Z. Angew. Math. Phys. (2020) 71:131

0044-2275/20/040001-22
published online July 16, 2020
https://doi.org/10.1007/s00033-020-01359-9

Zeitschrift für angewandte Mathematik und Physik ZAMP

Asymptotic analysis for a homogeneous bubbling regime Vlasov–Fokker–Planck/Navier–Stokes system

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Abstract. The evolution of a cloud of particles in a compressible fluid can be modeled with a Vlasov–Fokker–Planck equation for the distribution function of the particles coupled with Navier–Stokes or Euler equations for the density and velocity of the fluid. Formal calculations have established the convergence of solution to the mesoscopic model to solutions to the macroscopic Navier–Stokes or Euler model coupled with a Smoluchowski equation as the ratio of the settling time for the microscopic velocity fluctuation of the particles to the characteristic macroscopic time scale goes to zero. This paper provides a rigorous asymptotic analysis for a homogeneous mesoscopic fluid–particle interaction model for particles dispersed in a compressible fluid is provided for the bubbling regime. A relative entropy inequality for a mixed hyperbolic/parabolic system of equations is employed.

Mathematics Subject Classification. Primary 35Q99, Secondary 35B40.

Keywords. Fluid–particle interaction, Vlasov–Euler system, Vlasov–Navier–Stokes system.

1. Introduction

Fluid–particle interaction phenomena arise in several areas of science, including sedimentation analysis, biotechnology, medicine, waste-water recycling, mineral processing, atmospheric sciences, and combustion of fuel droplets [1,2,7,8,19,20]. In this paper, models for the both mesoscopic and macroscopic scaling are considered, in particular the relationship between the two models. In this paper, a compressible fluid, either viscous or inviscid, in which a cloud of identical particles is dispersed is considered. For these models, the particles are assumed to spheres of uniform mass density with radii small compared to the scale of spatial measurement. It is also assumed that the fluid and the particles exist in some fixed, bounded, spatial domain Ω in $\mathbb{R}^3$.

In these models, the fluid is described by two quantities: the nonnegative fluid density $\varrho(x,t)$ where $x \in \Omega$ and $t > 0$, and the fluid velocity field $\mathbf{u}(x,t)$. In the mesoscopic model, the particles are described by a distribution function $f(x,t,\xi)$ where $\xi \in \mathbb{R}^3$ is the microscopic fluctuation from the fluid velocity $\mathbf{u}$. In the macroscopic model, the particles are described by their density $\eta(x,t)$.

The physical interaction between the fluid and the particles manifests itself through the friction forces the particles and fluid exert mutually on each other, leading to a coupling to the fluid and kinetic equations. The models considered in this paper assume that the friction force follows Stokes' law and is proportional to the relative velocity $\mathbf{u} - \xi$.

The fluid is modeled with the Navier–Stokes equations in the viscous case and the Euler equations in the inviscid case, both of which model the evolution of the fluid density and velocity. In the mesoscopic model, the distribution function of the particles is modeled with a Vlasov–Fokker–Planck equation. This

J. B. gratefully acknowledges support from the National Science Foundation under Grant DMS-1401732.

system is

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{1.1a}$$

$$\begin{aligned}&\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x \varrho_\varepsilon^\gamma \\ &\quad = \mu \Delta_x \mathbf{u}_\varepsilon - \beta \varrho_\varepsilon \nabla_x \Phi + \int\limits_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - \mathbf{u}_\varepsilon \right) f_\varepsilon \, \mathrm{d}\xi\end{aligned} \tag{1.1b}$$

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \left[\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon \right] = \frac{1}{\varepsilon} \operatorname{div}_\xi \left[\left(\xi - \sqrt{\varepsilon} \mathbf{u}_\varepsilon \right) f_\varepsilon + \nabla_\xi f_\varepsilon \right]. \tag{1.1c}$$

The above model is valid for positive times t and for the domain Ω which is assumed to be $C^{2,\nu}$ for some $\nu > 0$. The unknown quantities are the fluid density ϱ_ε and the fluid velocity field $\mathbf{u}_\varepsilon$, which are functions of the space coordinate $x \in \Omega$ and time $t > 0$, and the mesoscopic particle density f_ε, which is a function of x, t, and the microscopic velocity fluctuation $\xi \in \mathbb{R}^3$. These unknown quantities are identified in terms of the positive ratio of settling time of the particles in the fluid to the characteristic observation time ε. The constant β represents the relative effect of the external potential Φ on the fluid *vis-a-vis* the effect on the particles, $\mu \geq 0$ is the viscosity of the fluid, which is taken to be constant, and γ is the adiabatic constant of the fluid, which is assumed to be greater than $\frac{3}{2}$.

In this paper, the interest is in the ratio between the settling time for the dissipation of the particles in the fluid and the characteristic observation time ε. In particular, this investigation is concerned with the asymptotic limit as $\varepsilon \to 0$, that is, when the settling time is small compared to the observation time scale. The mesoscopic model considered in the present work is for the so-called *bubbling regime*. In this scaling, the ratio of the characteristic time scale and length scale times the characteristic thermal speed is $\varepsilon^{-1/2}$. This contrasts with the *flowing regime* where this quantity is taken to be constant with respect to the ratio of the settling time to the characteristic observation time. The rigorous asymptotic analysis of the flowing regime is investigated in [18]. In the flowing regime scaling, the convergence is to solutions to a mixed hyperbolic/parabolic model where the parabolicity arises from the viscosity term for the fluid. In the bubbling regime, the convergence is to a mixed hyperbolic/parabolic model where in addition to the parabolicity due to the viscosity of the fluid, there is an additional viscosity in the particle equation, as seen below.

Through the mesoscopic particle distribution function, the macroscopic particle density η_ε is defined by integrating f_ε over the phase space in ξ, $\mathbb{R}^3$, that is,

$$\eta_\varepsilon(x,t) = \int\limits_{\mathbb{R}^3} f_\varepsilon(x,t,\xi) \, \mathrm{d}\xi. \tag{1.2}$$

In [11], the authors formally show that as $\varepsilon \to 0$, the solutions ϱ_ε, $\mathbf{u}_\varepsilon$, and η_ε to the mesoscopic model converge to the quantities ϱ, $\mathbf{u}$, and η, respectively, the latter three of which are solutions to the macroscopic system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \tag{1.3a}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho^\gamma + \eta) = -(\beta \varrho + \eta) \nabla_x \Phi + \mu \Delta_x \mathbf{u} \tag{1.3b}$$

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) = \Delta_x \eta. \tag{1.3c}$$

In order to get their result, [11, Theorem 4], the authors assume that in the sense of distributions, $\eta_\varepsilon \mathbf{u}_\varepsilon \to \eta \mathbf{u}$, $\int\limits_{\mathbb{R}^3} \xi f_\varepsilon \, \mathrm{d}\xi \otimes \mathbf{u}_\varepsilon \to 0$, $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u}$, and $\varrho_\varepsilon^\gamma \to \varrho^\gamma$. In the rigorous analysis in this current work, these assumptions are not made.

It is noted that in the case that $\mu > 0$, (1.3b) is a Navier–Stokes equation, and in the case of zero viscosity, $\mu = 0$, (1.3b) becomes an Euler equation. Thus, (1.3) is called the *Navier–Stokes–Smoluchowski* system in the viscous case $\mu > 0$ and called the *Euler–Smoluchowski* system in the inviscid case $\mu = 0$.

Well-posedness of the system (1.3) in both the viscous and inviscid cases has been analyzed in various previous work (see [3–6,10], for example).

In this work, it will be assumed that there is no external forcing term, that is, $\nabla_x \Phi \equiv 0$ on Ω. This will avoid complications of a flux term dependent explicitly on the spatial variable x. (Hyperbolic models with the complication are analyzed in [16].) Adopting the terminology for hyperbolic conservation laws (see [13], for example), this paper considers the homogeneous mesoscopic model

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{1.4a}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x \varrho_\varepsilon^\gamma = \mu \Delta_x \mathbf{u}_\varepsilon + \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - \mathbf{u}_\varepsilon \right) f_\varepsilon \, \mathrm{d}\xi \tag{1.4b}$$

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \xi \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} \operatorname{div}_\xi \left[\left(\xi - \sqrt{\varepsilon} \mathbf{u}_\varepsilon \right) f_\varepsilon + \nabla_\xi f_\varepsilon \right] \tag{1.4c}$$

and the homogeneous macroscopic model

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \tag{1.5a}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(\varrho^\gamma + \eta \right) = \mu \Delta_x \mathbf{u} \tag{1.5b}$$

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u}) = \Delta_x \eta \tag{1.5c}$$

and how solutions of (1.4) converge to solutions to (1.5).

