Z. Angew. Math. Phys. (2020) 71:157

0044-2275/20/050001-18
published online August 30, 2020
https://doi.org/10.1007/s00033-020-01388-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

Laplace stretch: Eulerian and Lagrangian formulations

Alan D. Freed, Shahla Zamani, László Szabó and John D. Clayton

Abstract. Two triangular factorizations of the deformation gradient tensor are studied. The first, termed the Lagrangian formulation, consists of an upper-triangular stretch premultiplied by a rotation tensor. The second, termed the Eulerian formulation, consists of a lower-triangular stretch postmultiplied by a different rotation tensor. The corresponding stretch tensors are denoted as the Lagrangian and Eulerian Laplace stretches, respectively. Kinematics (with physical interpretations) and work-conjugate stress measures are analyzed and compared for each formulation. While the Lagrangian formulation has been used in prior work for constitutive modeling of anisotropic and hyperelastic materials, the Eulerian formulation, which may be advantageous for modeling isotropic solids and fluids with no physically identifiable reference configuration, does not seem to have been used elsewhere in a continuum mechanical setting for the purpose of constitutive development, though it has been introduced before in a purely kinematic setting.

Mathematics Subject Classification. Primary 74A05; Secondary 15A90.

Keywords. Continuum mechanics, Kinematics, Finite strain, Gram–Schmidt factorization.

1. Introduction

Lagrangian formulations (i.e., constitutive models based on Lagrangian measures of strain) are typically preferred for modeling anisotropic solids, as well as certain isotropic solids that have a clearly defined initial, stress-free, or 'reference' state. This is readily apparent for single crystals, e.g., where a reference state is identified with a regular lattice geometry occupied by atoms in their minimum energy (ground) state. Hyperelasticity [1,2] is typically invoked in this context, where energy potentials that depend upon a Lagrangian strain are employed. Eulerian formulations (i.e., constitutive models based on Eulerian measures of strain), in contrast, are often preferred for modeling isotropic solids (and fluids) that have no obvious initial or reference state. Hypoelasticity [3,4] is often invoked in this context for the purpose of solving initial-boundary value problems numerically.

Motivation for this study is a continued need to develop constitutive models for biological tissues that can be understood and used by those who work in the medical profession. In vivo, soft tissues are under tension perpetually, and a stress-free reference state is never physically realized. In such cases, it becomes advantageous to choose a 'reference' state with clinical relevance, e.g., at max systole for cardiac analyses, or at total lung capacity for pulmonary analyses, etc. Consequently, an Eulerian formulation would be optimal in such cases. The intent of this paper is to create a theoretical framework suitable for such constitutive developments. It is not the intent of this paper to create said models nor to apply them. With regard to creating a framework capable of producing constitutive equations that can be understood by those working in the medical profession, we choose to extend the conjugate pair approach that comes from a Lagrangian decomposition of the deformation gradient whose stretch is triangular, seeking an analogous construction that will now be based upon an Eulerian decomposition of the deformation gradient that is triangular, too.

Birkhäuser

Deformation gradient $\mathbf{F}$ admits four different triangular decompositions: upper- and lower-triangular decompositions can be derived in terms of both the Lagrangian and Eulerian, Cauchy–Green, deformation tensors $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathsf{T}}$, respectively. In each case, the full deformation gradient tensor is decomposed into a product between an orthogonal tensor and a triangular stretch tensor. Restricting analysis to those deformation gradients with a positive determinant, each orthogonal tensor is a rotation, and each corresponding stretch, either upper- or lower-triangular, is unique for its corresponding rotation. Two of these four triangular decompositions are used here. They are selected so as to have physical attributes with like interpretations, but with different values. The first decomposition considered here splits deformation gradient $\mathbf{F}$ into an upper-triangular stretch $\boldsymbol{\mathcal{U}}$ followed (i.e., premultiplied) by a rotation tensor $\boldsymbol{\mathcal{R}}^L$ such that $\mathbf{F} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}}$. This kinematic construction is referred to here as the Lagrangian formulation of triangular decomposition, also known as a Gram–Schmidt factorization. The second decomposition studied here splits deformation gradient $\mathbf{F}$ into a rotation tensor $\boldsymbol{\mathcal{R}}^E$ followed (premultiplied) by a lower-triangular stretch tensor $\boldsymbol{\mathcal{V}}$ such that $\mathbf{F} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$. This construction is referred to as the Eulerian formulation of triangular decomposition. Consequently, $\mathbf{F} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$ where, in general, $\boldsymbol{\mathcal{R}}^L \neq \boldsymbol{\mathcal{R}}^E$.

The upper-triangular Lagrangian decomposition of $\mathbf{F}$ was first introduced in the context of continuum mechanics by McClellan [1,5] in 1976. Souchet [6] introduced the lower-triangular Lagrangian decomposition of $\mathbf{F}$ in 1993. In 2012, Srinivasa [7] used a Cholesky decomposition of the right Cauchy–Green tensor $\mathbf{C}$ to obtain components for the upper-triangular stretch tensor and found this stretch tensor to be very appealing for modeling anisotropic hyperelastic materials. Since then, the upper-triangular decomposition of $\mathbf{F}$ has found preference over its lower-triangular counterpart. Recent applications of this Lagrangian decomposition address: shape memory polymers [8], anisotropy [9] and composites [10], finite elasticity [7,9,11], biological membranes [12], soft biological tissues including viscoelastic and damage effects [13], and inelastic materials [8,14]. Even the conditions for compatibility have been established [15].

Both upper- and lower-triangular, Eulerian decompositions of the deformation gradient $\mathbf{F}$ were introduced into the literature by Boulanger and Hayes [16] in 2006 as 2D examples of their general 3D theory that they call *extended polar decomposition* of $\mathbf{F}$, of which there are an infinite number [17]. To the best of our knowledge, these 2D versions are the only appearance of an Eulerian triangular decomposition of $\mathbf{F}$ to be found in the literature. Their 3D theory was further generalized by Jarić et al. [18] through the introduction of projection operators. In Boulanger and Hayes' original paper [17], the authors studied unsheared triads and showed that the classic polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ (with rotation $\mathbf{R}$ being orthogonal, and with Lagrangian $\mathbf{U}$ and Eulerian $\mathbf{V}$ stretches being symmetric positive definite) has unsheared triads that associate with $\mathbf{U}$ before deformation and with $\mathbf{V}$ after deformation, and that these triads are orthonormal. They then introduced a general decomposition of $\mathbf{F}$ that obeys $\mathbf{F} = \mathbf{Q}\mathbf{G} = \mathbf{H}\mathbf{Q}$, where $\mathbf{Q}$ is an orthogonal rotation with $\mathbf{G}$ and $\mathbf{H}$ being non-symmetric descriptions for stretch. They proved that stretches $\mathbf{G}$ and $\mathbf{H}$ associate with unsheared triads that are oblique, of which there are an infinite number. It is worth pointing out that, in general, if $\mathbf{G}$ is triangular, then $\mathbf{H}$ is not, and vice versa. An exception is given Eqs. (10.13 & 10.14) of Ref. [16]: under simple shear, both $\mathbf{G}$ and $\mathbf{H}$ are lower triangular. Consequently, to map between Eulerian and Lagrangian triangular stretches requires knowing both the Eulerian and Lagrangian rotations, as determined herein. Also of interest, Freed and Zamani [19] took an orthonormal triad that deforms into an oblique triad and used this information to construct a convected metric for the deformation.

Advantages and drawbacks, when using an upper-triangular Lagrangian decomposition of the deformation gradient in constitutive construction, are discussed in the many papers by the authors cited here. Notably, triangular decompositions, unlike the classic and extended polar decompositions, require no eigenvector analysis to invoke. This is a consequence of the ability to apply Laplace's technique of successive orthogonal projections. Also, the set of all upper-triangular matrices with positive diagonal elements forms a group under multiplication [1], and therefore, the product of two upper-triangular matrices is an upper-triangular matrix. A like statement applies to the set of all lower-triangular matrices with positive diagonal elements, which is distinct from the group of upper-triangular matrices. However, closure under multiplication is not preserved for symmetric matrices, and as such, they do not constitute a group. Another useful feature resulting from triangular decompositions is that they lead to sets of six, independent, stress–strain, conjugate pairs that are scalar valued, whereas symmetric decompositions lead to, at best, scalar and deviatoric conjugate pairs. In addition, our conjugate pairs, which result from triangular decompositions, allow for a decoupling between simple and pure shears. This is not permitted when using symmetric deconstructions, and which is essential when modeling some materials, e.g., soft biological tissues. Furthermore, the components of triangular stretch have an obvious physical interpretation that facilitates direct and unambiguous parameterization of constitutive response data. Actually, there are 24, possible, physical interpretations of a triangular stretch in 3D [20], of which we have found two to be most useful, adopting one here.

This paper is organized as follows. Section 2 establishes components for the Eulerian and Lagrangian deformations $\mathbf{B}$ and $\mathbf{C}$ in terms of inner products between the row and column vectors of a deformation gradient $\mathbf{F}$. This is done in a manner that is useful for constructing the Eulerian and Lagrangian Laplace (triangular) stretches and their associated rotations. Section 3 introduces these Laplace stretches, whose components are quantified via Cholesky decompositions, with their associated rotations being determined via Gram factorizations. Section 4 assigns physical interpretations to the components of these Laplace stretches. A lower-triangular Eulerian stretch is chosen (over its upper-triangular counterpart) because its physical attributes are analogous to those of the upper-triangular Lagrangian stretch; however, their values are different—they are quantified in different coordinate systems. Section 5 provides three example problems to illustrate these kinematics. The 3D Eulerian kinematics presented in Sects. 3–5 are new to the literature. Section 6 presents one admissible set of conjugate pairs that the authors have found useful. Bijective maps between these physical attributes and components from their associated tensor fields are provided for both the Eulerian and Lagrangian fields, with the Eulerian results being new to the literature. It is in terms of these scalar-valued physical attributes for stress and strain (Eulerian or Lagrangian) that constitutive equations can be derived, thereby completing our theoretical framework for constitutive development. The fact that these attributes are scalar fields instead of tensor fields, and that they have unique physical interpretations, measurable in experiments, goes a long way toward making this constitutive development framework user-friendly for those in the medical community.

2. Deformation

Consider a body $\mathcal{B}$ embedded in a three-dimensional, Euclidean, point space oriented against a triad of orthogonal, unit, base vectors $(\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}})$. The motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ of some particle $\mathcal{P}$ located in $\mathcal{B}$ describes a homeomorphism that takes its original location $\boldsymbol{X} = X_1\vec{\boldsymbol{\imath}} + X_2\vec{\boldsymbol{\jmath}} + X_3\vec{\boldsymbol{k}}$ belonging to the body's reference configuration κ_r and places it into another location $\boldsymbol{x} = x_1\vec{\boldsymbol{\imath}} + x_2\vec{\boldsymbol{\jmath}} + x_3\vec{\boldsymbol{k}}$ where $\mathcal{P}$ resides in the body's current configuration κ_t.

