L^p-theory for a fluid–structure interaction model

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Abstract. We consider a fluid–structure interaction model for an incompressible fluid where the elastic response of the free boundary is given by a damped Kirchhoff plate model. Utilizing the Newton polygon approach, we first prove maximal regularity in L^p-Sobolev spaces for a linearized version. Based on this, we show existence and uniqueness of the strong solution of the nonlinear system for small data.

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1. Introduction and main result

We consider the system

$$\left.\begin{array}{rl}
\rho(\partial_t u + (u \cdot \nabla)u)) - \operatorname{div} T(u, q) = 0, & t > 0, \ x \in \Omega(t), \\
\operatorname{div} u = 0, & t > 0, \ x \in \Omega(t), \\
u = V_\Gamma, & t \geq 0, \ x \in \Gamma(t), \\
\frac{1}{\nu \cdot e_n} e_n^\tau T(u, q)\nu = \phi_\Gamma, & t \geq 0, \ x \in \Gamma(t), \\
\Gamma(0) = \Gamma_0, \quad V_\Gamma(0) = V_0, \quad u(0) = u_0, & x \in \Omega(0),
\end{array}\right\} \tag{1.1}$$

which represents a (one-phase) fluid–structure interaction model. The fluid with density $\rho > 0$ and viscosity $\mu > 0$ occupies at a time $t \geq 0$ the region $\Omega(t) \subseteq \mathbb{R}^n$ with boundary $\Gamma(t) = \partial\Omega(t)$. Furthermore, we assume the fluid to be incompressible, and we assume the stress to be given as

$$T(u, q) = 2\mu D(u) - q, \qquad D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\tau).$$

The unknowns in the model are the velocity u, the pressure q, and the interface Γ. We denote by ν the exterior unit normal field at Γ, by V_Γ the velocity of the boundary Γ, and by e_j the j-th standard basis vector in $\mathbb{R}^n$, i.e., $e_n = (0, \ldots, 0, 1)$.

The function ϕ_Γ describes the elastic response at Γ which is given by a damped Kirchhoff-type plate model. Throughout the paper, we assume that Γ is given as a graph of a function $\eta : \mathbb{R}_+ \times \mathbb{R}^{n-1} \to \mathbb{R}$, that is

$$\Gamma(t) = \left\{(x', \eta(t, x')); \ x' \in \mathbb{R}^{n-1}\right\}, \quad t \geq 0, \tag{1.2}$$

and that $\Gamma(t)$ is sufficiently flat. Thus, $\Omega(t)$ is a perturbed upper half-plane. In these coordinates, the elastic response is given as

$$\phi_\Gamma = m(\partial_t, \partial')\eta := \partial_t^2 \eta + \alpha(\Delta')^2\eta - \beta\Delta'\eta - \gamma\partial_t\Delta'\eta \tag{1.3}$$

for $\alpha, \gamma > 0$, $\beta \in \mathbb{R}$, where Δ' stands for the Laplacian in $\mathbb{R}^{n-1}$. Finally, the initial configuration and velocity of the interface resp. the initial fluid velocity are given by Γ_0 and V_0 resp. $u_0 = (u_0', u_0^n)$. Note that in addition to the initial position Γ_0 of the boundary, also its initial velocity V_0 has to be specified as

the equation is of second order with respect to time on the boundary. We remark that in the formulation of the boundary conditions in lines 3 and 4 of (1.1), one has to take into account that the Kirchhoff plate model is formulated in a Lagrangian setting, whereas for the fluid an Eulerian setting is used. This is discussed in more detail in the beginning of Sect. 2.

The symbol of $m(\partial_t, \partial')$ is given as

$$m(\lambda, \xi') = \lambda^2 + \alpha|\xi'|^4 + \beta|\xi'|^2 + \gamma\lambda|\xi'|^2, \quad \lambda \in \mathbb{C}, \ \xi' \in \mathbb{R}^{n-1},$$

which vanishes if

$$\lambda = -\frac{\gamma|\xi'|^2}{2} \pm \sqrt{\frac{\gamma^2|\xi'|^4}{4} - \alpha|\xi'|^4 - \beta|\xi'|^2}.$$

For $\gamma > 0$, the roots of $m(\cdot, \xi')$ lie in some sector which is a subset of $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$. This indicates that the term $-\gamma\partial_t\Delta'\eta$ in ϕ_Γ parabolizes the problem. Physically, one also speaks of structural damping of the plate.

We notice that basically the same results as proved in this note can be expected by considering layer like domains or rectangular type domains with periodic lateral boundary conditions. For simplicity, however, we restrict the approach given here to the just introduced geometry.

Model (1.1) was introduced in [23] in connection to applications to cardiovascular systems. In the 2D case, this system was investigated in [3] in the L^2-setting. In fact, in [3, Proposition 3.12] it is proved that the linear operator associated with (1.1) generates an analytic C_0-semigroup in a suitable Hilbert space setting. This exhibits the parabolic character of the problem. Therefore, it is reasonable to consider an L^p-theory for the system (1.1) which is the main purpose of this note.

Alternative approaches to system (1.1) in the L^2-setting also for the hyperbolic–parabolic case, i.e., $\gamma = 0$, are given, e.g., in [6,10,16,17,21], concerning weak solutions and, e.g., in [4,7,18,19] concerning (local) strong solutions. A more recent approach in a two-dimensional L^2-framework concerning global strong solutions is presented in [11]. Recently, in [20] the interaction between an incompressible fluid and a damped beam (which relates to the case of a one-dimensional boundary) was studied in the L^p-L^q-setting.

In the present paper, we develop an L^p-approach in general dimension for system (1.1). In order to formulate the main result, for $k, \ell \in \mathbb{N}_0$ non-cylindrical spaces are defined as

$$H_p^k(J, H_p^\ell(\Omega(t)))$$
$$:= \left\{ u : \bigcup_{t \in J}\{t\} \times \Omega(t) \to \mathbb{R}^n; \ u \text{ measurable and } \|u\|_{k,\ell} < \infty \right\},$$

where

$$\|u\|_{k,\ell}^p := \|u\|_{H_p^k(J,H_p^\ell(\Omega(t)))}^p := \sum_{m \le k} \sum_{|\alpha| \le \ell} \int_J \int_{\Omega(t)} \left|\partial_t^m \partial_x^\alpha u(t,x)\right|^p \, dx \, dt.$$

The space $L^p(J; \dot{H}_p^1(\Omega(t)))$ for the pressure is defined accordingly. We show the existence of strong solutions for small data and give a precise description of the maximal regularity spaces for the unknowns. More precisely, we prove the following main result for (1.1).

Theorem 1.1. *Let $n \ge 2$, $p \ge (n+2)/3$, $T > 0$, and $J = (0,T)$. Assume that*

$$\|u_0\|_{W_p^{2-2/p}(\Omega(0))} + \|\eta_0\|_{W_p^{5-3/p}(\mathbb{R}^{n-1})} + \|\eta_1\|_{W_p^{3-3/p}(\mathbb{R}^{n-1})} < \kappa,$$

for some $\kappa > 0$, where $\Gamma_0 = \text{graph}(\eta_0)$ and $V_0 = \{(0, \eta_1(x')); \ x' \in \mathbb{R}^{n-1}\}$ in (1.1). Then, there exists a unique solution (u, q, Γ) of system (1.1) such that $\Gamma = \text{graph}(\eta)$ and such that

$$u \in H_p^1(J; L^p(\Omega(t))) \cap L^p(J; H_p^2(\Omega(t))),$$

$$q \in L^p(J; \dot{H}_p^1(\Omega(t))),$$

$$\eta \in \mathbb{E}_\eta := W_p^{9/4 - 1/(4p)}(J; L^p(\mathbb{R}^{n-1})) \cap H_p^2(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))$$

$$\cap L^p(J; W_p^{5-1/p}(\mathbb{R}^{n-1})),$$

provided that $\kappa = \kappa(T)$ is small enough and that the following compatibility conditions are satisfied:

(1) $\text{div}\, u_0 = 0$,
(2) *if $p > \frac{3}{2}$, then $u_0'|_{\Gamma_0} = 0$ and $u_0^n|_{\Gamma_0} - \eta_1 = 0$ almost everywhere,*
(3) *there exists an $\eta_* \in \mathbb{E}_\eta$ with $\eta_*|_{t=0} = \eta_0$, $\partial_t \eta_*|_{t=0} = \eta_1$ and*

$$\partial_t \eta_* \in H_p^1(J; H_{p,0}^{-1}(\mathbb{R}_+^n)),$$

where

$$\partial_t \eta_*(\phi) := - \int_{\mathbb{R}^{n-1}} \partial_t \eta_* \phi\, dx', \quad \phi \in \dot{H}_{p'}^1(\mathbb{R}_+^n).$$

The solution depends continuously on the data.

Remark 1.2. (a) The compatibility conditions (1)–(3) are natural in the sense that they are also necessary for the existence of a strong solution. Condition (3) appears in a similar way for the two-phase Stokes problem, see, e.g., [22], Section 8.1. Note that the regularity for η^* in (3) does not follow from $\eta^* \in \mathbb{E}_\eta$.

