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Well-posedness results for a class of semilinear time-fractional diffusion equations

Bruno de Andrade, Vo Van Au, Donal O'Regan and Nguyen Huy Tuan

Abstract. In this paper, we discuss an initial value problem for the semilinear time-fractional diffusion equation. The local well-posedness (existence and regularity) is presented when the source term satisfies a global Lipschitz condition. The unique continuation of solution and finite time blowup result are presented when the reaction terms are logarithmic functions (local Lipschitz types).

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $(N \geq 1)$ be a bounded open set with boundary Ω^c. The main aim of the paper is to study the properties of the solutions of a class of time-fractional diffusion equations involving the so-called Riemann–Liouville (R–L) time-fractional derivative. More precisely, we consider the following initial value problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) = -\dfrac{\partial^{1-\alpha}}{\partial t}\mathcal{A}u(x,t) + F(u), & x \in \Omega, \quad 0 < t \leq T, \\ u(x,t) = 0, & x \in \Omega^c, \quad 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{$\mathbb{P}$}$$

where $T > 0$, $\alpha \in (0,1)$ is real number and $\dfrac{\partial^{1-\alpha}}{\partial t}u$ denotes the R–L time-fractional derivative of order $1-\alpha$ of the function u formally given by

$$\frac{\partial^{1-\alpha}}{\partial t}f(t) := \frac{\mathrm{d}}{\mathrm{d}}t\left(\mathcal{J}^{\alpha}f\right)(t), \quad t > 0, \tag{1.1}$$

where the Riemann–Liouville fractional integral operator $\mathcal{J}^{\alpha} : L^2(0,T) \to L^2(0,T)$ is defined by the formula (see, e.g., [1])

$$(\mathcal{J}^{\alpha}f)(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \tau^{\alpha-1}f(t-\tau)\mathrm{d}\tau, & 0 < \alpha < 1, \\ f(t), & \alpha = 0, \end{cases} \tag{1.2}$$

and $\Gamma(\cdot)$ is the Gamma function. The operator $\mathcal{A}$ is a linear, positive definite, self-adjoint operator with compact inverse in $L^2(\Omega)$, $u = u(x,t)$ is the state of the unknown function and $u_0(x)$ is a given function. The function F is a nonlinear source term which appears in some physical phenomena [2–4].

When $\alpha = 1$ and $\mathcal{A} = -\Delta$ problem $\mathbb{P}$ describes the nonlinear heat Eq. [2,5–7]

$$\frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = F(u). \tag{1.3}$$

If $\alpha \in (0,1)$, Problem $\mathbb{P}$ is called an initial value problem for the semilinear time-fractional diffusion equation; we refer the reader to [3,8–10] and the references therein. Many important physical models and practical problems require one to consider the diffusion model with a fractional derivative rather than a classical one, like physical models considering memory effects [2–4,11–13] and some corresponding engineering problems [2,3,14,15] with power-law memory (non-local effects) in time [4,8,16–20]. For nonlinearities of power-type $F(u) = |u|^{p-1}u$ for $p \geq 1$, Bruno de Andrade et al. [3] considered the fractional reaction–diffusion equation to discuss the global well-posedness and asymptotic behavior of solutions; see also [7,21] and the references therein. Studies of logarithmic nonlinearity have a long history in physics as they occur naturally in inflation cosmology, quantum mechanics, and nuclear physics [22] and PDEs with logarithmic nonlinearity have attracted many authors; see [23–26] and the references therein.

Results on initial value problems for R–L time-fractional diffusion equation with logarithmic nonlinearity are quite limited. The solution operator of our problem $E_{\alpha,1}(-\mathcal{A}t^\alpha)$ brings some difficulties in estimating and analyzing the solution (existence and regularity estimate of the solutions). We consider the model with the source terms $F_p(u) = \eta V_p(u)\log|u|$ and $V_p(u) = |u|^{p-2}u,\ p \geq 2, \eta > 0$ (locally Lipschitz type). To present the properties of the solutions in $W^{s,q}(\Omega)$, we need to consider the Lipschitz properties of the source function (both global Lipschitz property and local Lipschitz property). Based on the conditions of the constants s, q depending on the dimensions $N \geq 1$ and the constant $s > 0$, we set up the Sobolev embeddings $\mathbb{X}^s(\Omega) \hookrightarrow W^{s,q}(\Omega) \hookrightarrow L^p(\Omega)$ (see the definition of the spaces $W^{s,q}(\Omega)$ and $\mathbb{X}^s(\Omega)$ in (2.3) and (2.7) below).

In Sect. 2, we present some basic definitions and the setting for our work. Moreover, we obtain a precise representation of solutions using Mittag–Leffler operators. In Sect. 3, we first present local well-posedness results when the source term satisfies a global Lipschitz condition. Also local existence, continuation of solutions and finite time blowup results are presented when the source terms are logarithmic functions.

2. Notations and preliminaries

2.1. Relevant notations and the functional spaces

Given two positive quantities y, z, we write $y \lesssim z$ if there exists a constant $C > 0$ such that $y \leq Cz$. Let us recall that the spectral problem

$$\begin{cases} \mathcal{A}\phi_j(x) = \lambda_j \phi_j(x), & x \in \Omega, \quad \sigma \in (0,1], \\ \phi_j(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

admits a family of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots \nearrow \infty.$$

Given a Banach space B, let $C([0,T];B)$ be the set of all continuous functions which map $[0,T]$ into B. The norm of the function space $C^k([0,T];B)$, for $0 \leq k \leq \infty$ is denoted by

$$\|v\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \sup_{t\in[0,T]} \left\|v^{(i)}(t)\right\|_B < \infty. \tag{2.2}$$

For any real numbers $s > 0$ and $1 \leq p < \infty$, we recall the fractional Sobolev-type spaces $W^{s,p}(\Omega)$ via the Gagliardo approach (also called Aronszajn or Slobodeckij spaces). Fix a number $s \in (0,1)$ and for any $p \in [1, \infty)$, define $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \left\{ v \in L^p(\Omega) \quad \text{s.t.} \quad \frac{|v(x) - v(y)|}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega \times \Omega) \right\}. \tag{2.3}$$

For $0 < s < 1$, it can be said that $W^{s,p}(\Omega)$ is an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed the corresponding norm

$$\|v\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |v|^p \mathrm{d}x + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}, \tag{2.4}$$

where the seminorm

$$|v|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}, \tag{2.5}$$

denotes the Gagliardo (semi)norm of v. For $p = 2$ in (2.3), together with the norm $\|\cdot\|_{W^{s,2}(\Omega)}$ the space becomes a Hilbert space. Let us also set $W_0^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{s,2}(\Omega)}$. It is well known that if Ω is bounded, then we have the following continuous embeddings:

$$W_0^{s,2}(\Omega) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2s}}(\Omega), & \text{if} \quad s < \frac{N}{2}, \\ L^p(\Omega), & \text{if} \quad s = \frac{N}{2}, \\ C^{0, s-\frac{N}{2}}(\Omega), & \text{if} \quad s > \frac{N}{2}; \end{cases} \tag{2.6}$$

for more details on fractional Sobolev spaces see [17,27] and the references therein.

For each number $s \geq 0$, we define

$$\mathbb{X}^s(\Omega) := \left\{ v = \sum_{j=1}^\infty v_j \phi_j \in L^2(\Omega) : \|v\|^2_{\mathbb{X}^s(\Omega)} = \sum_{j=1}^\infty v_j^2 \lambda_j^s < \infty \right\}, \quad v_j = \int_\Omega v(x) \phi_j(x) \mathrm{d}x. \tag{2.7}$$

Let us denote by $H^s(\Omega)$ the Sobolev–Slobodecki space $W^{s,p}(\Omega)$ when $p = 2$, and by $H_0^s(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$. Throughout this paper, Ω is assumed to be smooth enough such that $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ for $0 < s < \frac{1}{2}$. This guarantees $H_0^s(\Omega) = H^s(\Omega)$. Moreover, it is well-known that

$$\mathbb{X}^s(\Omega) = \begin{cases} H_0^s(\Omega), & \text{for } 0 \leq s < \frac{1}{2}, \\ H_{00}^{1/2}(\Omega) \subsetneqq H_0^{1/2}(\Omega), & \text{for } s = \frac{1}{2}, \\ H_0^s(\Omega), & \text{for } \frac{1}{2} < s \leq 1, \\ H_0^1(\Omega) \cap H^s(\Omega), & \text{for } 1 < s \leq 2, \end{cases}$$

where we denote by $H_{00}^{1/2}(\Omega)$ the Lions–Magenes space. Let $\mathbb{X}^{-s}(\Omega)$ be the duality of $\mathbb{X}^s$ which corresponds to the dual inner product $(\cdot, \cdot)_{-s,s}$. Then, the operator $\mathcal{A}^s : \mathbb{X}^s(\Omega) \to \mathbb{X}^{-s}(\Omega)$ of the fractional power s can be defined by

$$\mathcal{A}^s v := \sum_{j=1}^\infty \lambda_j^s \left(v, \phi_j\right)_{-s,s} \phi_j, \quad \forall v \in \mathbb{X}^s.$$

The above settings can be found in [28] (Sect. 3) and [29] (Sect. 2). In the next lemmas, we present some useful embeddings between the spaces mentioned above.

Lemma 2.1. *Given $1 \leq p, p' < \infty$, $0 \leq s \leq s' < \infty$ and $s' - \frac{N}{p'} \geq s - \frac{N}{p}$. Then*

$$W^{s',p'}(\Omega) \hookrightarrow W^{s,p}(\Omega). \tag{2.8}$$

Lemma 2.2. *Let $0 \leq s \leq s' \leq 2$ and let $H^{-s}(\Omega)$ be the dual space of $H_0^s(\Omega)$. Then the following embeddings hold*

$$\mathbb{X}^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{X}^{-s}(\Omega), \tag{2.9}$$

and

$$\mathbb{X}^{s'}(\Omega) \hookrightarrow \mathbb{X}^s(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega) \hookrightarrow \mathbb{X}^{-s}(\Omega) \hookrightarrow \mathbb{X}^{-s'}(\Omega). \tag{2.10}$$

2.2. Properties of Mittag–Leffler functions and some related results

The Mittag–Leffler function is defined by (see [30])

$$E_{\alpha,\alpha'}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \alpha')}, \quad z \in \mathbb{C}, \tag{2.11}$$

where $\alpha > 0$ and $\alpha' \in \mathbb{R}$ are arbitrary constants, Γ is the usual gamma function.

Next, we give some properties of the Mittag–Leffler function. Let $\alpha' \in \mathbb{R}$, and $\alpha \in (0, 2)$, we have:

$$|E_{\alpha,\alpha'}(-z)| \leq \frac{C}{1 + |z|}, \quad \tau \leq \arg(z) \leq \pi,$$

where $C > 0$ depends on α, α', τ and $\frac{\pi\alpha'}{2} < \tau < \min\{\pi, \pi\alpha'\}$ (see e.g. [30]).

