Z. Angew. Math. Phys. (2020) 71:180

0044-2275/20/050001-9
published online October 7, 2020
https://doi.org/10.1007/s00033-020-01411-8

Zeitschrift für angewandte Mathematik und Physik ZAMP

Stoneley waves at the generalized Wiechert condition

S. V. Kuznetsov

Abstract. A generalization of the Wiechert condition by introducing two independent dimensionless parameters instead of one parameter in the original Wiechert condition is proposed. Variation of Stoneley wave velocity at varying two parameters of the generalized Wiechert condition at different Poisson's ratios is studied revealing a substantial discrepancy in Stoneley wave velocity profiles.

Mathematics Subject Classification. 35Q74, 74J15.

Keywords. Stoneley wave, Secular equation, Wiechert condition, Velocity.

1. Introduction

Herein, propagation of the interfacial Stoneley waves in layered media satisfying a more general condition, than the original Wiechert condition, is analyzed. (In more details, Wiechert condition is discussed in the next subsection.) The Wiechert condition [1,2] plays an important role in various applications in acoustics and, especially in studies of the interfacial Stoneley waves. In this respect, in the pioneering Stoneley work [3] existence of Stoneley waves propagating on an interface between two dissimilar isotropic halfspaces was studied at an assumption that physical properties of the halfspaces obey Wiechert condition; in [3] an explicit algebraic secular equation for Stoneley wave velocity was also derived.

The vast majority of the subsequent studies on Stoneley wave propagation in layered isotropic media were concerned with the Wiechert condition [4–16]. In [4,5], regions of existence for Stoneley waves plotted in terms of their relative physical parameters were constructed numerically, revealing that the Wiechert line belongs to the region of existence. Various forms of secular equations for Stoneley wave velocity were constructed in [6–12]. Several analytical methods for solving secular equations for Stoneley wave velocity were suggested in [13–15]. In [16], it was demonstrated that the studied regions of existence for Stoneley waves are multiply connected, instead of the previously assumed simply connected ones. Appearance of high-frequency Stoneley waves generated by propagation of Lamb waves in layered plates was studied numerically in [17–19] and by constructing high-frequency asymptotics in [20,21].

Stoneley waves propagating on an interface between anisotropic halfspaces were mainly studied by applying either three-dimensional formalism [22] or by complex sextic formalisms [23–25].

1.1. Wiechert condition

The original Wiechert condition asserts that the dimensionless physical parameters responsible for acoustic properties of the contacting media are proportional to a single parameter q. Consider inhomogeneous space consisting of two isotropic homogeneous halfspaces in a contact. Acoustical properties of the contacting halfspaces can be described by the following dimensionless parameters

$$\tilde{\rho} = \frac{\rho_1}{\rho_2}, \quad \tilde{\mu} = \frac{\mu_1}{\mu_2}, \quad \tilde{\lambda} = \frac{\lambda_1}{\lambda_2}, \tag{1.1}$$

where ρ_k are the material densities, and $\mu_k,\ \lambda_k,\ k=1,2$ are the corresponding Lame's constants of the contacting halfspaces.

The Wiechert condition imposes the following restriction on the dimensionless parameters

$$\tilde{\rho}=\tilde{\mu}=\tilde{\lambda}=q. \tag{1.2}$$

where q is the dimensionless variable, known also as the Wiechert parameter; see [15].

Taking into account definitions (1.1), Eq. (1.2) ensures

$$\frac{\mu_1}{\rho_1}=\frac{\mu_2}{\rho_2},\quad \frac{\lambda_1}{\rho_1}=\frac{\lambda_2}{\rho_2}. \tag{1.3}$$

Equation (1.3) ensures that the corresponding bulk wave velocities in the contacting halfspaces coincides

$$\alpha_1=\alpha_2,\quad \beta_1=\beta_2. \tag{1.4}$$

where α_k and β_k are correspondingly P and S wave velocities in the contacting halfspaces.

It can be easily shown that condition (1.4) implies also

$$c_{R_1}=c_{R_2}, \tag{1.5}$$

where $c_{R_k},\quad k=1,2$ are the corresponding Rayleigh wave velocities. Thus, Wiechert condition imposes strong restrictions on the possible values of P, S and Rayleigh wave velocities.

