\|_{\mathcal{H}}^2 + \epsilon\|\Psi\|_{L^2}^2 + R,\end{aligned} \tag{3.13}$$

where we used inequalities of the type

$$\frac{c}{|\lambda|^{1/2}}\|\theta_x\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}^{1/2}\|\Psi\|_{L^2} \leq \frac{c_\epsilon}{|\lambda|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + \epsilon\|\Psi\|_{L^2}^2 \leq c_\epsilon\|\theta_x\|_{L^2}^2 + \frac{\epsilon}{|\lambda|^2}\|U\|_{\mathcal{H}}^2 + \epsilon\|\Psi\|_{L^2}^2.$$

Recalling (3.2) and substitution of J_1 and J_2 into (3.9) yields

$$\int_0^L |\Psi|^2 \,\mathrm{d}x \leq c_{\epsilon_0}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{\epsilon_0}{|\lambda|^2}\|U\|_{\mathcal{H}}^2. \tag{3.14}$$

Using Lemma 3.2, we arrive to

$$\int_0^L |\mathcal{I}_\psi|^2 \,\mathrm{d}x \leq c_{\epsilon_0}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{c}{|\lambda|^2}\|U\|_{\mathcal{H}}^2 + R. \tag{3.15}$$

Multiplying $(3.1)_4$ by $\overline{S}$, we get

$$i\lambda\rho_2\int_0^L \Psi\overline{S}\,\mathrm{d}x - b\int_0^L \psi_{xx}\overline{S}\,\mathrm{d}x + \int_0^L |S|^2\,\mathrm{d}x + \sigma\int_0^L \theta_x\overline{S} = \rho_2\int_0^L f_4\overline{S}\,\mathrm{d}x.$$

Recalling the definition of S and using Eq. $(3.1)_2$ to rewrite G_0, we get

$$\begin{aligned}\int_0^L |S|^2\,\mathrm{d}x &= -\sigma\int_0^L \theta_x\overline{S}\,\mathrm{d}x + \rho_2 k\int_0^L \Psi[\overline{i\lambda(\varphi_x+\psi)}]\,\mathrm{d}x + b\psi_x\overline{S}\Big|_0^L - \overbrace{b\int_0^L \psi_x\overline{S}_x\,\mathrm{d}x}^{:=G_0} + R,\\ &= -\sigma\int_0^L \theta_x\overline{S}\,\mathrm{d}x + \rho_2 k\int_0^L \Psi\overline{\Phi_x} + |\Psi|^2\,\mathrm{d}x + b\rho_1\int_0^L \Psi_x\overline{\Phi}\,\mathrm{d}x + b\psi_x\overline{S}\Big|_0^L + R,\end{aligned} \tag{3.16}$$

where R is such that $|R| \leq C\|U\|\|F\|$. Therefore, we get

$$\int_0^L |S|^2\,\mathrm{d}x \leq \rho_2 k\int_0^L |\Psi|^2\,\mathrm{d}x + \underbrace{(\rho_2 k - \rho_1 b)}_{:=\chi_0}\int_0^L \Psi\overline{\Phi_x}\,\mathrm{d}x + \left(\rho_1 b\Psi\overline{\Phi} + b\psi_x\overline{S}\right)\Big|_0^L. \tag{3.17}$$

Using the observability inequalities (Lemma 3.1), we get

$$\begin{aligned}\left|(\rho_1 b\Psi\overline{\Phi}+b\psi_x\overline{S})\Big|_0^L\right| &\leq c\left(\mathcal{I}_\psi(0)+\mathcal{I}_\psi(L)\right)^{1/2}\left(\mathcal{I}(0)+\mathcal{I}(L)\right)^{1/2},\\ &\leq \frac{c}{\delta}\left(\mathcal{I}_\psi(0)+\mathcal{I}_\psi(L)\right)+\delta\|U\|_{\mathcal{H}}^2+c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},\end{aligned} \tag{3.18}$$

hence (3.14) implies

$$\mathcal{I}_\psi(0)+\mathcal{I}_\psi(L)\leq \delta\|U\|_{\mathcal{H}}^2+c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$$

for $\epsilon=\delta^2$. Substitution of the above expression in (3.18) and using (3.14) we get

$$\int_0^L |S|^2\,\mathrm{d}x\leq c|\chi_0|\int_0^L |\lambda||\Psi\overline{\varphi_x}|\,\mathrm{d}x+\delta\|U\|_{\mathcal{H}}^2+R. \tag{3.19}$$