(b) We remark that the maximal regularity space $\mathbb{E}_\eta$ for η describing the boundary is not a standard space. It is given as an intersection of three Sobolev spaces. This is due to the fact that the symbol of the complete system has an inherent inhomogeneous structure, and therefore the Newton polygon method is the correct tool to show maximal regularity. For the details, see Sect. 3.

c) We note that in the physically relevant situations $n = 2$ and $n = 3$, the case $p = 2$ is included. This might be of importance when considering the singular limit $\gamma \to 0$ for vanishing damping of the plate.

d) We formulated the result in the form of existence for fixed time and small data. By similar methods, one can also show short time existence for arbitrarily large data. This is more intricate, since then while estimating nonlinearities one has to carefully track the dependence of the constants on related smallness parameters. But we think that the known strategies, as elaborated, e.g., in [22], can be adapted.

The proof of Theorem 1.1 is based on several ingredients: First, we transform the system to a fixed domain and consider the linearization of the transformed system. By an application of the Newton polygon approach (see, e.g., [8] and [9]), we obtain maximal regularity for the linearized system. To deal with the nonlinearities, we employ embedding results on anisotropic Sobolev spaces given in [15].

Remark 1.3. a) The half-space model problem considered here can also be regarded as a first step towards an analysis on domains of more general geometry. By applying a suitable localization procedure, similar results are expected to hold, e.g., on bounded domains. On bounded domains, even global solvability for small data might be available.

b) An L^p-L^q-theory with $p \neq q$ might be available as well. For the linear theory, in particular concerning the Newton polygon approach, the use of [9] then has to be replaced by the generalized approach developed in [8], see the proof of Lemma 3.2. Concerning the nonlinear system, so far there is no L^p-L^q analogon of the results on multiplication in [15] available in the existing literature. For this purpose, the corresponding estimates of the nonlinearities then had to be derived by more direct methods.

2. The transformed system

We start with a short discussion of the boundary conditions, where the Eulerian approach for the fluid has to be coupled with the Lagrangian description for the plate (see also [17] and [10]). Let Γ be given as in (1.2) and assume that η is sufficiently smooth. Following the Kirchhoff plate model, in-plate motions are ignored, and the velocity of the plate at the point $(x', \eta(t, x'))^\tau$ is parallel to the vertical direction and given by $(0, \partial_t \eta(t, x'))^\tau = \partial_t \eta(t, x') e_n$. As the fluid is assumed to adhere to the plate, we have no-slip boundary conditions for the fluid, and the equality of the velocities yields the first boundary condition

$$u(t, x', \eta(t, x')) = \partial_t \eta(t, x') e_n \quad (t > 0, \, x' \in \mathbb{R}^n). \tag{2.1}$$

The exterior normal at the point $(x', \eta(t, x'))$ of the boundary $\Gamma(t)$ is given by

$$\nu = \nu(t, x') = \frac{1}{\sqrt{1 + |\nabla' \eta(t, x')|^2}} \begin{pmatrix} \nabla' \eta(t, x') \\ -1 \end{pmatrix}.$$

We define the transform of variables

$$\theta : J \times \mathbb{R}_+^n \to \bigcup_{t \in J} \{t\} \times \Omega(t), \; (t, x', x_n) \mapsto \theta(t, x', x_n) := (t, x', x_n + \eta(t, x')).$$

Obviously, we have $\theta^{-1}(t, x', y) = (t, x', y - \eta(t, x'))$. As it was discussed in [17], Section 1.2, the force F exerted by the fluid on the boundary is given by the evaluation of the stress tensor at the deformed boundary in the direction of the inner normal $-\nu(t, x')$. More precisely, we obtain ([17], Eq. (1.4))

$$F = -\sqrt{1 + |\nabla' \eta(t, x')|} \, e_n^\tau (T(u, q) \circ \theta(t, x)) \nu(t, x').$$

As $\sqrt{1 + |\nabla' \eta|^2} = -\nu(t, x') \cdot e_n$, the equality of the forces gives the second boundary condition

$$\frac{1}{\nu(t, x') \cdot e_n} \, e_n^\tau [T(u, q)](t, x', \eta(t, x')) \, \nu(t, x') = [m(\partial_t, \partial') \eta](t, x')$$
$$(t > 0, x' \in \mathbb{R}^{n-1}). \tag{2.2}$$

Conditions (2.1) and (2.2) are the precise formulation of the boundary conditions in (1.1).

To solve the problem (1.1), we first note that by a re-scaling argument we may assume that $\rho = \mu = 1$ for the density ρ and viscosity μ from now on. Next, we transform the problem (1.1) to a problem on the fixed half-space $\mathbb{R}_+^n$, using the above transformation θ. To this end, we set $J := (0, T)$ and write $x = (x', x_n) \in \mathbb{R}_+^n$ with $x' \in \mathbb{R}^{n-1}$. With the corresponding meaning, we write v', ∇', etc. The pull-back is then defined as

$$v := \Theta^* u := u \circ \theta, \qquad p := \Theta^* q := q \circ \theta,$$

and correspondingly the push-forward as

$$u := \Theta_* v := v \circ \theta^{-1}, \qquad q := \Theta_* p := p \circ \theta^{-1}.$$

We also set $\Gamma_0 = \Gamma(0) = \{(x', \eta_0(x')); \; x' \in \mathbb{R}^{n-1}\}$ and $V_0 = V_\Gamma(0) = (0, \eta_1(\cdot))^\tau$.

Applying the transform of variables to (1.1) leads to the following quasilinear system for (v, p, η):

$$\begin{aligned}
\partial_t v - \Delta v + \nabla p &= F_v(v, p, \eta) & \text{in} \quad & J \times \mathbb{R}_+^n, \\
\operatorname{div} v &= G(v, \eta) & \text{in} \quad & J \times \mathbb{R}_+^n, \\
v' &= 0 & \text{on} \quad & J \times \mathbb{R}^{n-1}, \\
\partial_t \eta - v^n &= 0 & \text{on} \quad & J \times \mathbb{R}^{n-1}, \\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta &= H_\eta(v, \eta) & \text{on} \quad & J \times \mathbb{R}^{n-1}, \\
v|_{t=0} &= v_0 & \text{in} \quad & \mathbb{R}_+^n, \\
\eta|_{t=0} &= \eta_0 & \text{in} \quad & \mathbb{R}^{n-1}, \\
\partial_t \eta|_{t=0} &= \eta_1 & \text{in} \quad & \mathbb{R}^{n-1}.
\end{aligned} \tag{2.3}$$

The nonlinear right-hand sides are given as

$$F_v(v,p,\eta) = (\partial_t \eta - \Delta'\eta)\partial_n v - 2(\nabla'\eta \cdot \nabla')\partial_n v + |\nabla'\eta|^2 \partial_n^2 v$$
$$-(v \cdot \nabla)v + (v' \cdot \nabla'\eta)\partial_n v + (\nabla'\eta, 0)^\tau \partial_n p,$$
$$G(v,\eta) = \nabla'\eta \cdot \partial_n v',$$
$$H_\eta(v,\eta) = -\nabla'\eta \cdot \partial_n v' - \nabla'\eta \cdot \nabla' v^n.$$

3. The linearized system

The aim of this section is to derive maximal regularity for the linearized system

$$\begin{aligned}
\partial_t v - \Delta v + \nabla p &= f_v \text{ in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
\operatorname{div} v &= g \text{ in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
v' &= 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
\partial_t \eta - v^n &= 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
-2\partial_n v^n + p - m(\partial_t, \partial')\eta &= f_\eta \text{ on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
v|_{t=0} &= v_0 \text{ in } \mathbb{R}_+^n, \\
\eta|_{t=0} &= \eta_0 \text{ in } \mathbb{R}^{n-1}, \\
\partial_t \eta|_{t=0} &= \eta_1 \text{ in } \mathbb{R}^{n-1}.
\end{aligned}$$
(3.1)