Lemma 2.3. (See [30,31]) *For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, there exist positive constants $\underline{C}, \overline{C}$, such that*

$$\text{(a)} \quad E_{\alpha,1}(-z) > 0, \quad \textit{for any} \quad z > 0; \tag{2.12a}$$

$$\text{(b)} \quad \frac{\underline{C}}{1 + z} \leq E_{\alpha,\alpha'}(-z) \leq \frac{\overline{C}}{1 + z}, \quad \textit{for} \quad \alpha' \in \mathbb{R},\ z > 0. \tag{2.12b}$$

Lemma 2.4. (See [31]) *Let α, λ, γ are positive constants, and for every $t > 0, n \in \mathbb{N}$, we have*

$$\text{(a)} \quad \frac{\mathrm{d}^n}{\mathrm{d}t^n} [E_{\alpha,1}(-\lambda t^\alpha)] = -\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^\alpha); \tag{2.13a}$$

$$\text{(b)} \quad \left|\lambda^\gamma t^{\alpha-1} E_{\alpha,\alpha'}(-\lambda t^\alpha)\right| \leq C t^{\alpha-1-\alpha\gamma}. \tag{2.13b}$$

Lemma 2.5. (See [32]) *The following equality holds*

$$E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(s) e^{-zs} \mathrm{d}s, \quad \textit{for} \quad z \in \mathbb{C}, \tag{2.14}$$

where we recall the definition of the Wright-type function

$$\mathcal{M}_\alpha(s) := \sum_{j=0}^{\infty} \frac{s^j}{j! \Gamma(-\alpha j + 1 - \alpha)}, \quad 0 < \alpha < 1. \tag{2.15}$$

Moreover, $\mathcal{M}_\alpha(s)$ is a probability density function, that is,

$$\mathcal{M}_\alpha(s) \geq 0, \quad \textit{for } s > 0; \quad \textit{and} \quad \int_0^\infty \mathcal{M}_\alpha(s) \mathrm{d}s = 1. \tag{2.16}$$

Lemma 2.6. (See [3], expression (6), for $\mathcal{A} = -\Delta$) *The function u is a mild solution of $\mathbb{P}$ if $u \in C([0,T];L^2(\Omega))$ and satisfies the following integral equation*

$$u(t) = E_{\alpha,1}(-t^\alpha \mathcal{A})u_0 + \int_0^t E_{\alpha,1}(-(t-\tau)^\alpha \mathcal{A})F(u)(\tau)\mathrm{d}\tau \tag{2.17}$$

for all $t < T$, and $\alpha \in (0,1)$.

Lemma 2.7. (a) (Weakly singular Grönwall's inequality, see [33], Theorem 1.2, page 2) *Let a, b, β, β' be non-negative constants and $\beta, \beta' < 1$. Assume that $\varphi \in L^1[0,T]$ satisfies*

$$\varphi(t) \le at^{-\beta} + b\int_0^t (t-s)^{-\beta'}\varphi(s)\mathrm{d}s, \quad \textit{for a.e.} \quad t \in (0,T]. \tag{2.18}$$

Then there exists a constant $C(b,\beta',T)$ such that

$$\varphi(t) \le C(b,\beta',T)\frac{at^{-\beta}}{1-\beta}, \quad \textit{for a.e.} \quad t \in (0,T]. \tag{2.19}$$

(b) (Fractional Grönwall's inequality, see [34], Corollary 2) *Assume $\beta > 0$, φ is nonnegative, locally integrable, and*

$$\varphi(t) \le a + b\int_0^t (t-\tau)^{\beta-1}\varphi(s)\mathrm{d}s,$$

on $(0,T)$, where a, b are positive constants. Then,

$$\varphi(t) \le aE_{\beta,1}\big(b\Gamma(\beta)t^\beta\big), \quad \textit{on } (0,T).$$

Lemma 2.8. (a) *For $z > 0$, then there exists a constant $C > 0$ depending on θ such that*

$$\begin{cases} \big|\log z\big| \le Cz^{-\theta}, & \textit{for} \quad \theta > 0, \quad 0 < z < 1, \\ \big|\log z\big| \le Cz^{\theta}, & \textit{for} \quad \theta > 0, \quad z \ge 1. \end{cases} \tag{2.20}$$

(b) (Hölder's inequality for negative exponents) (see [35]) *Let $k' < 0$, and $k \in \mathbb{R}$ be such that $\frac{1}{k'} + \frac{1}{k} = 1$ and $f(x), g(x) \ge 0$, $\forall x \in \Omega$ are Lebesgue measurable functions. Then*

$$\int_\Omega fg\mathrm{d}x \ge \left(\int_\Omega |f|^{k'}\mathrm{d}x\right)^{\frac{1}{k'}} \left(\int_\Omega |g|^{k}\mathrm{d}x\right)^{\frac{1}{k}}. \tag{2.21}$$

Proof. The proof of inequalities (2.20) and (2.21) are elementary, so we omit them here. □

Lemma 2.9. (See [17,27]) *Let $\Omega \subset \mathbb{R}^N$, $k, m \in \mathbb{N}$ with $k \ge m$ satisfying $(k-m)p < N$ and $1 \le p < \infty$. Then we have the following Sobolev embeddings*

$$\left.\begin{array}{lll} (SE1): W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega), & \textit{for} & 1 \le q < p^*_{k,m}, \\ (SE2): \mathbb{X}^s(\Omega) \hookrightarrow H^s(\Omega), & \textit{for} & s > 0, \\ (SE3): L^p(\Omega) \hookrightarrow \mathbb{X}^s(\Omega), & \textit{for} & -\frac{N}{2} < s \le 0, \quad p \ge 2^*_s, \\ (SE4): \mathbb{X}^s(\Omega) \hookrightarrow L^p(\Omega), & \textit{for} & 0 \le s < \frac{N}{2}, \quad p \le 2^*_s, \end{array}\right\} \tag{2.22}$$

where $p^*_{k,m}, 2^*_s$ are the so-called fractional Sobolev exponents, given by

$$\frac{1}{p^*_{k,m}} = \frac{1}{p} + \frac{m}{N} - \frac{k}{N}, \quad \textit{and} \quad \frac{1}{2^*_s} = \frac{1}{2} - \frac{s}{N}. \tag{2.23}$$

3. Main results

3.1. The case when the source terms are globally Lipchitz functions

In this section, we will study the existence and uniqueness of mild solutions to problem $\mathbb{P}$. First we assume the global Lipschitz continuity and the time Hölder continuity on the nonlinear term. More precisely, we suppose that $F : \mathbb{X}^p(\Omega) \to \mathbb{X}^q(\Omega)$, $F(0) = 0$, and

$$\big\|F(v_1) - F(v_2)\big\|_{\mathbb{X}^q(\Omega)} \le K\big\|v_1 - v_2\big\|_{\mathbb{X}^p(\Omega)}, \tag{3.1}$$

where $K : [0, T] \to \mathbb{R}_+$ and p, q are real numbers.

Our results in this section present the local well-posedness of the problem. Here, $\mathbb{Z}_{\beta,d}((0, T]; \mathbb{X}^q(\Omega))$ denotes the weighted space of all functions $v \in C((0, T]; \mathbb{X}^q(\Omega))$ such that

$$\|v\|_{\mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^q(\Omega))} := \sup_{t\in(0,T]} t^\beta e^{-dt}\|v(t, \cdot)\|_{\mathbb{X}^q(\Omega)} < \infty$$

where $\beta > 0,\ d > 0$. First we state the following lemma which will be useful in our main results. (This lemma can be found in [36], Lemma 8, page 9.)

Lemma 3.1. *Let $a > -1$, $b > -1$ such that $a + b \ge -1$, $h > 0$ and $t \in [0, T]$. For $\mu > 0$, the following limit holds*

$$\lim_{\mu\to\infty}\left(\sup_{t\in[0,T]} t^h \int_0^1 s^a(1 - s)^b e^{-\mu t(1-s)}\mathrm{d}s\right) = 0.$$

Now, we are in the position to introduce the main contributions of this work. Our main results address the existence and regularity of the mild solution.

Theorem 3.1. *Let $0 < \beta < 1$. Assume that $q - p < \min\left\{\frac{2(1-\beta)}{\alpha}, \frac{2\beta}{\alpha}\right\}$. Let $u_0 \in \mathbb{X}^{q-2\gamma}(\Omega))$ for any $0 < \gamma < \min\left\{\frac{\beta}{\alpha}; 1\right\}$. Then Problem $\mathbb{P}$ has a unique solution u in $\mathbb{Z}_{\beta,d_0}((0, T]; \mathbb{X}^q(\Omega))$ with some $d_0 > 0$. Moreover, there exist positive constant C independently of t, x and for $1/2 < \beta < 1$, $1 - \beta < \alpha < 1/2$ such that*

$$\big\|u(\cdot, t)\big\|_{\mathbb{X}^p(\Omega))} \le Ct^{-\beta}e^{dt}\big\|u_0\big\|_{\mathbb{X}^{q-2\gamma}(\Omega)}. \tag{3.2}$$

Proof. Define the mapping $\mathfrak{B} : \mathbb{Z}_{\beta,d}((0, T]; \mathbb{X}^p(\Omega)) \to \mathbb{Z}_{\beta,d}((0, T]; \mathbb{X}^p(\Omega))$, $d > 0$, by

$$\mathfrak{B}w(t) := E_{\alpha,1}(-t^\alpha\mathcal{A})u_0 + \int_0^t E_{\alpha,1}(-(t - \tau)^\alpha\mathcal{A})F(w)(\tau)\mathrm{d}\tau. \tag{3.3}$$

In what follows, we shall prove the existence of a unique solution of Problem $\mathbb{P}$. This is based on the Banach principal argument. First, since $0 < \gamma < 1$, we have

$$\begin{aligned}\Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|^2_{\mathbb{X}^p(\Omega))} &= \sum_{j=1}^\infty (u_0, \phi_j)^2 \left(E_{\alpha,1}\left(-\lambda_j t^\alpha\right)\right)^2 \lambda_j^p\\ &\le \sum_{j=1}^\infty (u_0, \phi_j)^2 \frac{C^2}{(1 + \lambda_j t^\alpha)^{2\gamma}}\lambda_j^p \le C^2t^{-2\alpha\gamma}\sum_{j=1}^\infty (u_0, \phi_j)^2 \lambda_j^{p-2\gamma}.\end{aligned} \tag{3.4}$$

It follows from the condition $\beta > \alpha\gamma$ that

$$t^\beta e^{-dt}\Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|_{\mathbb{X}^p(\Omega))} \le Ct^{\beta-\alpha\gamma}\|u_0\|_{\mathbb{X}^{p-2\gamma}(\Omega))} \le CT^{\beta-\alpha\gamma}\|u_0\|_{\mathbb{X}^{p-2\gamma}(\Omega))}. \tag{3.5}$$