1.1. Hyperbolic/parabolic systems

The proof of convergence relies upon relative entropy methods for mixed hyperbolic/parabolic systems. It is clear that (1.3) can be written as a mixed hyperbolic/parabolic system

$$\partial_t U + \sum_{i=1}^{3} \partial_{x_i} F_i(U, x) - \sum_{i=1}^{3} \partial_{x_i} (B(U) \partial_{x_i} DH(U)) = G(U, x) \tag{1.6}$$

where

$$U = \left[\varrho \ \mathbf{m}^T = \varrho \mathbf{u}^T \ \eta \right]^T$$

and the fluxes F_i, the viscosity matrix B, and the external forcing term G are appropriately defined. While the literature on relative entropy methods generally assumes a flux explicitly dependent on the solution only, the paper of Kruzhkov [16] explores relative entropy ideas for more general hyperbolic systems. These ideas along with their applications to the non-homogeneous problem are discussed in Sect. 4.

The homogeneous macroscopic model (1.5) can be written as

$$\partial_t U + \sum_{i=1}^{3} \partial_{x_i} F_i(U) - \sum_{i=1}^{3} \partial_{x_i} (B(U) \partial_{x_i} DH(U)) = 0 \tag{1.7}$$

with U defined as before,

$$B(U) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \eta \end{bmatrix},$$

and

$$F_1(U) = \left[\mathbf{m}_1 \ \ \frac{\mathbf{m}_1^2}{\varrho} + \varrho^\gamma + \eta \ \ \frac{\mathbf{m}_1 \mathbf{m}_2}{\varrho} \ \ \frac{\mathbf{m}_1 \mathbf{m}_3}{\varrho} \ \ \frac{\eta}{\varrho} \mathbf{m}_1 \right]^T,$$

$$F_2(U) = \begin{bmatrix} \mathbf{m}_2 & \frac{\mathbf{m}_1\mathbf{m}_2}{\varrho} & \frac{\mathbf{m}_2^2}{\varrho} + \varrho^\gamma + \eta & \frac{\mathbf{m}_2\mathbf{m}_3}{\varrho} & \frac{\eta}{\varrho}\mathbf{m}_2 \end{bmatrix}^T,$$

$$F_3(U) = \begin{bmatrix} \mathbf{m}_3 & \frac{\mathbf{m}_1\mathbf{m}_3}{\varrho} & \frac{\mathbf{m}_2\mathbf{m}_3}{\varrho} & \frac{\mathbf{m}_3^2}{\varrho} + \varrho^\gamma + \eta & \frac{\eta}{\varrho}\mathbf{m}_3 \end{bmatrix}^T.$$

It is clear that B is a positive semidefinite matrix for nonnegative μ and η, so this is indeed a mixed hyperbolic/parabolic system. Results using relative entropy inequalities for general mixed hyperbolic/parabolic systems have been obtained by Christoforou and Tzavaras in [12] and prior to that, Mellet and Vasseur investigated the special case where the viscosity term is of the form $B(U)\partial_{x_i} DH(U)$ where H is an entropy of the system (see [18]). However, in these results, the flux is a function of the solution U only and has no explicit dependence on the position x. But the flux in (1.6) does have such explicit dependence through the external forcing term in the flux of the Smoluchowski equation.

The goal is to show the convergence of solutions to the mesoscopic model (1.4) to solutions of the macroscopic model (1.5), noting that the interest of the convergence of the particle density is the convergence of the zero moment η_ε instead of the density function f_ε. The method followed here is based upon the method used in [18]: It uses a relative entropy inequality comparing the entropies to a smooth solution of the macroscopic model to the solutions to the mesoscopic model at level ε. However, relative entropies generally are used to compare weak solutions with smooth solutions and the derivations of the needed relative entropy inequalities rely on this fact (see [12,13], for instance). Here and in [18], the role of the weak solution is filled by functions that solve a related, but different problem. Thus, the derivation of the relative entropy inequality must be altered to reflect this fact.

1.2. Boundary conditions

In much of the previous work on the Navier–Stokes–Smoluchowski system (see [3–6,10]) and the Euler–Smoluchowski system (see [5]), the boundary conditions depend upon whether the fluid is viscous or inviscid. There are two types of boundary conditions on the spatial domain Ω: the no-slip conditions or the no-stick conditions. The no-slip conditions are for any positive t

$$\mathbf{u}|_{\partial\Omega} = \nabla_x \eta \cdot \mathbf{n} = 0 \tag{1.8}$$

where $\mathbf{n}$ is the outward normal to Ω on the boundary. For unbounded Ω, the analogous condition at infinity

$$\lim_{|x|\to\infty} \mathbf{u} = \lim_{|x|\to\infty} \nabla_x \eta = 0 \tag{1.9}$$

applies.

The no-stick boundary conditions are

$$\mathbf{u}\cdot\mathbf{n}|_{\Omega} = \nabla_x \eta \cdot \mathbf{n} = 0 \tag{1.10}$$

on the boundary $\partial\Omega$, or at infinity (taking the obvious limit) for unbounded Ω

$$\lim_{|x|\to\infty} \mathbf{u}\cdot\mathbf{n} = \lim_{|x|\to\infty} \nabla_x \eta \cdot \mathbf{n} = 0 \tag{1.11}$$

For the mesoscopic system (1.4), the boundary conditions for each $\varepsilon > 0$ are

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = \nabla_x \left(\int_{\mathbb{R}^3} f_\varepsilon \, \mathrm{d}\xi \right) = 0 \tag{1.12}$$

or

$$\mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x \left(\int_{\mathbb{R}^3} f_\varepsilon \, \mathrm{d}\xi \right) = 0 \tag{1.13}$$

for the no-slip and no-stick conditions, respectively, with the appropriate conditions at infinity for unbounded domains.

In terms of the system form of the model (1.7), the boundary conditions satisfy

$$\sum_{i=1}^{3} F_i(U)\mathbf{n}_i = \sum_{i=1}^{3} B(U)\partial_{x_i} DH(U)\mathbf{n}_i = 0 \tag{1.14}$$

on $\partial\Omega$ (or at infinity), in both the no-slip and no-stick conditions. These boundary conditions differ from the Dirichlet condition $U = 0$ on the spatial boundary for (1.7) in [18].

1.3. Admissible solutions and formal asymptotics

In this paper, so-called *admissible weak solutions* to (1.4) are considered.

Definition 1.1. (Admissible Weak Solutions) Let $\varepsilon > 0$ and consider initial data ϱ_0, $\mathbf{u}_0$, f_0. Let γ be a constant greater than $\frac{3}{2}$. The functions ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε are *admissible weak solutions* to (1.4) if and only if ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε solve (1.4) in the sense of distributions and they obey the entropy inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\gamma-1}\varrho_\varepsilon^\gamma \ \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_{\mathbb{R}^3} f_\varepsilon \ln f_\varepsilon + \frac{|\xi|^2}{2} f_\varepsilon \ \mathrm{d}\xi \ \mathrm{d}x \\ + \mu\int_\Omega |\nabla_x \mathbf{u}_\varepsilon|^2 \ \mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{R}^3} \left|(\xi - \sqrt{\varepsilon}\mathbf{u}_\varepsilon)f_\varepsilon + \nabla_\xi f_\varepsilon\right|^2 \frac{1}{f_\varepsilon} \ \mathrm{d}\xi \ \mathrm{d}x \le 0\end{aligned} \tag{1.15}$$

Remark 1.1. The entropy inequality is the energy inequality for (1.4). A basic calculation shows that smooth solutions to (1.4) obey (1.15) as an equality. The interested reader is referred also to [11] for more discussion on the mesoscopic energy inequality.

Remark 1.2. The energy for (1.4) will be denoted as

$$\mathcal{E}(\varrho, \mathbf{u}, f) \stackrel{\text{def}}{=} \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\varrho^\gamma + \int_{\mathbb{R}^3} f\ln f + \frac{|\xi|^2}{2} f \ \mathrm{d}\xi.$$

Next, equations for the moments of particle distribution f_ε are considered. First, integrating (1.4c) with respect to ξ yields

$$\partial_t \eta_\varepsilon + \operatorname{div}_x \mathbf{J}_\varepsilon = 0 \tag{1.16}$$

where

$$\mathbf{J}_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\int_{\mathbb{R}^3} \xi f_\varepsilon \ \mathrm{d}\xi$$

is the first moment of f_ε. Multiplying (1.4c) by ξ and integrating with respect to ξ gives

$$\varepsilon\partial_t \mathbf{J}_\varepsilon + \operatorname{div}_x \mathbb{P}_\varepsilon = \eta_\varepsilon \mathbf{u}_\varepsilon - \mathbf{J}_\varepsilon \tag{1.17}$$

where $\mathbb{P}_\varepsilon$ is the second moment of f_ε

$$\mathbb{P}_\varepsilon = \int_{\mathbb{R}^3} \xi\otimes\xi f_\varepsilon \ \mathrm{d}\xi.$$

It is noted that the admissible weak solutions in the sense of Definition 1.1 obey the above moment equations in the sense of distributions.