For convenience, we write these two position vectors as $\boldsymbol{X} = X_i\,\vec{\mathsf{e}}_i$ and $\boldsymbol{x} = x_i\,\vec{\mathsf{e}}_i$ by selecting an indexing strategy, e.g., $(\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}}) \mapsto (\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$, to ensure that the 1 material direction and the 12 material surface embed with the motion, as they are invariant under transformations of Laplace stretch [1]. How to select an appropriate indexing strategy is the topic of Ref. [21]. This selection technique has been applied to our example problems.

A deformation gradient $\mathbf{F}$ maps the set of all tangent vectors located at particle $\mathcal{P}$ in body $\mathcal{B}$ from its reference configuration κ_r into the current configuration κ_t. We assume that a body is simply connected and its motion $\boldsymbol{\chi}$ is sufficiently differentiable so that $\mathbf{F} = \partial\boldsymbol{\chi}(\boldsymbol{X}, t)/\partial\boldsymbol{X}$ exists, and therefore

$$F_{ij} = \frac{\partial\chi_i(\boldsymbol{X}, t)}{\partial X_j} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^r_1 \\ \boldsymbol{f}^r_2 \\ \boldsymbol{f}^r_3 \end{bmatrix} = \big[\, \boldsymbol{f}^c_1 \big| \boldsymbol{f}^c_2 \big| \boldsymbol{f}^c_3 \,\big] \tag{1}$$

where vectors $\boldsymbol{f}^r_i = F_{ij}\, \vec{\mathsf{e}}_j$ contain the rows of tensor $\mathbf{F} = F_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$, while vectors $\boldsymbol{f}^c_i = F_{ji}\, \vec{\mathsf{e}}_j$ contain its columns, $i = 1, 2, 3$, with repeated indices being summed according to Einstein's summation convention.

It follows straightaway that the right, Cauchy–Green, deformation tensor $\mathbf{C} := \mathbf{F}^{\mathsf{T}}\mathbf{F} = C_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$, which is a Lagrangian description of deformation, has components of

$$C_{ij} = \begin{bmatrix} \boldsymbol{f}^c_1 \cdot \boldsymbol{f}^c_1 & \boldsymbol{f}^c_1 \cdot \boldsymbol{f}^c_2 & \boldsymbol{f}^c_1 \cdot \boldsymbol{f}^c_3 \\ \boldsymbol{f}^c_2 \cdot \boldsymbol{f}^c_1 & \boldsymbol{f}^c_2 \cdot \boldsymbol{f}^c_2 & \boldsymbol{f}^c_2 \cdot \boldsymbol{f}^c_3 \\ \boldsymbol{f}^c_3 \cdot \boldsymbol{f}^c_1 & \boldsymbol{f}^c_3 \cdot \boldsymbol{f}^c_2 & \boldsymbol{f}^c_3 \cdot \boldsymbol{f}^c_3 \end{bmatrix} \tag{1.2a}$$

while the left, Cauchy–Green, deformation tensor $\mathbf{B} := \mathbf{F}\mathbf{F}^{\mathsf{T}} = B_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$, which is an Eulerian description of deformation, has components of

$$B_{ij} = \begin{bmatrix} \boldsymbol{f}^r_1 \cdot \boldsymbol{f}^r_1 & \boldsymbol{f}^r_1 \cdot \boldsymbol{f}^r_2 & \boldsymbol{f}^r_1 \cdot \boldsymbol{f}^r_3 \\ \boldsymbol{f}^r_2 \cdot \boldsymbol{f}^r_1 & \boldsymbol{f}^r_2 \cdot \boldsymbol{f}^r_2 & \boldsymbol{f}^r_2 \cdot \boldsymbol{f}^r_3 \\ \boldsymbol{f}^r_3 \cdot \boldsymbol{f}^r_1 & \boldsymbol{f}^r_3 \cdot \boldsymbol{f}^r_2 & \boldsymbol{f}^r_3 \cdot \boldsymbol{f}^r_3 \end{bmatrix} \tag{1.2b}$$

both of which are symmetric because, for example, $\boldsymbol{f}^r_1 \cdot \boldsymbol{f}^r_2 = \boldsymbol{f}^r_2 \cdot \boldsymbol{f}^r_1$ where $\boldsymbol{f}^r_1 \cdot \boldsymbol{f}^r_2 = F_{1i}F_{2i} = F_{11}F_{21} + F_{12}F_{22} + F_{13}F_{23}$, etc.

3. Laplace stretch

Laplace stretch, as it has been used in the literature to date, e.g., [1,5,7,8,10–14,19,21,22], derives from a Gram–Schmidt (or $\mathbf{QR}$) decomposition of the deformation gradient $\mathbf{F}$, where matrix $\mathbf{Q}$ is orthogonal, and matrix $\mathbf{R}$ is upper triangular.

Given a coordinate system with base vectors $(\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$, we denote such a decomposition as $\mathbf{F} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}$, where $\boldsymbol{\mathcal{R}} = \mathcal{R}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ has orthogonal components, and $\boldsymbol{\mathcal{U}} = \mathcal{U}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ has upper-triangular components. We select this calligraphic notation to illustrate its similarities and differences with the common polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{R} = R_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ has orthogonal components, and $\mathbf{U} = U_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ has symmetric components. Lagrangian fields $\mathbf{U}$ and $\boldsymbol{\mathcal{U}}$ are distinct measures for stretch.

A polar decomposition of the deformation gradient, i.e., $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, produces a Lagrangian measure for stretch (the right-stretch tensor $\mathbf{U}$) and an Eulerian measure for stretch (the left-stretch tensor $\mathbf{V}$) that share in a common, orthogonal, rotation tensor $\mathbf{R}$. An extended polar decomposition [17], viz., $\mathbf{F} = \mathbf{Q}\mathbf{G} = \mathbf{H}\mathbf{Q}$, produces non-symmetric, Lagrangian and Eulerian, measures for stretch $\mathbf{G}$ and $\mathbf{H}$, respectively, that also share in a common, orthogonal, rotation tensor $\mathbf{Q}$, different from $\mathbf{R}$. An objective of this document is to develop an Eulerian measure for stretch whose components populate a triangular matrix such that $\mathbf{F} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$, where $\boldsymbol{\mathcal{U}}$ is the Lagrangian Laplace stretch, and where $\boldsymbol{\mathcal{V}}$ is the Eulerian Laplace stretch, both with triangular elements. In contrast with the polar rotation $\mathbf{R}$ and the extended polar rotation $\mathbf{Q}$, the Lagrangian $\boldsymbol{\mathcal{R}}^L$ and Eulerian $\boldsymbol{\mathcal{R}}^E$ Gram rotations are distinct rotations. The Laplace stretches therefore relate via $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{R}}^{L^{\mathsf{T}}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$ and $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{R}}^{E^{\mathsf{T}}}$.

3.1. Lagrangian Laplace stretch

Here we describe a Gram–Schmidt factorization of the deformation gradient, i.e., $\mathbf{F} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}}$, wherein $\boldsymbol{\mathcal{U}} = \mathcal{U}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ is called the Lagrangian Laplace stretch or the right Laplace stretch.

Srinivasa [7] applied a Cholesky decomposition to the symmetric, positive-definite, right, Cauchy–Green, deformation tensor $\mathbf{C}$ to establish the components of his stretch tensor, denoted here as $\boldsymbol{\mathcal{U}} =$

$\mathcal{U}_{ij}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$; in particular,[1]

$$\begin{aligned}
&\mathcal{U}_{11} = \sqrt{C_{11}} \qquad \mathcal{U}_{12} = C_{12}/\mathcal{U}_{11} \qquad \mathcal{U}_{13} = C_{13}/\mathcal{U}_{11}\\
&\mathcal{U}_{21} = 0 \qquad \mathcal{U}_{22} = \sqrt{C_{22} - \mathcal{U}_{12}^2} \qquad \mathcal{U}_{23} = \big(C_{23} - \mathcal{U}_{12}\mathcal{U}_{13}\big)/\mathcal{U}_{22}\\
&\mathcal{U}_{31} = 0 \qquad \mathcal{U}_{32} = 0 \qquad \mathcal{U}_{33} = \sqrt{C_{33} - \mathcal{U}_{13}^2 - \mathcal{U}_{23}^2}
\end{aligned} \tag{2}$$

where components of the Lagrangian Laplace stretch $\mathcal{U}_{ij}$ are upper triangular. Its inverse $\boldsymbol{\mathcal{U}}^{-1} = \mathcal{U}_{ij}^{-1}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ follows straightaway, having components that are also upper-triangular, they being

$$\mathcal{U}_{ij}^{-1} = \begin{bmatrix} 1/\mathcal{U}_{11} & -\mathcal{U}_{12}/\mathcal{U}_{11}\mathcal{U}_{22} & (\mathcal{U}_{12}\mathcal{U}_{23} - \mathcal{U}_{13}\mathcal{U}_{22})/\mathcal{U}_{11}\mathcal{U}_{22}\mathcal{U}_{33} \\ 0 & 1/\mathcal{U}_{22} & -\mathcal{U}_{23}/\mathcal{U}_{22}\mathcal{U}_{33} \\ 0 & 0 & 1/\mathcal{U}_{33} \end{bmatrix} \tag{3}$$

thereby requiring each $\mathcal{U}_{ii}$, no sum on i, to be positive—a condition satisfied because of mass conservation. It is easily shown that the Lagrangian Laplace stretch $\mathcal{U}_{ij}$ belongs to a group under the operation of matrix multiplication. This group is comprised of all real, 3×3, upper-triangular matrices with positive diagonal elements [1]. Having a stretch tensor with this property has proven to be useful in applications, e.g., [1,14], as it does here.