In the sequel for $k \in \mathbb{N}_0$, $1 < p < \infty$, a domain $\Omega \subset \mathbb{R}^n$, and a Banach space X,

$$H_p^k(\Omega, X) := \left\{ u : \Omega \to X; \|u\|_{H_p^k} < \infty \right\}, \quad \|u\|_{H_p^k}^p := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}^p$$

denotes the standard X-valued Sobolev space. Here, $L^p(\Omega, X)$ denotes the standard Bochner–Lebesgue space. We also put $W_p^k(\Omega, X) := H_p^k(\Omega, X)$ for $k \in \mathbb{N}_0$. For $s > 0$, $s \notin \mathbb{N}$, Sobolev (or Bessel potential) and Sobolev–Slobodeckii spaces of fractional order are defined via complex and real interpolation, i.e., by

$$H_p^s(\Omega, X) := \left[H_p^k(\Omega, X), H_p^{k+1}(\Omega, X) \right]_{s-k} \text{ and}$$
$$W_p^s(\Omega, X) := \left(H_p^k(\Omega, X), H_p^{k+1}(\Omega, X) \right)_{s-k, p},$$

respectively, where $k < s < k+1$. Also as usual, we set $H_{p,0}^s(\Omega, X) := \overline{C_c^\infty(\Omega, X)}^{H_p^s(\Omega, X)}$ and $W_{p,0}^s(\Omega, X) := \overline{C_c^\infty(\Omega, X)}^{W_p^s(\Omega, X)}$, where C_c^∞ stands for the space of smooth and compactly supported functions in Ω. In case $X = \mathbb{R}^n$, corresponding dual spaces are defined as

$$H_p^{-k}(\Omega) := \left(H_{p',0}^k(\Omega) \right)' \quad \text{and} \quad H_{p,0}^{-k}(\Omega) := \left(H_{p'}^k(\Omega) \right)',$$

where $1/p + 1/p' = 1$. Accordingly, the spaces $W_p^{-k}(\Omega)$ and $W_{p,0}^{-k}(\Omega)$ are defined. If $\Omega = J = (0, T)$ is an interval, we also set

$$_0 H_p^s(J, X) := \overline{C_c^\infty((0, T], X)}^{H_p^s(\Omega, X)}$$

and $_0 W_p^s(J, X)$ accordingly. Observe that then we have $_0 H_p^1(J, X) = \{u \in H_p^1(J, X); u(0) = 0\}$. As references for vector-valued scales of Sobolev spaces, we mention [2], Chapter VII, and [13], Chapter 2.

We will consider system (3.1) in spaces with exponential weight with respect to the time variable. Let $\rho \in \mathbb{R}$ and X be a Banach space. For $u \in L^p(\mathbb{R}_+, X)$, we define Ψ_ρ as the multiplication operator with $e^{-\rho t}$, i.e., $\Psi_\rho u(t) := e^{-\rho t} u(t)$, $t \in \mathbb{R}_+$. The spaces with exponential weights are defined by

$$H_{p,\rho}^s(\mathbb{R}_+, X) := \Psi_{-\rho}(H_p^s(\mathbb{R}_+, X)),$$
$$W_{p,\rho}^s(\mathbb{R}_+, X) := \Psi_{-\rho}(W_p^s(\mathbb{R}_+, X))$$

with canonical norms $\|u\|_{H_{p,\rho}^s(\mathbb{R}_+, X)} := \|\Psi_\rho u\|_{H_p^s(\mathbb{R}_+, X)}$ and $\|u\|_{W_{p,\rho}^s(\mathbb{R}_+, X)} := \|\Psi_\rho u\|_{W_p^s(\mathbb{R}_+, X)}$. For $\rho \geq 0$ and $s > 0$, we define $_0 H_{p,\rho}^s(\mathbb{R}_+, X)$ and $_0 W_{p,\rho}^s(\mathbb{R}_+, X)$ analogously, replacing H_p^s and W_p^s by $_0 H_p^s$ and

$_0W_p^s$, respectively. For mapping properties and interpolation results under the condition that X is a UMD space, we refer, e.g., to [9], Lemma 2.2. We also make use of homogeneous spaces, e.g., for $\Omega \subset \mathbb{R}^n$ we set

$$\dot{H}_p^1(\Omega) := \{v \in L_{\mathrm{loc}}^1(\Omega) : \ \nabla v \in L^p(\Omega)\}, \quad \|v\|_{\dot{H}_p^1} := \|\nabla v\|_p,$$

and $\dot{H}_{p,0}^1(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\nabla \cdot\|_p}$. The corresponding dual spaces are defined as

$$\dot{H}_p^{-1}(\Omega) := \left(\dot{H}_{p',0}^1(\Omega)\right)' \quad \text{and} \quad \dot{H}_{p,0}^{-1}(\Omega) := \left(\dot{H}_{p'}^1(\Omega)\right)',$$

see [22] Section 7.2. The homogeneous Sobolev–Slobodeckii spaces $\dot{W}_p^s(\mathbb{R}^n)$ contain all functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|u\|_{\dot{W}_p^s}^p = \sum_{|\alpha|=[s]} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial^\alpha(v(x) - v(y))|^p}{|x-y|^{n+(s-[s])p}} \, dx \, dy < \infty,$$

where $[s] = \max\{k \in \mathbb{N}_0; \ k < s\}$, see [26]. Note that we have

$$\dot{W}_p^s(\mathbb{R}^n) = \dot{B}_{pp}^s(\mathbb{R}^n)$$

for $1 < p < \infty$, $n \in \mathbb{N}$, and $s \in \mathbb{R} \backslash \mathbb{Z}$, where the latter one denotes the homogeneous Besov space.

We refer to the pertinent monographs [1,5,24] for the scalar case and [2,13] for the X-valued case for properties, characterizations, and relations of the just introduced spaces.

In the following, we denote the time trace $u \mapsto \partial_t^k u|_{t=0}$ by γ_k^t and the trace to the boundary $u \mapsto \partial_n^k u|_{\mathbb{R}^{n-1}}$ by γ_k. We set $J = (0, T)$ for $T > 0$. The solution (v, p, η) of (3.1) will belong to the spaces

$$v \in \mathbb{E}_v := H_{p,\rho}^1(J; L^p(\mathbb{R}_+^n)) \cap L_\rho^p(J; H_p^2(\mathbb{R}_+^n)),$$

$$p \in \mathbb{E}_p := L_\rho^p(J; \dot{H}_p^1(\mathbb{R}_+^n)),$$

$$\eta \in \mathbb{E}_\eta := W_{p,\rho}^{9/4-1/(4p)}(J; L^p(\mathbb{R}^{n-1})) \cap H_{p,\rho}^2(J; W_p^{1-1/p}(\mathbb{R}^{n-1}))$$

$$\cap L_\rho^p(J; W_p^{5-1/p}(\mathbb{R}^{n-1})).$$

The function spaces for the right-hand side of (3.1) are given by

$$f_v \in \mathbb{F}_v := L_\rho^p(J; L^p(\mathbb{R}_+^n)),$$

$$g \in \mathbb{F}_g := H_{p,\rho}^1(J; \dot{H}_p^{-1}(\mathbb{R}_+^n)) \cap L_\rho^p(J; H_p^1(\mathbb{R}_+^n)),$$

$$f_\eta \in \gamma_0 \mathbb{E}_p := L_\rho^p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^{n-1})).$$

By trace results with respect to the time trace, the spaces for the initial values are given by

$$v_0 \in \gamma_0^t \mathbb{E}_v := W_p^{2-2/p}(\mathbb{R}_+^n),$$

$$\eta_0 \in \gamma_0^t \mathbb{E}_\eta := W_p^{5-3/p}(\mathbb{R}^{n-1}),$$

$$\eta_1 \in \gamma_1^t \mathbb{E}_\eta := W_p^{3-3/p}(\mathbb{R}^{n-1}),$$

see the proof of Theorem 3.1 below (necessity part). Note also that in this section we have $T = \infty$ and that we skipped indicating the ρ dependence in $\mathbb{E}_v$, $\mathbb{E}_p$, etc., since we only deal with weighted time-dependent spaces for the rest of this section. We will also need the following compatibility conditions:

(C1) $\operatorname{div} v_0 = g|_{t=0}$ in $\dot{H}_p^{-1}(\mathbb{R}_+^n)$.
(C2) If $p > \frac{3}{2}$, then $v_0'|_{\mathbb{R}^{n-1}} = 0$ almost everywhere in $\mathbb{R}^{n-1}$.
(C3) If $p > \frac{3}{2}$, then $v_0^n|_{\mathbb{R}^{n-1}} - \eta_1 = 0$ almost everywhere in $\mathbb{R}^{n-1}$.
(C4) There exists an $\eta_* \in \mathbb{E}_\eta$ with $\eta_*|_{t=0} = \eta_0$, $\partial_t \eta_*|_{t=0} = \eta_1$ and

$$(g, \partial_t \eta_*) \in H_{p,\rho}^1(J; \dot{H}_{p,0}^{-1}(\mathbb{R}_+^n)). \tag{3.2}$$

Here, we define

$$(g, \partial_t \eta_*)(\phi) := \int\limits_{\mathbb{R}^n_+} g\phi dx - \int\limits_{\mathbb{R}^{n-1}} \partial_t \eta_* \phi dx'$$

for $\phi \in \dot{H}^1_{p'}(\mathbb{R}^n_+)$. Additionally, we have $(g|_{t=0}, \eta_1) = (g|_{t=0}, v_0^n|_{\mathbb{R}^{n-1}})$ in $\dot{H}^{-1}_{p,0}(\mathbb{R}^n_+)$.

We remark that only (3.2) is an additional condition, as it was shown in [9], Theorem 4.5, that for every $\eta_0 \in \gamma_0^t \mathbb{E}_\eta$ and $\eta_1 \in \gamma_1^t \mathbb{E}_\eta$ there exists an $\eta_* \in \mathbb{E}_\eta$ with $\eta_*|_{t=0} = \eta_0$ and $\partial_t \eta_*|_{t=0} = \eta_1$.

The main result of this section is the following maximal regularity result.