Using Lemma 3.2 implies

$$\int_0^L \mathcal{I}_\varphi\,\mathrm{d}x\leq c|\chi_0|\int_0^L |\lambda||\Psi\overline{\varphi_x}|\,\mathrm{d}x+\delta\|U\|_{\mathcal{H}}^2+R. \tag{3.20}$$

Note that

$$c\chi_0\int_0^L |\lambda||\Psi\overline{\varphi_x}|\,\mathrm{d}x\leq c\chi_0^2|\lambda|^2\int_0^L |\Psi|^2\,\mathrm{d}x+\frac{1}{2}\|U\|_{\mathcal{H}}^2.$$

From (3.14) and (3.2), we get

$$\|U\|_{\mathcal{H}}^2=\int_0^L \mathcal{I}_\varphi+\mathcal{I}_\psi+\rho_3|\theta|^2\,\mathrm{d}x+\epsilon|v|^2+\epsilon|V|^2\leq c\chi_0^2|\lambda|^2\int_0^L |\Psi|^2\,\mathrm{d}x+\delta\|U\|_{\mathcal{H}}^2+\frac{1}{2}\|U\|_{\mathcal{H}}^2$$

if $\chi_0=0$, the exponential decays holds. Let us suppose that $\chi_0\neq 0$. Using (3.14), we have

$$\|U\|_{\mathcal{H}}^2\leq c_{\epsilon_0}|\lambda|^2\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+(\epsilon_0+\delta)\|U\|_{\mathcal{H}}^2+\frac{1}{2}\|U\|_{\mathcal{H}}^2.$$

So we have

$$\|U\|_{\mathcal{H}}^2\leq C_\delta|\lambda|^4\|F\|_{\mathcal{H}}^2,$$

therefore from Theorem 3.1, the polynomial decays hold. □

4. The semilinear problem

Here, we prove the well-posedness of the abstract semilinear problem and we show, under suitable conditions that the solution also decays polynomially to zero. Let $\mathcal{F}$ be a local Lipschitz function defined over a Hilbert space $\mathcal{H}$. Here, we assume that there exists a globally Lipschitz function $\widetilde{\mathcal{F}_R}$ such that for any ball $B_R=\{W\in\mathcal{H};\ \ \|W\|_{\mathcal{H}}\leq R\}$,

$$\mathcal{F}(0)=0,\quad \mathcal{F}(U)=\widetilde{\mathcal{F}_R}(U),\quad \forall U\in B_R. \tag{4.1}$$

Additionally, we assume that there exists a positive constant κ_0 such that

$$\int_0^t \left(\widetilde{\mathcal{F}_R}(U(s)),U(s)\right)_{\mathcal{H}}\,\mathrm{d}s\leq \kappa_0\|U(0)\|_{\mathcal{H}}^2,\qquad \forall U\in C([0,T];\mathcal{H}). \tag{4.2}$$

Under these conditions we present.

Theorem 4.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup of contraction, exponentially or polynomially stable with infinitesimal generator $\mathbb{A}$ over the phase space $\mathcal{H}$. Let $\mathcal{F}$ locally Lipschitz on $\mathcal{H}$ satisfying conditions (4.1) and (4.2). Then, there exists a global solution to*

$$U_t - \mathbb{A}U = \mathcal{F}(U), \quad U(0) = U_0 \in \mathcal{H}, \tag{4.3}$$

that decays exponentially or polynomially, respectively.

Proof. By hypotheses, there exist positive constants c_0 and γ such that $\|T(t)\| \leq c_0 e^{-\gamma t}$, and $\widetilde{\mathcal{F}_R}$ globally Lipschitz with Lipschitz constant K_0 verifying conditions (4.1) and (4.2). Let us consider the following space.

$$E_\mu = \left\{V \in L^\infty(0,\infty;\mathcal{H});\;\; t \mapsto e^{-\mu t}\|V(s)\| \in L^\infty(\mathbb{R})\right\}.$$

Using standard fixed point arguments, we can show that there exists only one global solution to

$$U_t^R - \mathbb{A}U^R = \widetilde{\mathcal{F}_R}(U^R), \quad U^R(0) = U_0 \in \mathcal{H}. \tag{4.4}$$

Multiplying the above equation by U^R, we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U^R(t)\|_{\mathcal{H}}^2 - (\mathbb{A}U^R, U^R)_{\mathcal{H}} = (\widetilde{\mathcal{F}_R}(U^R), U^R)_{\mathcal{H}}.$$