A Gram factorization of the deformation gradient $\mathbf{F} = F_{ij}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ produces a Lagrangian rotation tensor $\boldsymbol{\mathcal{R}}^L = \delta_{ij}\,\vec{\mathsf{e}}_i^L \otimes \vec{\mathsf{e}}_j = \mathcal{R}_{ij}^L\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ described by

$$\mathcal{R}_{ij}^L = \big[\, \vec{\mathsf{e}}_1^L \big| \vec{\mathsf{e}}_2^L \big| \vec{\mathsf{e}}_3^L \,\big] \tag{4a}$$

whose columns constitute unit base vectors that can be constructed via

$$\vec{\mathsf{e}}_1^L := \frac{\boldsymbol{f}_1^c}{\|\boldsymbol{f}_1^c\|} \tag{4b}$$

$$\vec{\mathsf{e}}_2^L := \frac{\boldsymbol{f}_2^c - (\boldsymbol{f}_2^c \cdot \vec{\mathsf{e}}_1^L)\vec{\mathsf{e}}_1^L}{\|\boldsymbol{f}_2^c - (\boldsymbol{f}_2^c \cdot \vec{\mathsf{e}}_1^L)\vec{\mathsf{e}}_1^L\|} \tag{4c}$$

$$\vec{\mathsf{e}}_3^L := \frac{\boldsymbol{f}_3^c - (\boldsymbol{f}_3^c \cdot \vec{\mathsf{e}}_1^L)\vec{\mathsf{e}}_1^L - (\boldsymbol{f}_3^c \cdot \vec{\mathsf{e}}_2^L)\vec{\mathsf{e}}_2^L}{\|\boldsymbol{f}_3^c - (\boldsymbol{f}_3^c \cdot \vec{\mathsf{e}}_1^L)\vec{\mathsf{e}}_1^L - (\boldsymbol{f}_3^c \cdot \vec{\mathsf{e}}_2^L)\vec{\mathsf{e}}_2^L\|} \tag{4d}$$

wherein Laplace's technique of removing successive orthogonal projections [23] is apparent, with norm $\|\boldsymbol{f}_1^c\| := \sqrt{\boldsymbol{f}_1^c \cdot \boldsymbol{f}_1^c}$, etc. It therefore follows that the Lagrangian Laplace stretch has components which can be expressed as

$$\mathcal{U}_{ij} = \begin{bmatrix} \vec{\mathsf{e}}_1^L \cdot \boldsymbol{f}_1^c & \vec{\mathsf{e}}_1^L \cdot \boldsymbol{f}_2^c & \vec{\mathsf{e}}_1^L \cdot \boldsymbol{f}_3^c \\ 0 & \vec{\mathsf{e}}_2^L \cdot \boldsymbol{f}_2^c & \vec{\mathsf{e}}_2^L \cdot \boldsymbol{f}_3^c \\ 0 & 0 & \vec{\mathsf{e}}_3^L \cdot \boldsymbol{f}_3^c \end{bmatrix} \tag{5}$$

that provide a means of geometric interpretation for this measure of stretch. The components $\mathcal{U}_{ij}$ of Lagrangian Laplace stretch $\boldsymbol{\mathcal{U}} = \mathcal{U}_{ij}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ evaluated in a reference frame $(\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$ are also projections of column vectors $\boldsymbol{f}_i^c$ extracted from a deformation gradient $\mathbf{F} = F_{ij}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ projected onto its Lagrangian coordinate axes $(\vec{\mathsf{e}}_1^L, \vec{\mathsf{e}}_2^L, \vec{\mathsf{e}}_3^L)$.

[1] Regarding Lagrangian stretches with triangular elements, McLellan [1,5] was the first to propose an upper-triangular decomposition of the deformation gradient. Later, Souchet [6] constructed a stretch tensor with lower-triangular components. We use Srinivasa's [7] approach for populating an upper-triangular stretch because, of these three Lagrangian approaches, this is the simplest framework to apply.

3.2. Eulerian Laplace stretch

Now we describe a Gram–Schmidt like factorization of the deformation gradient, viz., $\mathbf{F} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$, wherein $\boldsymbol{\mathcal{V}} = \mathcal{V}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ is called the Eulerian Laplace stretch, or the left Laplace stretch.

Applying a Cholesky factorization to the symmetric, positive-definite, left, Cauchy–Green, deformation tensor $\mathbf{B} := \mathbf{F}\mathbf{F}^{\mathsf{T}} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{V}}^{\mathsf{T}}$ with components $\mathbf{B} = B_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ allows one can construct a stretch tensor $\boldsymbol{\mathcal{V}} = \mathcal{V}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ whereby[2]

$$
\begin{aligned}
&\mathcal{V}_{11} = \sqrt{B_{11}} \qquad \mathcal{V}_{12} = 0 \qquad \mathcal{V}_{13} = 0 \\
&\mathcal{V}_{21} = B_{21}/\mathcal{V}_{11} \qquad \mathcal{V}_{22} = \sqrt{B_{22} - \mathcal{V}_{21}^2} \qquad \mathcal{V}_{23} = 0 \\
&\mathcal{V}_{31} = B_{31}/\mathcal{V}_{11} \qquad \mathcal{V}_{32} = \big(B_{32} - \mathcal{V}_{21}\mathcal{V}_{31}\big)/\mathcal{V}_{22} \qquad \mathcal{V}_{33} = \sqrt{B_{33} - \mathcal{V}_{31}^2 - \mathcal{V}_{32}^2}
\end{aligned}
\tag{6}
$$

where we now select the lower-triangular matrix from a Cholesky decomposition to quantify the components of our new stretch tensor. Its inverse $\boldsymbol{\mathcal{V}}^{-1} = \mathcal{V}_{ij}^{-1}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ follows straightaway, it having components of

$$
\mathcal{V}_{ij}^{-1} = \begin{bmatrix} 1/\mathcal{V}_{11} & 0 & 0 \\ -\mathcal{V}_{21}/\mathcal{V}_{11}\mathcal{V}_{22} & 1/\mathcal{V}_{22} & 0 \\ (\mathcal{V}_{32}\mathcal{V}_{21} - \mathcal{V}_{31}\mathcal{V}_{22})/\mathcal{V}_{11}\mathcal{V}_{22}\mathcal{V}_{33} & -\mathcal{V}_{32}/\mathcal{V}_{22}\mathcal{V}_{33} & 1/\mathcal{V}_{33} \end{bmatrix}
\tag{7}
$$

thereby requiring each $\mathcal{V}_{ii}$, no sum on i, to be positive—a condition satisfied because of mass conservation. It is easily shown that the Eulerian Laplace stretch $\mathcal{V}_{ij}$ belongs to a group under the operation of multiplication. This group is comprised of all real, 3×3, lower-triangular matrices with positive diagonal elements. The Eulerian and Lagrangian Laplace stretches belong to different mathematical groups.

A Gram-like[3] factorization of the deformation gradient $\mathbf{F} = F_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ can also describe an Eulerian rotation tensor $\boldsymbol{\mathcal{R}}^E = \delta_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j^E = \mathcal{R}_{ij}^E\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ constructed as

$$
\mathcal{R}_{ij}^E = \begin{bmatrix} \vec{\mathsf{e}}_1^E \\ \hline \vec{\mathsf{e}}_2^E \\ \hline \vec{\mathsf{e}}_3^E \end{bmatrix} = \left[\, \vec{\mathsf{e}}_1^E \,\middle|\, \vec{\mathsf{e}}_2^E \,\middle|\, \vec{\mathsf{e}}_3^E \,\right]^{\mathsf{T}}
\tag{8a}
$$

whose rows constitute unit base vectors that can be constructed via

$$
\vec{\mathsf{e}}_1^E := \frac{\boldsymbol{f}_1^r}{\|\boldsymbol{f}_1^r\|}
\tag{8b}
$$

$$
\vec{\mathsf{e}}_2^E := \frac{\boldsymbol{f}_2^r - (\boldsymbol{f}_2^r \cdot \vec{\mathsf{e}}_1^E)\vec{\mathsf{e}}_1^E}{\|\boldsymbol{f}_2^r - (\boldsymbol{f}_2^r \cdot \vec{\mathsf{e}}_1^E)\vec{\mathsf{e}}_1^E\|}
\tag{8c}
$$

$$
\vec{\mathsf{e}}_3^E := \frac{\boldsymbol{f}_3^r - (\boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_1^E)\vec{\mathsf{e}}_1^E - (\boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_2^E)\vec{\mathsf{e}}_2^E}{\|\boldsymbol{f}_3^r - (\boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_1^E)\vec{\mathsf{e}}_1^E - (\boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_2^E)\vec{\mathsf{e}}_2^E\|}
\tag{8d}
$$

where, again, Laplace's solution strategy of removing successive orthogonal projections [23] is apparent. It follows that the Eulerian Laplace stretch has components which can be expressed as

$$
\mathcal{V}_{ij} = \begin{bmatrix} \boldsymbol{f}_1^r \cdot \vec{\mathsf{e}}_1^E & 0 & 0 \\ \boldsymbol{f}_2^r \cdot \vec{\mathsf{e}}_1^E & \boldsymbol{f}_2^r \cdot \vec{\mathsf{e}}_2^E & 0 \\ \boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_1^E & \boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_2^E & \boldsymbol{f}_3^r \cdot \vec{\mathsf{e}}_3^E \end{bmatrix}
\tag{9}
$$

that provide a means of geometric interpretation for this measure of stretch. The components $\mathcal{V}_{ij}$ of Eulerian Laplace stretch $\boldsymbol{\mathcal{V}} = \mathcal{V}_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ evaluated in a reference frame $(\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$ are also projections

[2] The upper-left 2×2 matrix in Eq. (6) is equivalent to Eq. (9.3) in Boulanger and Hayes [16].

[3] The Gram factorization of a square matrix results in an orthogonal matrix and an upper-triangular matrix. Here we apply the same strategy, but we secure a different orthogonal matrix and a lower-triangular matrix; hence, the terminology 'Gram like.'

of row vectors $\boldsymbol{f}^r_i$ extracted from a deformation gradient $\mathbf{F} = F_{ij}\,\vec{\mathrm{e}}_i \otimes \vec{\mathrm{e}}_j$ projected onto its Eulerian coordinate axes $(\vec{\mathrm{e}}^E_1, \vec{\mathrm{e}}^E_2, \vec{\mathrm{e}}^E_3)$.

Obviously, rotations $\boldsymbol{\mathcal{R}}^L$ and $\boldsymbol{\mathcal{R}}^E$ are distinct, as are stretches $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{V}}$, given that the deformation gradient $\mathbf{F}$ decomposes as $\mathbf{F} = \boldsymbol{\mathcal{R}}^L\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$, and whose stretch tensors have triangular components $\mathcal{U}_{ij}$ and $\mathcal{V}_{ij}$ in $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$.

4. Physical interpretation of Laplace stretch components

Each Laplace stretch has six, independent, physical attributes. There are three, orthogonal, elongation ratios a, b and c, and there are three, orthogonal, simple shears α, β and γ. Their Lagrangian interpretations are quantified in a coordinate system with base vectors $(\vec{\mathrm{e}}^L_1, \vec{\mathrm{e}}^L_2, \vec{\mathrm{e}}^L_3)$, and are distinguished with an underline, viz., $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$. Their Eulerian interpretations are quantified in a coordinate system with base vectors $(\vec{\mathrm{e}}^E_1, \vec{\mathrm{e}}^E_2, \vec{\mathrm{e}}^E_3)$, and are distinguished with an overline, viz., $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$. In general, Lagrangian stretch attributes are distinct from their Eulerian counterparts. However, their geometric interpretations are the same. They differ only in their coordinate systems through which they are evaluated.