Theorem 3.1. *Let $p > 1$, $p \neq 3/2$, and $T = \infty$. Then, there exists a $\rho_0 > 0$ such that for every $\rho \geq \rho_0$, system (3.1) has a unique solution $(v, p, \eta) \in \mathbb{E}_v \times \mathbb{E}_p \times \mathbb{E}_\eta$ if and only if the data $f_v, g, f_\eta, v_0, \eta_0, \eta_1$ belong to the spaces above and satisfy the compatibility conditions* (C1)–(C4). *The solution depends continuously on the data.*

The proof of this theorem will be done in several steps and follows from Sects. 3.1–3.4.

3.1. Necessity

Let $(v, p, \eta) \in \mathbb{E}_v \times \mathbb{E}_p \times \mathbb{E}_\eta$ be a solution of (3.1). By standard continuity and trace results, the right-hand sides f_v, and g as well as the time trace v_0 belong to the spaces above. Noting that $\mathrm{div}: L^p(\mathbb{R}^n_+) \to \dot{H}^{-1}_p(\mathbb{R}^n_+)$ is continuous, we have $g = \mathrm{div}\, u \in H^1_{p,\rho}(\mathbb{R}_+; \dot{H}^{-1}_p(\mathbb{R}^n_+)) \subset C([0, \infty); \dot{H}^{-1}_p(\mathbb{R}^n_+))$, and as for all $p > 1$ we also have $v_0 \in W^{2-2/p}_p(\mathbb{R}^n_+) \subset L^p(\mathbb{R}^n_+)$, we obtain the compatibility condition (C1) for all $p > 1$ (see also [22], Theorem 7.2.1).

For f_η, note that we have $\mathbb{E}_\eta \subset H^1_{p,\rho}(\mathbb{R}_+; W^{3-1/p}_p(\mathbb{R}^{n-1}))$ by the mixed derivative theorem (see, e.g., [9], Lemma 4.3), and therefore

$$\partial_t \Delta' \eta \in L^p_\rho(\mathbb{R}_+; W^{1-1/p}_p(\mathbb{R}^{n-1})) \subset \gamma_0 \mathbb{E}_p.$$

It is easy to see that the other terms of $m(\partial_t, \partial')\eta$ belong to the same space. By standard trace results, we also obtain $\gamma_1 u \in \gamma_0 \mathbb{E}_p$. Concerning the pressure, we remark that $\gamma_0: \dot{H}^1_p(\mathbb{R}^n_+) \to \dot{W}^{1-1/p}_p(\mathbb{R}^{n-1})$ is a retraction, see, e.g., [14], Theorem 2.1, and therefore $\gamma_0 p \in \gamma_0 \mathbb{E}_p$. This yields $f_\eta \in \gamma_0 \mathbb{E}_p$. For the time traces of η, by putting $\mathcal{F} = \mathcal{K} = W$ and

$$s_1 = 0,\ r_1 = 5 - 1/p,\ s_2 = 2,\ r_2 = 1 - 1/p,\ s_3 = 9/4 - 1/4p,\ r_3 = 0,$$

we can apply [9], Theorem 4.5 which gives $\eta_0 \in \gamma_0^t \mathbb{E}_\eta$ and $\eta_1 \in \gamma_1^t \mathbb{E}_\eta$.

If $p > \frac{3}{2}$, then the boundary trace of v_0 exists in the space $W^{2-3/p}_p(\mathbb{R}^{n-1})$. This yields the compatibility conditions (C2) and (C3) as equality in the space $W^{2-3/p}_p(\mathbb{R}^{n-1})$, hence in particular as equality almost everywhere.

To show (C4), we can set $\eta_* := \eta$. For $\phi \in \dot{H}^1_{p'}(\mathbb{R}^n_+)$, we obtain

$$(g, \partial_t \eta)(\phi) = \int\limits_{\mathbb{R}^n_+} \mathrm{div}\, u\, \phi dx - \int\limits_{\mathbb{R}^{n-1}} u^n \phi dx' = -\int\limits_{\mathbb{R}^n_+} u \cdot \nabla \phi dx$$

and therefore $(g, \partial_t \eta) \in H^1_{p,\rho}(\mathbb{R}_+; \dot{H}^{-1}_{p,0}(\mathbb{R}^n_+))$. Setting $t = 0$, we obtain $(g|_{t=0}, \eta_1) = (g|_{t=0}, v_0^n)$ as equality in $\dot{H}^{-1}_{p,0}(\mathbb{R}^n_+)$.

3.2. Reductions

We can reduce some part of the right-hand side of (3.1) to zero by applying known results on the Stokes system. For this, let $(v^{(1)}, p^{(1)}) \in \mathbb{E}_v \times \mathbb{E}_p$ be the unique solution of the Stokes problem in the half space

$$
\begin{aligned}
\partial_t v^{(1)} - \Delta v + \nabla p^{(1)} &= f_v && \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
\operatorname{div} v^{(1)} &= g && \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
(v^{(1)})' &= 0 && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
(v^{(1)})^n &= \partial_t \eta_* && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
v^{(1)}|_{t=0} &= v_0 && \text{in } \mathbb{R}_+^n.
\end{aligned}
\tag{3.3}
$$

The unique solvability of (3.3) follows from [22], Theorem 7.2.1. To show that this theorem can be applied, we remark in particular that the compatibility condition (e) in [22, p. 324] holds because of (C4). Moreover, by the embedding $\mathbb{E}_\eta \subset H_{p,\rho}^{2-1/(2p)}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1}) \cap H_{p,\rho}^1(\mathbb{R}_+; W_p^{2-1/p}(\mathbb{R}^{n-1}))$ and the compatibility condition (C3), we see that also the compatibility condition (d0) in [22, p. 324] holds.

Let $\tilde{v} := v - v^{(1)}$, $\tilde{p} := p - p^{(1)}$, and $\tilde{\eta} := \eta - \eta_*$. Then, (v, p, η) is a solution of (3.1) if and only if $(\tilde{v}, \tilde{p}, \tilde{\eta})$ is a solution of

$$
\begin{aligned}
\partial_t \tilde{v} - \Delta \tilde{v} + \nabla \tilde{p} &= 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
\operatorname{div} \tilde{v} &= 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n, \\
\tilde{v}' &= 0 && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
\tilde{v}^n - \partial_t \tilde{\eta} &= 0 && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
-2\partial_n \tilde{v}^n + \tilde{p} - m(\partial_t, \partial')\tilde{\eta} &= \tilde{f}_\eta && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\
\tilde{v}|_{t=0} &= 0 && \text{in } \mathbb{R}_+^n, \\
\tilde{\eta}|_{t=0} &= 0 && \text{in } \mathbb{R}^{n-1}, \\
\partial_t \tilde{\eta}|_{t=0} &= 0 && \text{in } \mathbb{R}^{n-1}.
\end{aligned}
\tag{3.4}
$$

Here,

$$
\tilde{f}_\eta := f_\eta + 2\partial_n (v^{(1)})^n - p^{(1)} + m(\partial_t, \partial')\eta_*.
$$

By the trace results in Subsection 3.1, we have $\tilde{f}_\eta \in \gamma_0 \mathbb{E}_p$.

3.3. Solution operators for the reduced linearized problem

In the following, we show solvability for the reduced problem (3.4), omitting the tilde again. An application of the Laplace transform formally leads to the resolvent problem

$$
\begin{aligned}
\lambda v - \Delta v + \nabla p &= 0 && \text{in } \mathbb{R}_+^n, \\
\operatorname{div} v &= 0 && \text{in } \mathbb{R}_+^n, \\
v' &= 0 && \text{on } \partial\mathbb{R}_+^n, \\
v^n - \lambda \eta &= 0 && \text{on } \partial\mathbb{R}_+^n, \\
-2\partial_n v^n + p - m(\lambda, \partial')\eta &= f_\eta && \text{on } \partial\mathbb{R}_+^n
\end{aligned}
\tag{3.5}
$$

with

$$
m(\lambda, \partial')\eta = \lambda^2 \eta + \alpha(\Delta')^2 \eta - \beta \Delta' \eta - \gamma \lambda \Delta' \eta.
$$

We observe that the second and the third line of (3.5) imply that

$$
\partial_n v^n(\cdot, 0) = -\nabla' \cdot v'(\cdot, 0) = 0.
$$

Hence, the fifth line reduces to

$$
p - m(\lambda, \partial')\eta = f_\eta \quad \text{on } \partial\mathbb{R}_+^n.
$$

Applying partial Fourier transform in $x' \in \mathbb{R}^{n-1}$, we obtain the following system of ordinary differential equations in x_n for the transformed functions $\hat{v}$, $\hat{p}$ and $\hat{\eta}$:

$$\omega^2\hat{v} - \partial_n^2\hat{v} + (i\xi', \partial_n)^\tau\hat{p} = 0, \quad x_n > 0,$$
$$i\xi \cdot \hat{v}' + \partial_n\hat{v}^n = 0, \quad x_n > 0,$$
$$\hat{v}' = 0, \quad x_n = 0,$$
$$\lambda\hat{\eta} - \hat{v}^n = 0, \quad x_n = 0,$$
$$\hat{p} - m(\lambda, |\xi'|)\hat{\eta} = \hat{f}_\eta, \quad x_n = 0,$$