From the latter inequality, we deduce that $u_0 \in \mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))$. Indeed, for $w_1, w_2 \in \mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))$, we have

$$\begin{aligned}
&\big\|\mathfrak{B}w_1 - \mathfrak{B}w_2\big\|_{\mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))} \\
&\quad = \sup_{t\in(0,T]} t^\beta e^{-dt}\Big\|\int_0^t E_{\alpha,1}(-(t-\tau)^\alpha\mathcal{A})\big[F(w_1(\tau)) - F(w_2(\tau))\big]\mathrm{d}\tau\Big\|_{\mathbb{X}^p(\Omega)} \\
&\quad \le \sup_{t\in(0,T]} t^\beta e^{-dt}\int_0^t \Big\|E_{\alpha,1}(-(t-\tau)^\alpha\mathcal{A})\big[F(w_1(\tau)) - F(w_2(\tau))\big]\Big\|_{\mathbb{X}^p(\Omega)}\mathrm{d}\tau \\
&\quad \le C\sup_{t\in(0,T]} t^\beta e^{-dt}\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\Big\|F(w_1(\tau)) - F(w_2(\tau))\Big\|_{\mathbb{X}^q(\Omega)}\mathrm{d}\tau \\
&\quad \le CK\big\|v_1 - v_2\big\|_{\mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))}\sup_{t\in(0,T]} t^\beta\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-\beta}e^{-d(t-\tau)}\mathrm{d}\tau.
\end{aligned} \tag{3.6}$$

We derive the estimate

$$\big\|\mathfrak{B}w_1 - \mathfrak{B}w_2\big\|_{\mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))} \le \mathscr{L}_d\big\|v_1 - v_2\big\|_{\mathbb{Z}_{\beta,d}((0,T];\mathbb{X}^p(\Omega))},$$

where

$$\mathscr{L}_d = t^\beta\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-\beta}e^{-d(t-\tau)}\mathrm{d}\tau.$$

From the conditions of α, β, p, q, we find that

$$\beta - \frac{\alpha(q-p)}{2} > 0, \quad -\frac{\alpha(q-p)}{2} > -1, \quad -\beta > -1, \quad -\frac{\alpha(q-p)}{2} - \beta > -1.$$

Applying Lemma 3.1, we obtain that

$$\begin{aligned}
\lim_{d\to\infty}\mathscr{L}_d &:= K\lim_{\mu\to\infty}\left(\sup_{t\in(0,T]} t^\beta\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-\beta}e^{-(t-\tau)d}\mathrm{d}\tau\right) \\
&= K\lim_{d\to\infty}\left(\sup_{t\in(0,T]} t^{\beta-\frac{\alpha(q-p)}{2}}\int_0^1 (1-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-\beta}e^{-t(1-\tau)d}\mathrm{d}\tau\right) \\
&= 0.
\end{aligned} \tag{3.7}$$

Hence, there exists a positive $d > 0$ such that $\mathfrak{B}$ is a contraction mapping on $\mathbb{Z}_{\beta,d_0}((0,T];\mathbb{X}^p(\Omega))$. This together with (3.5) leads to $\mathfrak{B}w \in \mathbb{Z}_{\beta,d_0}((0,T];\mathbb{X}^p(\Omega))$ if $w \in \mathbb{Z}_{\beta,d_0}((0,T];\mathbb{X}^p(\Omega))$. Hence, we conclude that $\mathfrak{B}$ has a fixed point u in $\mathbb{Z}_{\beta,d_0}((0,T];\mathbb{X}^p(\Omega))$, i.e, u is a unique mild solution of Problem $\mathbb{P}$.

This and the technique in (3.6) yields

$$\begin{aligned}
\big\|u(t,\cdot)\big\|_{\mathbb{X}^p(\Omega)} &\le \Big\|E_{\alpha,1}(-t^\alpha\mathcal{A})u_0\Big\|_{\mathbb{X}^p(\Omega))} + \int_0^t \Big\|E_{\alpha,1}(-(t-\tau)^\alpha\mathcal{A})F(u(\tau))\Big\|_{\mathbb{X}^p(\Omega)}\mathrm{d}\tau \\
&\le Ct^{-\alpha\gamma}\|u_0\|_{\mathbb{X}^{p-2\gamma}(\Omega))} + CK\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\big\|u(\tau))\big\|_{\mathbb{X}^p(\Omega)}\mathrm{d}\tau.
\end{aligned} \tag{3.8}$$

Multiplying both sides to $t^\beta e^{-dt}$, we find that

$$t^\beta e^{-dt}\big\|u(t,\cdot)\big\|_{\mathbb{X}^p(\Omega)} \le Ce^{-dt}t^{\beta-\alpha\gamma}\|u_0\|_{\mathbb{X}^{p-2\gamma}(\Omega))} + CKt^\beta e^{-dt}\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\big\|u(\tau))\big\|_{\mathbb{X}^p(\Omega)}\mathrm{d}s. \tag{3.9}$$

By applying the Hölder inequality, and then using $e^{-2d(t-\tau)} < 1$, we can find some positive constant $\mathscr{M}$ such that

$$\begin{aligned}
&t^\beta e^{-dt}\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\big\|u(\tau))\big\|_{\mathbb{X}^p(\Omega)}\mathrm{d}\tau\\
&\le \left(t^{2\beta}\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-2\beta}e^{-2d(t-\tau)}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\left(\tau^\beta e^{-d\tau}\big\|u(s,\cdot))\big\|_{\mathbb{X}^p(\Omega)}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&\le \left(t^{2\beta}t^{-\frac{\alpha(q-p)}{2}}\int_0^1 (1-\tau)^{-\frac{\alpha(q-p)}{2}}\tau^{-2\beta}\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\left(\tau^\beta e^{-d\tau}\big\|u(\tau,\cdot))\big\|_{\mathbb{X}^p(\Omega)}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&\le \mathscr{M}\left(\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\left(\tau^\beta e^{-d\tau}\big\|u(\tau,\cdot))\big\|_{\mathbb{X}^\sigma(\Omega)}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

Taking the estimate (3.8), and (3.10) together gives that

$$\mathscr{U}_{\beta,d}(t) \le 2C^2e^{-2dt}t^{2\beta-2\alpha\gamma}\|u_0\|^2_{\mathbb{X}^{p-2\gamma}(\Omega))} + |\mathscr{M}|^2C^2K^2\int_0^t (t-\tau)^{-\frac{\alpha(q-p)}{2}}\mathscr{U}_{\beta,d}(\tau)\mathrm{d}\tau, \tag{3.11}$$

where

$$\mathscr{U}_{\beta,d}(t) := \left(t^\beta e^{-dt}\big\|u(t,\cdot)\big\|_{\mathbb{X}^p(\Omega)}\right)^2.$$

Applying Lemma 2.7(b), we deduce that

$$\mathscr{U}_{\beta,d}(t) \le 2C^2T^{2\beta-2\alpha\gamma}\|u_0\|^2_{\mathbb{X}^{p-2\gamma}(\Omega))}E_{1-\frac{\alpha(q-p)}{2},1}\left(|\mathscr{M}|^2C^2K^2\Gamma\left(1-\frac{\alpha(q-p)}{2}\right)t^{1-\frac{\alpha(q-p)}{2}}\right). \tag{3.12}$$

The proof of Theorem 3.1 is completed. □

3.2. The case when the source terms are locally Lipschitz functions

Next, we shall present the results when the source terms are logarithmic nonlinearities of the following type $F_p(u) = \eta V_p(u)\log|u|$ and $V_p(u) = |u|^{p-2}u$, for $p \ge 2, \eta > 0$.

Remark 3.1. For the source terms of polynomial type nonlinearities, i.e., $F_p(u) = V_p(u)$ a simpler result was considered in [2,5].

Lemma 3.2. *For $F_p(u)(x,t) = \eta V_p(u)\log|u| \in L^\infty(\Omega\times(0,T)\times\mathbb{R})$, $p \ge 2, \eta > 0$, there exists a positive constant C such that*

$$\begin{aligned}
|F_p(u) - F_p(w)| &\le C\left(\big|\log|u|\big|\,|u-w| + |u|^{p-2}\big|\log|u|\big|\,|u-w|\right)\\
&\quad + C\left(|w|^{p-2}\big|\log|u|\big|\,|u-w| + |w|^{p-2}\,|u-w|\right),
\end{aligned} \tag{3.13}$$

for all $(x,t) \in \Omega\times(0,T)$, $\forall u,w \in \mathbb{R}$.

Proof. For $(x,t) \in \Omega \times (0,T)$ and $u, w \in \mathbb{R}$, we have

$$\begin{aligned}\left|F_p(u) - F_p(w)\right| &= \eta\left|V_p(u)\log|u| - V_p(w)\log|w|\right| \\ &\leq \eta\left(\left|V_p(u) - V_p(w)\right|\left|\log|u|\right| + |V_p(u)|\left|\log|u| - \log|w|\right|\right). \end{aligned} \tag{3.14}$$

Thanks to the results in [5], we have that

$$\begin{aligned}\left|V_p(u) - V_p(w)\right| &= \left|u|^{p-2}u - |w|^{p-2}w\right| \\ &\leq C(1 + |u|^{p-2} + |w|^{p-2})|u - w|. \end{aligned} \tag{3.15}$$

Using the basic inequality $\log(1+z) < z$ for $z > 0$, one has

$$\begin{aligned}\left|\log|u| - \log|w|\right| &= \left|\log\left|1 + \frac{|u| - |w|}{|w|}\right|\right| \\ &< \left|\log\left(1 + \frac{|u-w|}{|w|}\right)\right| < \frac{|u-w|}{|w|}. \end{aligned} \tag{3.16}$$

From (3.14)–(3.16), we have the proof of Lemma 3.2. □

Theorem 3.2. (Local existence) *Let $\alpha \in (0,1)$, $N \geq 1, p \geq 2, 0 \leq s < s_2$, for $0 \leq s_2 < N/2$. Let $1 \leq q \leq \min\left\{2^\star_{s_2,s}; \frac{N\theta}{N+\theta s}\right\}$ with $2^\star_{s_2,s}$ satisfying $\frac{1}{2^\star_{s_2,s}} = \frac{1}{2} + \frac{s}{N} - \frac{s_2}{N}$ and $qs < N$. Let $u_0 \in \mathbb{X}^{s_2}(\Omega) \cap W^{s,q}(\Omega)$, and for the nonlinearity source of logarithmic function type*

$$F_p(u) = \eta V_p(u)\log|u|, \quad \text{for } V_p(u) = |u|^{p-2}u, \quad \text{with } p \geq 2, \eta > 0,$$

then there is a time constant $T > 0$ (depending only on u_0) such that Problem $\mathbb{P}$ has a unique mild solution belonging to $C([0,T]; W^{s,q}(\Omega))$.

Remark 3.2. In Theorem 3.2, for $N \geq 1$, and $0 \leq s_2 < N/2$ let us choose $N = 3, s_2 = 1$. From the conditions

$$\begin{cases} s < 2s_2, s \in \mathbb{N}, \\ 1 \leq q \leq \frac{2N}{N+2s-2s_2}, \end{cases} \quad \text{this implies that} \quad \begin{cases} s = 0, & q \in [1,6], \quad \text{or}, \\ s = 1, & q \in [1,2]. \end{cases} \tag{3.17}$$

Then, the Problem $\mathbb{P}$ has a unique mild solution $u \in C([0,T]; L^q(\Omega))$, $1 \leq q \leq 6$, or $u \in C([0,T]; W^{1,q}(\Omega))$, for $1 \leq q \leq 2$.