4.1. Lagrangian stretch attributes

The Lagrangian Laplace stretch has geometric interpretations that arise from Eq. (5) whereby one can assign [22]

$$\mathcal{U}_{ij} = \begin{bmatrix} \underline{a} & \underline{a}\underline{\gamma} & \underline{a}\underline{\beta} \\ 0 & \underline{b} & \underline{b}\underline{\alpha} \\ 0 & 0 & \underline{c} \end{bmatrix} = \begin{bmatrix} \underline{a} & 0 & 0 \\ 0 & \underline{b} & 0 \\ 0 & 0 & \underline{c} \end{bmatrix} \begin{bmatrix} 1 & 0 & \underline{\beta} \\ 0 & 1 & \underline{\alpha} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \underline{\gamma} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10a}$$

with an inverse of

$$\mathcal{U}^{-1}_{ij} = \begin{bmatrix} 1/\underline{a} & -\underline{\gamma}/\underline{b} & -(\underline{\beta} - \underline{\alpha}\underline{\gamma})/\underline{c} \\ 0 & 1/\underline{b} & -\underline{\alpha}/\underline{c} \\ 0 & 0 & 1/\underline{c} \end{bmatrix} \tag{10b}$$

whose constituents are measured in a coordinate frame with base vectors [19]

$$\vec{\mathrm{e}}^L_1 = \boldsymbol{f}^c_1 \;/\; \underline{a} \tag{11a}$$

$$\vec{\mathrm{e}}^L_2 = \left(\boldsymbol{f}^c_2 - \underline{\gamma}\boldsymbol{f}^c_1\right) \;/\; \underline{b} \tag{11b}$$

$$\vec{\mathrm{e}}^L_3 = \left(\boldsymbol{f}^c_3 - \underline{\alpha}\boldsymbol{f}^c_2 - (\underline{\beta} - \underline{\alpha}\underline{\gamma})\boldsymbol{f}^c_1\right) \;/\; \underline{c} \tag{11c}$$

all of which are described in terms of physical attributes defined as

$$\underline{a} := \mathcal{U}_{11}, \quad \underline{b} := \mathcal{U}_{22}, \quad \underline{c} := \mathcal{U}_{33}, \quad \underline{\alpha} := \frac{\mathcal{U}_{23}}{\mathcal{U}_{22}}, \quad \underline{\beta} := \frac{\mathcal{U}_{13}}{\mathcal{U}_{11}}, \quad \underline{\gamma} := \frac{\mathcal{U}_{12}}{\mathcal{U}_{11}} \tag{12}$$

where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are elongations, while $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ are magnitudes of shear, i.e., they are the extents of shear at unit elongation. From conservation of mass, the three elongations must be positive ($\underline{a} \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}_+$, $\underline{c} \in \mathbb{R}_+$), while the three shears may be of either sign ($\underline{\alpha} \in \mathbb{R}$, $\underline{\beta} \in \mathbb{R}$, $\underline{\gamma} \in \mathbb{R}$). Equation (10a) represents one of twenty-four, admissible, physical interpretations that one can assign to the components of Laplace stretch [20]. It is the interpretation that we find most useful.

According to Eq. (10), the Lagrangian Laplace stretch arises from the following sequence of deformations: it starts with an in-plane shear $\underline{\gamma}$, followed by two out-of-plane shears $\underline{\alpha}$ and $\underline{\beta}$, and then finishes with three elongations $\underline{a}$, $\underline{b}$ and $\underline{c}$, as illustrated in Fig. 1. Two vectors remain invariant under mappings

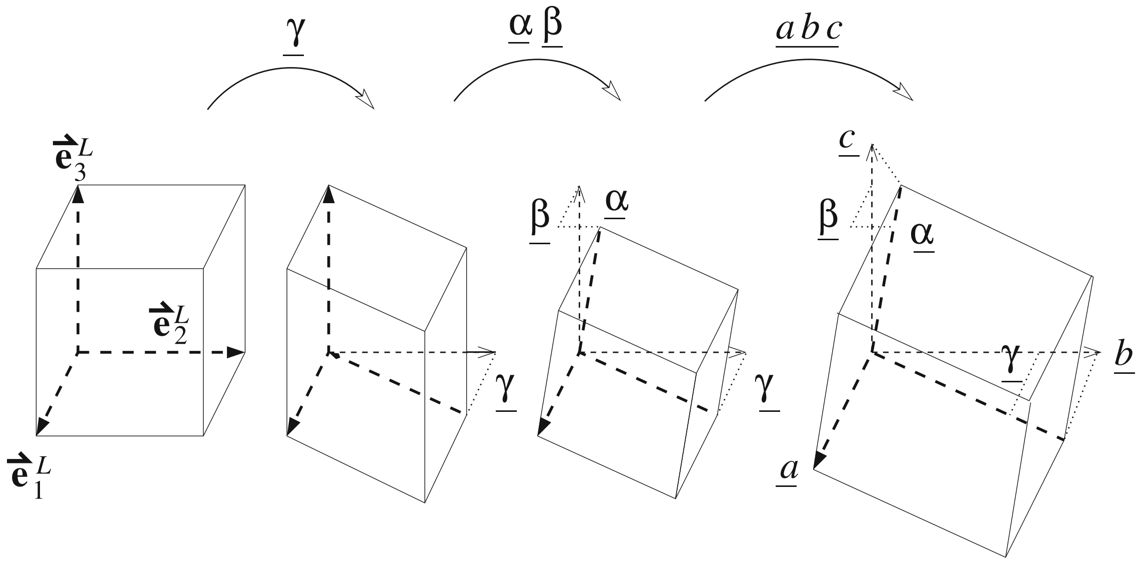

FIG. 1. A geometric interpretation for Lagrangian Laplace stretch

of the Lagrangian Laplace stretch; they are: vector $\vec{e}_1^L$ establishes the direction of in-plane shear, while vector $\vec{e}_1^L \times \vec{e}_2^L$ points normal to the plane of in-plane shear [1].

4.2. Eulerian stretch attributes

The Eulerian Laplace stretch has geometric interpretations that arise from Eq. (9) whereby one can assign

$$\mathcal{V}_{ij} = \begin{bmatrix} \overline{a} & 0 & 0 \\ \overline{a}\overline{\gamma} & \overline{b} & 0 \\ \overline{a}\overline{\beta} & \overline{b}\overline{\alpha} & \overline{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \overline{\gamma} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \overline{\beta} & \overline{\alpha} & 1 \end{bmatrix} \begin{bmatrix} \overline{a} & 0 & 0 \\ 0 & \overline{b} & 0 \\ 0 & 0 & \overline{c} \end{bmatrix} \tag{13a}$$

with an inverse of

$$\mathcal{V}_{ij}^{-1} = \begin{bmatrix} 1/\overline{a} & 0 & 0 \\ -\overline{\gamma}/\overline{b} & 1/\overline{b} & 0 \\ -(\overline{\beta} - \overline{\alpha}\overline{\gamma})/\overline{c} & -\overline{\alpha}/\overline{c} & 1/\overline{c} \end{bmatrix} \tag{13b}$$

whose constituents are measured in a coordinate frame with base vectors

$$\vec{e}_1^E = \boldsymbol{f}_1^r \,/\, \overline{a} \tag{14a}$$

$$\vec{e}_2^E = \left(\boldsymbol{f}_2^r - \overline{\gamma}\boldsymbol{f}_1^r\right) \,/\, \overline{b} \tag{14b}$$

$$\vec{e}_3^E = \left(\boldsymbol{f}_3^r - \overline{\alpha}\boldsymbol{f}_2^r - (\overline{\beta} - \overline{\alpha}\overline{\gamma})\boldsymbol{f}_1^r\right) \,/\, \overline{c} \tag{14c}$$

all of which are described in terms of physical attributes defined as

$$\overline{a} := \mathcal{V}_{11}, \quad \overline{b} := \mathcal{V}_{22}, \quad \overline{c} := \mathcal{V}_{33}, \quad \overline{\alpha} := \frac{\mathcal{V}_{32}}{\mathcal{V}_{22}}, \quad \overline{\beta} := \frac{\mathcal{V}_{31}}{\mathcal{V}_{11}}, \quad \overline{\gamma} := \frac{\mathcal{V}_{21}}{\mathcal{V}_{11}} \tag{15}$$

where $\overline{a}$, $\overline{b}$ and $\overline{c}$ are elongations, while $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are magnitudes of shear, i.e., they are the extents of shear at unit elongation. From conservation of mass, the three elongations must be positive ($\overline{a} \in \mathbb{R}_+$, $\overline{b} \in \mathbb{R}_+$, $\overline{c} \in \mathbb{R}_+$), while the three shears may be of either sign ($\overline{\alpha} \in \mathbb{R}$, $\overline{\beta} \in \mathbb{R}$, $\overline{\gamma} \in \mathbb{R}$).

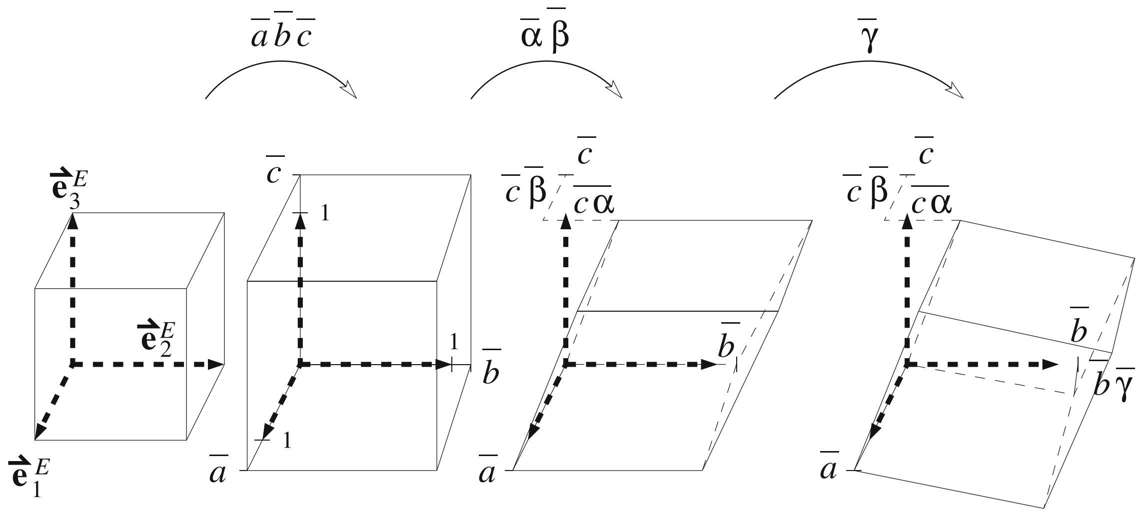

FIG. 2. A geometric interpretation for Eulerian Laplace stretch

According to Eq. (13), the Eulerian Laplace stretch arises from the following sequence of deformations: it starts with three elongations $\overline{a}$, $\overline{b}$ and $\overline{c}$, followed by two out-of-plane shears $\overline{\alpha}$ and $\overline{\beta}$, and then finishes with an in-plane shear $\overline{\gamma}$, as illustrated in Fig. 2. This sequence of deformations is the reverse of those occurring with the Lagrangian Laplace stretch, as one ought to expect. Two vectors remain invariant under mappings of the Eulerian Laplace stretch, too; they are: vector $\vec{\mathsf{e}}_1^E$ establishes the direction of in-plane shear, and vector $\vec{\mathsf{e}}_1^E \times \vec{\mathsf{e}}_2^E$ points normal to the plane of in-plane shear.

