



Boundedness in a two-dimensional Keller–Segel–Navier–Stokes system involving a rapidly diffusing repulsive signal

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Abstract. This paper is concerned with the Keller–Segel–Navier–Stokes system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n + \nabla \cdot (n \nabla c), & x \in \Omega, t > 0, \\ u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0, \\ u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \quad (\star)$$

with a given smooth gravitational potential Φ . It is shown that for all suitably regular initial data, a corresponding no-flux/no-flux/Dirichlet initial boundary value problem posed in a smoothly bounded planar domain admits a uniquely determined global classical solution (n, c, u, P) which has the additional property that n remains uniformly bounded. This partially goes beyond a recent result asserting global classical solvability, but without any boundedness information, in a related slightly more complex variant of (\star) accounting for parabolic evolution of the quantity c . In particular, the obtained outcome therefore provides further evidence indicating that the considered fluid interaction does not substantially reduce a certain explosion-avoiding character of the Keller–Segel-type chemorepulsion mechanisms, as known to form an essential feature of corresponding fluid-free analogues. The reasoning at its core relies on the use of a quasi-Lyapunov inequality which operates at regularity levels that seem rather unusual in this and related contexts, but which in the considered two-dimensional setting can be seen to serve as a starting point sufficient for a bootstrap-type series of arguments, finally providing global boundedness.

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1. Introduction

The interaction between populations of chemotactically migrating individuals and liquid environments has been the objective of considerable developments in the mathematical literature during the past decade. With to date persisting motivation originating from experimental findings that report nontrivial effects of such types of interplay, including support of pattern generation but also quantitative influences on spreading properties [6, 7, 28, 36, 37], analytical studies in this field, on the one hand, have been concerned with widely nontrivial questions related to corresponding existence theories, but beyond this, on the other hand, have partially been addressing aspects of qualitative solution behavior. Apart from some simulation-endorsed predictions on possibly blow-up preventing effects of fluid flows on Keller–Segel-type models [27], in certain particular cases characterized by suitably weak coupling, in which only the motion of the liquid influences migration of cells and signals, but not vice versa, some recent analytical results have rigorously confirmed nontrivial consequences of suitably chosen given fluid velocity fields on aggregation and spreading behavior in chemotaxis systems [16, 19–21].

In situations characterized by a genuinely mutual interaction in the sense that additionally also cells influence the fluid flow through buoyancy, as forming a core assumption in the modeling approaches e.g. from [37] and [1], in line with an accordingly increased complexity the knowledge is yet considerably

sparser, with a predominant focus on aspects from mere existence theory in many cases. Indeed, already for a model coupling a chemotaxis–*consumption* process to the Navier–Stokes equations in the context of the system

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), \\ c_t + u \cdot \nabla c &= \Delta c - nc, \\ u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, \end{cases} \quad (1.1)$$

for the cell population density n , the signal concentration c and the fluid variables u and P , establishing a reasonably comprehensive theory of global weak solvability has only been achieved quite recently, only after a series of partial results relying on various types of simplifications or restrictions on sizes of either the initial data or some key model ingredients [3, 9, 22, 42, 45], although the corresponding fluid-free nutrient taxis system is known to admit quite a straightforward existence theory according to a substantially dissipative character going along with the considered signal absorption mechanism [31].

Whereas beyond deriving similar existence statements for several more general chemotaxis–consumption–fluid systems [8, 10, 34, 38] the literature for (1.1) and some close relatives meanwhile even provides some results on large time stabilization toward homogeneous equilibria [24, 44, 46, 49, 52], the knowledge seems yet at a significantly more rudimentary level in cases when unlike in (1.1), the respective signal evolution is determined by *production* through cells, rather than consumption. In fact, rigorous conclusions concerned with accordingly obtained Keller–Segel–fluid systems of the form

$$\begin{cases} n_t + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - \nabla \cdot (n S(n) \nabla c) + f(u), \\ c_t + u \cdot \nabla c &= \Delta c - c + n, \\ u_t + \kappa(u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, \end{cases} \quad (1.2)$$

have so far mainly been restricted to statements on global solvability, with information on further properties such as boundedness limited to a few respective subcases, under appropriate hypotheses on smallness of initial data when $f \equiv 0$, $D \equiv S \equiv 1$ and $\kappa = 1$ [22], on suitably strong growth restrictions implied by considering logistic-type proliferation terms f of the form $f(n) = \rho n - \mu n^2$ with $\rho \in \mathbb{R}$ and $\mu > 0$ when $D \equiv S \equiv 1$ and either $\kappa = 0$ or $\kappa = 1$ [32, 33], or on adequately efficient enhancement of diffusion or saturation of taxis at large cell densities when $f \equiv 0$ [2, 26, 39–41, 47, 53]. Clearly, this lack of comprehensiveness with regard to unconditionally smooth solution behavior partially reflects the circumstance that in comparison with the respective fluid-free version of (1.1), the signal production mechanism in the Keller–Segel chemotaxis-only subsystem of (1.2) is known to enforce singularity formation in the sense of finite-time blow-up throughout various ranges of its ingredients [5, 18, 43].