Proof. For $N \geq 1, p \geq 2$, $0 < \theta \leq p-1$ (θ is defined in Lemma 2.8), we put

$$0 \leq s_2 < \min\left\{1; \frac{(p-1)N}{2p}\right\}, \quad s_1 = ps_2 - s^*, \tag{3.18}$$

$$s \in \mathbb{N} \quad \text{satisfies} \quad s < s_2, \quad 1 \leq q \leq \min\left\{2^*_{s_2,s}; \frac{N\theta}{N+\theta s}\right\}, \tag{3.19}$$

$$\max\{ps_2; Z(a,b)\} < s^* < \min\left\{ps_2 + \frac{N}{2}; 1 + (p-1)s_2\right\}, \tag{3.20}$$

where $2^*_{s_2,s} = \frac{1}{2} + \frac{s}{N} - \frac{s_2}{N}$, and $Z(a,b)$ be defined by

$$Z(a,b) = \frac{N(2pa - qb) + 2pq(s_2 b - sa)}{2qb}, \tag{3.21}$$

with the pairs (a,b) as follows:

$$(a,b) \in \{(1,1), (\theta,1), (p-2,1), (\theta,p-1), (\theta,p-1), (p-2,p-1)\}, \quad \text{for } \theta > 0.$$

Let $T>0$ and $R>0$ to be chosen later, and we consider the following space

$$\mathbb{W} :=\Big\{u\in C([0,T];W^{s,q}(\Omega)) : u(\cdot,0)=u_0,\ \text{and}\ \big\|u(\cdot,t)-u_0\big\|_{W^{m,q}(\Omega)}\le R\Big\}, \tag{3.22}$$

for $0<\alpha<1$, and we define the mapping $\mathbf{M}$ on $\mathbb{W}$ by

$$\begin{aligned}\mathbf{Mu}(t) &= E_{\alpha,1}(-t^{\alpha}\mathcal{A})u_0 + \int_0^t E_{\alpha,1}(-(t-\tau)^{\alpha}\mathcal{A})F_p(u)(\tau)\mathrm{d}\tau \\ &:= E_{\alpha,1}(-t^{\alpha}\mathcal{A})u_0 + J(F(u))(t).\end{aligned} \tag{3.23}$$

We show that $\mathbf{M}$ is invariant in $\mathbb{W}$ and $\mathbf{M}$ is a contraction.

- *Claim I* If $u_0\in\mathbb{X}^{s_2}(\Omega)\cap W^{s,q}(\Omega)$, *then* $\mathbf{M}$ *is* $\mathbb{W}$*-invariant.* In fact, from Lemma 2.3(b), we have

$$\begin{aligned}\|E_{\alpha,1}(-\mathcal{A}t^{\alpha})u_0-u_0\|^2_{\mathbb{X}^{s_2}(\Omega)} &= \sum_{j=1}^{\infty}(u_0,\phi_j)^2\left(E_{\alpha,1}\left(-\lambda_j t^{\alpha}\right)-1\right)^2\lambda_j^{s_2} \\ &\le 2(\overline{C}^2+1)\sum_{j=1}^{\infty}(u_0,\phi_j)^2\lambda_j^{s_2} \\ &\le C\|u_0\|^2_{\mathbb{X}^{s_2}(\Omega)}, \quad \forall t\in(0,T].\end{aligned} \tag{3.24}$$

From (3.19), one has $s<s_2$ and $1\le q\le 2^*_{s_2,s}$, and we have that $\mathbb{X}^{s_2}(\Omega)\hookrightarrow H^{s_2}(\Omega)\hookrightarrow W^{s,q}(\Omega)$ and then, we conclude from (3.24) that

$$\|E_{\alpha,1}\left(-\mathcal{A}t^{\alpha}\right)u_0-u_0\|_{W^{s,q}(\Omega)}\le C\|u_0\|_{\mathbb{X}^{s_2}(\Omega)}, \quad t\in(0,T]. \tag{3.25}$$

From (3.20), we have $ps_2<s^*<1+(p-1)s_2$, this implies that $0<s_2-s_1<1$ and for $ps_2<s^*<ps_2+\frac{N}{2}$, or $-\frac{N}{2}<ps_2-s^*<0$ thus $-\frac{N}{2}<s_1<0$. Taking $\frac{1}{2^*_{s_1}}=\frac{1}{2}-\frac{s_1}{N}$ and combine with Lemma 2.9, we obtain $L^{2^*_{s_1}}(\Omega)\hookrightarrow\mathbb{X}^{s_1}(\Omega)$. Using Lemma 2.3(b), we have for $t\in(0,T]$

$$\begin{aligned}\big\|J(F(u))(t)\big\|_{\mathbb{X}^{s_2}(\Omega)} &\le \int_0^t \|E_{\alpha,1}(-(t-\tau)^{\alpha}\mathcal{A})F_p(u)(\tau)\|_{\mathbb{X}^{s_2}(\Omega)}\mathrm{d}\tau \\ &\le \int_0^t\left(\sum_{j=1}^{\infty}\lambda_j^{s_2-s_1}\left(F_p(u)(\cdot,\tau),\phi_j\right)^2\lambda_j^{s_1}\left(\frac{\overline{C}}{1+\lambda_j(t-\tau)^{\alpha}}\right)^2\right)^{\frac{1}{2}}\mathrm{d}\tau \\ &\le \frac{\overline{C}}{\lambda_1^{\frac{2-s_2+s_1}{2}}}\int_0^t(t-\tau)^{-\alpha}\big\|F_p(u)(\cdot,\tau)\big\|_{\mathbb{X}^{s_1}(\Omega)}\mathrm{d}\tau \\ &\le C\int_0^t(t-\tau)^{-\alpha}\big\|F_p(u)(\cdot,\tau)\big\|_{L^{2^*_{s_1}}(\Omega)}\mathrm{d}\tau.\end{aligned} \tag{3.26}$$

Let us set $\Omega^- := \{x \in \Omega : |u(x)| < 1\}$ and $\Omega^+ := \{x \in \Omega : |u(x)| \geq 1\}$. Using Hölder's inequality, we have

$$\begin{aligned}
\int\limits_\Omega \left|F_p(u)\right|^{2^*_{s_1}} \mathrm{d}x &\leq \eta^{2^*_{s_1}} \int\limits_\Omega |u|^{(p-1)2^*_{s_1}} \left|\log |u|\right|^{2^*_{s_1}} \mathrm{d}x \\
&\leq C \left(\int\limits_\Omega \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int\limits_\Omega |u|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\leq C \left(\int\limits_{\Omega^-} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x + \int\limits_{\Omega^+} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int\limits_\Omega |u|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{p-1}{p}} \\
&\leq C \left(\left(\int\limits_{\Omega^-} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} + \left(\int\limits_{\Omega^+} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}}\right) \left(\int\limits_\Omega |u|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{p-1}{p}}, \qquad (3.27)
\end{aligned}$$

where we have used the elementary inequality $(a+b)^c \leq a^c + b^c$, for $0 < c < 1$. From the inequality (2.20) for $|u(x)| < 1,\ \forall x \in \Omega$, by applying Lemma 2.8b) for $k' = -\frac{1}{p2^*_{s_1}} < 0$, we have

$$\begin{aligned}
\left(\int\limits_{\Omega^-} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} &\leq C \left(\int\limits_{\Omega^-} |u(x)|^{-p\theta 2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq C \left(\left(\int\limits_{\Omega^-} |u(x)|^{-p\theta 2^*_{s_1}} \mathrm{d}x\right)^{-\frac{1}{p2^*_{s_1}}}\right)^{-2^*_{s_1}} \\
&\leq C \left(\left(\int\limits_{\Omega^-} |u(x)|^{\theta} \mathrm{d}x\right) \left(\int\limits_{\Omega^-} 1 \mathrm{d}x\right)^{-\frac{1+p2^*_{s_1}}{p2^*_{s_1}}}\right)^{-2^*_{s_1}} \\
&\leq C \|u\|^{-\theta 2^*_{s_1}}_{L^\theta(\Omega)} |\Omega|^{\frac{1+p2^*_{s_1}}{p}}, \qquad (3.28)
\end{aligned}$$

From the inequality (2.20) for $|u(x)| \geq 1$, we have

$$\left(\int\limits_{\Omega^+} \left|\log |u|\right|^{p2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} \leq C \left(\int\limits_{\Omega^+} |u(x)|^{p\theta 2^*_{s_1}} \mathrm{d}x\right)^{\frac{1}{p}} \leq C \|u\|^{\theta 2^*_{s_1}}_{L^{p\theta 2^*_{s_1}}(\Omega)}. \qquad (3.29)$$

From (3.27), (3.28) and (3.29), we conclude that

$$\|F_p(u)\|_{L^{2^*_{s_1}}(\Omega)} \leq C \left(\|u\|^{-\theta}_{L^\theta(\Omega)} + \|u\|^{\theta}_{L^{p\theta 2^*_{s_1}}(\Omega)}\right) \|u\|^{p-1}_{L^{p2^*_{s_1}}(\Omega)}. \qquad (3.30)$$

For $s > 0, qs < N$, and $q < \frac{N\theta}{N+\theta s}$, this implies that $q^*_s \leq \theta$ with q^*_s satisfies

$$\frac{1}{q^*_s} = \frac{1}{q} - \frac{s}{N},$$

we deduce from Lemma 2.9 that the following Sobolev embedding holds $L^\theta(\Omega) \hookrightarrow W^{s,q}(\Omega)$. Then we get that

$$\|u\|_{W^{s,q}(\Omega)} \leq C \|u\|_{L^\theta(\Omega)},$$

and for $\theta > 0$, we have that

$$\|u\|^{-\theta}_{L^{\theta}(\Omega)} \leq C\|u\|^{-\theta}_{W^{s,q}(\Omega)}.$$