5. Examples

5.1. Shear-free deformations

Any motion $\boldsymbol{\chi}(\boldsymbol{X}, t)$ described by the following deformation gradient quantified in an orthonormal coordinate system with base vectors $(\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$ is said to be shear free; specifically,

$$F_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \therefore \qquad B_{ij} = C_{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{16}$$

where λ_1, λ_2 and λ_3 are the three principal stretches that, in this case, obey $\underline{a} = \overline{a} = \lambda_1$, $\underline{b} = \overline{b} = \lambda_2$ and $\underline{c} = \overline{c} = \lambda_3$. The Laplace stretch tensors and their Gram rotations have components of

$$\mathcal{U}_{ij} = \mathcal{V}_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{with} \quad \mathcal{R}^L_{ij} = \mathcal{R}^E_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

Consequently, there is no distinction between the triangular Laplace stretches $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{V}}$ and the symmetric polar stretches $\mathbf{U}$ and $\mathbf{V}$ for this class of motions. The elongations a, b and c of Laplace stretch equate with the eigenvalues λ_1, λ_2 and λ_3 of polar stretch. This relationship between elongations and principal stretches disappears in the presence of shear [24].

5.2. Pure shear

Any motion $\boldsymbol{\chi}(\boldsymbol{X}, t)$ described by the following deformation gradient quantified in an orthonormal coordinate system with base vectors $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$ is said to be a pure shear [22], specifically

$$F_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \lambda & \lambda \\ 0 & -\lambda^{-1} & \lambda^{-1} \end{bmatrix} \tag{18}$$

where λ is the stretch of pure shear. This motion is described by Cauchy–Green deformation tensors with components of

$$B_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix} \quad \text{and} \quad C_{ij} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \lambda^2 + \lambda^{-2} & \lambda^2 - \lambda^{-2} \\ 0 & \lambda^2 - \lambda^{-2} & \lambda^2 + \lambda^{-2} \end{bmatrix} \tag{19}$$

that produce a Lagrangian Laplace stretch and its Gram rotation of

$$\mathcal{U}_{ij} = \frac{1}{\sqrt{\frac{1}{2}(\lambda^2 + \lambda^{-2})}} \begin{bmatrix} \sqrt{\frac{1}{2}(\lambda^2 + \lambda^{-2})} & 0 & 0 \\ 0 & \frac{1}{2}(\lambda^2 + \lambda^{-2}) & \frac{1}{2}(\lambda^2 - \lambda^{-2}) \\ 0 & 0 & 1 \end{bmatrix} \tag{20a}$$

and

$$\mathcal{R}^L_{ij} = \frac{1}{\sqrt{\lambda^2 + \lambda^{-2}}} \begin{bmatrix} \sqrt{\lambda^2 + \lambda^{-2}} & 0 & 0 \\ 0 & \lambda & \lambda^{-1} \\ 0 & -\lambda^{-1} & \lambda \end{bmatrix} \tag{20b}$$

along with an Eulerian Laplace stretch and its Gram rotation of

$$\mathcal{V}_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \quad \text{and} \quad \mathcal{R}^E_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \tag{21}$$

where $\boldsymbol{\mathcal{R}}^E$ rotates the Eulerian coordinate frame $(\vec{\mathrm{e}}^E_1, \vec{\mathrm{e}}^E_2, \vec{\mathrm{e}}^E_3)$ about the background frame $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$ by a fixed 45° in the 23 plane, whereas $\boldsymbol{\mathcal{R}}^L$ rotates the Lagrangian coordinate frame $(\vec{\mathrm{e}}^L_1, \vec{\mathrm{e}}^L_2, \vec{\mathrm{e}}^L_3)$ from the Eulerian frame $(\vec{\mathrm{e}}^E_1, \vec{\mathrm{e}}^E_2, \vec{\mathrm{e}}^E_3)$ at $\lambda = 1$ toward the background frame $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$ as $\lambda \to \infty$.

The above components for Eulerian Laplace stretch $\mathcal{V}_{ij}$ support Lodge's statement that pure shear is not a shearing deformation; it is a shear-free deformation in disguise [25,26]. Lodge justifies this position by pointing out that the eigenvectors for stretch do not rotate in a body during pure shears like they do during simple shears.

Here the elongations relate as $\underline{a} = \overline{a} = 1$, while $\underline{b} = \sqrt{(\lambda^2 + \lambda^{-2})/2}$ and $\overline{b} = \lambda$ with $\underline{c} = 1/\sqrt{(\lambda^2 + \lambda^{-2})/2}$ and $\overline{c} = \lambda^{-1}$, whereas the shears relate as $\underline{\alpha} = (\lambda^2 - \lambda^{-2})/(\lambda^2 + \lambda^{-2})$ and $\overline{\alpha} = 0$ with $\underline{\beta} = \overline{\beta} = \underline{\gamma} = \overline{\gamma} = 0$.

5.3. Simple shear

Any motion $\boldsymbol{\chi}(\boldsymbol{X}, t)$ described by the following deformation gradient quantified in an orthonormal coordinate system with base vectors $(\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3)$ constitutes a shearing motion; specifically,

$$F_{ij} = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

whose Cauchy–Green deformation tensors have components of

$$B_{ij} = \begin{bmatrix} 1+\beta^2 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \quad \text{and} \quad C_{ij} = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1+\beta^2 \end{bmatrix} \tag{23}$$

with its Lagrangian Laplace stretch and rotation having components of

$$\mathcal{U}_{ij} = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{R}^L_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

along with its Eulerian Laplace stretch and rotation having components of

$$\mathcal{V}_{ij} = \begin{bmatrix} \sqrt{1+\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ \beta/\sqrt{1+\beta^2} & 0 & 1/\sqrt{1+\beta^2} \end{bmatrix} \tag{25a}$$

and

$$\mathcal{R}^E_{ij} = \begin{bmatrix} 1/\sqrt{1+\beta^2} & 0 & \beta/\sqrt{1+\beta^2} \\ 0 & 1 & 0 \\ -\beta/\sqrt{1+\beta^2} & 0 & 1/\sqrt{1+\beta^2} \end{bmatrix} \tag{25b}$$

with the Eulerian Laplace stretch $\mathcal{V}_{ij}$ having diagonal elements akin to those of pure shear (cf. Eq. 21), plus an off-diagonal simple shearing that is attenuated by the extent of pure shearing present.

From a rheometric viewpoint, making stress a function of the Eulerian Laplace stretch would enable first- and second-normal stress differences to occur, with the first exceeding the second in magnitude, and they being of opposite sign. A Weisenberg effect would occur, because of a compressive stretch that would set up in the hoop direction. Furthermore, the shear stress would thin, because of an effect that $\sqrt{1+\gamma^2}$ would have on the shear strain $\gamma/\sqrt{1+\gamma^2}$. All of these 'rheometric effects' occur in polymeric liquids [27].

Here the elongations relate as $\underline{a} = 1$ and $\overline{a} = \sqrt{1+\beta^2}$, while $\underline{b} = \overline{b} = 1$ with $\underline{c} = 1$ and $\overline{c} = 1/\sqrt{1+\beta^2}$, whereas the shears relate as $\underline{\beta} = \beta$ and $\overline{\beta} = \beta/(1+\beta^2)$ with $\underline{\alpha} = \overline{\alpha} = \underline{\gamma} = \overline{\gamma} = 0$.

6. Frameworks for constitutive development

A time rate-of-change in the work being done at a particle by tractions applied to its body results in a source for internal power caused by stresses, often evaluated per unit mass. Here we construct sets of thermodynamic conjugate pairs for both the Lagrangian and Eulerian frameworks when using Laplace stretch as one's kinematic variable. The constituents of these pairs relate to one another via constitutive equations. To facilitate such endeavors, bijective maps are derived that convert stress and velocity-gradient tensor components into their associated thermodynamic stresses and strain rates, the latter of which are scalar fields.

6.1. Lagrangian stress–strain attributes

In terms of Lagrangian fields, stress power $\dot{W}$ can be written as $\frac{1}{\rho_0}\text{tr}(\mathbf{S}\dot{\mathbf{E}})$ wherein $\mathbf{S}$ is the second Piola–Kirchhoff stress, $\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{I})$ is the Green strain, and ρ_0 is the initial mass density at a particle of interest in a body.

It is easily verified that

$$\dot{W} = \tfrac{1}{\rho_0}\text{tr}(\mathbf{S}\dot{\mathbf{E}}) = \tfrac{1}{\rho_0}\text{tr}(\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{L}}^L) \quad \text{where} \quad \boldsymbol{\mathcal{S}} := \boldsymbol{\mathcal{U}}\mathbf{S}\boldsymbol{\mathcal{U}}^{\mathsf{T}}, \quad \boldsymbol{\mathcal{L}}^L := \dot{\boldsymbol{\mathcal{U}}}\boldsymbol{\mathcal{U}}^{-1} \tag{26}$$

given that $\mathbf{F} = \boldsymbol{\mathcal{R}}^L \boldsymbol{\mathcal{U}}$. The Lagrangian stress $\boldsymbol{\mathcal{S}}$ is symmetric because the second Piola–Kirchhoff stress $\mathbf{S}$ is symmetric, and the Lagrangian velocity gradient $\boldsymbol{\mathcal{L}}^L$ is upper-triangular—a consequence of the group that stretch $\boldsymbol{\mathcal{U}}$ belongs to. The above expression for stress power reduces to a sum of six scalar contributions; specifically

$$\rho_0 \dot{W} = \mathcal{S}_{11}\mathcal{L}^L_{11} + \mathcal{S}_{21}\mathcal{L}^L_{12} + \mathcal{S}_{22}\mathcal{L}^L_{22} + \mathcal{S}_{31}\mathcal{L}^L_{13} + \mathcal{S}_{32}\mathcal{L}^L_{23} + \mathcal{S}_{33}\mathcal{L}^L_{33} \tag{27}$$

wherein

$$\mathcal{L}^L_{ij} = \dot{\mathcal{U}}_{ik}\mathcal{U}^{-1}_{kj} = \begin{bmatrix} \dot{\underline{a}}/\underline{a} & \underline{a}\dot{\underline{\gamma}}/\underline{b} & \underline{a}(\dot{\underline{\beta}} - \underline{\alpha}\dot{\underline{\gamma}})/\underline{c} \\ 0 & \dot{\underline{b}}/\underline{b} & \underline{b}\dot{\underline{\alpha}}/\underline{c} \\ 0 & 0 & \dot{\underline{c}}/\underline{c} \end{bmatrix} \tag{28}$$

and we observe that the diagonal rates are logarithmic, while the off-diagonal rates are not logarithmic. (A very different triangular velocity gradient, viz., Eq. (36), arises in the Eulerian construction that follows.) How to construct proper finite differences to approximate derivatives for the physical attributes of Laplace stretch is discussed in Ref. [19].