Remarks 1.1. (A) Despite equal P, S and Rayleigh wave velocities of the halfspaces obeying Wiechert condition (1.2), the corresponding acoustic impedances $Z_k\equiv\alpha_k\rho_k$ and $Z_k^*\equiv\beta_k\rho_k,\quad k=1,2$ need not be equal.

(B) Condition (1.2) does not necessary require equal Poisson's ratios of the contacting media. However, if Poisson's ratios are identical, condition (1.2) implies:

$$\tilde{\rho}=\tilde{E}, \tag{1.6}$$

herein, $\tilde{E}$ is relative Young's modulus

$$\tilde{E}=\frac{E_1}{E_2} \tag{1.7}$$

and $E_k,\ k=1,2$ are Young's moduli of the contacted media.

(C) Considering 3D space defined by the dimensionless parameters $\tilde{\rho},\ \tilde{\lambda},\ \tilde{\mu}$, it can be observed that Wiechert condition in this space corresponds to a straight line with guide cosines

$$l_1=l_2=l_3=1/\sqrt{3}. \tag{1.8}$$

The line passing through the origin with guide cosines (1.8) is known as *Wiechert line* [16].

1.2. Generalization of the Wiechert condition

In [15], variation of Stoneley wave velocity along Wiechert line analyzed by the freezing coefficient method [26] revealed unimodal (single extremum) behavior with extremal value reached at $q\to 1\pm 0$ when both media have identical physical properties and Stoneley wave degenerates into S wave. However, Stoneley wave velocity variation along other directions in the 3D space defined by the dimensionless Lame's constants $\tilde{\lambda}$; $\tilde{\mu}$ and dimensionless density $\tilde{\rho}$, remains unexplored.

Herein, a natural generalization of the Wiechert condition by introducing two independent dimensionless variables

$$\tilde{\lambda}=\tilde{\mu}=q_1;\quad \tilde{\rho}=q_2 \tag{1.9}$$

is proposed, and in more details generalization (1.9) is discussed in Sec. 4.

Numerical analysis of Stoneley wave velocity variation under condition (1.9) is given in Sec. 5. Computations at different Poisson's ratios revealed almost identical regions of existence of Stoneley waves and similar shapes (not values) of the corresponding 3D plots. However, Stoneley wave velocity values heavily depend upon Poisson's ratios.

2. Stoneley secular equation

Stoneley waves propagate along a plane interface of two dissimilar halfspaces in a contact with constant velocity that depends solely on physical properties of the contacting media. The secular equation for Stoneley wave velocity constructed in [3] may be represented the following form

$$\begin{array}{l} P(c) \equiv c^4 \left((\rho_1 - \rho_2)^2 - (\rho_1 A_2 + \rho_2 A_1)(\rho_1 B_2 + \rho_2 B_1) \right) \\ \quad -2Kc^2 (\rho_1 A_2 B_2 - \rho_2 A_1 B_1 - \rho_1 + \rho_2) + K^2 (A_1 B_1 - 1)(A_2 B_2 - 1) = 0 \end{array}, \tag{2.1}$$

where c is the Stoneley wave velocity, and

$$K = 2\left(\rho_2 \beta_2^2 - \rho_1 \beta_1^2\right), \quad A_k = \sqrt{1 - \frac{c^2}{\alpha_k^2}}, \quad B_k = \sqrt{1 - \frac{c^2}{\beta_k^2}}, \quad k = 1, 2. \tag{2.2}$$

In Eqs. (2.1), (2.2) $\rho_k,\ k = 1, 2$ are material densities; $\alpha_k,\ \ \beta_k,\ \ k = 1, 2$ are, respectively, longitudinal and shear bulk wave velocities:

$$\alpha_k = \sqrt{\frac{\lambda_k + 2\mu_k}{\rho_k}}, \quad \beta_k = \sqrt{\frac{\mu_k}{\rho_k}}, \tag{2.3}$$

herein, $\lambda_k,\ \mu_k,\ \ k = 1, 2$ are the corresponding Lame's constants.