Now in contrast to the latter, Keller–Segel–production systems are known to exhibit significantly more regular solution behavior in cases when instead of attractive cross-diffusion mechanisms, as characterized by choices of positive S , rather *chemorepulsion* processes are accounted for. Fluid-free variants of (1.2) obtained on letting $S \equiv -1$, for instance, have been found to admit global classical and bounded solutions in two-dimensional domains in the paradigmatic setting when $D \equiv 1$ and $f \equiv 0$ for which the corresponding chemoattraction system with $S \equiv 1$ possesses some exploding solutions; even in its three-dimensional version, this repulsive Keller–Segel system still allows at least for global weak solvability, regardless of the size of the respective initial data [4]. Also in the presence of more general parameter functions D and S , some significantly relaxing effects of repulsion in comparison with attraction have been identified in the literature [11, 23, 35].

Absence of blow-up in two-dimensional chemorepulsion–Navier–Stokes systems. Main results. In such contexts of chemorepulsion, possible effects of fluid coupling in the style of (1.2) have recently been studied in [51], and the main outcome thereof indicated that some unconditionality of finite-time blow-up suppression in fact persists also in the presence of buoyancy-induced interaction with the Navier–Stokes system in spatially two-dimensional frameworks. Indeed, main results from [51] assert global classical solvability in an initial boundary value problem for the fully parabolic system (1.2) in bounded planar domains in the apparently most prototypical situation determined by the choices $D \equiv 1$, $f \equiv 0$ and $\kappa = 1$.

The present work intends to provide one step further in this direction by investigating the question how far apart from this, also *infinite-time blow-up*, well known as a possible feature of various attractive Keller–Segel-type systems [5, 25, 48] can be ruled out in two-dimensional chemorepulsion–Navier–Stokes systems. In order to make this advanced problem accessible to our analysis, we shall address this issue in the framework of a simplified variant of (1.2) in which the signal evolution is governed by an associated elliptic equation, which hence reflects the assumption that the considered chemical diffuses much faster than cells and fluid particles, as having formed technically essential fundamentals already in numerous precedents from the chemotaxis(–fluid) literature (cf., e.g., [29] and [21] for two representative examples).

Throughout the sequel, in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary we shall accordingly be concerned with the initial boundary value problem

$$\begin{cases} n_t + u \cdot \nabla n &= \Delta n + \nabla \cdot (n \nabla c), & x \in \Omega, t > 0, \\ u \cdot \nabla c &= \Delta c - c + n, & x \in \Omega, t > 0, \\ u_t + (u \cdot \nabla)u &= \Delta u + \nabla P + n \nabla \Phi, & \nabla \cdot u = 0, x \in \Omega, t > 0, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} &= 0, \quad u = 0, & x \in \partial\Omega, t > 0, \\ n(x, 0) &= n_0(x), \quad u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where our standing hypothesis on the gravitational potential Φ will be that

$$\Phi \in W^{2,\infty}(\Omega), \tag{1.4}$$

where concerning the initial data we shall assume that

$$\begin{cases} n_0 \in C^0(\overline{\Omega}) & \text{is nonnegative with } \bar{n}_0 > 0, & \text{and that} \\ u_0 \in W^{2,2}(\Omega; \mathbb{R}^2) \cap W_{0,\sigma}^{1,2}(\Omega). \end{cases} \tag{1.5}$$

Here and below, as usual we write $\bar{\varphi} := \frac{1}{|\Omega|} \int_{\Omega} \varphi$ for $\varphi \in L^1(\Omega)$, and let $W_{0,\sigma}^{1,2}(\Omega) := W_0^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_{\sigma}(\Omega)$, with $L^2_{\sigma}(\Omega) := \{\varphi \in L^2(\Omega; \mathbb{R}^2) \mid \nabla \cdot \varphi = 0 \text{ in } \mathcal{D}(\Omega)\}$ denoting the space of all solenoidal vector fields in $L^2(\Omega; \mathbb{R}^2)$. To complete our collection of notational conventions in this respect, let us furthermore announce that throughout the sequel, we let $A = -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2(\Omega; \mathbb{R}^2)$ with its domain given by $D(A) = W^{2,2}(\Omega; \mathbb{R}^2) \cap W_{0,\sigma}^{1,2}(\Omega)$ and with \mathcal{P} denoting the Helmholtz projection on $L^2(\Omega; \mathbb{R}^2)$, and for $\alpha > 0$ we let A^α represent the corresponding sectorial fractional powers.