Expressing Eq. (27) in terms of thermodynamic conjugate pairs is not a unique process, cf. Ref. [9]. Here we shall consider a pairing described by

$$\rho_0 \dot{W} = \underline{\pi}\dot{\underline{\delta}} + \sum_{i=1}^{3} \big(\underline{\sigma}_i \dot{\underline{\varepsilon}}_i + \underline{\tau}_i \dot{\underline{\gamma}}_i \big) \tag{29}$$

whose seven, conjugate, stress–strain pairs are defined as follows: a uniform bulk response is governed by a Lagrangian pressure $\underline{\pi}$ and a Lagrangian dilatation $\underline{\delta}$ defined by

$$\underline{\pi} := \mathcal{S}_{11} + \mathcal{S}_{22} + \mathcal{S}_{33} \quad \underline{\delta} := \ln \sqrt[3]{\frac{\underline{a}}{\underline{a}_0}\frac{\underline{b}}{\underline{b}_0}\frac{\underline{c}}{\underline{c}_0}} \quad \dot{\underline{\delta}} = \frac{1}{3}\left(\frac{\dot{\underline{a}}}{\underline{a}} + \frac{\dot{\underline{b}}}{\underline{b}} + \frac{\dot{\underline{c}}}{\underline{c}} \right) \tag{30a}$$

while the squeeze (pure shear) responses are governed by Lagrangian normal-stress differences $\underline{\sigma}_i$ and Lagrangian squeezes $\underline{\varepsilon}_i$ defined by

$$\underline{\sigma}_1 := \mathcal{S}_{11} - \mathcal{S}_{22} \quad \underline{\varepsilon}_1 := \ln \sqrt[3]{\frac{\underline{a}}{\underline{a}_0}\frac{\underline{b}_0}{\underline{b}}} \quad \dot{\underline{\varepsilon}}_1 = \frac{1}{3}\left(\frac{\dot{\underline{a}}}{\underline{a}} - \frac{\dot{\underline{b}}}{\underline{b}} \right) \tag{30b}$$

$$\underline{\sigma}_2 := \mathcal{S}_{22} - \mathcal{S}_{33} \quad \underline{\varepsilon}_2 := \ln \sqrt[3]{\frac{\underline{b}}{\underline{b}_0}\frac{\underline{c}_0}{\underline{c}}} \quad \dot{\underline{\varepsilon}}_2 = \frac{1}{3}\left(\frac{\dot{\underline{b}}}{\underline{b}} - \frac{\dot{\underline{c}}}{\underline{c}} \right) \tag{30c}$$

$$\underline{\sigma}_3 := \mathcal{S}_{33} - \mathcal{S}_{11} \quad \underline{\varepsilon}_3 := \ln \sqrt[3]{\frac{\underline{c}}{\underline{c}_0}\frac{\underline{a}_0}{\underline{a}}} \quad \dot{\underline{\varepsilon}}_3 = \frac{1}{3}\left(\frac{\dot{\underline{c}}}{\underline{c}} - \frac{\dot{\underline{a}}}{\underline{a}} \right) \tag{30d}$$

of which two are independent because $\underline{\sigma}_3 = -(\underline{\sigma}_1 + \underline{\sigma}_2)$ and $\underline{\varepsilon}_3 = -(\underline{\varepsilon}_1 + \underline{\varepsilon}_2)$, while the (simple) shear responses are governed by Lagrangian shear stresses $\underline{\tau}_i$ and Lagrangian shear strains $\underline{\gamma}_i$ defined by

$$\underline{\tau}_1 := \frac{\underline{b}}{\underline{c}}\mathcal{S}_{32} \qquad \underline{\gamma}_1 := \underline{\alpha} - \underline{\alpha}_0 \qquad \dot{\underline{\gamma}}_1 = \dot{\underline{\alpha}} \tag{30e}$$

$$\underline{\tau}_2 := \frac{\underline{a}}{\underline{c}}\mathcal{S}_{31} \qquad \underline{\gamma}_2 := \underline{\beta} - \underline{\beta}_0 \qquad \dot{\underline{\gamma}}_2 = \dot{\underline{\beta}} \tag{30f}$$

$$\underline{\tau}_3 := \frac{\underline{a}}{\underline{b}}\mathcal{S}_{21} - \frac{\underline{a}\underline{\alpha}}{\underline{c}}\mathcal{S}_{31} \qquad \underline{\gamma}_3 := \underline{\gamma} - \underline{\gamma}_0 \qquad \dot{\underline{\gamma}}_3 = \dot{\underline{\gamma}} \tag{30g}$$

wherein $\underline{a}_0$, $\underline{b}_0$ and $\underline{c}_0$ are their initial elongation ratios, and where $\underline{\alpha}_0$, $\underline{\beta}_0$ and $\underline{\gamma}_0$ are their initial shears.

Bijective maps exist to transform tensor components into thermodynamic stress–strain-rate attributes that, for isotropic materials,[4] are described by

$$\begin{Bmatrix} \underline{\pi} \\ \underline{\sigma}_1 \\ \underline{\sigma}_2 \\ \underline{\tau}_1 \\ \underline{\tau}_2 \\ \underline{\tau}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{b}/\underline{c} & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{a}/\underline{c} & 0 \\ 0 & 0 & 0 & 0 & -\underline{a}\underline{\alpha}/\underline{c} & \underline{a}/\underline{b} \end{bmatrix} \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{33} \\ \mathcal{S}_{32} \\ \mathcal{S}_{31} \\ \mathcal{S}_{21} \end{Bmatrix} \tag{31a}$$

with $\underline{\sigma}_3 = -\underline{\sigma}_1 - \underline{\sigma}_2$, and

$$\begin{Bmatrix} \dot{\underline{\delta}} \\ \dot{\underline{\varepsilon}}_1 \\ \dot{\underline{\varepsilon}}_2 \\ \dot{\underline{\gamma}}_1 \\ \dot{\underline{\gamma}}_2 \\ \dot{\underline{\gamma}}_3 \end{Bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & -1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{c}/\underline{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{c}/\underline{a} & \underline{b}\underline{\alpha}/\underline{a} \\ 0 & 0 & 0 & 0 & 0 & \underline{b}/\underline{a} \end{bmatrix} \begin{Bmatrix} \dot{\mathcal{U}}_{1i}\mathcal{U}^{-1}_{i1} \\ \dot{\mathcal{U}}_{2i}\mathcal{U}^{-1}_{i2} \\ \dot{\mathcal{U}}_{3i}\mathcal{U}^{-1}_{i3} \\ \dot{\mathcal{U}}_{2i}\mathcal{U}^{-1}_{i3} \\ \dot{\mathcal{U}}_{1i}\mathcal{U}^{-1}_{i3} \\ \dot{\mathcal{U}}_{1i}\mathcal{U}^{-1}_{i2} \end{Bmatrix} \tag{31b}$$

with $\dot{\underline{\varepsilon}}_3 = -\dot{\underline{\varepsilon}}_1 - \dot{\underline{\varepsilon}}_2$.

These strain-rate attributes can be integrated to get the Lagrangian thermodynamic strains $\underline{\delta}$, $\underline{\varepsilon}_1$, $\underline{\varepsilon}_2$, $\underline{\varepsilon}_3$, $\underline{\gamma}_1$, $\underline{\gamma}_2$ and $\underline{\gamma}_3$ by choosing initial conditions of $\underline{\delta}|_0 = \underline{\varepsilon}_1|_0 = \underline{\varepsilon}_2|_0 = \underline{\varepsilon}_3|_0 = \underline{\gamma}_1|_0 = \underline{\gamma}_2|_0 = \underline{\gamma}_3|_0 = 0$ provided that the initial elongation ratios have been specified as $\underline{a}_0$, $\underline{b}_0$ and $\underline{c}_0$ and that the initial magnitudes of shear have been specified as $\underline{\alpha}_0$, $\underline{\beta}_0$ and $\underline{\gamma}_0$.

At this juncture, constitutive equations between stress–strain attributes of the thermodynamic conjugate pairs $(\underline{\pi}, \underline{\delta})$, $(\underline{\sigma}_1, \underline{\varepsilon}_1)$, $(\underline{\sigma}_2, \underline{\varepsilon}_2)$, $(\underline{\tau}_1, \underline{\gamma}_1)$, $(\underline{\tau}_2, \underline{\gamma}_2)$ and $(\underline{\tau}_3, \underline{\gamma}_3)$ are to be introduced, e.g., Ref. [11], to solve for the Lagrangian thermodynamic stresses $\underline{\pi}$, $\underline{\sigma}_1$, $\underline{\sigma}_2$, $\underline{\sigma}_3$, $\underline{\tau}_1$, $\underline{\tau}_2$ and $\underline{\tau}_3$. These updated stress attributes map into our Lagrangian stress components $\mathcal{S}_{ij}$ as

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{33} \\ \mathcal{S}_{23} = \mathcal{S}_{32} \\ \mathcal{S}_{13} = \mathcal{S}_{31} \\ \mathcal{S}_{12} = \mathcal{S}_{21} \end{Bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & -1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & -1/3 & -2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{c}/\underline{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{c}/\underline{a} & 0 \\ 0 & 0 & 0 & 0 & \underline{b}\underline{\alpha}/\underline{a} & \underline{b}/\underline{a} \end{bmatrix} \begin{Bmatrix} \underline{\pi} \\ \underline{\sigma}_1 \\ \underline{\sigma}_2 \\ \underline{\tau}_1 \\ \underline{\tau}_2 \\ \underline{\tau}_3 \end{Bmatrix} \tag{32}$$

from which the second Piola–Kirchhoff stress $\mathbf{S} = S_{ij}\, \vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ is retrieved via $\mathbf{S} = \boldsymbol{\mathcal{U}}^{-1}\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{U}}^{-\mathsf{T}}$, i.e., $S_{ij} = \mathcal{U}^{-1}_{ik}\mathcal{S}_{k\ell}\mathcal{U}^{-1}_{j\ell}$, such that from here any commonly used stress tensor can be gotten.

Although $\underline{\sigma}_3$ and $\dot{\underline{\varepsilon}}_3$ are not needed from a constitutive perspective, they are required to correctly calculate stress power.

6.2. Eulerian stress–strain attributes

In terms of Eulerian fields, stress power $\dot{W}$ can be written as $\frac{1}{\rho_0}\mathrm{tr}(\boldsymbol{\tau}\mathbf{D})$ wherein $\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathsf{T}}$ is the Kirchhoff stress, which relates to Cauchy stress $\mathbf{T}$ via $\boldsymbol{\tau} := \det(\mathbf{F})\mathbf{T} = \frac{\rho_0}{\rho}\mathbf{T}$ with $\mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathsf{T}}) = \mathbf{F}^{-\mathsf{T}}\dot{\mathbf{E}}\mathbf{F}^{-1}$ being the symmetric part of the velocity gradient $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$, and ρ being the current mass density.

It can be shown that

$$\dot{W} = \tfrac{1}{\rho_0}\mathrm{tr}(\boldsymbol{\tau}\mathbf{D}) = \tfrac{1}{\rho_0}\mathrm{tr}\big(\boldsymbol{\tau}\boldsymbol{\mathcal{L}}^{E}\big) \tag{33a}$$

[4] See Ref. [9] for one way to extend this approach to anisotropic materials.

given that $\mathbf{F} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{R}}^E$, where this Eulerian velocity gradient $\boldsymbol{\mathcal{L}}^E$ is defined by

$$\boldsymbol{\mathcal{L}}^E := \overset{\circ}{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{-1} \quad \text{wherein} \quad \overset{\circ}{\boldsymbol{\mathcal{V}}} := \dot{\boldsymbol{\mathcal{V}}} + \boldsymbol{\mathcal{V}}\boldsymbol{\Omega}^E - \boldsymbol{\Omega}^E\boldsymbol{\mathcal{V}} \tag{33b}$$

with $\overset{\circ}{\boldsymbol{\mathcal{V}}}$ being an objective co-rotational derivative for this measure of stretch, and $\boldsymbol{\Omega}^E := \dot{\boldsymbol{\mathcal{R}}}^E\boldsymbol{\mathcal{R}}^{E^{\mathsf{T}}}$ being a spin of an Eulerian coordinate axes $(\vec{\mathsf{e}}_1^E, \vec{\mathsf{e}}_2^E, \vec{\mathsf{e}}_3^E)$ about the reference axes $(\vec{\mathsf{e}}_1, \vec{\mathsf{e}}_2, \vec{\mathsf{e}}_3)$.