Here, we have set $\omega := \omega(\lambda, \xi') := \sqrt{\lambda + |\xi'|^2}$ and

$$m(\lambda, \xi') := \lambda^2 + \alpha|\xi'|^4 + \gamma\lambda|\xi'|^2 + \beta|\xi'|^2.$$

Multiplying the first equation with $(i\xi', \partial_n)$ and combining it with the second one yields $(-|\xi'|^2 + \partial_n^2)\hat{p} = 0$ for $x_n > 0$. The only stable solution of this equation is given by

$$\hat{p}(\xi', x_n) = \hat{p}_0(\xi')e^{-|\xi'|x_n}, \quad \xi' \in \mathbb{R}^{n-1}, \; x_n > 0. \tag{3.6}$$

Putting the pressure term on the right-hand side, v formally solves a vector-valued heat equation. Hence, to solve the above system we employ the ansatz

$$\hat{v}'(\xi', x_n) = -\int_0^\infty k_+(\lambda, \xi', x_n, s)i\xi'\hat{p}(\xi', s)ds + \hat{\phi}'(\xi')e^{-\omega x_n}, \tag{3.7}$$

$$\hat{v}^n(\xi', x_n) = -\int_0^\infty k_-(\lambda, \xi', x_n, s)\partial_n\hat{p}(\xi', s)ds + \hat{\phi}^n(\xi')e^{-\omega x_n} \tag{3.8}$$

with the Green functions subject to Dirichlet resp. Neumann conditions

$$k_\pm(\lambda, \xi, x_n, s) := \frac{1}{2\omega}\left(e^{-\omega|x_n-s|} \pm e^{-\omega(x_n+s)}\right).$$

Here, the traces $\hat{p}_0$ and $\hat{\phi} = (\hat{\phi}', \hat{\phi}^n)^\tau$ still have to be determined. Note that by choosing k_+ in tangential and k_- in normal components, the integral parts in formulas (3.7) and (3.8) have vanishing divergence. This follows by a straight-forward calculation, see, e.g., [12], Section 2.6. Thus, $\text{div } v = 0$ enforces

$$i\xi' \cdot \hat{\phi}'(\xi') = \omega\hat{\phi}^n(\xi'). \tag{3.9}$$

The kinematic boundary condition instantly gives us

$$\lambda\hat{\eta} - \hat{\phi}^n = 0. \tag{3.10}$$

Next, by utilizing (3.6), from the tangential boundary condition we obtain

$$0 = \hat{v}'(\xi', 0) = -\int_0^\infty \frac{e^{-\omega s}}{\omega}i\xi'\hat{p}(\xi', s)ds + \hat{\phi}'(\xi'),$$

which implies

$$\frac{i\xi'}{\omega + |\xi'|}\hat{p}_0 = \omega\hat{\phi}'. \tag{3.11}$$

Multiplying this with $i\xi'$ and employing the relations (3.9), (3.10) yields

$$-\frac{|\xi'|^2}{\omega + |\xi'|}\hat{p}_0 = \omega^2\hat{\phi}^n = \lambda\omega^2\hat{\eta}. \tag{3.12}$$

Plugging this into the last line of the transformed system, we obtain

$$\left(\frac{\lambda \omega^2 (\omega + |\xi'|)}{|\xi'|^2} + m(\lambda, \xi') \right) \hat{\eta} = -\hat{f}_\eta. \tag{3.13}$$

This yields

$$\hat{\eta} = -\frac{|\xi'|^2}{N_L(\lambda, |\xi'|)} \hat{f}_\eta \tag{3.14}$$

with

$$N_L(\lambda, |\xi'|) = |\xi'|^2 m(\lambda, \xi') + \lambda \omega^2 (\omega + |\xi'|).$$

Formula (3.14) defines the solution operator for η as a function of f_η on the level of its Fourier–Laplace transform. The following result is based on the Newton polygon approach and shows that the solution operator is continuous on the related Sobolev spaces. In the following, we consider $(-\Delta')^{1/2}$ as an unbounded operator in $L_\rho^p(\mathbb{R}_+; L^p(\mathbb{R}^{n-1}))$ and define $N_L(\partial_t, (-\Delta')^{1/2})$ by the joint H^∞-calculus of ∂_t and $(-\Delta')^{1/2}$ (for details, we refer to, e.g., [9], Corollary 2.9). We will apply the Newton polygon approach on the Bessel potential scale H_p^s with respect to time and on the Besov scale B_{pp}^r with respect to space.

Lemma 3.2. *(a) There exists a $\rho_0 > 0$ such that for all $\rho \geq \rho_0$, the operator $N_L(\partial_t, (-\Delta')^{1/2}) \colon H_N \to L_\rho^p(\mathbb{R}_+; B_{pp}^{-1-1/p}(\mathbb{R}^{n-1}))$ is an isomorphism, where*

$$H_N := {}_0H_{p,\rho}^{5/2}(\mathbb{R}_+; B_{pp}^{-1-1/p}(\mathbb{R}^{n-1})) \cap {}_0H_{p,\rho}^2(\mathbb{R}_+; B_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$$

$$\cap L_\rho^p(\mathbb{R}_+; B_{pp}^{5-1/p}(\mathbb{R}^{n-1})).$$

(b) Let $\rho \geq \rho_0$. Then, for every $f_\eta \in \gamma_0 \mathbb{E}_p$, we have

$$\eta := \Delta' \big[N_L(\partial_t, (-\Delta')^{1/2}) \big]^{-1} f_\eta \in \mathbb{E}_\eta,$$

$$\phi^n := \partial_t \eta \in \gamma_0 \mathbb{E}_v,$$

$$p_0 := f_\eta + m(\partial_t, \partial')\eta \in \gamma_0 \mathbb{E}_p.$$

Proof. (a) We apply the Newton polygon approach developed in [9]. Replacing $z = |\xi'|$, the r-principle symbols, i.e., the leading terms of N_L associated with the relation $\lambda \sim z^r$ are easily calculated as

$$P_r(\lambda, z) = \begin{cases} \alpha z^6, & 0 < r < 2, \\ m_0(\lambda, z)z^2, & r = 2, \\ \lambda^2 z^2, & 2 < r < 4, \\ \lambda^2 z^2 + \lambda^{5/2}, & r = 4, \\ \lambda^{5/2}, & r > 4, \end{cases}$$

where $m_0 = m$ for $\beta = 0$, that is

$$m_0(\lambda, z) := \lambda^2 + \alpha z^4 + \gamma \lambda z^2.$$

In other words, the associated Newton polygon has the three relevant vertices $(6, 0)$, $(2, 2)$, and $(0, \frac{5}{2})$ and two relevant edges which again reflects the quasi-homogeneity of N_L.

Now, let $\varphi \in (0, \pi/2)$ and $\theta \in (0, \varphi/4)$ and put

$$\Sigma_\psi := \{z \in \mathbb{C} \backslash \{0\}; \ |\arg z| < \psi\}, \quad 0 < \psi < \pi.$$

For $r \neq 2$, we then obviously have

$$P_r(\lambda, z) \neq 0 \quad ((\lambda, z) \in \Sigma_{\pi - \varphi} \times \Sigma_\theta). \tag{3.15}$$

For $r = 2$, we deduce

$$P_2(\lambda, z) = 0 \quad \Leftrightarrow \quad \lambda = \frac{z^2}{2} \left(-\gamma \mp \sqrt{\gamma^2 - 4\alpha} \right).$$

By the fact that $\gamma > 0$, we see that

$$\varphi_0 := \pi - \arg\left(-\gamma \mp \sqrt{\gamma^2 - 4\alpha}\right) < \frac{\pi}{2}.$$

Thus, assuming $\varphi \in (\varphi_0, \pi/2)$ and $\theta \in (0, (\varphi - \varphi_0)/4)$ we see that (3.15) is satisfied for all $r > 0$. This allows for the application of [9, Theorem 3.3] (setting $s = 0$ and $r = -1 - 1/p$ in the notation of [9]) which yields (a).