In this form, (1.3) will turn out to enjoy a certain basic quasi-Lyapunov inequality which, though operating at yet quite low regularity levels, will enable us to adequately cope with the key challenge to establish some first time-independent estimates suitably going beyond the L^1 boundedness information gained from an evident mass conservation feature in (1.3). Specifically, our analysis will be based on the observation that thanks to some rudimentary but time-independent basic regularity features of c , available due to ellipticity of its subproblem in (1.3) (Sect. 3), for appropriately chosen $a > 0$ and $C > 0$ the quantities

$$\mathcal{F}(t) := - \int_{\Omega} \ln(n + 1) + a \int_{\Omega} |u|^2 \quad \text{and} \quad \mathcal{D}(t) := \int_{\Omega} n \ln(n + 1) + \int_{\Omega} |\nabla u|^2 \tag{1.6}$$

satisfy

$$\mathcal{F}'(t) + \frac{1}{C}\mathcal{F}(t) + \frac{1}{C}\mathcal{D}(t) \leq C$$

throughout the life span of a smooth solution to (1.3) (Lemma 4.4), the existence of which, as well as an associated extensibility criterion, is asserted in the preliminary Sect. 2. Accordingly implied a priori estimates for the dissipation rate functional \mathcal{D} (Lemma 4.5) will thereafter be seen to constitute a starting point for a series of arguments acting at successively higher stages of regularity, inter alia addressing the

time evolution of the functionals $\int_{\Omega} n \ln \frac{n}{n_0} + b \int_{\Omega} |\nabla u|^2 + 1$, for some $b > 0$ (Lemma 5.4), and $\int_{\Omega} n^2$ (Lemma 5.5), as seeming to be more closely related to approaches pursued in precedent studies.

This will finally reveal the following main result which indeed warrants boundedness of solutions irrespective of the size of their initial data, and hence excludes any possibility of infinite-time blow-up in (1.3):

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume that (1.4) holds and that n_0 and u_0 comply with (1.5). Then there exist functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ c \in C^{2,0}(\overline{\Omega} \times (0, \infty)), \\ u \in \bigcap_{\alpha \in (\frac{1}{2}, 1)} C^0([0, \infty); D(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty); \mathbb{R}^2) \quad \text{and} \\ P \in C^{1,0}(\overline{\Omega} \times (0, \infty)), \end{cases}$$

uniquely determined up to addition of constants to P , such that $n > 0$ and $c > 0$ in $\overline{\Omega} \times (0, \infty)$, that (n, c, u, P) solves (1.3) classically in $\Omega \times (0, \infty)$, and that

$$\sup_{t>0} \|n(\cdot, t)\|_{L^\infty(\Omega)} < \infty. \tag{1.7}$$

2. Local existence and extensibility

A theory of local existence in (1.3) can be obtained by adapting basically well-established arguments involving a contraction-type reasoning in a suitable function space framework. As we could not find a reference reasonably close to the situation addressed here, however, we include at least the essential part of a corresponding proof here.

Our approach will address the main difference in comparison with fully parabolic relatives, such as the class of problems considered in detail in [42], through the following statement on unique solvability in a variant of the elliptic subproblem contained in (1.3).

Lemma 2.1. *Let $\alpha \in (\frac{1}{2}, 1)$. Then there exists $C > 0$ with the property that whenever $T > 0$, for each $\widehat{n} \in C^0(\overline{\Omega} \times [0, T])$ and $\widehat{u} \in C^0([0, T]; D(A^\alpha))$ the problem*

$$\begin{cases} -\Delta c + c = \widehat{n} - \widehat{u} \cdot \nabla c, & x \in \Omega, \quad t \in [0, T], \\ \frac{\partial c}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in [0, T], \end{cases} \tag{2.1}$$

possesses a unique weak solution $c \in C^0([0, T]; W^{2,2}(\Omega))$, which is such that

$$\|c(\cdot, t)\|_{W^{2,2}(\Omega)} \leq C \cdot \sup_{s \in (0, T)} \|\widehat{n}(\cdot, s)\|_{L^\infty(\Omega)} \cdot \left\{ 1 + \sup_{s \in (0, T)} \|A^\alpha \widehat{u}(\cdot, s)\|_{L^2(\Omega)} \right\} \quad \text{for all } t \in [0, T]. \tag{2.2}$$