Since the semigroup is contractive, its infinitesimal generator is dissipative, therefore

$$\|U^R(t)\|_{\mathcal{H}}^2 \leq \|U_0\|_{\mathcal{H}}^2 + 2\int_0^t (\widetilde{\mathcal{F}_R}(U^R), U^R)_{\mathcal{H}}\,\mathrm{d}t.$$

Using (4.2), we get

$$\|U^R(t)\|_{\mathcal{H}}^2 \leq (1+k_0)\|U_0\|_{\mathcal{H}}^2.$$

Note that for $R > (1+k_0)\|U_0\|_{\mathcal{H}}^2$, we have that

$$\widetilde{\mathcal{F}_R}(V) = \mathcal{F}(V), \quad \forall\;\; \|V\|_{\mathcal{H}} \leq R.$$

In particular, we have

$$\widetilde{\mathcal{F}_R}(U^R(t)) = \mathcal{F}(U^R(t)).$$

This means that U^R is also solution of system (4.3) and because of the uniqueness we conclude that $U^R = U$. To show the exponential stability to system (4.3), it is enough to show the exponential decay to system (4.4). To do that, we use fixed points arguments.

$$\mathcal{T}(V) = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}_R}(V(s))\,\mathrm{d}s.$$

Note that $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$ for δ small, $(\gamma > \delta)$. In fact, for any $V \in E_{\gamma-\delta}$ we have

$$\begin{aligned}\|\mathcal{T}(V)\|_{\mathcal{H}} &\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \int_0^t \|\widetilde{\mathcal{F}_R}(V(s))\|_{\mathcal{H}} e^{-\gamma(t-s)}\, \mathrm{d}s,\\ &\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 \int_0^t \|V(s)\|_{\mathcal{H}} e^{-\gamma(t-s)}\, \mathrm{d}s,\\ &\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 e^{-\gamma t} \int_0^t e^{\delta s}\, \mathrm{d}s \sup_{s\in[0,t]} \left\{ e^{(\gamma-\delta)s} \|V(s)\|_{\mathcal{H}} \right\},\\ &\leq \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \frac{K_0 C}{\delta} e^{-(\gamma-\delta)t}.\end{aligned}$$

Hence, $\mathcal{T}(V) \in E_{\gamma-\delta}$. Using standard arguments, we show that $\mathcal{T}^n$ satisfies

$$\|\mathcal{T}^n(W_1) - \mathcal{T}^n(W_2)\| \leq \frac{(k_1 t)^n}{n!} \|W_1 - W_2\|_{\mathcal{H}}.$$

Therefore, we have a unique fixed point satisfying

$$\mathcal{T}^n(U) = U = T(t)U_0 + \int_0^t T(t-s)\widetilde{\mathcal{F}_R}(U(s))\, \mathrm{d}s.$$

That is U is a solution of (4.4), and since $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$, then the solution decays exponentially. To show the polynomial stability, we consider the space

$$E_p = \{V \in L^\infty(0,\infty;\mathcal{H});\;\; t \mapsto (1+t)^p \|V(s)\| \in L^\infty(\mathbb{R})\}$$

To show the invariance, we use

$$\sup_{t>0} (1+t)^p \int_0^t (1+t-s)^{-p}(1+s)^{-p}\, \mathrm{d}s < C$$

and use the same above reasoning. □

Let us consider the semilinear system

$$\begin{aligned}&\rho_1 \varphi^\epsilon_{tt} - k(\varphi^\epsilon_x + \psi^\epsilon)_x = 0, \quad \text{in } (0,L)\times(0,\infty)\\ &\rho_2 \psi^\epsilon_{tt} - b\psi^\epsilon_{xx} + k(\varphi^\epsilon_x + \psi^\epsilon) + \sigma\theta^\epsilon_x = 0, \quad \text{in } (0,L)\times(0,\infty)\\ &\rho_3 \theta^\epsilon_t - \tau\theta^\epsilon_{xx} + \sigma\psi^\epsilon_{xt} = 0, \quad \text{in } (0,L)\times(0,\infty)\\ &\epsilon v^\epsilon_{tt} + \epsilon v^\epsilon_t + \epsilon v^\epsilon + S^\epsilon(L,t) = -\tfrac{1}{\epsilon}\Big[(v^\epsilon - g_2)^+ - (g_1 - v^\epsilon)^+\Big].\end{aligned} \tag{4.5}$$