In [11], it is shown that in the bubbling regime, the particle distribution can be approximated by the Gaussian

$$f_\varepsilon \simeq \eta_\varepsilon \frac{1}{(2\pi)^{3/2}} e^{-|\xi|^2/2}. \tag{1.18}$$

For simplicity, the quantity

$$\eta \frac{1}{(2\pi)^{3/2}} e^{-|\xi|^2/2}$$

will be denoted as M_η. Additionally, the second moment of f_ε,

$$\mathbb{P}_\varepsilon = \int\limits_{\mathbb{R}^3} \xi \otimes \xi f_\varepsilon \, \mathrm{d}\xi$$

can be approximated by

$$\mathbb{P}_\varepsilon \simeq \eta_\varepsilon \mathbb{I}_3.$$

Using a minimization principle from [9],

$$\mathcal{E}(\varrho, \mathbf{u}, M_\eta) \leq \mathcal{E}(\varrho, \mathbf{u}, f)$$

if

$$\int\limits_{\mathbb{R}^3} f \, \mathrm{d}\xi + \mathcal{E}(\varrho, \mathbf{u}, f) < \infty,$$

which means after a straightforward calculation,

$$H(\varrho, \mathbf{u}, \eta) \leq \mathcal{E}(\varrho, \mathbf{u}, f) \tag{1.19}$$

where

$$\eta = \int\limits_{\mathbb{R}^3} f \, \mathrm{d}\xi.$$

Remark 1.3. It is noted that admissible weak solutions to (1.1) in the sense of Definition 1.1 also obey the zero and first moment equations (1.16) and (1.17) in the sense of distributions.

1.4. Plan of the paper

The main result of the paper is as follows.

Theorem 1.1. *Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, f_\varepsilon\}_{\varepsilon>0}$ be admissible weak solutions to (1.4) where $\gamma > \frac{3}{2}$ in the sense of Definition 1.1 obeying one of the sets of boundary conditions in Sect. 1.2 with initial data $\{\varrho_0, \mathbf{u}_0, f_0\}$ and define η_ε with (1.2) with*

$$\eta_0(x) \overset{def}{=} \int\limits_{\mathbb{R}^3} f_0(x, \xi) \, \mathrm{d}\xi. \tag{1.20}$$

If the system (1.4) with the initial data $\{\varrho_0, \mathbf{u}_0, \eta_0\}$ admits a smooth solution $\{\varrho, \mathbf{u}, \eta\}$ to (1.5) where ϱ is bounded from below away from zero, then

$$\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\} \to \{\varrho, \mathbf{u}, \eta\}$$

as $\varepsilon \to 0$.

As part of the work, it will be shown that

$$\partial_t U_\varepsilon + \sum_{i=1}^{3} \partial_{x_i} F_i(U_\varepsilon) - \partial_{x_i}(B(U_\varepsilon)\partial_{x_i} DH(U_\varepsilon)) \to 0 \tag{1.21}$$

in the sense of distributions as $\varepsilon \to 0$.

The rest of the paper is devoted to proving Theorem 1.1 and is organized as follows.

(1) In Sect. 2, the relative entropy inequality that forms the key to the analysis in the problem is derived. Much of the work follows the work in [18], but the relative viscosity terms provide more difficulty in the current problem, as such, with the inclusion of these terms, the calculations need to be redone. A relative entropy inequality for general hyperbolic/parabolic systems (1.7) is derived which is used to find a relative entropy inequality for the homogeneous macroscopic model (1.5).

(2) In Sect. 3, bounds on terms in the relative entropy inequality are calculated, and the convergence of the kinetic approximation term in (1.21) is shown. These results are used in conjunction with Gronwall's inequality to prove Theorem 1.1.

(3) In Sect. 4, discussion on the issues with a non-homogeneous problem is provided.

2. Relative entropy inequality

The key tool in showing the convergence of mesoscopic solutions to (1.4) to solutions to the homogeneous Navier–Stokes/Euler–Smoluchowski equations (1.7) is a relative entropy inequality. This part parallels the work in [18] which uses the fact that the viscosity term of the system is of the form

$$\sum_{i=1}^{3} \partial_{x_i}(B(U)\partial_{x_i} DH(U))$$

where B is a positive semidefinite matrix and H is the mechanical entropy.

Since the solution U under consideration is in $\mathbb{R}^5$, it is well known that the only convex entropy is the mechanical entropy

$$\begin{aligned} H(U) &\stackrel{\text{def}}{=} H(\varrho, \mathbf{m}, \eta) \stackrel{\text{def}}{=} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\gamma-1}\varrho^\gamma + \eta\ln\eta - \frac{3}{2}\eta\ln(2\pi) \\ &= H(\varrho, \mathbf{u}, \eta) \stackrel{\text{def}}{=} \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\varrho^\gamma + \eta\ln\eta - \frac{3}{2}\eta\ln(2\pi). \end{aligned} \tag{2.1}$$

This means that

$$DH(U) = \begin{bmatrix} -\frac{|\mathbf{m}|^2}{2\varrho^2} + \frac{\gamma}{\gamma-1}\varrho^{\gamma-1} \\ \frac{\mathbf{m}}{\varrho} \\ \ln\eta + 1 - \frac{3}{2}\ln(2\pi) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma}{\gamma-1}\varrho^{\gamma-1} \\ \mathbf{u} \\ \ln\eta + 1 - \frac{3}{2}\ln(2\pi) \end{bmatrix} \tag{2.2}$$

and

$$D^2H(U) = \begin{bmatrix} -\frac{|\mathbf{m}|^2}{\varrho^3} + \gamma\varrho^{\gamma-2} & -\frac{\mathbf{m}^T}{\varrho^2} & 0 \\ \frac{\mathbf{m}}{\varrho^2} & \frac{1}{\varrho^2}\mathbb{I}_3 & 0 \\ 0 & 0 & \frac{1}{\eta} \end{bmatrix} = \begin{bmatrix} -\frac{|\mathbf{u}|^2}{\varrho} + \gamma\varrho^{\gamma-2} & -\frac{\mathbf{u}^T}{\varrho} & 0 \\ \frac{\mathbf{u}}{\varrho} & \frac{1}{\varrho}\mathbb{I}_3 & 0 \\ 0 & 0 & \frac{1}{\eta} \end{bmatrix}. \tag{2.3}$$

This means that after some straightforward calculations involving the definition of the entropy flux that

$$\begin{aligned} Q_i(U) \stackrel{\text{def}}{=} Q_i(\varrho, \mathbf{m}, \eta) &= \left[\frac{|\mathbf{m}|^2}{2\varrho^2} + \frac{\gamma}{\gamma-1} + \frac{\eta}{\varrho}(\ln\eta + 1)\right]\mathbf{m}_i \\ &= Q(\varrho, \mathbf{u}, \eta) = \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\gamma}{\gamma-1}\varrho^\gamma + \eta(\ln\eta+1)\right]\mathbf{u}_i. \end{aligned} \tag{2.4}$$

In light of the boundary conditions (1.14), the boundary condition

$$\sum_{i=1}^{3} Q_i(U)\mathbf{n}_i = 0 \tag{2.5}$$

arises.

The relative entropy compares the entropies of two solutions. More specifically, the relative entropy can be seen as the difference of the entropy for the first solution and the first-order Taylor approximation for the entropy of the first solution centered at the second solution. It is defined as

$$\mathcal{H}(U|\overline{U}) \overset{\text{def}}{=} H(U) - H(\overline{U}) - DH(\overline{U}) \cdot (U - \overline{U}) \tag{2.6}$$

meaning for the current problem

$$\begin{aligned}&\mathcal{H}(\varrho, \mathbf{u}, \eta|\overline{\varrho}, \overline{\mathbf{u}}, \overline{\eta})\\ &\quad = \frac{1}{2}\varrho|\mathbf{u} - \overline{\mathbf{u}}|^2 + \frac{1}{\gamma - 1}\left[\varrho^\gamma - \overline{\varrho}^\gamma - \gamma\overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho})\right]\\ &\qquad + \eta \ln \eta - \overline{\eta} \ln \overline{\eta} - (\ln \overline{\eta} + 1)(\eta - \overline{\eta}).\end{aligned} \tag{2.7}$$

First, a relative entropy inequality for the general hyperbolic/parabolic system (1.7) is found.

Proposition 2.1. (Relative Entropy Inequality) *Let U be a C^1 function and let $\overline{U}$ be a smooth solution to (1.7) such that both U and $\overline{U}$ satisfy the boundary condition (1.14). Then, the relative entropy inequality*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{H}(U|\overline{U})\ \mathrm{d}x\\ &\quad + \int_\Omega \sum_{i=1}^{3} \left[B(U)\partial_{x_i}(DH(U) - DH(\overline{U}))\right] \cdot \partial_{x_i}\left[DH(U) - DH(\overline{U})\right]\ \mathrm{d}x\\ &\leq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(U)\ \mathrm{d}x + \int_\Omega \sum_{i=1}^{3}\left[B(U)\partial_{x_i}DH(U)\right] \cdot \partial_{x_i}DH(U)\ \mathrm{d}x\\ &\quad - \int_\Omega DH(\overline{U}) \cdot \left[\partial_t U + \sum_{i=1}^{3}\partial_{x_i}F_i(U) - \sum_{i=1}^{3}\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)\right)\right]\ \mathrm{d}x\\ &\quad - \int_\Omega \sum_{i=1}^{3}\partial_{x_i}\left[DH(\overline{U})\right] \cdot \left[F_i(U) - F_i(\overline{U}) - DF_i(\overline{U})(U - \overline{U})\right]\ \mathrm{d}x\\ &\quad + \int_\Omega \sum_{i=1}^{3} D\mathcal{H}(U|\overline{U}) \cdot \partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\ \mathrm{d}x\\ &\quad - \int_\Omega \sum_{i=1}^{3}\left[(B(U) - B(\overline{U}))\partial_{x_i}DH(\overline{U})\right] \cdot \partial_{x_i}\left[DH(U) - DH(\overline{U})\right]\ \mathrm{d}x\end{aligned} \tag{2.8}$$

holds.