Proof. Given any such T, \widehat{n} and \widehat{u} , recalling that

$$D(A^\alpha) \hookrightarrow C^0(\overline{\Omega}; \mathbb{R}^2) \tag{2.3}$$

[14, 17] we see that for each fixed $t \in [0, T]$ letting

$$B(\varphi, \psi) := \int_{\Omega} \nabla \varphi \cdot \nabla \psi + \int_{\Omega} \varphi \psi + \int_{\Omega} (\widehat{u}(\cdot, t) \cdot \nabla \varphi) \psi, \quad \varphi \in W^{1,2}(\Omega), \quad \psi \in W^{1,2}(\Omega),$$

introduces a well-defined and continuous bilinear form on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. By solenoidality of $\widehat{u}(\cdot, t)$ and the fact that $\widehat{u}(\cdot, t) = 0$ on $\partial\Omega$, B moreover is coercive, because

$$\begin{aligned} B(\varphi, \varphi) &= \int_{\Omega} |\nabla\varphi|^2 + \int_{\Omega} \varphi^2 + \frac{1}{2} \int_{\Omega} \widehat{u}(\cdot, t) \cdot \nabla\varphi^2 \\ &= \int_{\Omega} |\nabla\varphi|^2 + \int_{\Omega} \varphi^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega), \end{aligned} \quad (2.4)$$

whence according to the Lax–Milgram lemma there exists a unique $c(\cdot, t) \in W^{1,2}(\Omega)$ such that

$$\int_{\Omega} \nabla c(\cdot, t) \cdot \nabla\varphi + \int_{\Omega} c(\cdot, t)\varphi = \int_{\Omega} \widehat{n}(\cdot, t)\varphi - \int_{\Omega} \left(\widehat{u}(\cdot, t) \cdot \nabla c(\cdot, t) \right) \varphi \quad \text{for all } \varphi \in W^{1,2}(\Omega). \quad (2.5)$$

As a first estimate for $c(\cdot, t)$, in view of (2.4) we obtain from (2.5) and Young’s inequality that

$$\int_{\Omega} |\nabla c(\cdot, t)|^2 + \int_{\Omega} c^2(\cdot, t) = \int_{\Omega} \widehat{n}(\cdot, t)c(\cdot, t) \leq \frac{1}{2} \int_{\Omega} c^2(\cdot, t) + \frac{|\Omega|}{2} \|\widehat{n}(\cdot, t)\|_{L^\infty(\Omega)}^2,$$

which implies that

$$\int_{\Omega} |\nabla c(\cdot, t)|^2 + \int_{\Omega} c^2(\cdot, t) \leq |\Omega| \cdot R_1^2, \quad (2.6)$$

where $R_1 := \sup_{s \in (0, T)} \|\widehat{n}(\cdot, s)\|_{L^\infty(\Omega)}$.

Apart from that, (2.4) and (2.5) ensure that for arbitrary $t \in [0, T]$ and $s \in [0, T]$, $d := c(\cdot, t) - c(\cdot, s)$ satisfies

$$\begin{aligned} \int_{\Omega} |\nabla d|^2 + \int_{\Omega} d^2 &= \int_{\Omega} \left(\widehat{n}(\cdot, t) - \widehat{n}(\cdot, s) \right) d - \int_{\Omega} \left\{ \left(\widehat{u}(\cdot, t) - \widehat{u}(\cdot, s) \right) \cdot \nabla c(\cdot, t) \right\} d - \int_{\Omega} \left(\widehat{u}(\cdot, s) \cdot \nabla d \right) d \\ &= \int_{\Omega} \left(\widehat{n}(\cdot, t) - \widehat{n}(\cdot, s) \right) d - \int_{\Omega} \left\{ \left(\widehat{u}(\cdot, t) - \widehat{u}(\cdot, s) \right) \cdot \nabla c(\cdot, t) \right\} d \\ &\leq \frac{1}{2} \int_{\Omega} d^2 + \|\widehat{n}(\cdot, t) - \widehat{n}(\cdot, s)\|_{L^2(\Omega)}^2 + \|\widehat{u}(\cdot, t) - \widehat{u}(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, t)\|_{L^2(\Omega)}^2, \end{aligned}$$

which together with (2.6) entails that

$$c \in C^0([0, T]; W^{1,2}(\Omega)), \quad (2.7)$$

because our assumptions in particular guarantee that

$$\widehat{n} \in C^0([0, T]; L^2(\Omega)) \quad \text{and} \quad \widehat{u} \in C^0(\overline{\Omega} \times [0, T]; \mathbb{R}^2) \quad (2.8)$$

due to (2.3).