From (3.21), the constant $s^* > Z(1,1)$, [for $Z(1,1)$ defined in (3.21)] and observe that

$$p2^*_{s_1} = \frac{2Np}{N-2s_1} = \frac{2Np}{N+2s^*-2ps_2} < \frac{2Np}{N+2Z(1,1)-2ps_2} = \frac{Nq}{N-sq} = q^*_s, \tag{3.31}$$

then we also obtain $W^{s,q}(\Omega) \hookrightarrow L^{p2^*_{s_1}}(\Omega)$. For $s^* > Z(\theta,1)$ [for $Z(\theta,1)$ defined in (3.21)], we infer that

$$p\theta 2^*_{s_1} = p\theta\frac{2N}{N-2s_1} = \frac{2Np}{N+2s^*-2ps_2} = p\theta\frac{2N}{N+2Z(\theta,1)-2ps_2} \leq q^*_s,$$

this implies that $W^{s,q}(\Omega) \hookrightarrow L^{p\theta 2^*_{s_1}}(\Omega)$. This implies that

$$\|F_p(u)\|_{L^{2^*_{s_1}}(\Omega)} \leq C\left(\|u\|^{-\theta}_{W^{s,q}(\Omega)} + \|u\|^{\theta}_{W^{s,q}(\Omega)}\right)\|u\|^{p-1}_{W^{s,q}(\Omega)},$$

and from (3.26), and for $0<\theta<p-1$, we have

$$\begin{aligned}
\underline{\text{The (RHS) of}}\ (3.26) &\leq C\int_0^t (t-\tau)^{-\alpha}\left(\|u(\cdot,\tau)\|^{-\theta}_{W^{s,q}(\Omega)} + \|u(\cdot,\tau)\|^{\theta}_{W^{s,q}(\Omega)}\right)\|u(\cdot,\tau)\|^{p-1}_{W^{s,q}(\Omega)}\mathrm{d}\tau \\
&\leq C\int_0^t (t-\tau)^{-\alpha}\left(\|u(\cdot,\tau)\|^{p-1-\theta}_{W^{s,q}(\Omega)} + \|u(\cdot,\tau)\|^{p-1+\theta}_{W^{s,q}(\Omega)}\right)\mathrm{d}\tau \\
&\leq C\left(\left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\int_0^t (t-\tau)^{-\alpha}\mathrm{d}\tau \\
&\leq C\left(\left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\frac{t^{1-\alpha}}{1-\alpha},
\end{aligned} \tag{3.32}$$

where from (3.22), we have that $\|u(\cdot,\tau)\|_{W^{s,q}(\Omega)} \leq R+\|u_0\|_{W^{s,q}(\Omega)}$, for all $\tau \in [0,T]$. From (3.26), (3.32), we obtain that for $t \in (0,T]$

$$\|J(F(u))(t)\|_{\mathbb{X}^{s_2}(\Omega)} \leq C\left(\left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)t^{1-\alpha}. \tag{3.33}$$

For the constants s,q satisfying (3.19), we have that $\mathbb{X}^{s_2}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ and $\alpha \in (0,1)$, and for all $t \in [0,T]$, we get

$$\|J(F(u))(t)\|_{W^{s,q}(\Omega)} \leq C\left(\left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)T^{1-\alpha}. \tag{3.34}$$

Hence, from (3.24) and (3.34), for every $t \in (0,T]$,

$$\begin{aligned}
&\|\mathbf{M}\mathbf{u}(t) - u_0\|_{W^{s,q}(\Omega)} \\
&\quad\leq C\|u_0\|_{\mathbb{X}^{s_2}(\Omega)} + C\left(\left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R+\|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)T^{1-\alpha}.
\end{aligned} \tag{3.35}$$

Therefore we see that if $R = 2C \|u_0\|_{\mathbb{X}^{s_2}(\Omega)}$ and for the constant $C > 0$, $\theta < p-1$ such that

$$R \geq 2C \left(R + \|u_0\|_{W^{s,q}(\Omega)} \right)^{p-1-\theta} T^{1-\alpha},$$

and

$$R \geq 2C \left(R + \|u_0\|_{W^{s,q}(\Omega)} \right)^{p-1+\theta} T^{1-\alpha}.$$

Then, we imply $\mathbf{M}$ is invariant in $\mathbb{W}$.

• *Claim II* $\mathbf{M} : \mathbb{W} \to \mathbb{W}$ *is a contraction map.* Let $u, w \in \mathbb{W}$, and similar to (3.26) and using Lemma 2.3(b), one has for every $t \in (0, T]$,

$$\begin{aligned} \|\mathbf{Mu}(t) - \mathbf{Mw}(t)\|_{\mathbb{X}^{s_2}(\Omega)} &= \|J(F(u))(t) - J(F(w))(t)\|_{\mathbb{X}^{s_2}(\Omega)} \\ &\leq C \int_0^t (t-\tau)^{-\alpha} \left\| F_p(u)(\cdot,\tau) - F_p(w)(\cdot,\tau) \right\|_{\mathbb{X}^{s_1}(\Omega)} \mathrm{d}\tau \\ &\leq C \int_0^t (t-\tau)^{-\alpha} \left\| F_p(u)(\cdot,\tau) - F_p(w)(\cdot,\tau) \right\|_{L^{2^*_{s_1}}(\Omega)} \mathrm{d}\tau, \end{aligned} \tag{3.36}$$

in which we used the Sobolev embedding $L^{2^*_{s_1}}(\Omega) \hookrightarrow \mathbb{X}^{s_1}(\Omega)$, for $\frac{1}{2^*_{s_1}} = \frac{1}{2} - \frac{s_1}{N}$ and $-\frac{N}{2} < s_1 \leq 0$. By recalling Lemma 3.2, we arrive at

$$\begin{aligned} \|F_p(u) - F_p(w)\|_{L^{2^*_{s_1}}(\Omega)} \leq{}& C \Big\| |\log|u|| \, |u-w| \Big\|_{L^{2^*_{s_1}}(\Omega)} + C \Big\| |u|^{p-2} |\log|u|| \, |u-w| \Big\|_{L^{2^*_{s_1}}(\Omega)} \\ &+ C \Big\| |w|^{p-2} |\log|u|| \, |u-w| \Big\|_{L^{2^*_{s_1}}(\Omega)} + C \Big\| |w|^{p-2} |u-w| \Big\|_{L^{2^*_{s_1}}(\Omega)}. \end{aligned} \tag{3.37}$$

For the constant $2^*_{s_1} \geq 1$, using Hölder's inequality, we get

$$\begin{aligned} \Big\| |\log|u|| \, |u-w| \Big\|_{L^{2^*_{s_1}}(\Omega)}^{2^*_{s_1}} &= \int_\Omega \left(|\log|u|| |u-w| \right)^{2^*_{s_1}} \mathrm{d}x = \int_\Omega |\log|u||^{2^*_{s_1}} |u-w|^{2^*_{s_1}} \mathrm{d}x \\ &\leq \left(\int_\Omega |\log|u||^{\frac{p2^*_{s_1}}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |u-w|^{p2^*_{s_1}} \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\Omega^-} |\log|u||^{\frac{p2^*_{s_1}}{p-1}} \mathrm{d}x + \int_{\Omega^+} |\log|u||^{\frac{p2^*_{s_1}}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_\Omega |u-w|^{p2^*_{s_1}} \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \left(\left(\int_{\Omega^-} |\log|u||^{\frac{p2^*_{s_1}}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} + \left(\int_{\Omega^+} |\log|u||^{\frac{p2^*_{s_1}}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \right) \left(\int_\Omega |u-w|^{p2^*_{s_1}} \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned} \tag{3.38}$$

From the inequality (2.20) for $|u(x)|<1,\ \forall x\in\Omega$, we have

$$\begin{aligned}&\left(\int\limits_{\Omega^-}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\\&\le C\left(\int\limits_{\Omega^-}\big|u(x)\big|^{-\frac{p\theta 2^*_{s_1}}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\le\left(\left(\int\limits_{\Omega^-}\big|u(x)\big|^{-\frac{p\theta 2^*_{s_1}}{p-1}}\mathrm{d}x\right)^{-\frac{p-1}{p2^*_{s_1}}}\right)^{-2^*_{s_1}}\\&\le C\left(\left(\int\limits_{\Omega^-}|u(x)|^{\theta}\mathrm{d}x\right)\left(\int\limits_{\Omega^-}1\mathrm{d}x\right)^{-\frac{p-1+p2^*_{s_1}}{p2^*_{s_1}}}\right)^{-2^*_{s_1}}\le C\big\|u\big\|^{-\theta 2^*_{s_1}}_{L^\theta(\Omega)}|\Omega|^{\frac{p-1+p2^*_{s_1}}{p}},\end{aligned}\tag{3.39}$$

where we have chosen $k'=-\frac{p-1}{p2^*_{s_1}}<0$ in Lemma 2.8(b). For $|u(x)|\ge 1,\ \forall x\in\Omega$, we have

$$\left(\int\limits_{\Omega^+}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\le C\left(\int\limits_{\Omega^+}\big|u(x)\big|^{\frac{p\theta 2^*_{s_1}}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\le C\big\|u\big\|^{\theta 2^*_{s_1}}_{L^{\frac{p\theta 2^*_{s_1}}{p-1}}(\Omega)}.\tag{3.40}$$

From (3.38), (3.39) and (3.40), we conclude that

$$\Big\|\big|\log|u|\big||u-w|\Big\|_{L^{2^*_{s_1}}(\Omega)}\le C\left(\|u\|^{-\theta}_{L^\theta(\Omega)}+\|u\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-1}}(\Omega)}\right)\|u-w\|_{L^{p2^*_{s_1}}(\Omega)}.\tag{3.41}$$

Thanks to Hölder's inequality, we get that

$$\begin{aligned}&\int\limits_{\Omega}\left(|u|^{p-2}\big|\log|u|\big||u-w|\right)^{2^*_{s_1}}\mathrm{d}x=\int\limits_{\Omega}|u|^{(p-2)2^*_{s_1}}\big|\log|u|\big|^{2^*_{s_1}}\big|u-w\big|^{2^*_{s_1}}\mathrm{d}x\\&\le\left(\int\limits_{\Omega}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int\limits_{\Omega}|u|^{p(p-2)2^*_{s_1}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int\limits_{\Omega}\big|u-w\big|^{p2^*_{s_1}}\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}\tag{3.42}$$

Similar to (3.39) and (3.40), we have the following estimate

$$\begin{aligned}\left(\int\limits_{\Omega}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}&\le\left(\int\limits_{\Omega^-}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}+\left(\int\limits_{\Omega^+}\big|\log|u|\big|^{\frac{p2^*_{s_1}}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{p}}\\&\le C\left(\big\|u\big\|^{-\theta 2^*_{s_1}}_{L^\theta(\Omega)}+\big\|u\big\|^{\theta 2^*_{s_1}}_{L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)}\right).\end{aligned}\tag{3.43}$$

Combining (3.42) and (3.43), we get that

$$\begin{aligned}&\big\||u|^{p-2}\big|\log|u|\big||u-w|\big\|_{L^{2^*_{s_1}}(\Omega)}\\&\le C\left(\big\|u\big\|^{-\theta}_{L^\theta(\Omega)}+\big\|u\big\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)}\right)\big\|u\big\|^{p-2}_{L^{p(p-2)2^*_{s_1}}(\Omega)}\big\|u-w\big\|_{L^{p2^*_{s_1}}(\Omega)}.\end{aligned}\tag{3.44}$$

Similarly,

$$\begin{aligned} &\left\| |w|^{p-2} \left| \log |u| \right| |u-w| \right\|_{L^{2^*_{s_1}}(\Omega)} \\ &\leq C \left(\|u\|^{-\theta}_{L^{\theta}(\Omega)} + \|u\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)} \right) \|w\|^{p-2}_{L^{p(p-2)2^*_{s_1}}(\Omega)} \|u-w\|_{L^{p2^*_{s_1}}(\Omega)}. \end{aligned} \tag{3.45}$$