(b) By (3.14), we have

$$\eta = \left[N_L(\partial_t, (-\Delta')^{1/2})\right]^{-1} \Delta' f_\eta.$$

As Δ' is an isomorphism from $\dot{H}_p^{2+t}(\mathbb{R}^{n-1})$ to $\dot{H}_p^t(\mathbb{R}^{n-1})$ for each $t \in \mathbb{R}$, by real interpolation of these spaces (see [14], Lemma 1.1) we see that it is also an isomorphism from $\dot{B}_{pp}^t(\mathbb{R}^{n-1})$ to $\dot{B}_{pp}^{t-2}(\mathbb{R}^{n-1})$ for each $t \in \mathbb{R}$. In particular, $\Delta' f_\eta \in L_\rho^p(\mathbb{R}_+; \dot{B}_{pp}^{-1-1/p}(\mathbb{R}^{n-1}))$. Using the fact that for $s < 0$ the embedding $\dot{B}_{pp}^s(\mathbb{R}^{n-1}) \subset B_{pp}^s(\mathbb{R}^{n-1})$ holds (see [25, p. 104, (3.339)], [26, Section 3.1]), we obtain the embedding

$$L_\rho^p(\mathbb{R}_+; \dot{B}_{pp}^{-1-1/p}(\mathbb{R}^{n-1})) \subset L_\rho^p(\mathbb{R}_+; B_{pp}^{-1-1/p}(\mathbb{R}^{n-1})).$$

An application of (a) yields

$$\eta \in {}_0H_{p,\rho}^{5/2}(\mathbb{R}_+; B_{pp}^{-1-1/p}(\mathbb{R}^{n-1})) \cap {}_0H_{p,\rho}^2(\mathbb{R}_+; B_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$$
$$\cap L_\rho^p(\mathbb{R}_+; B_{pp}^{5-1/p}(\mathbb{R}^{n-1})).$$

Now, the mixed derivative theorem in mixed scales (see [8], Proposition 2.76) implies

$${}_0H_{p,\rho}^{5/2}(\mathbb{R}_+; B_{pp}^{-1-1/p}(\mathbb{R}^{n-1})) \cap {}_0H_{p,\rho}^2(\mathbb{R}_+; B_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$$
$$\subset B_{pp,\rho}^{9/4-1/(4p)}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1}))$$

and we obtain $\eta \in \mathbb{E}_\eta$.

For $u^n := \partial_t \eta$, we immediately get

$$u^n \in W_{p,\rho}^{5/4-1/(4p)}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1})) \cap L_\rho^p(\mathbb{R}_+; W_p^{3-1/p}(\mathbb{R}^{n-1}) \subset \gamma_0 \mathbb{E}_v.$$

Finally, the fact that $m(\partial_t, \partial')\eta \in \gamma_0 \mathbb{E}_p$ for $\eta \in \mathbb{E}_\eta$ was already remarked in Subsection 3.1. $\square$

Due to the last result, we obtain the existence of a solution (v, p, η) of (3.4). In fact, for η, ϕ^n, and p_0 defined as in Lemma 3.2(b), we can define p and v by (the Laplace and Fourier inverse transform of) (3.6) and (3.7)–(3.8), respectively. Here, ϕ' is given by (3.11). As we know that ϕ^n and p_0 belong to the canonical spaces by Lemma 3.2(b), we get $v \in \mathbb{E}_v$ and $p \in \mathbb{E}_p$ by standard results on the Stokes equation (see, e.g., [12], Section 2.6, and [22], Section 7.2). By construction, (v, p, η) is a solution of (3.4).

3.4. Uniqueness of the solution

To show that the solution of (3.1) is unique, let (v, p, η) be a solution with zero right-hand side and zero initial data. Then, the Laplace transform in t and partial Fourier transform in x' is well-defined, and the calculations above show, in particular, that

$$\hat{\eta} = -\frac{|\xi'|^2}{N_L(\lambda, |\xi'|)} \hat{f}_\eta = 0$$

for almost all $\xi' \in \mathbb{R}^{n-1}$. Therefore, $\eta = 0$ which implies that (v, p) is the solution of the Dirichlet Stokes system with zero data. Therefore, $v = 0$ and $p = 0$.

This finishes the proof of Theorem 3.1.

Remark 3.3. Theorem 3.1 was formulated on the infinite time interval $(0, \infty)$ with exponentially weighted spaces with respect to t. As usual in the theory of maximal regularity, we obtain the same results on finite time intervals $t \in J = (0, T)$ with $T < \infty$ without weights, i.e., with $\rho = 0$. This is due to the fact that on finite time intervals the weighted and unweighted norms are equivalent and that there exists an extension operator from $(0, T)$ to $(0, \infty)$ acting on all spaces above.

Therefore, the results of Theorem 3.1 hold with $\rho = 0$ on the finite interval $J = (0, T)$. As we consider the nonlinear equation on a finite time interval, we will replace the function spaces above by $\mathbb{E}_v := H^1(J; L^p(\mathbb{R}^n_+)) \cap L^p(J; H^2_p(\mathbb{R}^n_+))$, etc., keeping the same notation.

4. The nonlinear system

To prove mapping properties of the nonlinearities, we employ sharp estimates for anisotropic function spaces provided in [15]. In fact, we can proceed very similar as in [15, Section 5.2, Proposition 5.6]. For $\omega_j \in \mathbb{N}_0$, $j = 1, \ldots, \nu$, we define a weight vector as $\omega := (\omega_1, \ldots, \omega_\nu)$ and denote by $\dot{\omega} := \mathrm{lcm}\{\omega_1, \ldots, \omega_\nu\}$ the lowest common multiple. Further, for $n = (n_1, \ldots, n_\nu) \in \mathbb{N}^\nu$ we write

$$\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_\nu}.$$

The (generalized) Sobolev index of an E-valued anisotropic function space then reads as

$$\frac{1}{\dot{\omega}}\left(s - \frac{\omega \cdot n}{p}\right) =: \begin{cases} \mathrm{ind}(B^{s,\omega}_{p,q}(\mathbb{R}^n, E)), \ s \in \mathbb{R}, \ 1 < p < \infty, \ 1 \le q \le \infty, \\ \mathrm{ind}(H^{s,\omega}_p(\mathbb{R}^n, E)), \quad -\infty < s < \infty, \ 1 < p < \infty, \\ \mathrm{ind}(W^{s,\omega}_p(\mathbb{R}^n, E)), \quad\ \ 0 \le s < \infty, \ 1 \le p < \infty, \end{cases}$$

where $\omega \cdot n = \sum_{j=1}^\nu \omega_j n_j$. Note that we have the corresponding definition, if $\mathbb{R}^n$ is replaced by a Cartesian product of intervals. For an introduction to anisotropic spaces above, we refer to [2,15] and the references cited therein. In particular, it is possible to represent the anisotropic spaces as intersections (see [2, Section VII.3.5]). In the situation considered here, we always have $\omega = (2, 1)$, and we obtain the equality

$$H^{s,(2,1)}_p(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))$$
$$= H^{s/2}_p(J, L^p(\mathbb{R}^{n-1}, L^p(\mathbb{R}_+))) \cap L^p(J, H^s_p(\mathbb{R}^{n-1}, L^p(\mathbb{R}_+)))$$

for $s > 0$, which can be seen as the definition of the anisotropic space. The analog representation holds for the anisotropic scales $W^{s,(2,1)}_p$ and $B^{s,(2,1)}_{p,q}$.

$$\mathrm{ind}\big(H^{1,(2,1)}_p(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))\big) = \frac{1}{2}\left(1 - \frac{2+n-1}{p}\right) = \frac{1}{2} - \frac{n+1}{2p}.$$

Now, let $J = (0, T)$. By the mixed derivative theorem, see, e.g., [9, Lemma 4.3], we have

$$H^2_p(J, W^{1-1/p}_p(\mathbb{R}^{n-1})) \cap L^p(J, W^{5-1/p}_p(\mathbb{R}^{n-1})) \hookrightarrow H^1_p(J, W^{3-1/p}_p(\mathbb{R}^{n-1})).$$

This yields

$$\partial_t \eta \in W^{5/4-1/4p}_p(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W^{3-1/p}_p(\mathbb{R}^{n-1}))$$
$$\hookrightarrow W^{1-1/2p}_p(J, L^p(\mathbb{R}^{n-1})) \cap L^p(J, W^{2-1/p}_p(\mathbb{R}^{n-1})) \tag{4.1}$$
$$= W^{2-1/p,(2,1)}_p(J \times \mathbb{R}^{n-1})$$

for $\eta \in \mathbb{E}_3$. Again by the mixed derivative theorem, we have

$$H^2_p(J, W^{1-1/p}_p(\mathbb{R}^{n-1})) \cap L^p(J, W^{5-1/p}_p(\mathbb{R}^{n-1})) \hookrightarrow W^{2-1/2p}_p(J, H^1_p(\mathbb{R}^{n-1})),$$

which gives us

$$\partial_j \eta \in W^{2-1/2p}_p(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W^{4-1/p}_p(\mathbb{R}^{n-1}))$$
$$= W^{4-1/p,(2,1)}_p(J \times \mathbb{R}^{n-1}) \tag{4.2}$$

for $\eta \in \mathbb{E}_3$ and $j = 1, \ldots, n-1$. Analogously, we obtain that

$$
\begin{aligned}
\partial_j \partial_k \eta &\in W_p^{3/2-1/2p}(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W_p^{3-1/p}(\mathbb{R}^{n-1})) \\
&= W_p^{3-1/p,(2,1)}(J \times \mathbb{R}^{n-1}) \\
&\hookrightarrow W_p^{2-1/p,(2,1)}(J \times \mathbb{R}^{n-1})
\end{aligned}
\tag{4.3}
$$

for $\eta \in \mathbb{E}_3$ and $j, k = 1, \ldots, n-1$.