We next recall standard elliptic regularity theory [15] to fix $C_1 > 0$ with the property that if $f \in L^2(\Omega)$ and $v \in W^{1,2}(\Omega)$ are such that $\int_{\Omega} \nabla v \cdot \nabla\varphi + \int_{\Omega} v\varphi = \int_{\Omega} f\varphi$ for all $\varphi \in W^{1,2}(\Omega)$, then actually $v \in W^{2,2}(\Omega)$

with

$$\|v\|_{W^{2,2}(\Omega)} \leq C_1 \|f\|_{L^2(\Omega)}. \quad (2.9)$$

In view of (2.6), a first application of this yields, as our second estimate for c , the inequality

$$\begin{aligned} \|c(\cdot, t)\|_{W^{2,2}(\Omega)} &\leq C_1 \left\| \widehat{n}(\cdot, t) - \widehat{u}(\cdot, t) \cdot \nabla c(\cdot, t) \right\|_{L^2(\Omega)} \\ &\leq C_1 \|\widehat{n}(\cdot, t)\|_{L^2(\Omega)} + C_1 \|\widehat{u}(\cdot, t)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, t)\|_{L^2(\Omega)} \\ &\leq C_1 |\Omega|^{\frac{1}{2}} R_1 + C_1 C_2 R_1 R_2 \quad \text{for all } t \in [0, T], \end{aligned} \quad (2.10)$$

with $R_2 := \sup_{s \in (0, T)} \|A^\alpha \widehat{u}(\cdot, s)\|_{L^2(\Omega)}$ and with $C_2 := |\Omega|^{\frac{1}{2}} \cdot \sup_{0 \neq \varphi \in D(A^\alpha)} \frac{\|\varphi\|_{L^\infty(\Omega)}}{\|A^\alpha \varphi\|_{L^2(\Omega)}}$ being finite due to (2.3). This establishes (2.2), so that it remains to verify the claimed continuity property. To this end, we note that as a second consequence of (2.9) we obtain that for all $t \in [0, T]$ and $s \in [0, T]$,

$$\begin{aligned} \|c(\cdot, t) - c(\cdot, s)\|_{W^{2,2}(\Omega)} &\leq C_1 \left\| \widehat{n}(\cdot, t) - \widehat{n}(\cdot, s) - \widehat{u}(\cdot, t) \cdot \nabla c(\cdot, t) + \widehat{u}(\cdot, s) \cdot \nabla c(\cdot, s) \right\|_{L^2(\Omega)} \\ &\leq C_1 \|\widehat{n}(\cdot, t) - \widehat{n}(\cdot, s)\|_{L^2(\Omega)} \\ &\quad + C_1 \|\widehat{u}(\cdot, t) - \widehat{u}(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, s)\|_{L^2(\Omega)} \\ &\quad + C_1 \|\widehat{u}(\cdot, t)\|_{L^\infty(\Omega)} \|\nabla c(\cdot, t) - \nabla c(\cdot, s)\|_{L^2(\Omega)}, \end{aligned}$$

so that relying on (2.7) and once more on our assumption we conclude that indeed c is continuous on $[0, T]$ as a $W^{2,2}(\Omega)$ -valued function. \square

We can therefore establish a local theory by means of a rather straightforward reasoning based on Banach’s fixed point theorem.

Lemma 2.2. *Suppose that (1.4) and (1.5) hold. Then there exist $T_{\max} \in (0, \infty]$ and functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ c \in C^{2,0}(\overline{\Omega} \times (0, T_{\max})), \\ u \in \bigcap_{\alpha \in (\frac{1}{2}, 1)} C^0([0, T_{\max}); D(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}); \mathbb{R}^2) \\ P \in C^{1,0}(\Omega \times (0, T_{\max})), \end{cases} \quad \text{and} \quad (2.11)$$

uniquely determined up to addition of constants to P , such that $n > 0$ and $c > 0$ in $\overline{\Omega} \times (0, T_{\max})$, that (n, c, u, P) forms a classical solution of (1.3) in $\Omega \times (0, T_{\max})$, and that

$$\text{if } T_{\max} < \infty, \text{ then } \limsup_{t \nearrow T_{\max}} \left\{ \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot, t)\|_{L^2(\Omega)} \right\} \text{ for all } \alpha \in \left(\frac{1}{2}, 1\right). \quad (2.12)$$

Moreover,

$$\int_{\Omega} n(\cdot, t) = \int_{\Omega} n_0 \quad \text{for all } t \in (0, T_{\max}) \quad (2.13)$$

and

$$\int_{\Omega} c(\cdot, t) = \int_{\Omega} n_0 \quad \text{for all } t \in (0, T_{\max}). \quad (2.14)$$