A more convenient form of the secular equation was proposed by Scholte [6–8] in the dimensionless form

$$P(\tilde{c}) \equiv L\tilde{c}^4 + 2M\,\tilde{c}^2 + N = 0, \tag{2.4}$$

where $\tilde{c}$ is the dimensionless velocity

$$\tilde{c} = \frac{c}{\beta_2} \tag{2.5}$$

and

$$\begin{array}{l} L = \left((1 - \tilde{\rho})^2 - \left(\tilde{\rho}\tilde{A}_2 + \tilde{A}_1 \right)\left(\tilde{\rho}\tilde{B}_2 + \tilde{B}_1 \right) \right) \\ M = \tilde{K}\left(1 + \tilde{\rho}\tilde{A}_2\tilde{B}_2 - \tilde{A}_1\tilde{B}_1 - \tilde{\rho} \right) \\ N = \tilde{K}^2 \left(1 - \tilde{A}_1\tilde{B}_1 \right)\left(1 - \tilde{A}_2\tilde{B}_2 \right) \end{array}. \tag{2.6}$$

In (2.6), coefficient $\tilde{K}$ has the form

$$\tilde{K} = 2\left(1 - \tilde{\rho}\tilde{\beta}^2\right) \tag{2.7}$$

and

$$\tilde{A}_k = \sqrt{1 - \tilde{c}^2 \frac{\beta_2^2}{\alpha_k^2}}, \qquad \tilde{B}_k = \sqrt{1 - \tilde{c}^2 \frac{\beta_2^2}{\beta_k^2}}, \qquad k = 1, 2. \tag{2.8}$$

Secular equation in a form (2.4) will be used in the further analysis.

3. Generalized Wiechert condition

A better suited for modeling real geophysical formations is the generalized Wiechert condition containing two independent dimensionless parameters:

$$\tilde{\rho}, \quad \tilde{\lambda} = \tilde{\mu}. \tag{3.1}$$

It will also be assumed that both media have identical Poisson's ratios.

Taking into account expressions (2.3) for bulk waves yields

$$\tilde{\alpha} \equiv \frac{\alpha_1}{\alpha_2} = \chi, \quad \tilde{\beta} \equiv \frac{\beta_1}{\beta_2} = \chi, \tag{3.2}$$

where

$$\chi = \sqrt{\frac{\tilde{\mu}}{\tilde{\rho}}}. \tag{3.3}$$

Taking into account Eqs. (3.2), the adjacent media can have different velocities of bulk waves, and consequently different Rayleigh wave velocities, obeying the analogous relation

$$\tilde{c}_R \equiv \frac{c_{R_1}}{c_{R_2}} = \chi. \tag{3.4}$$

The natural physical restrictions imply

$$\tilde{\rho} > 0, \quad \tilde{\mu} > 0, \tag{3.5}$$

and in view of $(3.1)_2$ both media may have either positive or negative, but identical Poisson's ratios; thus, the case when one medium has positive Poisson's ratio, while another negative one, is prohibited.

Remarks 3.1. (A) Condition (3.1) defines a Q-plane in the 3D space of dimensionless parameters $\tilde{\rho}$, $\tilde{\lambda}$, $\tilde{\mu}$. It is easy to show that the Q-plane contains a straight line defined by the Wiechert condition (1.2); see Remark 1.1.C.

(B) The following condition that is actually due to Stoneley [3] ensures attenuation of Stoneley wave with depth in both halfspaces

$$\tilde{c} < \min(\chi;\ 1). \tag{3.6}$$

4. Stoneley secular equation at the generalized Wiechert condition

At conditions (3.1) coefficients (2.6) of the secular equation (2.4) take the form

$$\begin{aligned} L &= \left((1-\tilde{\rho})^2 - \left(\tilde{\rho}\tilde{A}_2 + \tilde{A}_1\right)\left(\tilde{\rho}\tilde{B}_2 + \tilde{B}_1\right)\right) \\ M &= \tilde{K}\left(1 + \tilde{\rho}\tilde{A}_2\tilde{B}_2 - \tilde{A}_1\tilde{B}_1 - \tilde{\rho}\right) \\ N &= \tilde{K}^2\left(1 - \tilde{A}_1\tilde{B}_1\right)\left(1 - \tilde{A}_2\tilde{B}_2\right) \end{aligned} . \tag{4.1}$$

In (4.1) coefficient $\tilde{K}$ becomes

$$\tilde{K} = 2\left(1 - \tilde{\rho}\tilde{\chi}^2\right) \tag{4.2}$$

and

$$\tilde{A}_1 = \sqrt{1 - \frac{\tilde{c}^2}{\chi^2\gamma^2}}, \quad \tilde{A}_2 = \sqrt{1 - \frac{\tilde{c}^2}{\gamma^2}}, \quad \tilde{B}_1 = \sqrt{1 - \frac{\tilde{c}^2}{\chi^2}}, \quad , \tilde{B}_2 = \sqrt{1 - \tilde{c}^2}, \tag{4.3}$$