Remark 2.1. The attentive reader will notice that U is not assumed to be a solution of any system of equations. This differs from the standard use of the relative entropy where U is a weak solution. However, for the present work, the idea is that U will represent

$$\left[\varrho_\varepsilon, \varrho_\varepsilon\mathbf{u}_\varepsilon^T, \eta_\varepsilon\right]^T$$

where ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε is a solution of (1.4) and η_ε defined through (1.2). Thus, it is not expected that U would be a weak solution of (1.7). Also, this proposition is independent of the definitions of F, B, and H.

Proof. The proof starts with taking U and $\overline{U}$ to be smooth functions of x and t. After performing the calculations and letting $\overline{U}$ be a solution to (1.7), a standard density argument is taken to allow U to be an arbitrary function with the regularity of an admissible solution to (1.4).

Letting U and $\overline{U}$ be smooth functions of x and t, taking the time derivative of the relative entropy gives

$$\begin{aligned}\partial_t\mathcal{H}(U|\overline{U}) &= \partial_t H(U) - \partial_t H(\overline{U})\\ &\quad - \left[D^2H(\overline{U})\partial_t\overline{U}\right]\cdot(U-\overline{U}) - DH(\overline{U})\cdot(\partial_t U - \partial_t\overline{U})\\ &= \partial_t H(U) - \partial_t H(\overline{U})\\ &\quad - \left[D^2H(\overline{U})\left(\partial_t\overline{U} + \sum_{i=1}^3\partial_{x_i}F_i(\overline{U}) - \sum_{i=1}^3\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\right)\right]\cdot(U-\overline{U})\\ &\quad - DH(\overline{U})\cdot\left[\partial_t U + \sum_{i=1}^3\partial_{x_i}F_i(U) - \sum_{i=1}^3\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)\right)\right]\\ &\quad + DH(\overline{U})\cdot\left[\partial_t\overline{U} + \sum_{i=1}^3\partial_{x_i}F_i(\overline{U}) - \sum_{i=1}^3\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\right]\\ &\quad + \left[D^2H(\overline{U})\sum_{i=1}^3\partial_{x_i}F_i(\overline{U})\right]\cdot(U-\overline{U})\\ &\quad - \left[D^2H(\overline{U})\sum_{i=1}^3\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\right]\cdot(U-\overline{U})\\ &\quad + DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}F_i(U) - DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)\right)\\ &\quad - DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}F_i(\overline{U}) + DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right).\end{aligned} \tag{2.9}$$

Next, to help rewrite the previous equation, it is noted following the work in [18] that for each i,

$$\begin{aligned}&\partial_{x_i}\left[DQ_i(\overline{U})\cdot(U-\overline{U})\right]\\ &\quad = DH(\overline{U})\cdot\partial_{x_i}\left[DF_i(\overline{U})(U-\overline{U})\right] + \left[D^2H(\overline{U})\partial_{x_i}F_i\overline{U})\right]\cdot(U-\overline{U})\end{aligned} \tag{2.10}$$

and using the work in [18] along with some rearranging of viscosity terms

$$\begin{aligned}&-\left[D^2H(\overline{U})\sum_{i=1}^3\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U})\right]\cdot(U-\overline{U})\\ &\quad -DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}(B(U)\partial_{x_i}DH(U)) + DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\\ &= -DH(U)\cdot\sum_{i=1}^3\partial_{x_i}(B(U)\partial_{x_i}DH(U)) + DH(\overline{U})\cdot\sum_{i=1}^3\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\\ &\quad + \left[DH(U) - DH(\overline{U})\right]\cdot\sum_{i=1}^3\partial_{x_i}\left(B(U)\partial_{x_i}DH(U) - B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\end{aligned}$$

$$+D\mathcal{H}(U|\overline{U})\cdot\sum_{i=1}^{3}\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U})), \tag{2.11}$$

where

$$D\mathcal{H}(U|\overline{U})\overset{\text{def}}{=}D_UH(U)-D_{\overline{U}}H(\overline{U})-D^2_{\overline{U}}H(\overline{U})(U-\overline{U}).$$

Using the appropriate substitutions into (2.9) from the previous two equations yields

$$\begin{aligned}\partial_t\mathcal{H}(U|\overline{U}) &= \partial_tH(U)-\partial_tH(\overline{U})\\ &\quad-\left[D^2H(\overline{U})\left(\partial_t\overline{U}+\sum_{i=1}^{3}\partial_{x_i}F_i(\overline{U})-\sum_{i=1}^{3}\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\right)\right]\cdot(U-\overline{U})\\ &\quad-DH(\overline{U})\cdot\left[\partial_tU+\sum_{i=1}^{3}\partial_{x_i}F_i(U)-\sum_{i=1}^{3}\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)\right)\right]\\ &\quad+DH(\overline{U})\cdot\left[\partial_t\overline{U}+\sum_{i=1}^{3}\partial_{x_i}F_i(\overline{U})-\sum_{i=1}^{3}\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\right]\\ &\quad+\sum_{i=1}^{3}\partial_{x_i}\left[DQ_i(\overline{U})\cdot(U-\overline{U})\right]\\ &\quad+DH(\overline{U})\cdot\sum_{i=1}^{3}\partial_{x_i}\left[F_i(U)-F_i(\overline{U})-DF_i(\overline{U})(U-\overline{U})\right]\\ &\quad-DH(U)\cdot\sum_{i=1}^{3}\partial_{x_i}(B(U)\partial_{x_i}DH(U))\\ &\quad+DH(\overline{U})\cdot\sum_{i=1}^{3}\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\\ &\quad+\left[DH(U)-DH(\overline{U})\right]\cdot\sum_{i=1}^{3}\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)-B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\\ &\quad+D\mathcal{H}(U|\overline{U})\cdot\sum_{i=1}^{3}\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U})).\end{aligned} \tag{2.12}$$

The attentive reader will notice that up to this point, the proof has only relied upon U and $\overline{U}$ being smooth. In order to continue the proof, the next step is to assume that $\overline{U}$ is a classical solution of (1.4). With this assumption, the above becomes

$$\begin{aligned}\partial_t\mathcal{H}(U|\overline{U}) &= \partial_tH(U)+\sum_{i=1}^{3}\partial_{x_i}Q_i(\overline{U})-DH(U)\cdot\sum_{i=1}^{3}\partial_{x_i}(B(U)\partial_{x_i}DH(\overline{U}))\\ &\quad-DH(\overline{U})\cdot\left[\partial_tU+\sum_{i=1}^{3}\partial_{x_i}F_i(U)-\sum_{i=1}^{3}\partial_{x_i}\left(B(U)\partial_{x_i}DH(U)\right)\right]\\ &\quad+\sum_{i=1}^{3}\partial_{x_i}\left[DQ_i(\overline{U})\cdot(U-\overline{U})\right]\\ &\quad+DH(\overline{U})\cdot\sum_{i=1}^{3}\partial_{x_i}\left[F_i(U)-F_i(\overline{U})-DF_i(\overline{U})(U-\overline{U})\right]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{3}\partial_{x_i}\left[B(U)\partial_{x_i}(DH(U)-DH(\overline{U}))\right]\cdot\left[DH(U)-DH(\overline{U})\right]\\
&+\sum_{i=1}^{3}\left[(B(U)-B(\overline{U}))\partial_{x_i}DH(\overline{U})\right]\cdot\left[DH(U)-DH(\overline{U})\right]\\
&\times\sum_{i=1}^{3}\partial_{x_i}\left(B(\overline{U})\partial_{x_i}DH(\overline{U})\right)\cdot D\mathcal{H}(U|\overline{U}).
\end{aligned}\tag{2.13}
$$

Using the boundary conditions after integrating the above equation over Ω and standard density arguments allowing the use of U which has weak derivatives completes the proof. □

Next, the result of the previous lemma is applied to the particular problem considered in this paper. From this point forward, $\overline{U}$ represents a smooth solution to the homogeneous version of (1.3) and U will be replaced by U_ε which represents an admissible weak solution of (1.4). In the next proposition, the main result of this section of the paper, a specific relative entropy inequality comparing a solution of the mesoscopic model with a smooth solution to the macroscopic model, is stated.