The above system can be written as

$$U_t - \mathcal{A}U = \mathcal{F}(U), \qquad U(0) = U_0,$$

where $\mathcal{A}$ is given by (2.3) and $\mathcal{F}$ is given by

$$\mathcal{F}(U) = (0,0,0,0,0,0,f(v))^\top, \quad f(v) = -\frac{1}{\epsilon^2}\Big[(v-g_2)^+ - (g_1 - v)^+\Big]. \tag{4.6}$$

Note that $\mathcal{F}$ is a Lipschitz function verifying hypothesis (4.1)–(4.2). In fact, $\mathcal{F}(0) = 0$. Moreover,

$$\begin{aligned}\int_0^t \big(\mathcal{F}(U(s)), U(s)\big)_{\mathcal{H}}\, \mathrm{d}s &= -\int_0^t \frac{1}{\epsilon^2}\Big[(v-g_2)^+ - (g_1-v)^+\Big] v_t\, \mathrm{d}s,\\ &= -\frac{1}{2\epsilon^2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\Big[|(v-g_2)^+|^2 + |(g_1-v)^+|^2\Big]\, \mathrm{d}s,\\ &\le \frac{1}{2\epsilon^2}\Big[|(v_0-g_2)^+|^2 + |(g_1-v_0)^+|^2\Big].\end{aligned}$$

Theorem 4.2. *The nonlinear semigroup defined by system (4.5) is exponentially stable, provided $\chi_0 = 0$. Otherwise the solution decays polynomially as established in Theorem 3.2.*

Proof. It is a direct consequence of Theorem 4.1.
Let us introduce the functionals

$$\mathcal{I}(x,t) = \rho_2 b|\psi_t(x,t)|^2 + |M(x,t)|^2 + \rho_1 k|\varphi_t(x,t)|^2 + |S(x,t)|^2,$$
$$\mathcal{L}(t) = \int_0^L \rho_2 q_x|\psi_t|^2 + q_x|M|^2 + \rho_1 q_x|\varphi_t|^2 + q_x|S|^2\, \mathrm{d}x - \int_0^L q\rho_1 k\Phi\overline{\Psi} - qS\overline{M}\, \mathrm{d}x,$$

where q is as in (3.3) hence there exist positive constants C_0 and C_1 such that

$$C_0\int_0^L \mathcal{I}(x,t)\, \mathrm{d}x \le \mathcal{L}(t) \le C_1\int_0^L \mathcal{I}(x,t)\, \mathrm{d}x. \tag{4.7}$$

Under the above conditions, we establish the observability inequalities to the evolution system. □

Lemma 4.1. *The solution of system (4.5) satisfies*

$$\left|\int_0^t \mathcal{I}(L,s)\ ds - \int_0^t \mathcal{L}(s)\ ds\right| \le cE(0),$$
$$\left|\int_0^t \mathcal{I}(0,s)\ ds - \int_0^t \mathcal{L}(s)\ ds\right| \le cE(0).$$

Proof. Multiply Eq. $(4.5)_1$ by $q\overline{S}$ and Eq. $(4.5)_2$ by $q\overline{M}$ summing up and performing integration by parts and use the same approach as in the proof of Lemma 3.3. To achieve the second inequality, we use q_0 instead of q given by (3.3). □

Theorem 4.3. *For any initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0) \in \mathcal{H}$, there exists a weak solution to Signorini's problem (1.1)–(1.5), which decays as establish in Theorem 4.2.*

Proof. From Theorem 4.1, there exists only one solution to system (4.5) verifying

$$E(t, \varphi^\epsilon, \psi^\epsilon, \theta^\epsilon) + \tau\int_0^t\int_0^L |\theta_x^\epsilon|^2 \mathrm{d}x\mathrm{d}s \le E(0, \varphi^\epsilon, \psi^\epsilon, \theta^\epsilon), \tag{4.8}$$

where

$$2E(t) = \int_0^L \Big[\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + k|\varphi_x + \psi|^2 + b|\psi_x|^2 + \rho_3|\theta|^2\Big]\mathrm{d}x + \frac{1}{\epsilon}\mathcal{N}(t) + \epsilon|v_t|^2 + \epsilon|v|^2.$$

and

$$\mathcal{N}(t) := |(\varphi(L,t) - g_2)^+|^2 + |(g_1 - \varphi(L,t))^+|^2$$

In particular, from Lemma 4.1 we have that

$$\mathcal{I}_\epsilon(L,t) \quad \textit{uniformly bounded in} \quad L^2(0,T), \quad \forall \epsilon > 0, \tag{4.9}$$