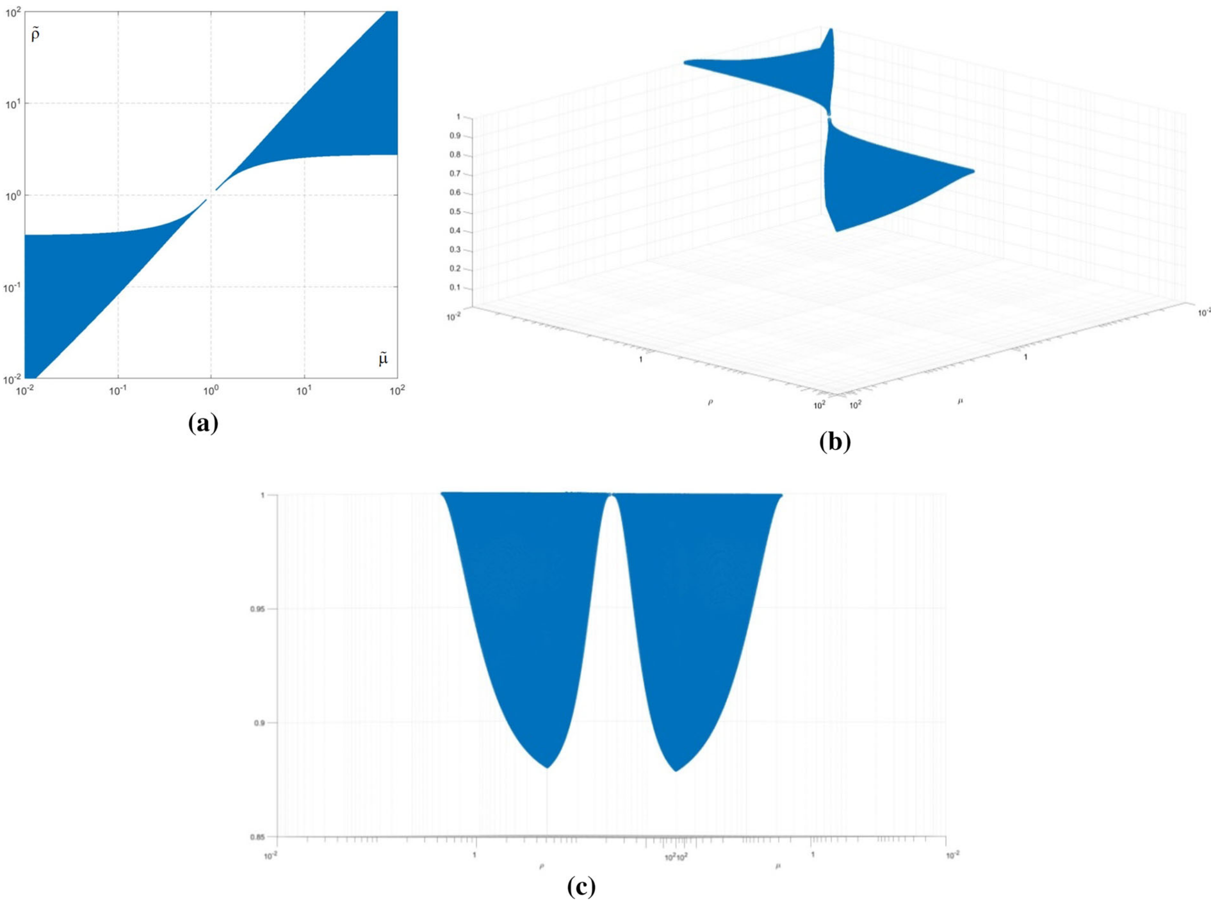

FIG. 1. Poisson's ratio 0.0; **a** region of existence; **b** 3D plot for velocity variation; **c** projection onto the Wiechert plane

where

$$\gamma = \sqrt{2}\sqrt{\frac{1-\nu}{1-2\nu}}. \tag{4.4}$$

In (4.4), ν is the common Poisson's ratio of both media; at $\nu \in (-1;\ 0.5)$parameter $\gamma \in \left(\frac{2}{\sqrt{3}};\ \infty\right)$.