Proposition 2.2. *For each $\varepsilon > 0$, let $U = U_\varepsilon = \left[\varrho_\varepsilon \ \varrho_\varepsilon\mathbf{u}_\varepsilon \ f_\varepsilon\right]^T$ where $\varrho_\varepsilon, \mathbf{u}_\varepsilon, f_\varepsilon$ is an admissible weak solution to (1.4) with*

$$
\eta_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \ \mathrm{d}\xi
$$

and let $\overline{U} = \left[\overline{\varrho} \ \overline{\varrho}\overline{\mathbf{u}}^T \ \overline{\eta}\right]^T$ where $\overline{\varrho}$, $\overline{\mathbf{u}}$, $\overline{\eta}$ is a smooth solution to (1.5). Then,

$$
\begin{aligned}
&\int_{\Omega}\mathcal{H}(U_\varepsilon|\overline{U})(T)\ \mathrm{d}x+\mu\int_0^T\int_{\Omega}|\nabla_x\mathbf{u}_\varepsilon-\nabla_x\overline{\mathbf{u}}|^2\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T\int_{\Omega}\eta_\varepsilon|\nabla_x\ln\eta_\varepsilon-\nabla_x\ln\overline{\eta}|^2\ \mathrm{d}x\ \mathrm{d}t\\
&\quad\le-\int_0^T\int_{\Omega}DH(\overline{U})\cdot\left[\partial_tU_\varepsilon+\sum_{i=1}^{3}\partial_{x_i}F_i(U_\varepsilon)-\partial_{x_i}(B(U_\varepsilon)\partial_{x_i}DH(U_\varepsilon))\right]\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_0^T\int_{\Omega}\sum_{i=1}^{3}\partial_{x_i}[DH(\overline{U})]\cdot\left[F_i(U_\varepsilon)-F_i(\overline{U})-DF_i(\overline{U})(U_\varepsilon-\overline{U})\right]\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+\int_0^T\int_{\Omega}\sum_{i=1}^{3}D\mathcal{H}(U_\varepsilon|\overline{U})\cdot\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_0^T\int_{\Omega}(\eta_\varepsilon-\overline{\eta})\nabla_x\ln\overline{\eta}\cdot\nabla_x(\ln\eta_\varepsilon-\ln\overline{\eta})\ \mathrm{d}x\ \mathrm{d}t.
\end{aligned}\tag{2.14}
$$

Proof. Using the definitions of H and B with $U = U_\varepsilon$ and $\overline{U}$ defined as mentioned in the proposition, (2.8) becomes after integrating over $[0,T]$

$$
\int_{\Omega}\mathcal{H}(U_\varepsilon|\overline{U})(T)\ \mathrm{d}x+\mu\int_0^T\int_{\Omega}|\nabla_x\mathbf{u}_\varepsilon-\nabla_x\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega \eta_\varepsilon|\nabla_x \ln\eta_\varepsilon - \nabla_x \ln\overline{\eta}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \mathcal{H}(U_\varepsilon|\overline{U})(0)\,\mathrm{d}x\\
&+\int_\Omega H(U_\varepsilon)(T)\,\mathrm{d}x - \int_\Omega H(U_\varepsilon)(0)\,\mathrm{d}x\\
&+\mu\int_0^T\int_\Omega |\nabla_x \mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \eta_\varepsilon\left|\nabla_x \ln\eta_\epsilon\right|^2\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega DH(\overline{U})\cdot\left[\partial_t U_\varepsilon + \sum_{i=1}^3 \partial_{x_i}F_i(U_\varepsilon) - \partial_{x_i}(B(U)\partial_{x_i}DH(U_\varepsilon))\right]\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \sum_{i=1}^3 \partial_{x_i}[DH(\overline{U})]\cdot\left[F_i(U_\varepsilon) - F_i(\overline{U}) - DF_i(\overline{U})(U_\varepsilon-\overline{U})\right]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega \sum_{i=1}^3 D\mathcal{H}(U_\varepsilon|\overline{U})\cdot\partial_{x_i}(B(\overline{U})\partial_{x_i}DH(\overline{U}))\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega (\eta_\varepsilon-\overline{\eta})\nabla_x\ln\overline{\eta}\cdot\nabla_x(\ln\eta_\varepsilon - \ln\overline{\eta})\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.15)
\end{aligned}
$$

It is noted that using the initial conditions, $\mathcal{H}(U_\varepsilon|\overline{U})(0) = 0$ a.e. on Ω. Next, the term

$$
\int_\Omega H(U_\varepsilon)(T)\,\mathrm{d}x - \int_\Omega H(U_\varepsilon)(0)\,\mathrm{d}x + \mu\int_0^T\int_\Omega |\nabla_x\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t
$$

is shown to be non-positive. Indeed, with a couple of clever additions of zero,

$$
\begin{aligned}
&\int_\Omega H(U_\varepsilon)(T) - H(U_\varepsilon)(0)\,\mathrm{d}x\\
&\quad+\mu\int_0^T\int_\Omega |\nabla_x\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \eta_\varepsilon|\nabla_x\ln\eta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_\Omega H(U_\varepsilon)(T) - \mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon,f_\varepsilon)(T)\,\mathrm{d}x + \int_0^T\int_\Omega \eta_\varepsilon|\nabla_x\ln\eta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{\varepsilon}\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|(\xi-\sqrt{\varepsilon}\mathbf{u}_\varepsilon)\sqrt{f_\varepsilon} + 2\nabla_\xi\sqrt{f_\varepsilon}\right|^2\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mathcal{E}(\varrho_\varepsilon,\mathbf{u}_\varepsilon,f_\varepsilon)(T)\,\mathrm{d}x + \mu\int_\Omega |\nabla_x\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\\
&\quad+\frac{1}{\varepsilon}\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|(\xi-\sqrt{\varepsilon}\mathbf{u}_\varepsilon)\sqrt{f_\varepsilon} + 2\nabla_\xi\sqrt{f_\varepsilon}\right|^2\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t - \mathcal{E}(\varrho_0,\mathbf{u}_0,f_0)
\end{aligned}
$$

$$+\mathcal{E}(\varrho_0, \mathbf{u}_0, f_0) - H(U_\varepsilon)(0). \tag{2.16}$$

By following the minimization principle from Bouchut [9], the first two lines of the right side of (2.16) are non-positive. By the energy inequality for solutions to the mesoscopic problem (1.15),

$$\begin{aligned}\int_\Omega \mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, f_\varepsilon) - \mathcal{E}(\varrho_0, \mathbf{u}_0, f_0)\ \mathrm{d}x + \mu \int_0^T \int_\Omega |\nabla_x \mathbf{u}_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t \\ + \frac{1}{\varepsilon} \int_0^T \int_\Omega \int_{\mathbb{R}^3} \left| (\xi - \sqrt{\varepsilon}\mathbf{u}_\varepsilon)\sqrt{f_\varepsilon} + 2\nabla_\xi \sqrt{f_\varepsilon} \right|^2\ \mathrm{d}\xi\ \mathrm{d}x\ \mathrm{d}t \le 0.\end{aligned} \tag{2.17}$$

Additionally, the initial conditions give

$$\mathcal{E}(\varrho_0, \mathbf{u}_0, f_0) - H(U_\varepsilon)(0) = 0,$$

which after plugging into (2.15) completes the proof. □

3. Convergence

This section is dedicated to bounding the right side of (2.14), which can be broken down into the kinetic approximation term, the relative flux term, and the relative viscosity term, respectively. The bulk of the analysis for this paper is in showing the kinetic approximation term converges to zero as ε tends to zero. This analysis is saved for the last subsection of Sect. 3. The relative flux term and the relative viscosity term are shown to be bounded by the relative entropy, allowing the use of Gronwall's inequality to show that if the relative entropy is zero initially, the relative entropy at any time T converges to zero as $\varepsilon \to 0$. This will prove Theorem 1.1 in light of the following lemma illustrating properties of the relative entropy following from taking derivatives and recalling that $\gamma > \frac{3}{2}$.

Lemma 3.1. *Consider the relative entropy*

$$\mathcal{H}(U_\varepsilon|\overline{U}) = \mathcal{H}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon|\overline{\varrho}, \overline{\mathbf{u}}, \overline{\eta}) = \frac{1}{2}\varrho|\mathbf{u}_\varepsilon - \overline{\mathbf{u}}|^2 + H_F(\varrho_\varepsilon, \overline{\varrho}) + H_P(\eta_\varepsilon, \overline{\eta})$$

where

$$H_F(\varrho, \overline{\varrho}) \stackrel{def}{=} \frac{1}{\gamma - 1}\left[\varrho^\gamma - \overline{\varrho}^\gamma - \gamma\overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho})\right]$$

and

$$H_P(\eta, \overline{\eta}) \stackrel{def}{=} \eta \ln \eta - \overline{\eta} \ln \overline{\eta} - (\ln \overline{\eta} + 1)(\eta - \overline{\eta}).$$

Then, for fixed $\overline{\varrho}$ and $\overline{\eta}$ both nonnegative, $H_F(\cdot, \overline{\varrho})$ has an absolute minimum on $[0, \infty)$ of zero at $\varrho = \overline{\varrho}$ and $H_P(\cdot, \overline{\eta})$ has an absolute minimum on $(0, \infty)$ of zero at $\eta = \overline{\eta}$. Additionally, H_F and H_P are convex for positive ϱ and η, respectively.