Proof. As our reasoning follows quite a well-established line of arguments, we restrict ourselves to sketching the main aspects. For fixed $\alpha \in (\frac{1}{2}, 1)$ and with $T \in (0, 1)$ to be chosen below and

$$R := \|n_0\|_{L^\infty(\Omega)} + \|A^\alpha u_0\|_{L^2(\Omega)} + 1,$$

on the closed subset

$$S := \left\{ (n, u) \in X \mid \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot, t)\|_{L^2(\Omega)} \leq R \text{ for all } t \in [0, T] \right\}$$

of the Banach space $X := C^0([0, T]; C^0(\overline{\Omega}) \times D(A^\alpha))$, equipped with the norm defined by setting $\|(n, u)\|_X := \sup_{t \in (0, T)} \left\{ \|n(\cdot, t)\|_{L^\infty(\Omega)} + \|A^\alpha u(\cdot, t)\|_{L^\infty(\Omega)} \right\}$ for $(u, v) \in X$, we introduce a mapping $\Psi = (\Psi_1, \Psi_2)$ by letting

$$(\Psi_1[\widehat{n}, \widehat{u}])(\cdot, t) := e^{t\Delta} n_0 + \int_0^t e^{(t-s)\Delta} \nabla \cdot \left\{ \widehat{n}(\cdot, s) \left(\nabla c(\cdot, s) - \widehat{u}(\cdot, s) \right) \right\} ds$$

and

$$(\Psi_2[\widehat{n}, \widehat{u}])(\cdot, t) := e^{-tA}u_0 - \int_0^t e^{-(t-s)A} \mathcal{P} \left\{ (\widehat{u}(\cdot, s) \cdot \nabla) \widehat{u}(\cdot, s) - \widehat{n}(\cdot, s) \nabla \Phi \right\} ds$$

for $(\widehat{n}, \widehat{u}) \in S$ and $t \in [0, T]$, where $(e^{t\Delta})_{t \geq 0}$ and $(e^{-tA})_{t \geq 0}$ represent the Neumann heat semigroup and the Dirichlet Stokes semigroup on Ω , and where according to Lemma 2.1, $c \in C^0([0, T]; W^{2,2}(\Omega))$ denotes the correspondingly obtained weak solution of (2.1).

To see that Ψ in fact maps S into itself, we fix any $r > 2$ and observe that by well-known regularization properties of $(e^{t\Delta})_{t \geq 0}$ [13], by (2.2) and by continuity of the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ and $D(A^\alpha) \hookrightarrow L^r(\Omega; \mathbb{R}^2)$, we can find $C_1 > 0$ and $C_2 = C_2(R) > 0$ such that regardless of our subsequent choice of $T \in (0, 1)$,

$$\begin{aligned} \left\| (\Psi_1[\widehat{n}, \widehat{u}])(\cdot, t) \right\|_{L^\infty(\Omega)} &\leq \|n_0\|_{L^\infty(\Omega)} + C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}} \left\| \widehat{n}(\cdot, t) \left(\nabla c(\cdot, s) - \widehat{u}(\cdot, s) \right) \right\|_{L^r(\Omega)} ds \\ &\leq \|n_0\|_{L^\infty(\Omega)} \\ &\quad + C_1 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}} \|\widehat{n}(\cdot, s)\|_{L^\infty(\Omega)} \cdot \left\{ \|\nabla c(\cdot, s)\|_{L^r(\Omega)} + \|\widehat{u}(\cdot, s)\|_{L^r(\Omega)} \right\} ds \\ &\leq \|n_0\|_{L^\infty(\Omega)} + C_2 T^{\frac{1}{2}-\frac{1}{r}} \quad \text{for all } t \in [0, T]. \end{aligned} \tag{2.15}$$

Likewise, the smoothing action of the semigroup $(e^{-tA})_{t \geq 0}$ [12] along with the continuity of the embeddings $D(A^\alpha) \hookrightarrow L^\infty(\Omega; \mathbb{R}^2)$ and $D(A^\alpha) \hookrightarrow W^{1,2}(\Omega; \mathbb{R}^2)$ as well as the boundedness of $\nabla \Phi$ ensures the existence of $C_3 > 0$ and $C_4 = C_4(R) > 0$, again independent of $T \in (0, 1)$, such that

$$\begin{aligned} &\left\| (A^\alpha \Psi_2[\widehat{n}, \widehat{u}])(\cdot, t) \right\|_{L^2(\Omega)} \\ &\leq \|A^\alpha u_0\|_{L^2(\Omega)} + C_3 \int_0^t (t-s)^{-\alpha} \left\| \mathcal{P} \left\{ (\widehat{u}(\cdot, s) \cdot \nabla) \widehat{u}(\cdot, s) - \widehat{n}(\cdot, s) \nabla \Phi \right\} \right\|_{L^2(\Omega)} ds \\ &\leq \|A^\alpha u_0\|_{L^2(\Omega)} + C_3 \int_0^t (t-s)^{-\alpha} \left\{ \|\widehat{u}(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla \widehat{u}(\cdot, s)\|_{L^2(\Omega)} + \|\widehat{n}(\cdot, s)\|_{L^2(\Omega)} \|\nabla \Phi\|_{L^\infty(\Omega)} \right\} ds \\ &\leq \|A^\alpha u_0\|_{L^2(\Omega)} + C_4 T^{1-\alpha} \quad \text{for all } t \in [0, T]. \end{aligned} \tag{2.16}$$