With coefficients (4.1)–(4.3), the considered secular equation (2.4) becomes a three parametric one with parameters $\tilde{\rho}$, $\tilde{\chi}$ and γ or ν.

5. Stoneley waves at the generalized Wiechert condition

This section concerns with numerical analysis of the real and positive root of secular equation (2.4) with coefficients defined by Eq. (4.1). In view of the non-monotonic behavior of the algebraic function $P(\tilde{c})$ in the left-hand side of Eq. (2.4), search of the appropriate root was done by the dichotomy algorithm coupled with the secant line method at the interval where the negative product

$$P(\tilde{c}_j)P(\tilde{c}_{j+1}) < 0 \tag{5.1}$$

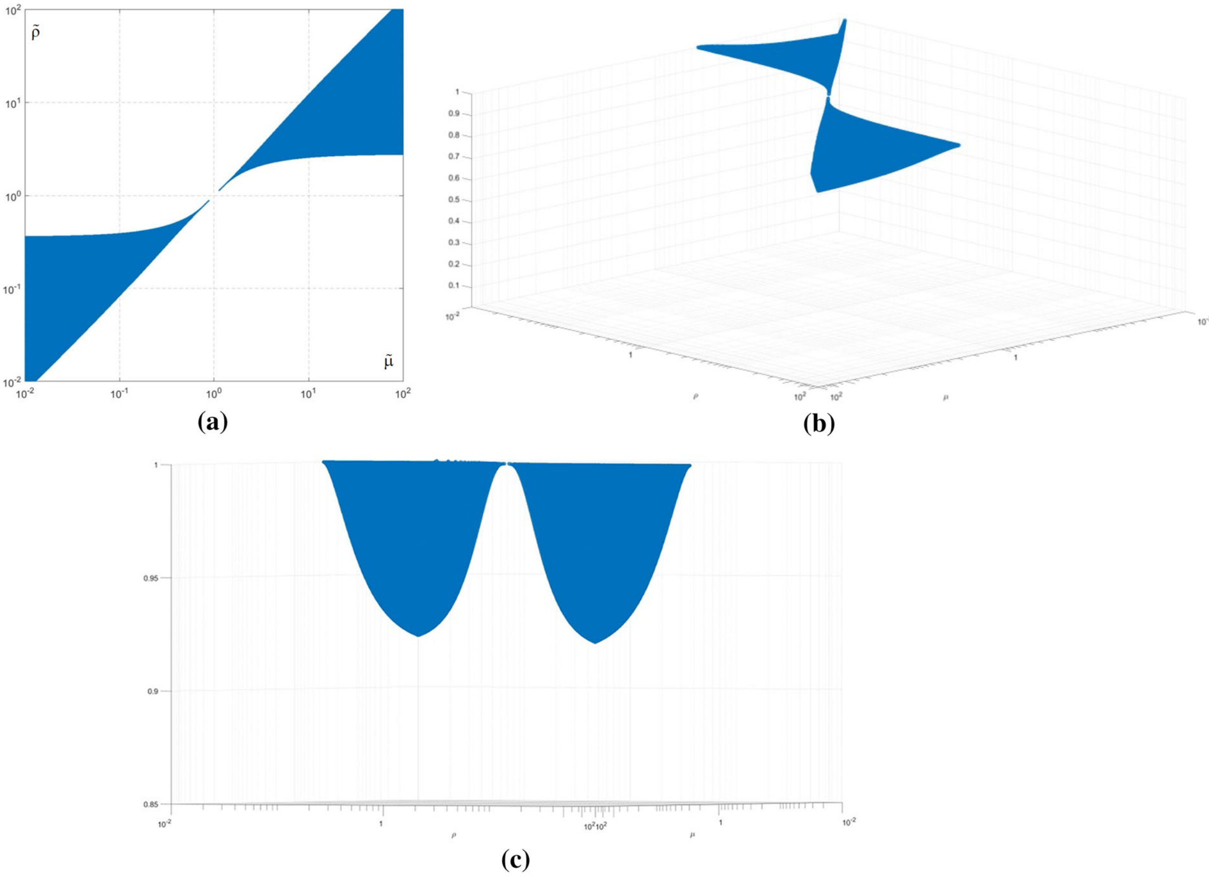

Fig. 2. Poisson's ratio 0.25; **a** region of existence; **b** 3D plot for velocity variation; **c** projection onto the Wiechert plane