Since the kinetic fluid energy part of the relative entropy, $\frac{1}{2}|\mathbf{u} - \overline{\mathbf{u}}|^2$, is always positive except for being zero when $\mathbf{u} = \overline{\mathbf{u}}$, in light of Lemma 3.1, the three parts of the relative entropy—the kinetic fluid energy, H_F, and H_P—can be considered separately.

3.1. Relative flux

We first look to control the relative flux term

$$\int_0^T \int_\Omega \sum_{i=1}^{3} \partial_{x_i}[DH(\overline{U})] \cdot [F_i(U_\varepsilon) - F_i(\overline{U}) - DF_i(\overline{U})(U - \overline{U})] \; \mathrm{d}x \; \mathrm{d}t.$$

Control of the relative flux term by the relative entropy is a well-established result (see [6,14] for more specific examples and [12,13] for more general applications), and so the bounds are only summarized here. A series of straightforward calculations shows that

$$\begin{aligned}
&\sum_{i=1}^{3} \partial_{x_i}[DH(\overline{U})] \cdot [F_i(U_\varepsilon) - F_i(\overline{U}) - DF_i(\overline{U})(U - \overline{U})] \\
&\quad = \varrho_\varepsilon[(\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \otimes (\mathbf{u}_\varepsilon - \overline{\mathbf{u}})] : \nabla_x \overline{\mathbf{u}} + H_F(\varrho_\varepsilon, \overline{\varrho}) \operatorname{div}_x \overline{\mathbf{u}} \\
&\qquad + \left(\frac{\eta_\varepsilon}{\overline{\eta}} - \frac{\varrho_\varepsilon}{\overline{\varrho}} \right) (\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \cdot \nabla_x \overline{\eta}.
\end{aligned} \tag{3.1}$$

From standard techniques from previous work (see [6,14], for example) using the fact that the derivatives of $\overline{\mathbf{u}}$ are bounded on $[0,T] \times \Omega$,

$$\left| \sum_{i=1}^{3} \partial_{x_i}[DH(\overline{U})] \cdot [F_i(U_\varepsilon) - F_i(\overline{U}) - DF_i(\overline{U})(U - \overline{U})] \right| \leq C \varrho_\varepsilon |\mathbf{u}_\varepsilon - \overline{\mathbf{u}}|^2 \tag{3.2}$$

for some positive constant C independent of ε.

Additionally, it is clear in light of Lemma 3.1 that

$$|H_F(\varrho_\varepsilon, \overline{\varrho}) \operatorname{div}_x \overline{\mathbf{u}}| \leq C H_F(\varrho_\varepsilon, \overline{\varrho}) \tag{3.3}$$

for some positive constant C independent of ε.

Lastly, the above sources provide the results to give the existence of a positive constant C independent of ε such that

$$\left| \left(\frac{\eta_\varepsilon}{\overline{\eta}} - \frac{\varrho_\varepsilon}{\overline{\varrho}} \right) (\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \cdot \nabla_x \overline{\eta} \right| \leq C(\varrho |\mathbf{u}_\varepsilon - \overline{\mathbf{u}}|^2 + H_P(\eta_\varepsilon, \overline{\eta})) \tag{3.4}$$

provided the derivatives of $\overline{\mathbf{u}}$ and $\overline{\eta}$ are bounded on $[0,T] \times \Omega$.

Combining (3.2)–(3.3) and invoking Lemma 3.1, the following lemma is obtained.

Lemma 3.2. *Let $\overline{U}$ represent a smooth solution to (1.7) and let U_ε represent an admissible weak solution to (1.4). Assume that the first derivatives of $\overline{\mathbf{u}}$ and $\overline{\eta}$ are bounded on $[0,T] \times \Omega$. Then, there exists some constant C independent of ε, but dependent on $\overline{U}$ such that*

$$\begin{aligned}
&\left| \sum_{i=1}^{3} \int_0^T \int_\Omega \left[F_i(U_\varepsilon) - F_i(\overline{U}) - DF(\overline{U})(U_\varepsilon - \overline{U}) \right] \cdot \partial_{x_i} DH(\overline{U}) \; \mathrm{d}x \; \mathrm{d}t \right| \\
&\quad \leq C \int_0^T \int_\Omega \mathcal{H}(U_\varepsilon | \overline{U}) \; \mathrm{d}x \; \mathrm{d}t.
\end{aligned} \tag{3.5}$$

3.2. Relative entropy viscosity term

The next step is to develop coercivity estimates for the relative entropy inequality's viscosity term. The main lemma of this subsection is as follows.

Lemma 3.3. *Let U be a smooth solution to (1.7) and let U_ε represent an admissible solution to (1.4). Assume that $\overline{\varrho}$ is bounded strictly below by some positive number $\overline{\varrho}_1$. Then, there exists some constant C independent of ε, but dependent on $\overline{U}$ such that*

$$\Bigg| \int_0^T \int_\Omega D\mathcal{H}(U_\varepsilon|\overline{U}) \cdot \sum_{i=1}^3 \partial_{x_i}(B(\overline{U})\partial_{x_i} DH(\overline{U}))\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega (\eta_\varepsilon - \overline{\eta})\nabla_x \ln \overline{\eta} \cdot \nabla_x(\ln \eta_\varepsilon - \ln \overline{\eta})\ \mathrm{d}x\ \mathrm{d}t \Bigg| \leq C \int_0^T \int_\Omega \mathcal{H}(U_\varepsilon|\overline{U})\ \mathrm{d}x\ \mathrm{d}t + \frac{\mu}{2} \int_0^T \int_\Omega |\nabla_x \mathbf{u}_\varepsilon - \nabla_x \overline{\mathbf{u}}|^2\ \mathrm{d}x\ \mathrm{d}t. \tag{3.6}$$

Proof. It is first noted that a standard calculation shows that

$$\begin{aligned} &D\mathcal{H}(U_\varepsilon|\overline{U}) \cdot \sum_{i=1}^3 \partial_{x_i}(B(\overline{U})\partial_{x_i} DH(\overline{U})) \\ &= \mu(\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \cdot \Delta_x \overline{\mathbf{u}} - \frac{\mu}{\overline{\varrho}}(\varrho_\varepsilon - \overline{\varrho})\overline{\mathbf{u}} \cdot \Delta_x \overline{\mathbf{u}} - \frac{1}{\overline{\varrho}}(\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \\ &\quad + (\ln \eta_\varepsilon - \ln \overline{\eta})\Delta_x \overline{\eta} - \frac{1}{\overline{\eta}}(\eta_\varepsilon - \overline{\eta})\Delta_x \overline{\eta}. \end{aligned} \tag{3.7}$$

First, it is noted that the term

$$\int_0^T \int_\Omega (\ln \eta_\varepsilon - \ln \overline{\eta})\Delta_x \overline{\eta} - \frac{1}{\overline{\eta}}(\eta_\varepsilon - \overline{\eta})\Delta_x \overline{\eta}\ \mathrm{d}x\ \mathrm{d}t$$

cancels with the second integral on the left of (3.6) after some integration by parts and rearranging of terms (using the fact that η_ε has one weak spatial derivative. Thus,

$$\begin{aligned} &\int_0^T \int_\Omega D\mathcal{H}(U_\varepsilon|\overline{U}) \cdot \sum_{i=1}^3 \partial_{x_i}(B(\overline{U})\partial_{x_i} DH(\overline{U}))\ \mathrm{d}x\ \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega (\eta_\varepsilon - \overline{\eta})\nabla_x \ln \overline{\eta} \cdot \nabla_x(\ln \eta_\varepsilon - \ln \overline{\eta})\ \mathrm{d}x\ \mathrm{d}t \\ &= \int_0^T \int_\Omega \mu(\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \cdot \Delta_x \overline{\mathbf{u}} - \frac{\mu}{\overline{\varrho}}(\varrho_\varepsilon - \overline{\varrho})\overline{\mathbf{u}} \cdot \Delta_x \mathbf{u} - \frac{1}{\overline{\varrho}}(\mathbf{u}_\varepsilon - \overline{\mathbf{u}}) \cdot \Delta_x \overline{\mathbf{u}}\ \mathrm{d}x\ \mathrm{d}t. \end{aligned} \tag{3.8}$$

These remaining terms are handled with the use of Hölder's, Young's, and Poincare's inequalities to find the bound proving the lemma. □

3.3. Kinetic approximation

Before handling the kinetic approximation term, bounds arising from the mesoscopic energy inequality (1.15) need to be determined. However, the energy functional contains the quantity $f_\varepsilon \ln f_\varepsilon$ that is not always nonnegative. However, the negative part of this quantity can be controlled as the following classical lemma (proven, example in [15]) shows.