We use the Hölder's inequality to obtain that

$$\begin{aligned} \left\| |w|^{p-2} |u-w| \right\|^{2^*_{s_1}}_{L^{2^*_{s_1}}(\Omega)} &= \int_{\Omega} \left(|w|^{p-2} |u-w| \right)^{2^*_{s_1}} \mathrm{d}x \\ &\leq \left(\int_{\Omega} |w|^{\frac{p(p-2)2^*_{s_1}}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |u-w|^{p2^*_{s_1}} \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leq \|w\|^{(p-2)2^*_{s_1}}_{L^{\frac{p(p-2)2^*_{s_1}}{p-1}}(\Omega)} \|u-w\|^{2^*_{s_1}}_{L^{p2^*_{s_1}}(\Omega)}. \end{aligned} \tag{3.46}$$

Combining the results obtained in (3.37), (3.41), (3.44), (3.45) and (3.46), we have

$$\begin{aligned} \left\| F_p(u)(\cdot,t) - F_p(w)(\cdot,t) \right\|_{L^{2^*_{s_1}}(\Omega)} &\leq C \left(\|u\|^{-\theta}_{L^{\theta}(\Omega)} + \|u\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-1}}(\Omega)} \right) \|u-w\|_{L^{p2^*_{s_1}}(\Omega)} \\ &+ C \left(\|u\|^{-\theta}_{L^{\theta}(\Omega)} + \|u\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)} \right) \|u\|^{p-2}_{L^{p(p-2)2^*_{s_1}}(\Omega)} \|u-w\|_{L^{p2^*_{s_1}}(\Omega)} \\ &+ C \left(\|u\|^{-\theta}_{L^{\theta}(\Omega)} + \|u\|^{\theta}_{L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)} \right) \|w\|^{p-2}_{L^{p(p-2)2^*_{s_1}}(\Omega)} \|u-w\|_{L^{p2^*_{s_1}}(\Omega)} \\ &+ C\, \|w\|^{p-2}_{L^{\frac{p(p-2)2^*_{s_1}}{p-1}}(\Omega)} \|u-w\|_{L^{p2^*_{s_1}}(\Omega)}. \end{aligned} \tag{3.47}$$

From (3.18)–(3.21) we have the following:

- ▷ For q^*_s satisfying $\frac{1}{q^*_s} = \frac{1}{q} - \frac{s}{N}$, for $q \leq \frac{N\theta}{N+\theta s}$ and $sq < N$, then $q^*_s \leq \theta$ and we deduce from Lemma 2.9 that $L^{\theta}(\Omega) \hookrightarrow W^{s,q}(\Omega)$. This implies that

$$\|u\|^{-\theta}_{L^{\theta}(\Omega)} \leq C \|u\|^{-\theta}_{W^{s,q}(\Omega)}, \quad \text{for } 0 < \theta \leq p-1.$$

- ▷ For $s^* > Z(1,1)$, a similar argument with (3.31) and we observe that $q^*_s \geq p2^*_{s_1}$, and then we deduce from Lemma 2.9 that the following Sobolev embedding holds $W^{s,q}(\Omega) \hookrightarrow L^{p2^*_{s_1}}(\Omega)$.
- ▷ For $s^* > Z(p-2,1)$, implies that

$$p(p-2)2^*_{s_1} = \frac{2p(p-2)N}{N-2s_1} < \frac{2p(p-2)N}{N+2Z(p-2,1)-2ps_2} \leq q^*_s, \tag{3.48}$$

  and we infer that $W^{s,q}(\Omega) \hookrightarrow L^{p(p-2)2^*_{s_1}}(\Omega)$.
- ▷ For $s^* > Z(\theta, p-1)$, we observe that $\frac{p\theta 2^*_{s_1}}{p-1} \leq q^*_s$, then we get $W^{s,q}(\Omega) \hookrightarrow L^{\frac{p\theta 2^*_{s_1}}{p-1}}(\Omega)$.
- ▷ For $s^* > Z(\theta, p-2)$, we have $\frac{p\theta 2^*_{s_1}}{p-2} \leq q^*_s$, and this implies that $W^{s,q}(\Omega) \hookrightarrow L^{\frac{p\theta 2^*_{s_1}}{p-2}}(\Omega)$.
- ▷ For $s^* > Z(p-2, p-1)$, implies $\frac{p(p-2)2^*_{s_1}}{p-1} \leq q^*_s$, and we infer that $W^{s,q}(\Omega) \hookrightarrow L^{\frac{p(p-2)2^*_{s_1}}{p-1}}(\Omega)$.

We can now combine the results above together with (3.47) to deduce that

$$\left\| F_p(u)(\cdot,t) - F_p(w)(\cdot,t) \right\|_{L^{2^*_{s_1}}(\Omega)} \leq K(R,u_0) \left\| u(\cdot,t) - w(\cdot,t) \right\|_{W^{s,q}(\Omega)}, \tag{3.49}$$

for all $t \in (0, T]$, we have used that $\max\left\{\|u\|_{W^{s,q}(\Omega)}; \|w\|_{W^{s,q}(\Omega)}\right\} \le R + \|u_0\|_{W^{s,q}(\Omega)}$, and for the constant $K(R, u_0) := K(N, p, \theta, s_1, R, \|u_0\|_{W^{s,q}(\Omega)})$ but independent of t. From this, one observes that

$$\begin{aligned}
\underline{\text{The (RHS) of }} (3.36) &\le CK(R,u_0) \int_0^t (t-\tau)^{-\alpha} \|u(\cdot,\tau) - w(\cdot,\tau)\|_{W^{s,q}(\Omega)} \, \mathrm{d}\tau \\
&\le CK(R,u_0) \int_0^t (t-\tau)^{-\alpha} \left(\|u(\cdot,\tau) - w(\cdot,\tau)\|_{W^{s,q}(\Omega)} \right) \mathrm{d}\tau \\
&\le CK(R,u_0) \|u(\cdot,\tau) - w(\cdot,\tau)\|_{C([0,T];W^{s,q}(\Omega))} \int_0^t (t-\tau)^{-\alpha} \mathrm{d}\tau \\
&\le CK(R,u_0) \|u - w\|_{C([0,T];W^{s,q}(\Omega))} \frac{t^{1-\alpha}}{1-\alpha}.
\end{aligned} \tag{3.50}$$

Inserting the result of (3.50) into (3.36), we obtain that

$$\|\mathbf{Mu}(t) - \mathbf{Mw}(t)\|_{\mathbb{X}^{s_2}(\Omega)} \le CK(R,u_0) T^{1-\alpha} \|u - w\|_{C([0,T];W^{s,q}(\Omega))}.$$

For the constants s, q satisfying (3.19), we have that $\mathbb{X}^{s_2}(\Omega) \hookrightarrow W^{s,q}(\Omega)$, and

$$\|\mathbf{M}u - \mathbf{Mw}\|_{C([0,T];W^{s,q}(\Omega))} \le CK(R,u_0) T^{1-\alpha} \|u - w\|_{C([0,T];W^{s,q}(\Omega))}. \tag{3.51}$$

Choosing $T, K(R, u_0)$ small enough such that $CK(R, u_0)T^{1-\alpha} < 1$, it follows that $\mathbf{M}$ is a contraction map on $\mathbb{W}$. So, we invoke the contraction mapping principle to conclude that the map $\mathbf{M}$ has a unique fixed point u in $\mathbb{W}$. The proof of Theorem 3.2 is completed. $\square$

Since we already know that the mild solution of $\mathbb{P}$ does exist, the question is whether it will continue (continuation to a bigger interval of existence) and in what situation it is non-continuation by blowup.

Definition 3.1. Given a mild solution $u \in C([0, T]; W^{s,q}(\Omega))$ of $\mathbb{P}$ for $\alpha \in (0, 1)$, we say that $u^\star$ is a *continuation* of u in $(0, T^\star]$ for $T^\star > T$ if it is satisfies

$$\begin{cases} u^\star \in C([T, T^\star]; W^{s,q}(\Omega)) \text{ is a mild solution of } (\mathbb{P}) \text{ for all } t \in [T, T^\star], \\ u^\star(x,t) = u(x,t) \quad \text{whenever} \quad t \in [0,T], x \in \Omega. \end{cases} \tag{3.52}$$

Theorem 3.3. (Continuation) *Suppose that the assumptions of Theorem 3.2 are satisfied. Then, the mild solution (unique) on* $(0, T]$ *of Problem* $\mathbb{P}$ *can be extended to the interval* $(0, T^\star]$*, for some* $T^\star > T$*, so that, the extended function is also the mild solution (unique) of Problem* $\mathbb{P}$ *on* $(0, T^\star]$.

Proof. Let $u : [0, T] \to W^{s,q}(\Omega)$ be a mild solution of Problem $\mathbb{P}$ (T is the time from Theorem 3.2). Fix $R^\star > 0$, and for $T^\star > T$, ($T^\star$ depending on $R^\star$), we shall prove that $u^\star : [0, T^\star] \to W^{s,q}(\Omega)$ is a mild solution of Problem $\mathbb{P}$. Assume the following estimates hold:

$$CT^{-\alpha}(T^\star)^\alpha \|u_0\|_{\mathbb{X}^{s_2}(\Omega)} \leq \frac{R^\star}{4}, \tag{3.53}$$

$$C\left(R^\star + \|u(\cdot,T)\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} (T^\star)^{1-\alpha} \leq \frac{R^\star}{8}, \tag{3.54}$$

$$C\left(R^\star + \|u(\cdot,T)\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} (T^\star)^{1-\alpha} \leq \frac{R^\star}{8}, \tag{3.55}$$

$$C\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} (T^\star)^{\frac{\alpha s_1-\alpha s_2+2}{2}} < \frac{R^\star}{8}, \tag{3.56}$$

$$C\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} (T^\star)^{\frac{\alpha s_1-\alpha s_2+2}{2}} < \frac{R^\star}{8}, \tag{3.57}$$

$$CK(R,u_0)(T^\star)^{\alpha+\alpha\gamma-2\alpha\gamma} \leq \frac{R^\star}{4}, \tag{3.58}$$

where $0<\theta\leq p-1$ and $K(R,u_0)$ is defined in the proof of Theorem 3.2. For $T^\star \geq T>0$ and $R^\star>0$, let us define

$$\mathbb{W}^\star := \left\{u^\star \in C([0,T^\star];W^{s,q}(\Omega)) : \mid \begin{array}{ll} u^\star(\cdot,t)=u(\cdot,t), & \forall t\in(0,T], \\ \|u^\star(\cdot,t)-u(\cdot,T)\|_{C([T,T^\star];W^{s,q}(\Omega))} \leq R^\star, & \forall t\in[T,T^\star]. \end{array}\right\} \tag{3.59}$$

• *Step I We show that* $\mathbf{M}$ *defined as in* (3.23) *is the operator on* $\mathbb{W}^\star$. Let $u^\star\in\mathbb{W}^\star$ and we consider two cases.

∗ If $t\in(0,T]$, then by virtue of Theorem 3.2, we have the Problem $\mathbb{P}$ has a unique solution and we also have $u^\star(\cdot,t)=u(\cdot,t)$. Thus $\mathbf{M}u^\star(t)=\mathbf{M}u(t)=u(\cdot,t)$ for all $t\in(0,T]$.