at two adjacent values $\tilde{c}_j;\ \tilde{c}_{j+1}$ is achieved along with condition $ImP(\tilde{c}_j) = 0;\ \ ImP(\tilde{c}_{j+1}) = 0$. In view of condition (3.6), search of a root was done in the interval $(0;\ \min(\chi;\ 1))$; at the first dichotomy stage, the speed interval was divided into 10^3 subintervals; and 10–20 iterations at second stage of root refinement by the secant line method. With these parameters, accuracy of the root determination

$$|P(\tilde{c}_{\text{root}})| \leq \varepsilon \tag{5.2}$$

was about 10^{-20}. To minimize effect of possible round off errors, all computations were performed with long mantissas having more than 60 decimal digits [27,28].

Plots in Figs. 1, 2 and 3 show (a) regions of existence in a 2D space defined by those values of parameters $\tilde{\mu},\ \tilde{\rho}$ that ensure existence of Stoneley wave; (b) 3D plots of dimensionless Stoneley wave velocity variation $\tilde{c}$ (vertical axis) vs dimensionless parameters $\tilde{\mu},\ \tilde{\rho}$; and (c) projection of the corresponding 3D plots onto vertical plane containing vertical axis $\tilde{c}$ and *Wiechert line* [15]; the latter is defined as line $\tilde{\mu} = \tilde{\rho}$; see Figs. 1c, 2c and 3c. This vertical plane will be called *Wiechert plane.*

The plots correspond to three Poisson's values $\nu = 0;\ 0.25;\ 0.49$

The plots in Figs. 1a, 2a and 3a reveal that the regions of existence have almost identical shapes; large similarity can be observed at comparing shapes (not values) of the corresponding 3D velocity plots (Figs. 1b, 2b, 3b), while projections of the velocity plots onto the Wiechert plane are substantially different, see Figs. 1c, 2c and 3c.