which means that the first order energy is uniformly bounded for any $\epsilon > 0$. Standard procedures implies that the solution of system (4.5) converges in the distributional sense to system (1.1). It remains to show that condition (1.4) holds. Using Theorem 4.1, we get that $\varphi_t^\epsilon(L,t)$ and $S^\epsilon(L,t)$ are bounded in $L^2(0,T)$ for any $\epsilon > 0$, so is v_{tt}^ϵ. Using $(4.5)_4$, we get

$$\int_0^T \Big[\epsilon v_{tt} + \epsilon v_t + \epsilon v + S^\epsilon(L,t)\Big][u-v]\,\mathrm{d}t = -\frac{1}{\epsilon}\int_0^T \Big[(v-g_2)^+ - (g_1-v)^+\Big][u-v]\,\mathrm{d}t,$$

for any $u \in L^2(0,T;\mathcal{K}) \cap H^1(0,T;L^2(0,L))$, where $\mathcal{K} = \{w \in H^1(0,L), \quad g_1 \leq w(L) \leq g_2\}$. It is no difficult to see that

$$\lim_{\epsilon\to 0}\int_0^T \Big[\epsilon v_{tt}^\epsilon + \epsilon v_t^\epsilon + \epsilon v^\epsilon\Big][u-v^\epsilon]\,\mathrm{d}t = 0.$$

In fact, from $(4.5)_4$ ϵv_{tt}^ϵ is bounded by a constant depending on ϵ, in $L^2(0,T)$, from (4.9) v_t^ϵ is also uniformly bounded in $L^2(0,T)$. Therefore, v_t^ϵ is a continuous function, uniformly bounded in $L^\infty(0,T)$. Making an integration by parts, we get

$$\int_0^T \epsilon v_{tt}^\epsilon[u(t) - v^\epsilon]\,\mathrm{d}t = \epsilon\, v_t^\epsilon[u(t) - v^\epsilon]\big|_0^T - \int_0^T \epsilon v_t^\epsilon[u_t(t) - v_t^\epsilon]\,\mathrm{d}t \quad \to \quad 0.$$

Hence,

$$\lim_{\epsilon\to 0}\int_0^T S^\epsilon(L,t)[u-v]\,\mathrm{d}t = \lim_{\epsilon\to 0}\int_0^T -\frac{1}{\epsilon}\Big[(v-g_2)^+ - (g_1-v)^+\Big][u-v]\,\mathrm{d}t. \tag{4.10}$$

Note that

$$\begin{aligned}\int_0^T (v-g_2)^+[u(t)-v(t)]\,\mathrm{d}t &= \int_0^T (v-g_2)^+[u(t)-g_2]\ \mathrm{d}t - \int_0^T (v-g_2)^+(v-g_2)\ \mathrm{d}t,\\ &= \int_0^T (v-g_2)^+[u(t)-g_2]\ \mathrm{d}t - \int_0^T (v-g_2)^+(v-g_2)^+\ \mathrm{d}t \leq 0,\end{aligned}$$

for any $g_1 \leq u(L,t) \leq g_2$. Similarly, we get

$$-\int_0^T [(g_1-v)^+[u(t)-v(t)]\,\mathrm{d}t \leq 0.$$

Therefore, from the last two inequalities we arrive to

$$\int_0^T \frac{1}{\epsilon}\Big[(v-g_2)^+ - (g_1-v)^+\Big][u(t)-v(t)]\,\mathrm{d}t \leq 0, \quad \forall \epsilon > 0,$$

for any $u \in H^1(0,T;L^2(0,L))$ such that $g_1 \leq u(L,t) \leq g_2$. Letting $\epsilon \to 0$ and recalling that $v = \varphi(L,t)$ we get

$$\int_0^T S(L,t)[u(L,t) - \varphi(L,t)]\,\mathrm{d}t \geq 0, \quad \forall u \in L^2(0,T;\mathcal{K}).$$

From this relation, we get (1.5). The proof of the existence is now complete. To show the asymptotic behaviour, we use Theorem 4.2 to get

$$E(t,\varphi^\epsilon,\psi^\epsilon,\theta^\epsilon) \leq CE(0,\varphi^\epsilon,\psi^\epsilon,\theta^\epsilon)e^{-\gamma t}.$$

So, using the semicontinuity of the norm and noting that $\mathcal{N}(0) = 0$, we obtain

$$E(t,\varphi,\psi,\theta) \leq \liminf_{\epsilon\to 0^+} E(t,\varphi^\epsilon,\psi^\epsilon,\theta^\epsilon) \leq C\Big\{\lim_{\epsilon\to 0^+} E(0,\varphi^\epsilon,\psi^\epsilon,\theta^\epsilon)\Big\}e^{-\gamma t} \leq CE(0,\varphi,\psi,\theta)e^{-\gamma t}$$

where C is a positive constant independent of parameter ϵ. Thus, we conclude the exponential stability of the Signorini's problem. Similarly, we get the polynomial stability. □

Remark 4.1. We believe that the polynomial rate of decay is optimal in the sense that it is the same rate obtained in [10,11] where the authors show the optimality.