Consequently, stress power $\rho_0\dot{W} = \mathrm{tr}\big(\boldsymbol{\tau}\boldsymbol{\mathcal{L}}^E\big)$ arises from two sources in this Eulerian construction, viz. $\dot{W} = \dot{W}_1 + \dot{W}_2$. The first is energetic, i.e.,

$$\dot{W}_1 := \tfrac{1}{\rho_0}\mathrm{tr}\big(\boldsymbol{\tau}\dot{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{-1}\big) \tag{34a}$$

while the second satisfies objectivity, viz.,

$$\dot{W}_2 := \tfrac{1}{\rho_0}\mathrm{tr}\big(\boldsymbol{\tau}\boldsymbol{\mathcal{V}}\boldsymbol{\Omega}^E\boldsymbol{\mathcal{V}}^{-1}\big) \tag{34b}$$

noting that $\mathrm{tr}(\boldsymbol{\tau}\boldsymbol{\Omega}^E) = 0$. Thermodynamic stress–strain conjugate pairs can be established in terms of energetic expression (34a). Objective correction (34b) is required to quantify the work being done, but it plays no role when creating our Eulerian stress–strain attributes, as every term in this sum has a component of spin in it; therefore, $\dot{W}_2 = 0$ whenever $\boldsymbol{\Omega}^E = \mathbf{0}$.

Because $\dot{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{-1} = \dot{\mathcal{V}}_{ik}\mathcal{V}_{kj}^{-1}\,\vec{\mathsf{e}}_i \otimes \vec{\mathsf{e}}_j$ has components that are lower triangular, a consequence of the group that tensor $\boldsymbol{\mathcal{V}}$ belongs to, the first contribution to stress power put forward in Eq. (34a) reduces to a sum of six scalar contributions; specifically,

$$\rho_0\dot{W}_1 = \tau_{11}\dot{\mathcal{V}}_{1i}\mathcal{V}_{i1}^{-1} + \tau_{12}\dot{\mathcal{V}}_{2i}\mathcal{V}_{i1}^{-1} + \tau_{13}\dot{\mathcal{V}}_{3i}\mathcal{V}_{i1}^{-1} + \tau_{22}\dot{\mathcal{V}}_{2i}\mathcal{V}_{i2}^{-1} + \tau_{23}\dot{\mathcal{V}}_{3i}\mathcal{V}_{i2}^{-1} + \tau_{33}\dot{\mathcal{V}}_{3i}\mathcal{V}_{i3}^{-1} \tag{35}$$

wherein

$$\dot{\mathcal{V}}_{ik}\mathcal{V}_{kj}^{-1} = \begin{bmatrix} \frac{\dot{\overline{a}}}{\overline{a}} & 0 & 0 \\ \dot{\overline{\gamma}} + \overline{\gamma}\left(\frac{\dot{\overline{a}}}{\overline{a}} - \frac{\dot{\overline{b}}}{\overline{b}}\right) & \frac{\dot{\overline{b}}}{\overline{b}} & 0 \\ \dot{\overline{\beta}} - \overline{\gamma}\,\dot{\overline{\alpha}} + \overline{\beta}\left(\frac{\dot{\overline{a}}}{\overline{a}} - \frac{\dot{\overline{c}}}{\overline{c}}\right) - \overline{\alpha\gamma}\left(\frac{\dot{\overline{b}}}{\overline{b}} - \frac{\dot{\overline{c}}}{\overline{c}}\right) & \dot{\overline{\alpha}} + \overline{\alpha}\left(\frac{\dot{\overline{b}}}{\overline{b}} - \frac{\dot{\overline{c}}}{\overline{c}}\right) & \frac{\dot{\overline{c}}}{\overline{c}} \end{bmatrix} \tag{36}$$

which is strikingly different from that of its Lagrangian counterpart $\dot{\boldsymbol{\mathcal{U}}}\boldsymbol{\mathcal{U}}^{-1}$ found in Eq. (28).[5] Present here are the squeeze rates $\dot{\overline{\varepsilon}}_1 = \frac{1}{3}\big(\dot{\overline{a}}/\overline{a} - \dot{\overline{b}}/\overline{b}\big)$, etc., which appear in the off-diagonal terms, along with their corresponding shear rates, e.g., $\dot{\overline{\gamma}}$, thereby substantiating our selection of conjugate pairs.

In Eq. (36), a clear delineation exists between pure and simple shearing deformations. Such a delineation does not arise whenever one uses symmetric measures for stretch, where an isotropic–deviatoric decomposition is the extent to which such fields can be deconstructed.

Expressing Eq. (35) in terms of Eulerian, thermodynamic, conjugate pairs, analogous to those considered for the Lagrangian frame, one can write

$$\rho_0\dot{W}_1 = \overline{\pi}\dot{\overline{\delta}} + \sum_{i=1}^{3}\big(\overline{\sigma}_i\dot{\overline{\varepsilon}}_i + \overline{\tau}_i\dot{\overline{\gamma}}_i\big) \tag{37}$$

whose seven, conjugate, stress–strain pairs are defined as follows: a uniform bulk response is governed by an Eulerian pressure $\overline{\pi}$ and an Eulerian dilatation $\overline{\delta}$ defined by

$$\overline{\pi} := \tau_{11} + \tau_{22} + \tau_{33} \qquad \overline{\delta} := \ln\sqrt[3]{\frac{\overline{a}}{\overline{a}_0}\frac{\overline{b}}{\overline{b}_0}\frac{\overline{c}}{\overline{c}_0}} \qquad \dot{\overline{\delta}} = \frac{1}{3}\left(\frac{\dot{\overline{a}}}{\overline{a}} + \frac{\dot{\overline{b}}}{\overline{b}} + \frac{\dot{\overline{c}}}{\overline{c}}\right) \tag{38a}$$

[5] Curiously, $\boldsymbol{\mathcal{U}}^{-1}\dot{\boldsymbol{\mathcal{U}}}$ has components akin to Eq. (36), except its components are upper triangular instead of lower triangular, and are expressed in terms of the Lagrangian stretch attributes instead of their Eulerian counterparts.

while the squeeze (pure shear) responses are governed by Eulerian normal-stress differences $\overline{\sigma}_i$ and Eulerian squeezes $\overline{\varepsilon}_i$ defined by

$$\overline{\sigma}_1 := \tau_{11} - \tau_{22} + 3\overline{\gamma}\tau_{12} \qquad \overline{\varepsilon}_1 := \ln \sqrt[3]{\frac{\overline{a}}{\overline{a}_0}\frac{\overline{b}_0}{\overline{b}}} \qquad \dot{\overline{\varepsilon}}_1 = \frac{1}{3}\left(\frac{\dot{\overline{a}}}{\overline{a}} - \frac{\dot{\overline{b}}}{\overline{b}}\right) \tag{38b}$$

$$\overline{\sigma}_2 := \begin{cases} \tau_{22} - \tau_{33} \\ +3\overline{\alpha}(\tau_{23} - \overline{\gamma}\tau_{13}) \end{cases} \qquad \overline{\varepsilon}_2 := \ln \sqrt[3]{\frac{\overline{b}}{\overline{b}_0}\frac{\overline{c}_0}{\overline{c}}} \qquad \dot{\overline{\varepsilon}}_2 = \frac{1}{3}\left(\frac{\dot{\overline{b}}}{\overline{b}} - \frac{\dot{\overline{c}}}{\overline{c}}\right) \tag{38c}$$

$$\overline{\sigma}_3 := -\tau_{11} + \tau_{33} - 3\overline{\beta}\tau_{13} \qquad \overline{\varepsilon}_3 := \ln \sqrt[3]{\frac{\overline{c}}{\overline{c}_0}\frac{\overline{a}_0}{\overline{a}}} \qquad \dot{\overline{\varepsilon}}_3 = \frac{1}{3}\left(\frac{\dot{\overline{c}}}{\overline{c}} - \frac{\dot{\overline{a}}}{\overline{a}}\right) \tag{38d}$$

of which only two are independent, while the (simple) shear responses are governed by Eulerian shear stresses $\overline{\tau}_i$ and strains $\overline{\gamma}_i$ defined by

$$\overline{\tau}_1 := \tau_{23} - \overline{\gamma}\tau_{13} \qquad \overline{\gamma}_1 := \overline{\alpha} - \overline{\alpha}_0 \qquad \dot{\overline{\gamma}}_1 = \dot{\overline{\alpha}} \tag{38e}$$

$$\overline{\tau}_2 := \tau_{13} \qquad \overline{\gamma}_2 := \overline{\beta} - \overline{\beta}_0 \qquad \dot{\overline{\gamma}}_2 = \dot{\overline{\beta}} \tag{38f}$$

$$\overline{\tau}_3 := \tau_{12} \qquad \overline{\gamma}_3 := \overline{\gamma} - \overline{\gamma}_0 \qquad \dot{\overline{\gamma}}_3 = \dot{\overline{\gamma}} \tag{38g}$$

wherein $\overline{a}_0$, $\overline{b}_0$ and $\overline{c}_0$ are their initial elongation ratios, and where $\overline{\alpha}_0$, $\overline{\beta}_0$ and $\overline{\gamma}_0$ are their initial shear offsets.

The sets of thermodynamic conjugate pairs for the Lagrangian and Eulerian frameworks are taken to be the same. Each set is composed of three modes: one pair to describe uniform dilatation, three pairs to describe pure shears and three pairs to describe simple shears. In both cases, only two of the three pure-shear pairs are independent, thereby resulting in sets of six, independent, conjugate pairs that have direct connections with the six independent components of stress and stretch rate.