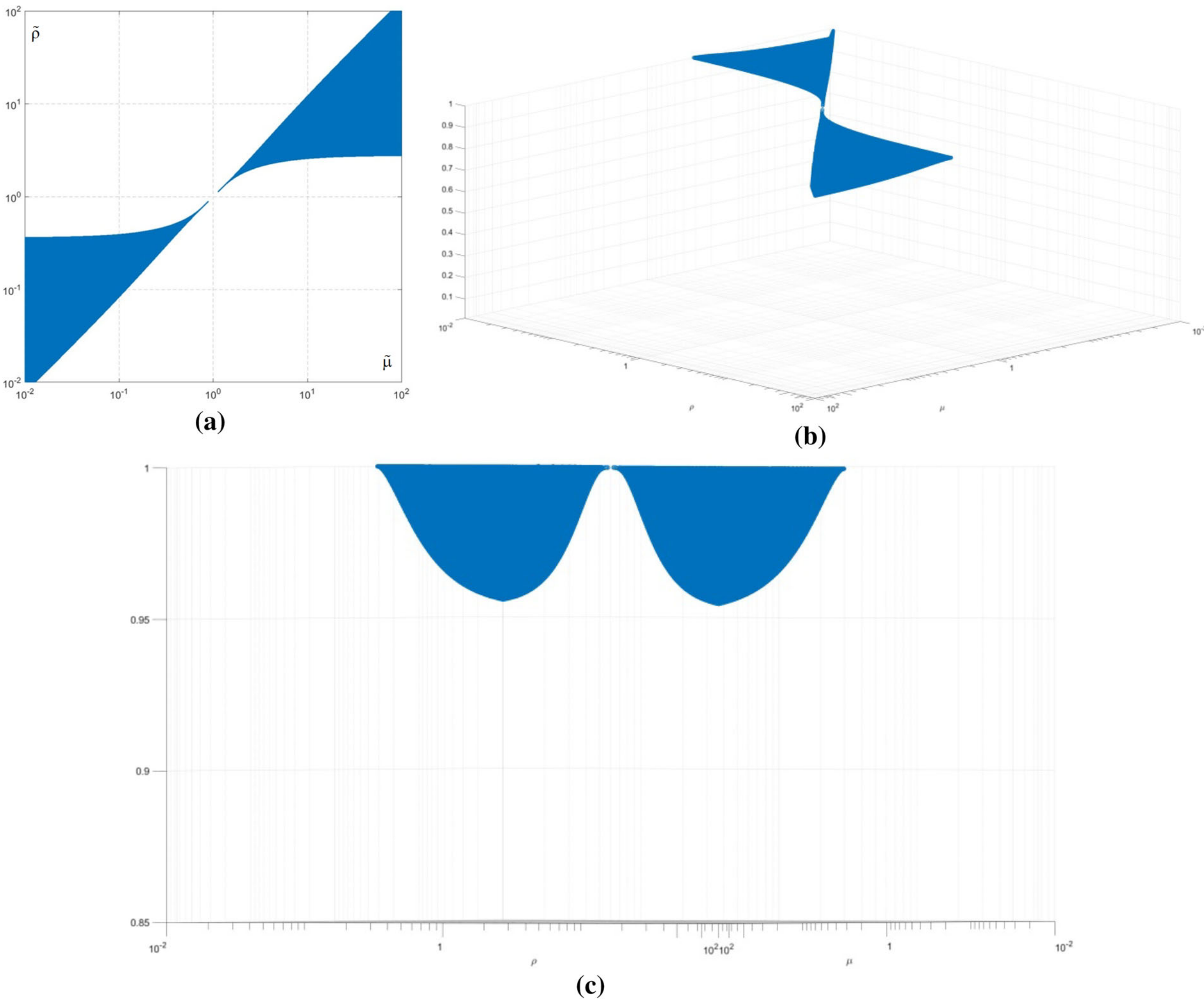

FIG. 3. Poisson's ratio 0.49 **a** region of existence; **b** 3D plot for velocity variation; **c** projection onto the Wiechert plane

6. Concluding remarks

A generalization of the Wiechert condition by introducing two independent dimensionless parameters $\tilde{\lambda} = \tilde{\mu} = q_1; \quad \tilde{\rho} = q_2$ instead of a single parameter $\tilde{\lambda} = \tilde{\mu} = \tilde{\rho} = q$ in the original Wiechert condition is proposed.

Variation of Stoneley wave velocity at varying two parameters of the generalized Wiechert condition at different Poisson's ratios reveals a substantial discrepancy in the Stoneley wave velocity profiles. While regions of existence (Figs. 1a, 2a, 3a) are visually almost undistinguishable, the Stoneley wave velocity profiles show a substantial discrepancy (Figs. 1c, 2c, 3c) in both shape and values.

Considering 3D plots showing variation of Stoneley wave velocity $\tilde{c}$ vs two varying parameters $\tilde{\lambda} = \tilde{\mu}$ and $\tilde{\rho}$ (Figs. 1b, 2b, 3b), it should be noted that their shapes are similar; however, velocity values are different, as plots in Figs. 1c, 2c and 3c show.

And the final remark concerns an assertion [6,7] that Poisson's ratio does not considerably affect Stoneley wave velocity. Such a conclusion was inspired by comparison of regions of existence at different

Poisson ratios. However, as our analysis reveals, Stoneley wave velocity heavily depends upon Poisson's ratio; see Figs. 1c, 2c and 3c.

Acknowledgements

The work was supported by the Russian Science Foundation Grant 20-11-20133.