For the velocity, we have

$$
v \in H_p^{2,(2,1)}(J \times \mathbb{R}_+^n) \hookrightarrow H_p^{2,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+)).
\tag{4.4}
$$

Another application of the mixed derivative theorem yields

$$
\partial_j v \in H_p^{1,(2,1)}(J \times \mathbb{R}_+^n) \hookrightarrow H_p^{1,(2,1)}(J \times \mathbb{R}^{n-1}, L_p(\mathbb{R}_+)),
\tag{4.5}
$$

$$
\partial_j \partial_k v \in L_p(J \times \mathbb{R}_+^n) = L_p(J \times \mathbb{R}^{n-1}, L_p(\mathbb{R}_+)),
\tag{4.6}
$$

for $j, k = 1, \ldots, n$. Taking trace, this also implies

$$
\begin{aligned}
v|_{\partial \mathbb{R}_+^n} &\in W_p^{1-1/2p}(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W_p^{2-1/p}(\mathbb{R}^{n-1})) \\
&= W_p^{2-1/p,(2,1)}(J \times \mathbb{R}^{n-1})
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\partial_j v|_{\partial \mathbb{R}_+^n} &\in W_p^{1/2-1/2p}(J, L_p(\mathbb{R}^{n-1})) \cap L_p(J, W_p^{1-1/p}(\mathbb{R}^{n-1})) \\
&= W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1})
\end{aligned}
\tag{4.8}
$$

for $j = 1, \ldots, n$.

Now, we denote by L the linear operator on the left-hand side of system (2.3) and by $N = (F_v, G, 0, 0, H_\eta, 0, 0, 0)$ its nonlinear right-hand side. Then, (2.3) is reformulated as

$$
L(v, p, \eta) = N(v, p, \eta) + (0, 0, 0, 0, 0, v_0, \eta_0, \eta_1).
$$

We also set

$$
\widetilde{\mathbb{E}} := \mathbb{E}_v \times \mathbb{E}_p \times \mathbb{E}_\eta,
$$

$$
\widetilde{\mathbb{F}} := \mathbb{F}_v \times \mathbb{F}_g \times \{0\} \times \{0\} \times \gamma_0 \mathbb{E}_p \times \gamma_0^t \mathbb{E}_v \times \gamma_0^t \mathbb{E}_\eta \times \gamma_1^t \mathbb{E}_\eta.
$$

The nonlinearity admits the following properties.

Theorem 4.1. *Let $p \geq (n+2)/3$. Then, $N \in C^\omega(\widetilde{\mathbb{E}}, \widetilde{\mathbb{F}})$, $N(0) = 0$, and we have $DN(0) = 0$ for the Fréchet derivative of N.*

Proof. Mapping properties of F_v. Gathering (4.1), (4.3), and (4.5), we can estimate the term

$$
(\partial_t \eta - \Delta' \eta) \, \partial_n v,
$$

as desired, provided the vector-valued embedding

$$
\underbrace{W_p^{2-1/p,(2,1)}(J \times \mathbb{R}^{n-1})}_{\mathrm{ind}_1 = 1 - \frac{n+2}{2p}} \cdot \underbrace{H_p^{1,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\mathrm{ind}_2 = \frac{1}{2} - \frac{n+1}{2p}}
$$

$$
\hookrightarrow \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\mathrm{ind} = -\frac{n+1}{2p}}
\tag{4.9}
$$

does hold. Applying [15, Theorem 1.7], this readily follows if at least one of the two indices ind_1, ind_2 is non-negative. The strictest condition to be fulfilled by [15, Theorem 1.7], however, is $\mathrm{ind}_1 + \mathrm{ind}_2 \geq \mathrm{ind}$ in case that both of the indices on the left-hand side are negative which can occur for small p. It is easily seen that this condition is equivalent to

$$
p \geq \frac{n+2}{3}.
\tag{4.10}
$$

For the terms

$$2(\nabla'\eta \cdot \nabla')\,\partial_n v, \quad |\nabla'\eta|^2 \partial_n^2 v, \quad (\nabla'\eta, 0)^\tau \,\partial_n p,$$

we employ (4.2), (4.6) and the vector-valued embeddings

$$\underbrace{\left[W_p^{4-1/p,(2,1)}(J \times \mathbb{R}^{n-1}) \right]^m}_{\text{ind}_1 = 2 - \frac{n+2}{2p}} \cdot \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind}_2 = -\frac{n+1}{2p}}$$
$$\hookrightarrow \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind} = -\frac{n+1}{2p}} \tag{4.11}$$

for $m = 1, 2$. Due to [15, Theorem 1.9], the above embeddings are valid, provided that $\text{ind}_1 > 0$ or, equivalently,

$$p > \frac{n+2}{4}. \tag{4.12}$$

Next, (4.4) and (4.5) show that we obtain the desired estimate of the term $(v \cdot \nabla)v$, if

$$\underbrace{H_p^{2,(2,1)}(J \times \mathbb{R}_+^n)}_{\text{ind}_1 = 1 - \frac{n+2}{2p}} \cdot \underbrace{H_p^{1,(2,1)}(J \times \mathbb{R}_+^n)}_{\text{ind}_2 = \frac{1}{2} - \frac{n+2}{2p}} \hookrightarrow \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}_+^n)}_{\text{ind} = -\frac{n+2}{2p}}.$$

This is guaranteed by [15, Theorem 1.7] if $\max\{\text{ind}_1, \text{ind}_2\} \geq 0$. Again, for small values of p both of the indices on the left-hand side can become negative. Then, [15, Theorem 1.7] implies the embedding above if $\text{ind}_1 + \text{ind}_2 \geq \text{ind}$, which is equivalent to (4.10).

Thanks to (4.2) and (4.5), the term $(v' \cdot \nabla'\eta)\partial_n v$ can be estimated by utilizing the embedding

$$\underbrace{H_p^{1,(2,1)}(J \times \mathbb{R}^{n-1}, H_p^1(\mathbb{R}_+))}_{\text{ind}_1 = \frac{1}{2} - \frac{n+1}{2p}} \cdot \underbrace{W_p^{4-1/p,(2,1)}(J \times \mathbb{R}^{n-1})}_{\text{ind}_2 = 2 - \frac{n+2}{2p}}$$
$$\cdot \underbrace{H_p^{1,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind}_3 = \frac{1}{2} - \frac{n+1}{2p}} \hookrightarrow \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind} = -\frac{n+1}{2p}}. \tag{4.13}$$

Note that here we also employ

$$H_p^{2,(2,1)}(J \times \mathbb{R}_+^n) \hookrightarrow H_p^{1,(2,1)}(J \times \mathbb{R}^{n-1}, H_p^1(\mathbb{R}_+))$$

and $H_p^1(\mathbb{R}_+) \cdot L^p(\mathbb{R}_+) \hookrightarrow L^p(\mathbb{R}_+)$ which is valid due to the Sobolev embedding $H_p^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$ for $p > 1$. Thanks to [15, Theorem 1.7] (4.13) holds, if $\min\{\text{ind}_1, \text{ind}_2, \text{ind}_3\} \geq 0$. If at least one of the three indices on the left-hand side is negative, then the sum of the negative indices on the left-hand side has to exceed the index on the right-hand side. The most restrictive constraint hence results from $\text{ind}_1 + \text{ind}_2 + \text{ind}_3 \geq \text{ind}$, which is fullfilled if

$$p \geq \frac{2n+3}{6}. \tag{4.14}$$

Consequently, by our assumptions F_v has the desired mapping properties, since (4.10) also yields (4.12) and (4.14).

Mapping properties of G. First, we show $G(v, \eta) \in H_p^1(J, \dot{H}_p^{-1}(\mathbb{R}_+^n))$. Integration by parts yields $\partial_n \in \mathscr{L}(L_p(J \times \mathbb{R}_+^n), L_p(J, \dot{H}_p^{-1}(\mathbb{R}_+^n)))$. Using this property and the fact that η does not depend on x_n, it is sufficient to estimate the terms

$$\partial_t \nabla'\eta \cdot v', \ \nabla'\eta \cdot \partial_t v'$$

in $L_p(J \times \mathbb{R}_+^n)$. Thanks to (4.1) and the mixed derivative theorem, we know

$$\partial_t \nabla'\eta \in W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1}).$$

The first term can thus be estimated by the vector-valued embedding

$$\underbrace{W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1})}_{\text{ind}_1 = \frac{1}{2} - \frac{n+2}{2p}} \cdot \underbrace{H_p^{2,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind}_2 = 1 - \frac{n+1}{2p}}$$

$$\hookrightarrow \underbrace{H_p^{0,(2,1)}(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))}_{\text{ind} = -\frac{n+1}{2p}}.$$

According to [15, Theorem 1.7], this embedding is again valid, if we have $\max\{\text{ind}_1, \text{ind}_2\} \geq 0$ or if $\text{ind}_1 + \text{ind}_2 \geq \text{ind}$ in case that both indices on the left-hand side are negative. The latter condition is again equivalent to (4.10).