Bijective maps exist to transform tensor components into thermodynamic stress–strain-rate attributes that, for isotropic materials, are described by

$$\begin{Bmatrix} \overline{\pi} \\ \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\tau}_1 \\ \overline{\tau}_2 \\ \overline{\tau}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 3\overline{\gamma} \\ 0 & 1 & -1 & 3\overline{\alpha} & -3\overline{\alpha}\overline{\gamma} & 0 \\ 0 & 0 & 0 & 1 & -\overline{\gamma} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{32} \\ \tau_{31} \\ \tau_{21} \end{Bmatrix} \tag{39a}$$

with

$$\overline{\sigma}_3 = -\overline{\sigma}_1 - \overline{\sigma}_2 + 3\left(\overline{\alpha}\overline{\tau}_1 - \overline{\beta}\overline{\tau}_2 + \overline{\gamma}\overline{\tau}_3\right) \tag{39b}$$

which arises from the constraint equation

$$\begin{Bmatrix} \overline{\sigma}_1 - 3\overline{\gamma}\overline{\tau}_3 \\ \overline{\sigma}_2 - 3\overline{\alpha}\overline{\tau}_1 \\ \overline{\sigma}_3 + 3\overline{\beta}\overline{\tau}_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{Bmatrix}$$

and where

$$\begin{Bmatrix} \dot{\overline{\delta}} \\ \dot{\overline{\varepsilon}}_1 \\ \dot{\overline{\varepsilon}}_2 \\ \dot{\overline{\gamma}}_1 \\ \dot{\overline{\gamma}}_2 \\ \dot{\overline{\gamma}}_3 \end{Bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & -1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & -1/3 & 0 & 0 & 0 \\ 0 & -\overline{\alpha} & \overline{\alpha} & 1 & 0 & 0 \\ -\overline{\beta} & 0 & \overline{\beta} & \overline{\gamma} & 1 & 0 \\ -\overline{\gamma} & \overline{\gamma} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\mathcal{V}}_{1i}\mathcal{V}^{-1}_{i1} \\ \dot{\mathcal{V}}_{2i}\mathcal{V}^{-1}_{i2} \\ \dot{\mathcal{V}}_{3i}\mathcal{V}^{-1}_{i3} \\ \dot{\mathcal{V}}_{2i}\mathcal{V}^{-1}_{i3} \\ \dot{\mathcal{V}}_{1i}\mathcal{V}^{-1}_{i3} \\ \dot{\mathcal{V}}_{1i}\mathcal{V}^{-1}_{i2} \end{Bmatrix} \tag{39c}$$

with

$$\dot{\bar{\varepsilon}}_3 = -\dot{\bar{\varepsilon}}_1 - \dot{\bar{\varepsilon}}_2. \tag{39d}$$

These strain rates can be integrated to get the Eulerian thermodynamic strains $\overline{\delta}$, $\overline{\varepsilon}_1$, $\overline{\varepsilon}_2$, $\overline{\varepsilon}_3$, $\overline{\gamma}_1$, $\overline{\gamma}_2$ and $\overline{\gamma}_3$ by using initial conditions of $\overline{\delta}|_0 = \overline{\varepsilon}_1|_0 = \overline{\varepsilon}_2|_0 = \overline{\varepsilon}_3|_0 = \overline{\gamma}_1|_0 = \overline{\gamma}_2|_0 = \overline{\gamma}_3|_0 = 0$ provided that the initial elongation ratios have been specified as $\overline{a}_0$, $\overline{b}_0$ and $\overline{c}_0$ and that the initial magnitudes of shear have been specified as $\overline{\alpha}_0$, $\overline{\beta}_0$ and $\overline{\gamma}_0$.

At this juncture, constitutive equations between the Eulerian thermodynamic conjugate pairs $(\overline{\pi}, \overline{\delta})$, $(\overline{\sigma}_1, \overline{\varepsilon}_1)$, $(\overline{\sigma}_2, \overline{\varepsilon}_2)$, $(\overline{\tau}_1, \overline{\gamma}_1)$, $(\overline{\tau}_2, \overline{\gamma}_2)$ and $(\overline{\tau}_3, \overline{\gamma}_3)$ are to be introduced (a topic for future work) to solve for the Eulerian thermodynamic stresses $\overline{\pi}$, $\overline{\sigma}_1$, $\overline{\sigma}_2$, $\overline{\tau}_1$, $\overline{\tau}_2$ and $\overline{\tau}_3$. After the thermodynamic stresses have been updated, they can be mapped back into the components of Kirchhoff stress τ_{ij} via

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} = \tau_{32} \\ \tau_{13} = \tau_{31} \\ \tau_{12} = \tau_{21} \end{Bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 1/3 & -\overline{\alpha} & 0 & -2\overline{\gamma} \\ 1/3 & -1/3 & 1/3 & -\overline{\alpha} & 0 & \overline{\gamma} \\ 1/3 & -1/3 & -2/3 & 2\overline{\alpha} & 0 & \overline{\gamma} \\ 0 & 0 & 0 & 1 & \overline{\gamma} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{\pi} \\ \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\tau}_1 \\ \overline{\tau}_2 \\ \overline{\tau}_3 \end{Bmatrix} \tag{40}$$

from which any commonly used stress tensor can be easily gotten.

Although $\overline{\sigma}_3$ and $\dot{\bar{\varepsilon}}_3$ are not needed from a constitutive perspective, they are required to calculate stress power. Also, to correctly compute stress power, Eqs. (34a or 37 and 34b) must both contribute, the former because of straining and the latter because of coordinate spin. A numerical strategy based upon quaternion theory to acquire spin tensors from rotation tensors by using finite difference schemes can be found in Ref. [14]. Alternatively, Haller [28] extended the idea of unsheared triads introduced by Boulanger and Hayes [17] and constructed triads along which the stretch rates are free from shear.

7. Conclusions

Lagrangian and Eulerian triangular decompositions of deformation have been analyzed and compared. Physically observable stretch/strain components comprising the triangular Laplace stretch of each decomposition have been derived and then highlighted in several example problems involving homogeneous deformations. Consideration of stress power, i.e., rate of working done by each stretch rate, has enabled the derivation of work conjugate stress–stretch tensors, as well as thermodynamically conjugate scalar pairs that are stress–strain attributes with physical meaning. The current results provide a theoretical foundation for constructing constitutive models using either an Eulerian or a Lagrangian approach, as deemed appropriate by the problem to be undertaken.

Acknowledgements

This research was inspired by a conversation that ADF had with Prof. Michael Sacks at the 2019 Annual Meeting of the Society for Engineering Science held at Washington University in St. Louis, where he encouraged the author to develop of an Eulerian constitutive framework suitable for biomechanics. The authors also thank anonymous reviewers for their constructive comments.

References

[1] McLellan, A.G.: The Classical Thermodynamics of Deformable Materials, Cambridge Monographs in Physics. Cambridge University Press, Cambridge (1980)
[2] Clayton, J.D.: Nonlinear Mechanics of Crystals. Springer, Dordrecht (2011)
[3] Truesdell, C.: Hypoelasticity. J. Ration. Mech. Anal. **4**, 83–133 (1955)
[4] Ericksen, J.L.: Hypo-elastic potentials. Q. J. Mech. Appl. Math. **11**, 67–72 (1958)
[5] McLellan, A.G.: Finite strain coordinate and the stability of solid phases. J. Phys. C Solid State Phys. **9**, 4083–4094 (1976)
[6] Souchet, R.: Concerning the polar decomposition of the deformation gradient. Int. J. Eng. Sci. **31**, 1499–1506 (1993)
[7] Srinivasa, A.R.: On the use of the upper triangle (or QR) decomposition for developing constitutive equations for Green-elastic materials. Int. J. Eng. Sci. **60**, 1–12 (2012)
[8] Ghosh, P., Srinivasa, A.R.: Development of a finite strain two-network model for shape memory polymers using QR decomposition. Int. J. Eng. Sci. **81**, 177–191 (2014)
[9] Freed, A.D.: A note on stress/strain conjugate pairs: explicit and implicit theories of thermoelasticity for anisotropic materials. Int. J. Eng. Sci. **120**, 155–171 (2017)
[10] Erel, V., Freed, A.D.: Stress/strain basis pairs for anisotropic materials. Compos. Part B Eng. **120**, 152–158 (2017)
[11] Freed, A.D., Zamani, S.: Elastic Kelvin–Poisson-Poynting solids described through scalar conjugate stress/strain pairs derived from a QR factorization of F. J. Mech. Phys. Solids **129**, 278–293 (2019)
[12] Freed, A.D., Erel, V., Moreno, M.R.: Conjugate stress/strain base pairs for planar analysis of biological tissues. J. Mech. Mater. Struct. **12**, 219–247 (2017)
[13] Clayton, J.D., Freed, A.D.: A constitutive model for lung mechanics and injury applicable to static, dynamic, and shock loading. Mech. Soft Mater. **2**, 3 (2020)
[14] Freed, A.D., le Graverend, J.B., Rajagopal, K.R.: A decomposition of Laplace stretch with applications in inelasticity. Acta Mechanica **230**, 3423–3429 (2019)
[15] Paul, S., Freed, A.D.: A simple and practical representation of compatibility condition derived using a QR decomposition of the deformation gradient. Acta Mechanica **231**, 3289–3304 (2020)
[16] Boulanger, P., Hayes, M.: Extended polar decompositions for plane strain. J. Elast. **83**, 29–64 (2006)
[17] Boulanger, P., Hayes, M.: Unsheared triads and extended polar decompositions of the deformation gradient. Int. J. Non-Linear Mech. **36**, 399–420 (2001)
[18] Jarić, J., Stamenović, D., Djordjević, V.D.: On extended polar decomposition. J. Elast. **83**, 277–289 (2006)
[19] Freed, A.D., Zamani, S.: On the use of convected coordinate systems in the mechanics of continuous media derived from a QR factorization of F. Int. J. Eng. Sci. **127**, 145–161 (2018)
[20] Gao, X.-L., Li, Y.Q.: The upper triangular decomposition of the deformation gradient: possible decompositions of the distortion tensor. Acta Mechanica **229**, 1927–1948 (2018)
[21] Paul, S., Freed, A.D.: On coordinate indexing when using Laplace stretch (in review)
[22] Freed, A.D., Srinivasa, A.R.: Logarithmic strain and its material derivative for a QR decomposition of the deformation gradient. Acta Mechanica **226**, 2645–2670 (2015)
[23] Leon, S.J., Björck, Å., Gander, W.: Gram-Schmidt orthogonalization: 100 years and more. Numer. Linear Algebra Appl. **20**, 492–532 (2013)
[24] Rosakis, P.: Ellipticity and deformations with discontinuous gradients in finite elastostatics. Arch. Ration. Mech. Anal. **109**, 1–37 (1990)
[25] Lodge, A.S.: Elastic Liquids: An Introductory Vector Treatment of Finite-Strain Polymer Rheology. Academic Press, London (1964)
[26] Lodge, A.S.: Body Tensor Fields in Continuum Mechanics: With Applications to Polymer Rheology. Academic Press, New York (1974)
[27] Bird, R.B., Armstrong, R.C., Hassager, O.: Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd edn. Wiley, New York (1987)
[28] Haller, G.: Dynamic rotation and stretch tensors from a dynamic polar decomposition. J. Mech. Phys. Solids **86**, 70–93 (2016)

Alan D. Freed and Shahla Zamani
Department of Mechanical Engineering
Texas A&M University
College Station TX77843
USA
e-mail: zamani.shahla@tamu.edu

Alan D. Freed and John D. Clayton
Impact Physics Branch
U.S. Army Research Laboratory
Aberdeen Proving Ground
Aberdeen MD21005
USA
e-mail: afreed@tamu.edu

John D. Clayton
e-mail: john.d.clayton1.civ@mail.mil

László Szabó
Budapest University of Technology and Economics
Budapest 1521
Hungary
e-mail: szabo@mm.bme.hu

(Received: March 17, 2020; revised: August 16, 2020)