Remark 4.2. The uniqueness of the solution to Signorini's problem (1.1)–(1.4) remains an open question.

The same approach can be used to show existence of the semilinear problem

$$\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu_1\varphi|\varphi|^\alpha = 0,\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \sigma\theta_x + \mu_2\psi|\psi|^\beta = 0,\\
&\rho_3\theta_t - \tau\theta_{xx} + \sigma\psi_{xt} = 0.
\end{aligned} \tag{4.11}$$

Theorem 4.4. *Under the same hypothesis from Theorem 4.3, there is at least one solution to Signorini's problem (4.11) satisfying (1.2)–(1.5).*

Proof. As in Theorem 4.3, we consider the function

$$\mathcal{F}(U) = (0, -\mu_1\varphi^\epsilon|\varphi^\epsilon|^\alpha, 0, -\mu_2\psi^\epsilon|\psi^\epsilon|^\beta, 0, 0, f(v))^\top,$$

where f is given by (4.6). Note that $\mathcal{F}(0) = 0$. Using the mean value theorem to $g(s) = |s|^\alpha s$, we obtained the inequality

$$\Big| s|s|^\alpha - r|r|^\alpha \Big| \leq (|s|^\alpha + |r|^\alpha)|s - r|.$$

Taking the norm in $\mathcal{H}$ and since φ_i^ϵ and ψ_i^ϵ belong to $H^1(0,L) \subset L^\infty(0,L)$, then we get

$$\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\|_{\mathcal{H}} \leq C\|U_1 - U_2\|_{\mathcal{H}}.$$

Therefore, $\mathcal{F}$ is locally Lipschitz. Since

$$(\mathcal{F}U,U)_{\mathcal{H}} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{\mu_1}{1+\alpha}|\varphi^\epsilon|^{\alpha+2}\,\mathrm{d}x + \frac{\mu_2}{1+\beta}|\psi^\epsilon|^{\beta+2}\,\mathrm{d}x$$

then

$$\int_0^t (\mathcal{F}U,U)_{\mathcal{H}} \leq \int_0^L \frac{\mu_1}{1+\alpha}|\varphi^\epsilon(0)|^{\alpha+2}\,\mathrm{d}x + \frac{\mu_2}{1+\beta}|\psi^\epsilon(0)|^{\beta+2}\,\mathrm{d}x.$$

Thus, there exists a positive constant c_0 such that

$$\int_0^t (\mathcal{F}U,U)_{\mathcal{H}} \leq c_0\|U\|_{\mathcal{H}}^2.$$

Note that for this function, there exists the cut-off function

$$f_{1,R_2} = \begin{cases} \mu_1 x|x|^{\alpha} & if\ x \leq R_2, \\ \mu_1 x|R_2|^{\alpha} & if\ x \geq R_2. \end{cases} \qquad f_{2,R_2} = \begin{cases} \mu_2 x|x|^{\beta} & if\ x \leq R_2, \\ \mu_2 x|R_2|^{\beta} & if\ x \geq R_2. \end{cases}$$

It is not difficult to check that

$$\widetilde{\mathcal{F}}_{R_2} = (0, f_{1,R_2}, 0, f_{2,R_2}, 0, 0, 0)^{\top}$$

is globally Lipschitz. Using Theorem 4.1, our conclusion follows. □

5. Numerical approach

In this section, we consider the numerical solution of the penalized problem (4.5). We use the finite element methods over $(0, L)$ and the finite difference in time.