Therefore, if we fix $T_1 \in (0, 1)$ small enough fulfilling $\max\{C_2 T^{\frac{1}{2}-\frac{1}{r}}, C_4 T^{1-\alpha}\} \leq \frac{1}{2}$, then for any choice of $T \in (0, T_1)$ we infer from (2.15) and (2.16) that indeed

$$\|\Psi[\widehat{n}, \widehat{u}]\|_X \leq \left\{ \|n_0\|_{L^\infty(\Omega)} + \frac{1}{2} \right\} + \left\{ \|A^\alpha u_0\|_{L^2(\Omega)} + \frac{1}{2} \right\} = R \quad \text{for all } (\widehat{n}, \widehat{u}) \in S.$$

By quite straightforward modification of the above procedure, it can furthermore readily be shown that there exists $T_2 \in (0, T_1)$ such that whenever $T \in (0, T_2)$, Ψ even becomes a contraction on S , meaning that fixing any such T we obtain a unique $(n, u) \in S$ such that $(n, u) = \Psi[n, u]$. Standard prolongation and regularity arguments thereafter show that (n, u) can be extended up to some $T_{\max} \in (0, \infty]$ fulfilling (2.12), that actually n, c and u satisfy the inclusions in (2.11) and that (n, c, u, P) solves (1.3) classically in $\Omega \times (0, T_{\max})$ with some appropriately chosen $P \in C^{1,0}(\Omega \times (0, T_{\max}))$ enjoying the claimed uniqueness property [30].

The statements on positivity and the mass conservation features in (2.13) and (2.14), finally, result from applications of the parabolic and elliptic strong maximum principles, and from integration of the first two equations in (1.3). \square

3. Making use of ellipticity: basic time-independent estimates for c

The following lemma documents an observation of quite basic character, but of essential importance for our approach, by stating a regularity feature of c which is implied by nonnegativity of n and c and by solenoidality of u , and which due to ellipticity of the second equation in (1.3) is independent with respect to time.

Lemma 3.1. *Let $p \in (0, 1)$. Then there exists $C(p) > 0$ such that*

$$\int_{\Omega} \left| \nabla \left(c(\cdot, t) + 1 \right)^{\frac{p}{2}} \right|^2 \leq C(p) \quad \text{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. We multiply the second equation in (1.3) by $(c + 1)^{p-1}$ to see that since $\nabla \cdot u = 0$ as well as $n \geq 0$ and $c \geq 0$, due to (2.14) we have

$$\begin{aligned} (1 - p) \int_{\Omega} (c + 1)^{p-2} |\nabla c|^2 &= \int_{\Omega} c(c + 1)^{p-1} - \int_{\Omega} n(c + 1)^{p-1} + \frac{1}{p} \int_{\Omega} u \cdot \nabla (c + 1)^p \\ &= \int_{\Omega} c(c + 1)^{p-1} - \int_{\Omega} n(c + 1)^{p-1} + \frac{1}{p} \int_{\Omega} u \cdot \nabla (c + 1)^p \\ &\leq \int_{\Omega} c(c + 1)^{p-1} \\ &\leq \int_{\Omega} c \\ &= \int_{\Omega} n_0 \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

from which (3.1) follows. \square

By straightforward interpolation relying on (2.14), this implies time-independent bounds for c in any L^q space with finite q .

Lemma 3.2. *For all $q \in (1, \infty)$, there exists $C(q) > 0$ such that*

$$\|c(\cdot, t)\|_{L^q(\Omega)} \leq C(q) \quad \text{for all } t \in (0, T_{\max}). \tag{3.2}$$