Lemma 3.4. *Let Ω be bounded and ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε be an admissible solution to (1.4). Then, there is some positive constant C such that for any $\varepsilon > 0$*

$$\begin{aligned}&\int_\Omega \int_{\mathbb{R}^3} \frac{|\xi|^2}{4} f_\varepsilon + |f_\varepsilon \ln f_\varepsilon| \ \mathrm{d}\xi \ \mathrm{d}x \ \mathrm{d}t(T) \\ &\quad + \frac{1}{\varepsilon} \int_0^T \int_\Omega \int_{\mathbb{R}^3} \left| \sqrt{\varepsilon}\mathbf{u}_\varepsilon - \xi)\sqrt{f_\varepsilon} - 2\nabla_\xi \sqrt{f_\varepsilon} \right|^2 \ \mathrm{d}\xi \ \mathrm{d}x \ \mathrm{d}t \le C.\end{aligned} \tag{3.9}$$

Proof. For any $s \ge 0$, it is clear that

$$|s \ln s| = s \ln s - 2s \ln s \chi_{[0,1]}(s)$$

where $\chi_{[0,1]}$ is the indicator function on $[0, 1]$. It is also noted that there is some constant $C_1 > 0$ such that for any $s \ge 0$ and $\omega > 0$,

$$- s \ln s \chi_{[0,1]}(s) \le s\omega + C_1 \sqrt{s} \chi_{[0,e^{-\omega}]}(s) \le s\omega + C_1 e^{-\omega/2}. \tag{3.10}$$

Then, fixing $T > 0$ and taking $s = f_\varepsilon$ and $\omega = \frac{|\xi|^2}{8}$ and integrating over $\Omega \times \mathbb{R}^3$, the previous equation yields

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^3} |f_\varepsilon \ln f_\varepsilon| \ \mathrm{d}\xi \ \mathrm{d}x(T) &\le \int_\Omega \int_{\mathbb{R}^3} f_\varepsilon \ln f_\varepsilon \ \mathrm{d}\xi \ \mathrm{d}x \\ &\quad + \frac{1}{4} \int_\Omega \int_{\mathbb{R}^3} |\xi|^2 f_\varepsilon \ \mathrm{d}\xi \ \mathrm{d}x + 2C_1 \int_\Omega \int_{\mathbb{R}^3} e^{-|\xi|^2/16} \ \mathrm{d}\xi \ \mathrm{d}x.\end{aligned} \tag{3.11}$$

Using this and the fact that Ω is bounded, there is some positive constant C_2 such that,

$$\begin{aligned}&\int_\Omega \int_{\mathbb{R}^3} \frac{|\xi|^2}{4} f_\varepsilon + |f_\varepsilon \ln f_\varepsilon| \ \mathrm{d}\xi \ \mathrm{d}x \ \mathrm{d}t(T) \\ &\quad + \frac{1}{\varepsilon} \int_0^T \int_\Omega \int_{\mathbb{R}^3} \left| \sqrt{\varepsilon}\mathbf{u}_\varepsilon - \xi)\sqrt{f_\varepsilon} - 2\nabla_\xi \sqrt{f_\varepsilon} \right|^2 \ \mathrm{d}\xi \ \mathrm{d}x \ \mathrm{d}t \\ &\le \int_\Omega \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\gamma - 1} \varrho_\varepsilon^\gamma \ \mathrm{d}x(T) + \int_\Omega \int_{\mathbb{R}^3} f_\varepsilon \ln f_\varepsilon + \frac{|\xi|^2}{2} f_\varepsilon \ \mathrm{d}\xi \ \mathrm{d}x(T) \\ &\quad + \frac{1}{\varepsilon} \int_0^T \int_\Omega \int_{\mathbb{R}^3} \left| (\sqrt{\varepsilon}\mathbf{u}_\varepsilon - \xi)\sqrt{f_\varepsilon} - 2\nabla_\xi \sqrt{f_\varepsilon} \right|^2 \ \mathrm{d}\xi \ \mathrm{d}x \ \mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \mu \left| \nabla_x \mathbf{u}_\varepsilon \right|^2 \ \mathrm{d}x \ \mathrm{d}t + C_2 \le C\end{aligned} \tag{3.12}$$

where (1.15) has been used. □

Lemma 3.5. *For each $\varepsilon > 0$, let ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε be admissible weak solutions to (1.4) in the sense of Definition 1.1. Then, the following quantities are bounded by some constant C dependent only on T, Ω, and the initial data ϱ_0, $\mathbf{u}_0$, f_0 in the following spaces.*

$$(\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon) \ \text{in} \ L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \tag{3.13}$$

$$(\varrho_\varepsilon) \ \text{in} \ L^\infty(0, T; L^\gamma(\Omega)) \tag{3.14}$$

$$(\mathbf{u}_\varepsilon) \text{ in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \text{ and thus in } L^2(0,T;L^6(\Omega;\mathbb{R}^3)) \tag{3.15}$$

$$\left(\frac{1}{\varepsilon f_\varepsilon}\left[(\xi-\sqrt{\varepsilon}\mathbf{u}_\varepsilon)f_\varepsilon+\nabla_\xi f_\varepsilon\right]\right) \text{ in } L^2(0,T;L^2(\Omega\times\mathbb{R}^3;\mathbb{R}^3)). \tag{3.16}$$

From these bounds, particularly the last bound in the previous lemma, the following result is obtained.

Lemma 3.6. *Let $(\varrho_\varepsilon,\mathbf{u}_\varepsilon,f_\varepsilon)$ be an admissible solution to (1.4). Then, there is some constant C dependent on T, Ω, and the initial data such that*

$$\left|\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)f_\varepsilon\ \mathrm{d}\xi\ \mathrm{d}x\ \mathrm{d}t\right|\le C\sqrt{\varepsilon}. \tag{3.17}$$

Proof. Noting that

$$\int_{\mathbb{R}^3}\nabla_\xi f_\varepsilon\ \mathrm{d}\xi=0,$$

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)f_\varepsilon\ \mathrm{d}\xi\right|&=\left|\int_{\mathbb{R}^3}(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)f_\varepsilon-\nabla_\xi f_\varepsilon\ \mathrm{d}\xi\right|\\&\le\left(\int_{\mathbb{R}^3}f_\varepsilon\ \mathrm{d}\xi\right)^{1/2}\left(\int_{\mathbb{R}^3}\left|(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)f_\varepsilon-\nabla_\xi f_\varepsilon\right|^2\frac{1}{f_\varepsilon}\ \mathrm{d}\xi\right)^{1/2}\end{aligned} \tag{3.18}$$

by Hölder's inequality. The result follows by noticing that the first integral on the right of the previous inequality is just η_ε and the second integral is bounded by $C\sqrt{\varepsilon}$ by (3.16). □

The last term to be bounded is the kinetic approximation term,

$$\int_0^T\int_\Omega DH(\overline{U})\cdot\left[\partial_tU_\varepsilon+\sum_{i=1}^3\partial_{x_i}F_i(U_\varepsilon)-\sum_{i=1}^3\partial_{x_i}(B(U_\varepsilon)\partial_{x_i}DH(U_\varepsilon))\right]\ \mathrm{d}x\ \mathrm{d}t$$

which has to be understood in the sense of distributions since U_ε represents a weak solution to (1.4).

It is first noted that in the sense of distributions,

$$\begin{aligned}&\partial_tU_\varepsilon+\sum_{i=1}^3\partial_{x_i}F_i(U_\varepsilon)-\sum_{i=1}^3\partial_{x_i}(B(U_\varepsilon)\partial_{x_i}DH(U_\varepsilon))\\&=\begin{bmatrix}0\\ \int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{\varepsilon}}-\mathbf{u}_\varepsilon\right)f_\varepsilon+\nabla_xf_\varepsilon\ \mathrm{d}\xi\\ -\operatorname{div}_x\int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{\varepsilon}}-\mathbf{u}_\varepsilon\right)f_\varepsilon+\nabla_xf_\varepsilon\ \mathrm{d}\xi\end{bmatrix}.\end{aligned} \tag{3.19}$$

Thus, the kinetic approximation term can be written as

$$\begin{aligned}&\int_0^T\int_\Omega DH(\overline{U})\cdot\left[\partial_tU_\varepsilon+\sum_{i=1}^3\partial_{x_i}F_i(U_\varepsilon)-\sum_{i=1}^3\partial_{x_i}(B(U_\varepsilon)\partial_{x_i}DH(U_\varepsilon))\right]\ \mathrm{d}x\ \mathrm{d}t\\&=-\int_0^T\int_\Omega[D_{\mathbf{m}}(\overline{U})-\nabla_xD_\eta H(\overline{U})]\cdot\int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{\varepsilon}}-\mathbf{u}_\varepsilon\right)f_\varepsilon+\nabla_xf_\varepsilon\ \mathrm{d}\xi\ \mathrm{d}x\ \mathrm{d}t\end{aligned} \tag{3.20}$$

noting that $D_{\mathbf{m}}H(\overline{U}) = \overline{\mathbf{u}}$ and $D_\eta H(\overline{U}) = \ln\overline{\eta}+1$. It is noted that the quantity in the inner-most integral in the previous equation is the term

$$\nabla_x \eta_\varepsilon - \eta_\varepsilon \mathbf{u}_\varepsilon + \mathbf{J}_\varepsilon \tag{3.21}$$

where $\mathbf{J}_\varepsilon$ is the first moment of f_ε. Note that using the equation for the first moment of f_ε (1.17), the previous term can be written as

$$\nabla_x \eta_\varepsilon + \varepsilon\partial_t \mathbf{J}_\varepsilon + \operatorname{div}_x \mathbb{P}_\varepsilon, \tag{3.22}$$

noting again that these equations and terms are to be understood in the sense of distributions. Using this fact, the kinetic approximation term becomes

$$\begin{aligned}
&\int_0^T \int_\Omega DH(\overline{U}) \cdot \left[\partial_t U_\varepsilon + \sum_{i=1}^3 \partial_{x_i} F_i(U_\varepsilon) - \sum_{i=1}^3 \partial_{x_i}(B(U_\varepsilon)\partial_{x_i} DH(U_\varepsilon))\right] \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \sqrt{\varepsilon}(\partial_t \overline{\mathbf{u}} - \partial_t \nabla_x \ln\overline{\eta}) \cdot \int_{\mathbb{R}^3} \xi f_\varepsilon \,\mathrm{d}\xi \\
&\qquad + (\nabla_x \overline{\mathbf{u}} - \nabla_x(\nabla_x \ln\overline{\eta})) : \int_{\mathbb{R}^3} (\xi\otimes\xi - \mathbb{I}_3) f_\varepsilon \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.23}$$

Thus, the problem becomes to bound the term ξf_ε in the appropriate space by a constant and to bound the term $(\xi\times\xi)f_\varepsilon$ in the appropriate space by a positive power of ε. This in turn will force the kinetic approximation term to zero as $\varepsilon \to 0$.