The second term may be estimated by employing (4.2), the vector-valued embedding (4.11) for $m = 1$, and $\partial_t v \in L^p(J \times \mathbb{R}^{n-1}, L^p(\mathbb{R}_+))$ under constraint (4.12).

To see that also $G(v, \eta) \in L^p(J, H_p^1(\mathbb{R}_+^n))$, we estimate the terms

$$\partial_j \nabla' \eta \cdot \partial_n v', \ \nabla' \eta \cdot \partial_j \partial_n v', \ \nabla' \eta \cdot \partial_n^2 v', \qquad j = 1, \ldots, n-1,$$

in $L^p(J \times \mathbb{R}_+^n)$. Similar as above, this may be accomplished by utilizing (4.2), (4.3), (4.5), (4.6) in combination with the vector-valued embeddings (4.9), and (4.11). Once more, this is feasible if (4.10) holds.

Mapping properties of H_η. Note that $W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1})) \hookrightarrow \gamma_0 \mathbb{E}_p$. Hence, according to (4.2) and (4.5) we can estimate the terms

$$-\nabla' \eta \cdot \partial_n v', \ -\nabla' \eta \cdot \nabla' v^n$$

as desired provided that the embedding

$$\left[\underbrace{W_p^{4-1/p,(2,1)}(J \times \mathbb{R}^{n-1})}_{\text{ind}_1 = 2 - \frac{n+2}{2p}}\right] \cdot \underbrace{W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1}))}_{\text{ind}_2 = \frac{1}{2} - \frac{n+2}{2p}}$$

$$\hookrightarrow \underbrace{W_p^{1-1/p,(2,1)}(J \times \mathbb{R}^{n-1}))}_{\text{ind} = \frac{1}{2} - \frac{n+2}{2p}}$$

is at our disposal. By [15, Theorem 1.9], this is the case if $\text{ind}_1 > 0$. Hence, the nonlinearity H_η has the desired mapping properties, provided that $p > (n+2)/4$. This, in turn, is true since (4.10) is satisfied.

Altogether we have proved the asserted embeddings, i.p. that $N(\widetilde{\mathbb{E}}) \subset \widetilde{\mathbb{F}}$. The claimed smoothness of N as well as $N(0) = 0$ and $DN(0) = 0$ follows obviously by the fact that N consists of polynomial nonlinearities which are of quadratic or higher order. $\qquad \square$

For a Banach space E, we denote by $B_E(x, r)$ the open ball in E with radius $r > 0$ centered in $x \in E$. Based on Theorems 3.1 and 4.1, we can derive well-posedness of (2.3) for small data. For simplicity, we also set

$$\mathbb{E} := \left\{ (v, p, \eta) \in \mathbb{E}_v \times \mathbb{E}_p \times \mathbb{E}_\eta; \ \partial_t \eta = v^n, \ v' = 0 \text{ on } \partial \mathbb{R}_+^n \right\},$$

$$\mathbb{F} := \Bigg\{ (f_v, g, 0, 0, f_\eta, v_0, \eta_0, \eta_1) \in \widetilde{\mathbb{F}}; \ f_v, g, 0, 0, f_\eta, v_0, \eta_0, \eta_1 \text{ satisfy}$$

$$\text{the compatibility conditions (C1)-(C4)} \Bigg\}.$$

Theorem 4.2. *Let $p \geq (n+2)/3$ and $T > 0$. Then, there is a $\kappa = \kappa(T) > 0$ such that for $(f_v, g, 0, 0, f_\eta, v_0, \eta_0, \eta_1) \in B_{\widetilde{\mathbb{F}}}(0, \kappa)$ satisfying the compatibility conditions (C2)–(C4) and*

$$\text{div } v_0 = \nabla' \eta_0 \cdot \partial_n v_0' + g|_{t=0} \quad \text{in } \dot{H}_p^{-1}(\mathbb{R}_+^n) \tag{4.15}$$

there is a unique solution $(v, p, \eta) \in \mathbb{E}$ of system (2.3). The solution depends continuously on the data.

Proof. We pick $f := (f_v, g, 0, 0, f_\eta, v_0, \eta_0, \eta_1)$ as assumed. System (2.3) (including exterior forces) reads as

$$L(v, p, \eta) = N(v, p, \eta) + f. \tag{4.16}$$

We first have to verify that the right-hand side belongs to $\mathbb{F}$. Observe that (4.15) gives (C1). Hence, by our assumptions the compatibility conditions (C1)–(C3) are satisfied. To see compatibility condition (C4), we have to verify that there exists an $\eta_* \in \mathbb{E}_\eta$ satisfying $(\eta_*, \partial_t \eta_*)|_{t=0} = (\eta_0, \eta_1)$ and

$$(g + G(v, \eta), \partial_t \eta_*) \in H_p^1(J; \dot{H}_{p,0}^{-1}(\mathbb{R}_+^n))$$

for every triple $(v, p, \eta) \in \mathbb{E}$ such that $(v, \eta, \partial_t \eta)|_{t=0} = (v_0, \eta_0, \eta_1)$. Note that by assumption there is an extension $\eta_* \in \mathbb{E}_\eta$ with the prescribed traces such that

$$(g, \partial_t \eta_*) \in H_p^1(J; \dot{H}_{p,0}^{-1}(\mathbb{R}_+^n)).$$

Hence, it suffices to prove that

$$(\nabla'\eta \cdot \partial_n v', 0) \in H_p^1(J; \dot{H}_{p,0}^{-1}(\mathbb{R}_+^n)) \tag{4.17}$$

For $\phi \in \dot{H}_p^1(\mathbb{R}_+^n)$, we observe that thanks to $v'(x', 0) = 0$ we obtain

$$\int_{\mathbb{R}_+} \phi(x) \nabla'\eta(x') \cdot \partial_n v'(x)\, dx_n = -\int_{\mathbb{R}_+} \nabla'\eta(x') \cdot v'(x) \partial_n \phi(x)\, dx_n.$$

In order to deduce (4.17), it hence suffices to prove that

$$\nabla'\eta \cdot v' \in H_p^1(J; L^p(\mathbb{R}_+^n)).$$

Thanks to (4.2) and (4.4), this follows from the embedding

$$W_p^{4-1/p,(2,1)}(J \times \mathbb{R}^{n-1}) \cdot H_p^{2,(2,1)}(J \times \mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$$
$$\hookrightarrow H_p^{2,(2,1)}(J \times \mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) \hookrightarrow H_p^1(J; L^p(\mathbb{R}_+^n)).$$

Applying once again [15, Theorem 1.9], we see that this is fulfilled if $\mathrm{ind}\big(W_p^{4-1/p,(2,1)}(J \times \mathbb{R}^{n-1})\big) > 0$. This, in turn, holds if $p > (n+2)/4$ which is implied by our assumption $p \geq (n+2)/3$. Thus, (4.17) follows.

Altogether we have proved that $(f_v, g, 0, 0, f_\eta, v_0, \eta_0, \eta_1) \in B_{\widetilde{\mathbb{F}}}(0, \kappa)$ satisfying the compatibility conditions (C2)–(C4) and (4.15) implies that $N(w) + f \in \mathbb{F}$ for $w \in \overline{B_{\mathbb{E}}(0, r)}$. Hence, the right-hand side of (4.16) belongs to $\mathbb{F}$, and we can define

$$K(w) = L^{-1}(N(w) + f), \quad w \in \overline{B_{\mathbb{E}}(0, r)}.$$

We now prove that K is a contraction on $\overline{B_{\mathbb{E}}(0, r)}$ for $r > 0$ small enough. Theorem 3.1 yields that $L \in \mathscr{L}_{is}(\mathbb{E}, \mathbb{F})$. This and the mean value theorem imply

$$\|K(w) - K(z)\|_{\mathbb{E}} \leq C\|N(w) - N(z)\|_{\mathbb{E}}$$
$$\leq C \sup_{v \in B_{\mathbb{E}}(0, r)} \|DN(v)\|_{\mathscr{L}(\mathbb{E}, \widetilde{\mathbb{F}})} \|w - z\|_{\mathbb{E}} \quad (w, z \in \overline{B_{\mathbb{E}}(0, r)}).$$

Fixing $r > 0$ such that $\sup_{v \in B_{\mathbb{E}}(0,r)} \|DN(v)\|_{\mathscr{L}(\mathbb{E},\widetilde{\mathbb{F}})} \leq 1/2C$, which is possible thanks to Theorem 4.1, we see that K is contractive. The estimate above and Theorem 4.1 also imply

$$\|K(w)\|_{\mathbb{E}} \leq \|K(w) - K(0)\|_{\mathbb{E}} + C\|f\|_{\mathbb{F}}$$
$$\leq \frac{r}{2} + C\kappa \quad (w \in \overline{B_{\mathbb{E}}(0, r)}).$$

Choosing $\kappa \leq r/2C$, we see that K is indeed a contraction on $\overline{B_{\mathbb{E}}(0, r)}$. The contraction mapping principle gives the result. $\square$

By the equivalence of the systems (1.1) and (2.3) given through the diffeomorphic transform introduced in Sect. 2, it is clear that Theorem 4.2 implies our main result Theorem 1.1.

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