5.1. Algorithms and numerical experiment

Let X_h be a partition over the interval $\Omega = (0, L)$, that is, $X_h = \{0 = x_0 < x_1 < \cdots < x_N = L\}$, $\Omega_{j+1} = (x_j, x_{j+1})$, where N_e is the number of the elements obtained of partition. We consider the finite-dimensional $S_1^h = \{u \in C(0, L); u\Big|_{\Omega_e} \in P_1(\Omega_e)\}$, where P_1 is the set of linear polynomials over Ω_e, and $U^h = \{u^h \in S_1^h; u^h(0) = 0\}$ and $V^h = \{v^h \in S_1^h; v^h(L) = 0\}$. We use a representation of the functions φ^h and ψ^h as in [16], so we have

$$u^h(t, x) = \sum_{i=1}^{2N} d_i(t)\phi_i(x), \quad v^h(t, x) = \sum_{i=1}^{N} \theta_i(t)\omega_i(x)$$

where $\phi_i(x),\ i = 1, \ldots, 2N$, and $\omega_i(x),\ i = 1, \ldots, N$, are the global vector interpolation functions. So, we obtain the following dynamical problem in $\mathbb{R}^N \times \mathbb{R}^{2N}$.

$$\begin{aligned} &\mathbf{M}_1\dot{\theta}(t) + \mathbf{K}_1\theta(t) + \mathbf{C}_1^{\top}\dot{\mathbf{d}}(t) = \mathbf{F}_1(t), \\ &\mathbf{M}_2\ddot{\mathbf{d}}(t) + \mathbf{K}_2(\mathbf{d}(t)) + \mathbf{C}_1\theta(t) = \mathbf{F}_2(t), \\ &\theta(0) = \theta_0, \quad \mathbf{d}(0) = \mathbf{d}_0 \quad \text{and} \quad \dot{\mathbf{d}}(0) = \mathbf{d}_1 \end{aligned}$$

where $\mathbf{M}_1$ is the thermal capacity matrix, $\mathbf{K}_1$: the conductivity matrix, $\mathbf{C}_1$: the coupled matrix and $\mathbf{F}_1$: the heat source vector. $\mathbf{M}_2$: the consistent mass matrix, $\mathbf{K}_2(\mathbf{d}(t))$: the vector of consistent nodal elastic stiffness at time t, and $\mathbf{F}_2(t)$: the vector of consistent nodal applied forces generalized at time t. Furthermore, θ_0, $\mathbf{d}_0$ and $\mathbf{d}_1$ are temperature, displacement and velocities, nodal initial.

To solve the above system, we introduce a partition P of the time domain $[0, T]$ into M intervals of length Δt such that $0 = t_0 < t_1 < \cdots < t_M = T$, with $t_{n+1} - t_n = \Delta t$ and we use the well-known Trapezoidal generalized rules and Newmark's methods (see [17,18]). In our problem, we have a nonlinear system. Thus, our numerical scheme becomes

$$\begin{aligned} &\mathbf{M}_1\dot{\theta}_{n+1} + \mathbf{K}_1\theta_{n+1} + \mathbf{C}_1^{\top}\dot{\mathbf{d}}_{n+1} = \mathbf{F}_1^{n+1} \\ &\mathbf{M}_2\ddot{\mathbf{d}}_{n+1} + \mathbf{K}_2\mathbf{d}_{n+1} + \mathbf{C}_1\theta_{n+1} = \mathbf{F}_2^{n+1} + \tilde{\mathbf{K}}_2(\mathbf{d}_{n+1}) \\ &\quad \theta_{n+1} = \theta_n + \Delta t\dot{\theta}_{n+\alpha}, \\ &\quad \dot{\theta}_{n+\alpha} = (1 - \alpha)\dot{\theta}_n + \alpha\dot{\theta}_{n+1}, \\ &\quad \mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t\dot{\mathbf{d}}_n + \frac{\Delta t^2}{2}[(1 - 2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}] \\ &\quad \dot{\mathbf{d}}_{n+1} = \dot{\mathbf{d}}_n + \Delta t[(1 - \gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}] \end{aligned}$$