∗ If $t\in[T,T^\star]$, we have

$$\begin{aligned} &\|\mathbf{M}u^\star(t)-u(\cdot,T)\|_{W^{s,q}(\Omega)} \\ &\quad\leq \|(E_{\alpha,1}(-t^\alpha\mathcal{A})-E_{\alpha,1}(-T^\alpha\mathcal{A}))u_0\|_{W^{s,q}(\Omega)} \\ &\qquad+\int_T^t \|E_{\alpha,1}(-(t-\tau)^\alpha\mathcal{A})F_p(u^\star)(\tau)\|_{W^{s,q}(\Omega)}\,\mathrm{d}\tau \\ &\qquad+\int_0^T \|(E_{\alpha,1}(-(t-\tau)^\alpha\mathcal{A})-E_{\alpha,1}(-(T-\tau)^\alpha\mathcal{A}))F_p(u^\star)(\tau)\|_{W^{s,q}(\Omega)}\,\mathrm{d}\tau \\ &\quad=: \|J_2(u_0)(t)\|_{W^{s,q}(\Omega)}+\|J_3(u^\star)(t)\|_{W^{s,q}(\Omega)}+\|J_4(u^\star)(t)\|_{W^{s,q}(\Omega)}. \end{aligned} \tag{3.60}$$

Estimating the term $\|J_2(u_0)(t)\|_{W^{s,q}(\Omega)}$, using Lemma 2.5, we have for all $t\in[T,T^\star]$,

$$\begin{aligned} \|J_2(u_0)(t)\|^2_{\mathbb{X}^{s_2}(\Omega)} &= \sum_{j=1}^\infty \lambda_j^{s_2}(u_0,\phi_j)^2\left(E_{\alpha,1}(-\lambda_j t^\alpha)-E_{\alpha,1}(-\lambda_j T^\alpha)\right)^2 \\ &= \sum_{j=1}^\infty \lambda_j^{s_2}(u_0,\phi_j)^2\left(\int_0^\infty \mathcal{M}_\alpha(z)\left|e^{-z\lambda_j t^\alpha}-e^{-z\lambda_j T^\alpha}\right|\mathrm{d}z\right)^2 \\ &\leq \sum_{j=1}^\infty \lambda_j^{s_2}(u_0,\phi_j)^2\left(\int_0^\infty \mathcal{M}_\alpha(z)e^{-z\lambda_j T^\alpha}\left|e^{-z\lambda_j(t^\alpha-T^\alpha)}-1\right|\mathrm{d}z\right)^2. \end{aligned}$$

For $z>0$, using the inequality $1-e^{-z}\le z$, and $ze^{-z}\le 1$, one obtains

$$\begin{aligned}\|J_2(u_0)(t)\|^2_{\mathbb{X}^{s_2}(\Omega)} &\le \sum_{j=1}^{\infty}\lambda_j^{s_2}(u_0,\phi_j)^2\left(\lambda_j(t^\alpha-T^\alpha)\int_0^\infty \mathcal{M}_\alpha(z)\left(z\lambda_jT^\alpha\right)^{-1}z\mathrm{d}z\right)^2\\ &\le \sum_{j=1}^{\infty}\lambda_j^{s_2}(u_0,\phi_j)^2\left((t^\alpha-T^\alpha)T^{-\alpha}\int_0^\infty \mathcal{M}_\alpha(z)\mathrm{d}z\right)^2\\ &\le (t-T)^{2\alpha}T^{-2\alpha}\|u_0\|^2_{\mathbb{X}^{s_2}(\Omega)}, \end{aligned}\tag{3.61}$$

where we have use the inequalities

$$a^c-b^c\le(a-b)^c,\quad \text{for } a>b>0, c\in(0,1),\quad \text{and} \int_0^\infty \mathcal{M}_\alpha(z)\mathrm{d}z=1.$$

For the constants s,q satisfying (3.19), we have that $\mathbb{X}^{s_2}(\Omega)\hookrightarrow W^{s,q}(\Omega)$. Hence, we get that

$$\|J_2(u_0)(t)\|_{W^{s,q}(\Omega)}\le C(t-T)^\alpha T^{-\alpha}\|u_0\|_{\mathbb{X}^{s_2}(\Omega)}\le C(T^\star)^\alpha T^{-\alpha}\|u_0\|_{\mathbb{X}^{s_2}(\Omega)}.\tag{3.62}$$

From (3.53), this implies that the following estimate holds

$$\|J_2(u_0)\|_{C([0,T^\star];W^{s,q}(\Omega))}\le CT^{-\alpha}(T^\star)^\alpha\|u_0\|_{\mathbb{X}^{s_2}(\Omega)}\le\frac{R^\star}{4}.\tag{3.63}$$

Similar to (3.32), we have the following estimate for all $t\in[T,T^\star]$ (note that we can choose $T^\star>T$ and close enough to T)

$$\begin{aligned}\|J_3(u^\star)(t)\|_{W^{s,q}(\Omega)}&\le C\big\|J_3(u)(t)\big\|_{\mathbb{X}^{s_2}(\Omega)}\\ &\le C\left(\left(R^\star+\big\|u(\cdot,T)\big\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R^\star+\big\|u(\cdot,T)\big\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)(t-T)^{1-\alpha},\end{aligned}\tag{3.64}$$

where from (3.59), for all $t\in[T,T^\star]$, we have used that

$$\|u^\star(\cdot,t)\|_{W^{s,q}(\Omega)}\le R^\star+\big\|u(\cdot,T)\big\|_{W^{s,q}(\Omega)}.$$

Using (3.54) and (3.55), we infer that

$$\begin{aligned}&\|J_3(u^\star)\|_{C([0,T^\star];W^{s,q}(\Omega))}\\ &\quad\le C\left(\left(R^\star+\big\|u(\cdot,T)\big\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R^\star+\big\|u(\cdot,T)\big\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)(T^\star)^{1-\alpha}\le\frac{R^\star}{4}.\end{aligned}\tag{3.65}$$

We continue with the estimate on the third term of (3.60), and using Lemma 2.3(b) and Lemma 2.4, we obtain for all $t\in[T,T^\star]$

$$\begin{aligned}\left|E_{\alpha,1}\left(-\lambda_j(t-\tau)^\alpha\right)-E_{\alpha,1}\left(-\lambda_j(T-\tau)^\alpha\right)\right| &= \left|\int_{T-\tau}^{t-\tau}-\lambda_jz^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_jz^\alpha\right)\mathrm{d}z\right|\\ &=\left|\int_{T-\tau}^{t-\tau}\lambda_j^{1+\frac{s_2-s_1}{2}}z^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_jz^\alpha\right)\mathrm{d}z\right|\lambda_j^{\frac{s_1-s_2}{2}}\\ &\le C\left|\int_{T-\tau}^{t-\tau}z^{\frac{\alpha s_1-\alpha s_2}{2}-1}\mathrm{d}z\right|\lambda_j^{\frac{s_1-s_2}{2}}\le C\lambda_j^{\frac{s_1-s_2}{2}}(T-\tau)^{\frac{\alpha s_1-\alpha s_2}{2}}.\end{aligned}\tag{3.66}$$

For the constant s_1 satisfying $-\frac{N}{2} < s_1 \le 0$ and $\frac{1}{2^\star_{s_1}} = \frac{1}{2} - \frac{s_1}{N}$, from Lemma 2.9, we obtain $L^{2^\star_{s_1}}(\Omega) \hookrightarrow \mathbb{X}^{s_1}(\Omega)$. Hence, we deduce that

$$
\begin{aligned}
&\|J_4(u^\star)(t)\|_{\mathbb{X}^{s_2}(\Omega)} \\
&\quad \le \int_0^T \left(\sum_{j=1}^\infty \lambda_j^{s_2} \left(F_p(u^\star)(\cdot,\tau), \phi_j\right)^2 \left| E_{\alpha,1}\left(-\lambda_j (t-\tau)^\alpha\right) - E_{\alpha,1}\left(-\lambda_j (T-\tau)^\alpha\right)\right|^2 \right)^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad \le C \int_0^T \left(\sum_{j=1}^\infty \lambda_j^{s_1} (T-\tau)^{\alpha s_1 - \alpha s_2} \left(F_p(u^\star)(\cdot,\tau), \phi_j\right)^2 \right)^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad \le C \int_0^T (T-\tau)^{\frac{\alpha s_1 - \alpha s_2}{2}} \|F_p(u^\star)(\cdot,\tau)\|_{\mathbb{X}^{s_1}(\Omega)} \mathrm{d}\tau \\
&\quad \le C \int_0^T (T-\tau)^{\frac{\alpha s_1 - \alpha s_2}{2}} \|F_p(u^\star)(\cdot,\tau)\|_{L^{2^\star_{s_1}}(\Omega)} \mathrm{d}\tau,
\end{aligned}
\tag{3.67}
$$

In the same way as in (3.32), and from (3.59), one obtains

$$
\begin{aligned}
&\underline{\text{The (RHS) of }} (3.67) \\
&\quad \le C \int_0^T (T-\tau)^{\frac{\alpha s_1 - \alpha s_2}{2}} \left(\|u(\cdot,\tau)\|^{-\theta}_{W^{s,q}(\Omega)} + \|u(\cdot,\tau)\|^{\theta}_{W^{s,q}(\Omega)} \right) \|u(\cdot,\tau)\|^{p-1}_{W^{s,q}(\Omega)} \mathrm{d}\tau \\
&\quad \le C \left(\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} \right) \int_0^T (T-\tau)^{\frac{\alpha s_1 - \alpha s_2}{2}} \mathrm{d}\tau \\
&\quad \le C \left(\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} \right) T^{\frac{\alpha s_1 - \alpha s_2 + 2}{2}},
\end{aligned}
\tag{3.68}
$$

for $\theta < p-1$. From (3.67), we have that $\mathbb{X}^{s_2}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ and obtain

$$
\|J_4(u^\star)(t)\|_{W^{s,q}(\Omega)} \le C \left(\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} \right) T^{\frac{\alpha s_1 - \alpha s_2 + 2}{2}}. \tag{3.69}
$$

Thus, for $t \in (T, T^\star]$, we obtain

$$
\|J_4(u^\star)(t)\|_{W^{s,q}(\Omega)} \le C \left(\left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta} + \left(R + \|u_0\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta} \right) (T^\star)^{\frac{\alpha s_1 - \alpha s_2 + 2}{2}}. \tag{3.70}
$$

From (3.56) and (3.57), we get

$$
\|J_4(u^\star)\|_{C([0,T^\star];W^{s,q}(\Omega))} \le \frac{R^\star}{4}. \tag{3.71}
$$

It follows from (3.63), (3.65), (3.71) that, for every $t \in [T, T^\star]$

$$
\|\mathbf{M}u^\star - u(\cdot,T)\|_{C([0,T^\star];W^{s,q}(\Omega))} \le \frac{R^\star}{4} + \frac{R^\star}{4} + \frac{R^\star}{4} \le R^\star.
$$

We have shown that $\mathbf{M}$ is a map $\mathbb{W}^\star$ into $\mathbb{W}^\star$.