References

[1] Wiechert, E., Geiger, L.: Bestimmung des Weges der Erdbebenwellen im Erdinnern. Phys. Zeit. **II**, 294 (1910)
[2] Wiechert, E., Zöppritz, K.: Our present knowledge of the Earth. In: Report of the Board of Regents of the Smithsonian Institution, 431 (1908)
[3] Stoneley, R.: Elastic waves at the surface of separation of two solids. Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **106**, 416 (1924)
[4] Sezawa, K.: Formation of boundary waves at the surface of a discontinuity within the Earth's crust. Bull. Earthq. Res. Inst. Tokyo Univ. **16**, 504 (1938)
[5] Sezawa, K., Kanai, K.: The range of possible existence of Stoneley waves, and some related problems. Bull. Earthq. Res. Inst. Tokyo Univ. **17**, 1 (1939)
[6] Scholte, J.G.: On the Stoneley wave equation. I. Proceedings/ Koninklijke Nederlandse Akademie van Wetenschappen **45**, 20 (1942)
[7] Scholte, J.G.: On the Stoneley wave equation II. Proceedings/Koninklijke Nederlandsche Akademie van Weten-schappen **45**, 159 (1942)
[8] Scholte, J.G.: The range of existence of Rayleigh and Stoneley waves. Geophys. J. Int. **5**, 120 (1947)
[9] Cagniard, L.: Reflexion et Refraction des Ondes Seismiques Progressive. Gauthier- Villard, Paris (1939)
[10] Ginzbarg, A.S., Strick, E.: Stoneley-wave velocities for a solid-solid interface. Bull. Seismol. Soc. Am. **48**(1), 51 (1958)
[11] Murty, G.S.: Wave propagation at an unbounded interface between two elastic half-spaces. J. Acous. Soc. Am. **58**, 1094 (1975)
[12] Murty, G.S.: A theoretical model for the attenuation and dispersion of Stoneley waves at the loosely bonded interface of elastic half spaces. Phys. Earth Planet. Inter. **11**, 65 (1975)
[13] Vinh, P.C., Giang, P.T.H.: On formulas for the velocity of Stoneley waves propagating along the loosely bonded interface of two elastic half-spaces. Wave Motion **48**, 646 (2011)
[14] Vinh, P.C., Malischewsky, P.G., Giang, P.T.H.: Formulas for the speed and slowness of Stoneley waves in bonded isotropic elastic half-spaces with the same bulk wave velocities. Int. J. Eng. Sci. **60**, 53 (2012)
[15] Kuznetsov, S.V.: Stoneley waves at the Wiechert condition. Z. Angew. Math. Phys. **71**, 114 (2020)
[16] Ilyashenko, A.V.: Stoneley waves in a vicinity of the Wiechert condition. Int. J. Dyn. Control (2020). https://doi.org/10.1007/s40435-020-00625-y
[17] Bostron, J.H., Rose, J.L., Moose, C.A.: Ultrasonic guided interface waves at a soft-stiff boundary. J. Acoust. Soc. Am. **134**, 4351 (2013)
[18] Bing, L., Ming-hang, L., Tong, L.: Interface waves in multilayered plates. J Acoust. Soc. Am. **143**, 2541 (2018)
[19] Kuznetsov, S.V.: Abnormal dispersion of Lamb waves in stratified media. Z. Angew. Math. Phys. **70**, 175 (2019)
[20] Kaplunov, J., Prikazchikov, D.: Asymptotic theory for Rayleigh and Rayleigh-type waves. Adv. Appl. Mech. **50**, 1–106 (2017)
[21] Wootton, P.T., Kaplunov, J., Prikazchikov, D.: A second-order asymptotic model for Rayleigh waves on a linearly elastic half plane. IMA J. Appl. Math. **85**(1), 113 (2020)
[22] Lim, T.C., Musgrave, M.J.P.: Stoneley waves in anisotropic media. Nature **225**, 372 (1970)
[23] Barnett, D.M., Lothe, J., Gavazza, S.D., Musgrave, M.J.P.: Consideration of the existence of interfacial (Stoneley) waves in bonded anisotropic elastic half-spaces. Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **412**, 153 (1985)
[24] Chadwick, P., Borejko, P.: Existence and uniqueness of Stoneley waves. Geophys. J. Int. **118**, 279 (1994)
[25] Kuznetsov, S.V.: Cauchy formalism for Lamb waves in functionally graded plates. J. Vib. Control **25**, 1227 (2018)
[26] Kuznetsov, S.V.: Fundamental and singular solutions of Lame equations for media with arbitrary elastic anisotropy. Q. Appl. Math. **63**(3), 455 (2005)

[27] Berger, C.F., et al.: An automated implementation of on-shell methods for one-loop amplitudes. Phys. Rev. Ser. D **78**, 036003 (2008)
[28] Bailey, H., Barrio, R., Borwein, J.M.: High precision computation, mathematical physics and dynamics. Appl. Math. Comput. **218**, 10106–10121 (2012)

S. V. Kuznetsov
Ishlinsky Institute for Problems in Mechanics
RAS
Moscow
Russia

S. V. Kuznetsov
Moscow State University of Civil Engineering
Moscow
Russia
e-mail: kuzn-sergey@yandex.ru

(Received: July 8, 2020; revised: September 24, 2020; accepted: September 27, 2020)