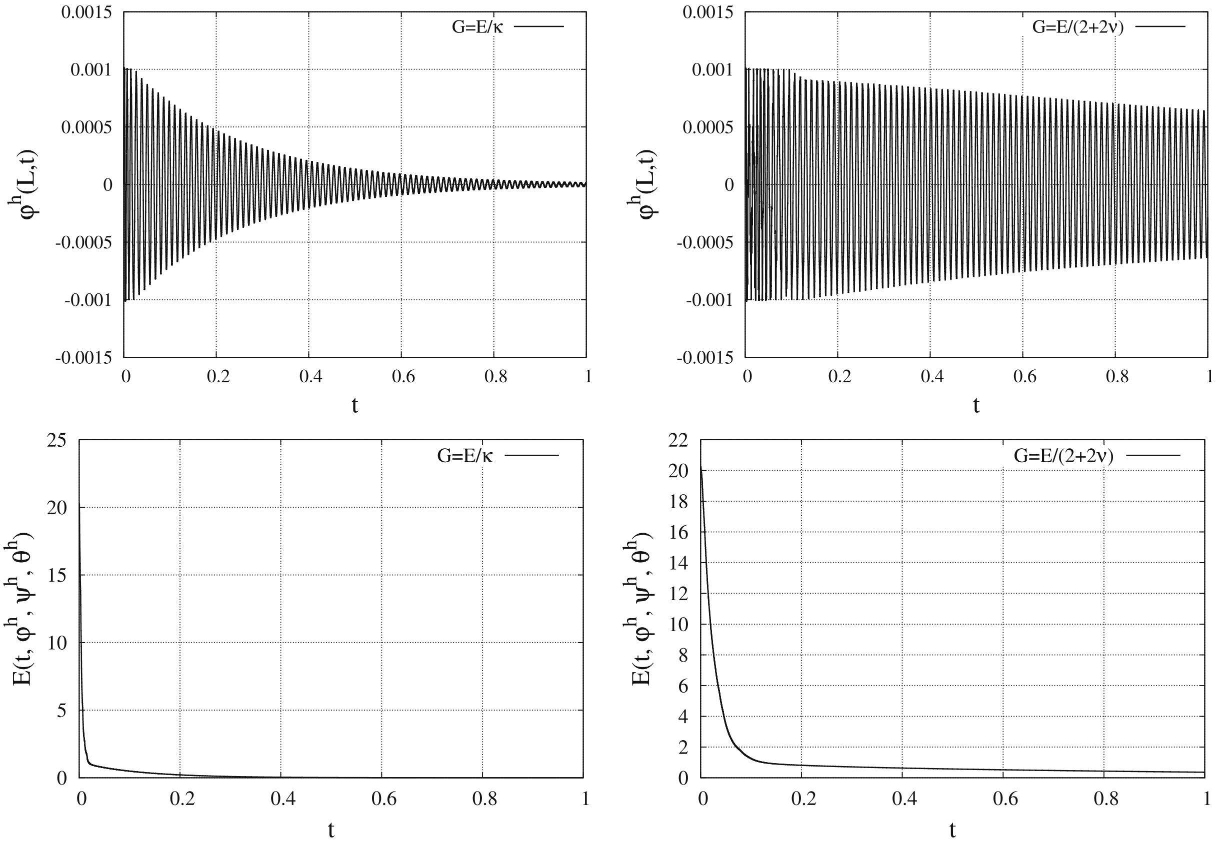

FIG. 2. Beam's oscillations at the end $x = L : \varphi^h(L,t)$ and the asymptotic behaviour of the energy at time 1 s. In this case, we performed the experiment to $\chi_0 = 0$ (this equality is equivalent to $G = \frac{E}{\kappa}$) and $\chi_0 \neq 0$,, respectively

where

$$\tilde{\mathbf{K}}_2(\mathbf{d}_{n+1}) = \frac{1}{\epsilon}(0, 0, \ldots, (d_{2N-1}(t) - g_2)^+ - (g_1 - d_{2N-1}(t))^+, 0)^\top$$

and β, γ and α are parameters that govern the stability and accuracy of the methods.

Remark 5.1. A typical numerical problem to the Timoshenko system is the shear locking. Numerical alternatives were performed in the literature, we indicate the classical reference by Arnold [19], Hughes et al. [20] and Prathap and Bhashyam [21].

Remark 5.2. To get computational results, we use the implemented code in Language C. The graphics were developed using GNUplot.

5.1.1. Numerical experiment. To verify the asymptotic behaviour of the numerical solutions, we consider the parameter from algorithms $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. In these experiments, we consider the following initial conditions:

$$\varphi(x,0) = 0,\ \psi(x,0) = 0,\ \psi_t(x,0) = 0 \text{ and } \theta(x,0) = x^2 - 2x^3 + x^4.$$

Also, we take a finite element mesh with $h = 0.00125$ and $\Delta t = 10^{-6}$ s.

Experiment We consider a rectangular beam with $L = 1.0$ m, thickness 0.1 m, width 0.1 m, $E = 69.10^9$ N/m^2 $\rho = 2700$ Kg/m^3, $\nu = 0.3$ (Poisson ratio), and $\tau = 42$ W/m K and $\varphi_t(x, 0) = 1 - \cos(\frac{2\pi}{L}x)$. The penalization parameter $\epsilon = 10^{-9}$ and $g_2 = -g_1 = 0.001$ m (Fig. 2).

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