- *Step II We show that* $\mathbf{M}$ *is a contraction on* $\mathbb{W}^\star$. Let $u, w \in \mathbb{W}^\star$, and we have that for $0 \le t \le T^\star$,

$$\mathbf{M}u(t) - \mathbf{M}w(t) = \int_0^t E_{\alpha,1}\left(-\mathcal{A}\,(t-\tau)^{\alpha}\right)\left(F_p(u)(\tau) - F_p(w)(\tau)\right) \mathrm{d}\tau, \tag{3.72}$$

where we note that $\mathbf{Mu}(t) - \mathbf{Mw}(t) = 0$, vanishes in $\mathbb{W}^\star$ for all $t \in (0,T]$. Then, for all $t \in [0,T^\star]$, proceeding as in Claim (2) of the last theorem, we have

$$\left\|\mathbf{Mu}(t) - \mathbf{Mw}(t)\right\|_{\mathbb{X}^{s_2}(\Omega)} \le CK(R,u_0)(T^\star)^{1-\alpha}\left\|u-w\right\|_{C([0,T^*];W^{s,q}(\Omega))} \le \frac{R^\star}{4}\left\|u-w\right\|_{C([0,T^*];W^{s,q}(\Omega))}.$$

Thus, using the Sobolev embedding $\mathbb{X}^{s_2}(\Omega) \hookrightarrow W^{s,q}(\Omega)$ with s,q satisfying (3.19), for all $T^\star > 0$, so without loss of generality, we may assume that $0 \le R^\star < 4$, and we infer that

$$\left\|\mathbf{Mu} - \mathbf{Mw}\right\|_{C([0,T^\star];W^{s,q}(\Omega))} \le \frac{R^\star}{4}\left\|u-v\right\|_{C([0,T^\star];W^{s,q}(\Omega))}. \tag{3.73}$$

This implies that $\mathbf{M}$ is a $\frac{R^\star}{4}$-contraction. By the Banach contraction principle it follows that $\mathbf{M}$ has a unique fixed point $u^\star$ of $\mathbf{M}$ in $\mathbb{W}^\star$, which is a continuation of u. This finishes the proof. □

The next results are on global existence or non-continuation by a blowup.

Definition 3.2. Let $u(x,t)$ be a solution of $\mathbb{P}$. We define the maximal existence time $T_{\max}$ of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \le t < \infty$, then $T_{\max} = \infty$.
(ii) If there exists $T \in (0,\infty)$ such that $u(x,t)$ exists for $0 \le t < T$, but does not exist at $t = T$, then $T_{\max} = T$.

Definition 3.3. Let $u(x,t)$ be a solution of $\mathbb{P}$. We say $u(x,t)$ blows up in finite time if the maximal existence time $T_{\max}$ is finite and

$$\lim_{t\to T_{\max}^-}\sup \|u(\cdot,t)\|_{W^{s,q}(\Omega)} = \infty. \tag{3.74}$$

Theorem 3.4. (Global existence or finite time blowup) *For* $N \ge 1, p \ge 2, 0 \le s < 2s_2$, *for* $0 \le s_2 < N/2$ *and* $1 \le q \le \min\left\{2^\star_{s_2,s}; \frac{N\theta}{N+\theta s}\right\}$ *with* $2^\star_{s_2,s}$ *satisfying* $\frac{1}{2^\star_{s_2,s}} = \frac{1}{2} + \frac{s}{N} - \frac{s_2}{N}$ *and* $qs < N$. *For* $u_0 \in \mathbb{X}^{s_2}(\Omega) \cap W^{s,q}(\Omega)$, *there exists a maximal time* $T_{\max} > 0$ *such that* $u \in C([0,T_{\max}];W^{s,q}(\Omega))$ *is the mild solution of* $\mathbb{P}$. *Thus, either Problem* $\mathbb{P}$ *has a unique global mild solution on* $[0,\infty)$ *or there exists a maximal time* $T_{\max} < \infty$ *such that*

$$\lim_{t\to T_{\max}^-}\sup \|u(\cdot,t)\|_{W^{s,q}(\Omega)} = \infty.$$

Proof. Let $u_0 \in \mathbb{X}^{s_2}(\Omega) \cap W^{s,q}(\Omega)$ and define

$$T_{\max} := \sup\left\{T > 0 : \text{there exits a solution on } (0,T]\right\}.$$

Assume that $T_{\max} < \infty$, and $\|u(\cdot,t)\|_{\mathbb{X}^{s_2}(\Omega)} \le R_0$, for some $R_0 > 0$. Now suppose there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0,T_{\max})$ such that $t_n \to T_{\max}$ and $\{u(\cdot,t_n)\}_{n\in\mathbb{N}} \subset \mathbb{X}^{s_2}(\Omega)$. Let us show that $\{u(\cdot,t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{X}^{s_2}(\Omega)$. Indeed, given $\epsilon > 0$, fix $N \in \mathbb{N}$ such that for all $n,m > N$, $0 < t_n < t_m <$

$T_{\max}$, we have

$$\begin{aligned}
&\left\|u(\cdot,t_m)-u(\cdot,t_n)\right\|_{W^{s,q}(\Omega)}\\
&\quad\le \left\|\left(E_{\alpha,1}\left(-t_m^\alpha\mathcal{A}\right)-E_{\alpha,1}\left(-t_n^\alpha\mathcal{A}\right)\right)u_0\right\|_{W^{s,q}(\Omega)}\\
&\qquad+\int_{t_n}^{t_m}\left\|E_{\alpha,1}\left(-\left(t_m-\tau\right)^\alpha\mathcal{A}\right)F_p(u)(\tau)\right\|_{W^{s,q}(\Omega)}\mathrm{d}\tau\\
&\qquad+\int_0^{t_n}\left\|\left(E_{\alpha,1}\left(-\left(t_n-\tau\right)^\alpha\mathcal{A}\right)-E_{\alpha,1}\left(-\left(T_{\max}-\tau\right)^\alpha\mathcal{A}\right)\right)F_p(u)(\tau)\right\|_{W^{s,q}(\Omega)}\mathrm{d}\tau\\
&\qquad+\int_0^{t_m}\left\|\left(E_{\alpha,1}\left(-\left(T_{\max}-\tau\right)^\alpha\mathcal{A}\right)-E_{\alpha,1}\left(-\left(t_m-\tau\right)^\alpha\right)\mathcal{A}\right)F_p(u)(\tau)\right\|_{W^{s,q}(\Omega)}\mathrm{d}\tau\\
&\quad=\left\|J_5(u_0)\right\|_{W^{s,q}(\Omega)}+\left\|J_6(u)\right\|_{W^{s,q}(\Omega)}+\left\|J_7(u)\right\|_{W^{s,q}(\Omega)}+\left\|J_8(u)\right\|_{W^{s,q}(\Omega)}.
\end{aligned}\tag{3.75}$$

Similar to (3.61), and using the Sobolev embedding $\mathbb{X}^{s_2}(\Omega)\hookrightarrow W^{s,q}(\Omega)$, we have that

$$\left\|J_5(u_0)\right\|_{W^{s,q}(\Omega)}\le C\left\|J_5(u_0)(t)\right\|_{\mathbb{X}^{s_2}(\Omega)}\le C\left|t_m-t_n\right|^\alpha t_n^{-\alpha}\left\|u_0\right\|_{\mathbb{X}^{s_2}(\Omega)}.\tag{3.76}$$

In the same way as in (3.32), we get

$$\left\|J_6(u)(t)\right\|_{W^{s,q}(\Omega)}\le C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|t_m-t_n\right|^{1-\alpha}.\tag{3.77}$$

Similar to (3.68), we have

$$\left\|J_7(u)\right\|_{W^{s,q}(\Omega)}\le C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|T_{\max}-t_n\right|^{\frac{\alpha s_1-\alpha s_2+2}{2}},\tag{3.78}$$

and

$$\left\|J_8(u)\right\|_{W^{s,q}(\Omega)}\le C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|T_{\max}-t_m\right|^{\frac{\alpha s_1-\alpha s_2+2}{2}}.\tag{3.79}$$

Thus, since $\{t_n\}_{n\in\mathbb{N}^*}$ is convergent we can take $N:=N(\epsilon)\in\mathbb{N}^*$ with $m\ge n\ge N$ such that $|t_m-t_n|$ is as small as we want, and we have

$$\begin{aligned}
&C\left|t_m-t_n\right|^\alpha t_n^{-\alpha}\left\|u_n\right\|_{\mathbb{X}^{s_2}(\Omega)}<\frac{\epsilon}{4},\\
&C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|t_m-t_n\right|^{1-\alpha}<\frac{\epsilon}{4},\\
&C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|T_{\max}-t_m\right|^{\frac{\alpha s_1-\alpha s_2+2}{2}}<\frac{\epsilon}{4},
\end{aligned}$$

and

$$C\left(\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1-\theta}+\left(R_0+\left\|u_0\right\|_{W^{s,q}(\Omega)}\right)^{p-1+\theta}\right)\left|T_{\max}-t_n\right|^{\frac{\alpha s_1-\alpha s_2+2}{2}}<\frac{\epsilon}{4}.$$

Hence, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left\|u(\cdot, t_m) - u(\cdot, t_n)\right\|_{W^{s,q}(\Omega)} < \epsilon, \quad \text{for } m, n \geq N. \tag{3.80}$$

It follows that $\{u(\cdot, t_n)\}_{n\in\mathbb{N}} \subset W^{s,q}(\Omega)$ is a Cauchy sequences and for $\{t_n\}_{n\in\mathbb{N}^*}$ arbitrary we have proved the existence of the limit

$$\lim_{t\to T_{\max}^-} \|u(\cdot, t)\|_{W^{s,q}(\Omega)} < \infty.$$

FRom our previos result we deduce that the solution can extended to some larger interval (u can be continued beyond $T_{\max}$), and this contradict the definition of $T_{\max}$. Thus, either $T_{\max} = \infty$ or if $T_{\max} < \infty$ then $\lim_{t\to T_{\max}^-} \|u(\cdot, t)\|_{W^{s,q}(\Omega)} = \infty$. The proof of Theorem 3.4 is finished. □

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Bruno de Andrade
Departamento de Matemática
Universidade Federal de Sergipe
Avenue Rosa Else
São Cristóvão SE
Brazil
e-mail: bruno@mat.ufs.br

Vo Van Au
Institute of Fundamental and Applied Sciences
Duy Tan University
Ho Chi Minh City 700000
Vietnam
e-mail: vovanau@duytan.edu.vn

Vo Van Au
Faculty of Natural Sciences
Duy Tan University
Da Nang 550000
Vietnam

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway
Ireland
e-mail: donal.oregan@nuigalway.ie

Nguyen Huy Tuan
Department of Mathematics and Computer Science
University of Science
Ho Chi Minh City
Vietnam

Nguyen Huy Tuan
Vietnam National University
Ho Chi Minh City
Vietnam
e-mail: nhtuan@hcmus.edu.vn

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