First, the term

$$\int_0^T \int_\Omega \sqrt{\varepsilon}(\partial_t \overline{\mathbf{u}} - \partial_t \nabla_x \ln\overline{\eta}) \cdot \int_{\mathbb{R}^3} \xi f_\varepsilon \,\mathrm{d}\xi$$

is bounded. It is noted that

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \xi f_\varepsilon \,\mathrm{d}\xi\right| &\le \int_{|\xi|\le 1} |\xi| f_\varepsilon \,\mathrm{d}\xi + \int_{|\xi|>1} |\xi| f_\varepsilon \,\mathrm{d}\xi \\
&\le \int_{|\xi|\le 1} f_\varepsilon \,\mathrm{d}\xi + \int_{|\xi|>1} |\xi|^2 f_\varepsilon \le \int_{\mathbb{R}^3} f_\varepsilon \,\mathrm{d}\xi + \int_{\mathbb{R}^3} |\xi|^2 f_\varepsilon \,\mathrm{d}\xi
\end{aligned} \tag{3.24}$$

which is bounded by some positive constant C independent of ε by the boundary conditions and (1.15). Thus,

$$\left|\int_{\mathbb{R}^3} \xi f_\varepsilon \,\mathrm{d}\xi\right| \le C\sqrt{\varepsilon} \tag{3.25}$$

where C depends on the initial data, Ω, T, $\partial_t \overline{\mathbf{u}}$ and $\partial_t \nabla_x \ln\overline{\eta}$.

Next the term

$$\int_0^T \int_\Omega (\nabla_x \overline{\mathbf{u}} - \nabla_x(\nabla_x \ln\overline{\eta})) : \int_{\mathbb{R}^3} (\xi\otimes\xi - \mathbb{I}_3) f_\varepsilon \,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

is considered. First, it is noted that by adding zero by adding and subtracting the terms $\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon f_\varepsilon$, $\sqrt{\varepsilon}\mathbf{u}_\varepsilon\otimes\xi f_\varepsilon$, and $\sqrt{\varepsilon}\mathbf{u}_\varepsilon\sqrt{f_\varepsilon}\otimes 2\nabla_\xi\sqrt{f_\varepsilon}$ gives that

$$\begin{aligned}\int_{\mathbb{R}^3}(\xi\otimes\xi-\mathbb{I}_3)f_\varepsilon\,\mathrm{d}\xi &= \int_{\mathbb{R}^3}\varepsilon(\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)f_\varepsilon\,\mathrm{d}\xi\\ &\quad-\int_{\mathbb{R}^3}\sqrt{\varepsilon}\mathbf{u}_\varepsilon\sqrt{f_\varepsilon}\otimes\left[(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}-2\nabla_\xi\sqrt{f_\varepsilon}\right]\\ &\quad+\left[(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}-2\nabla_\xi\sqrt{f_\varepsilon}\right]\otimes\xi\sqrt{f_\varepsilon}\,\mathrm{d}\xi.\end{aligned}\tag{3.26}$$

Thus, there is some positive constant C dependent on the quantities $\overline{\varrho}$, $\overline{\mathbf{u}}$, and $\overline{\eta}$ such that

$$\begin{aligned}&\left|\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\nabla_x\overline{\mathbf{u}}-\nabla_x(\nabla_x\ln\overline{\eta})):\int_{\mathbb{R}^3}(\xi\otimes\xi-\mathbb{I}_3)f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\le C\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\varepsilon|\mathbf{u}_\varepsilon|^2+|\xi|^2)f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\ &\quad\cdot\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}-2\nabla_\xi\sqrt{f_\varepsilon}\right|^2\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\ &\le C\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\varepsilon|\mathbf{u}_\varepsilon|^2+|\xi|^2)f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\sqrt{\varepsilon}.\end{aligned}\tag{3.27}$$

The task is now to bound the quantity

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\varepsilon|\mathbf{u}_\varepsilon|^2+|\xi|^2)f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

Clearly from the energy inequality (1.15),

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3}|\xi|^2f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

is bounded by a constant uniformly in ε. Thus, it is enough to show that

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi\right|^2f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t$$

is bounded uniformly in ε. Following the idea of the process in [18], this term can be written as

$$\begin{aligned}&\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi\right|^2f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}\cdot\left[(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}-2\nabla_\xi\sqrt{f_\varepsilon}\right]\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega\int_{\mathbb{R}^3}(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\cdot\nabla_\xi f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\\
&\leq\int_0^T\int_\Omega\int_{\mathbb{R}^3} f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t+\left(\int_0^T\int_\Omega\int_{\mathbb{R}}\left|\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi\right|^2 f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\cdot\left(\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|(\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi)\sqrt{f_\varepsilon}-2\nabla_\xi\sqrt{f_\varepsilon}\right|^2\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2},
\end{aligned}
\tag{3.28}
$$

which after a use of Young's inequality, some rearranging of terms, and (1.15) gives

$$
\left|\int_0^T\int_\Omega\int_{\mathbb{R}^3}\left|\sqrt{\varepsilon}\mathbf{u}_\varepsilon-\xi\right|^2 f_\varepsilon\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\right|\leq C(1+\sqrt{\varepsilon})
\tag{3.29}
$$

The work in this subsection leads to the following lemma.

Lemma 3.7. *Let ϱ_ε, $\mathbf{u}_\varepsilon$, f_ε be an admissible solution to (1.1) and $\overline{\varrho}$, $\overline{\mathbf{u}}$, $\overline{\eta}$ a smooth solution to (1.3) where $\Phi\equiv 0$ and with the same corresponding initial data. Then,*

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega DH(\overline{U})\right.\\
&\qquad\left.\cdot\left[\partial_t U_\varepsilon+\sum_{i=1}^3\partial_{x_i}F_i(U_\varepsilon)-\sum_{i=1}^3\partial_{x_i}(B(U_\varepsilon)\partial_{x_i}DH(U_\varepsilon))\right]\,\mathrm{d}x\,\mathrm{d}t\right|\leq C\sqrt{\varepsilon}
\end{aligned}
\tag{3.30}
$$

where the left side of (3.30) is understood in the sense of distributions and C is some positive constant independent of ε.

4. Concluding comments

4.1. Weak solutions to the mesoscopic model

One avenue of future work is to prove the existence of admissible weak solutions to the mesoscopic model (1.4) for each $\varepsilon>0$. This was done for the related flowing-regime model in [17]. The flowing-regime model is similar to (1.4). However, in the bubbling regime, the ratio of the thermal speed to the characteristic velocity scale depends on ε, unlike the flowing regime in which this ratio is independent of ε.

4.2. Inhomogeneous case

In this paper, the external potential is assumed to be zero, leading to a lack of modeling of any external forces such as gravity and buoyancy. While these effects are easily introduced into the model (see [6,10,11], for example), the mathematical analysis of this term is not so trivial. Much of the work with relative entropies for arbitrary systems relies on the flux being a function of the solution only. However, adding the effects of external forces gives a Smoluchowski equation for the macroscopic model with a flux of $\eta\mathbf{u}-\eta\nabla_x\Phi$. Since Φ is generally taken to be a function of the position, the flux of (1.3) now depends explicitly on x. Thus, much of the work in deriving the relative entropy inequality (2.8) is complicated by the fact that

$$\partial_{x_i} F_i(U, x) = DF_i \partial_{x_i} U + F_{x_i}.$$

In [16], problems involving hyperbolic equations with fluxes dependent on the independent variables of the problem are shown to have an L^1 contraction property; however, the work does not consider using a general entropy and the corresponding relative entropy. In other papers, work has been done on the inhomogeneous (1.3) using a relative entropy to obtain weak–strong uniqueness [6], but in this case both solutions obey the equation, albeit one only weakly.

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(Received: October 8, 2019; revised: June 29, 2020)