Proof. Since the two-dimensional Gagliardo–Nirenberg inequality provides $C_1 = C_1(q) > 0$ fulfilling

$$\begin{aligned} \|c\|_{L^q(\Omega)}^{\frac{q}{2(q-1)}} &\leq \|c + 1\|_{L^q(\Omega)}^{\frac{q}{2(q-1)}} \\ &= \left\| (c + 1)^{\frac{1}{4}} \right\|_{L^{4q}(\Omega)}^{\frac{2q}{q-1}} \\ &\leq C_1 \left\| \nabla (c + 1)^{\frac{1}{4}} \right\|_{L^2(\Omega)}^2 \left\| (c + 1)^{\frac{1}{4}} \right\|_{L^4(\Omega)}^{\frac{2}{q-1}} + C_1 \left\| (c + 1)^{\frac{1}{4}} \right\|_{L^4(\Omega)}^{\frac{2q}{q-1}} \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

and since

$$\left\| (c + 1)^{\frac{1}{4}} \right\|_{L^4(\Omega)}^4 = \int_{\Omega} (c + 1) = \int_{\Omega} n_0 + |\Omega| \quad \text{for all } t \in (0, T_{\max})$$

by (2.14), this is a consequence of Lemma 3.1 when applied to $p := \frac{1}{2}$. □

Again combined with Lemma 3.1, this in fact enables us to derive bounds for ∇c , without any weight functions, with respect to the norm in $L^r(\Omega)$ for arbitrary $r < 2$.

Lemma 3.3. *For each $r \in [1, 2)$, one can find $C(r) > 0$ with the property that*

$$\|\nabla c(\cdot, t)\|_{L^r(\Omega)} \leq C(r) \quad \text{for all } t \in (0, T_{\max}). \tag{3.3}$$

Proof. Using that $r < 2$, we may employ Young’s inequality to estimate

$$\begin{aligned} \int_{\Omega} |\nabla c|^r &= \int_{\Omega} \left\{ (c + 1)^{-\frac{3}{2}} |\nabla c|^2 \right\}^{\frac{r}{2}} \cdot (c + 1)^{\frac{3r}{4}} \\ &\leq \int_{\Omega} (c + 1)^{-\frac{3}{2}} |\nabla c|^2 + \int_{\Omega} (c + 1)^{\frac{3r}{2(2-r)}} \\ &\leq \int_{\Omega} (c + 1)^{-\frac{3}{2}} |\nabla c|^2 + 2^{\frac{3r}{2(2-r)}} \int_{\Omega} c^{\frac{3r}{2(2-r)}} + 2^{\frac{3r}{2(2-r)}} |\Omega| \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

whence another application of Lemma 3.1 to $p := \frac{1}{2}$ entails (3.3). □

4. Basic boundedness features relying on a quasi-energy inequality

Next approaching the core of our analysis, let us state the direct outcome of a testing procedure which addresses the time evolution of the first contribution to the functional \mathcal{F} in (1.6), and which again relies on the elliptic character of the second equation in (1.3).

Lemma 4.1. *We have*

$$\begin{aligned} &-\frac{d}{dt} \int_{\Omega} \ln(n + 1) + \int_{\Omega} \frac{|\nabla n|^2}{(n + 1)^2} + \int_{\Omega} n \ln(n + 1) + \int_{\Omega} \frac{n}{n + 1} \\ &= \int_{\Omega} \left\{ \ln(n + 1) + \frac{1}{n + 1} \right\} \cdot (c + u \cdot \nabla c) \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \tag{4.1}$$

Proof. According to the first equation in (1.3) and the identity $\nabla \cdot u = 0$, we see that

$$\begin{aligned} -\frac{d}{dt} \int_{\Omega} \ln(n + 1) + \int_{\Omega} \frac{|\nabla n|^2}{(n + 1)^2} &= -\int_{\Omega} \frac{n}{(n + 1)^2} \nabla n \cdot \nabla c + \int_{\Omega} u \cdot \nabla \ln(n + 1) \\ &= -\int_{\Omega} \frac{n}{(n + 1)^2} \nabla n \cdot \nabla c \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \tag{4.2}$$

where rewriting

$$\frac{n}{(n + 1)^2} \nabla n = \nabla \ln(n + 1) + \nabla \frac{1}{n + 1} \quad \text{in } \Omega \times (0, T_{\max})$$

we may integrate by parts once more and use the second equation from (1.3) to obtain that

$$\begin{aligned} \int_{\Omega} \frac{n}{(n+1)^2} \nabla n \cdot \nabla c &= \int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\} \cdot \Delta c \\ &= \int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\} \cdot (-n + c + u \cdot \nabla c) \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

Inserted into (4.2) this yields (4.1). □

As for the evolution of the respective second integral in (1.6), we may import the following basic information from the literature concerned with the fully parabolic analogue of (1.3), without repeating details of a proof which, according to the identical form of the respective Navier–Stokes subproblem, can in fact be copied word by word from the considered precedent.

Lemma 4.2. *There exists $C > 0$ such that*

$$\frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 \leq C \int_{\Omega} n \ln \frac{n}{n_0} + C \quad \text{for all } t \in (0, T_{\max}). \tag{4.3}$$

Proof. On the basis of a functional inequality from [50, Lemma 2.2], this can be seen by precisely repeating the argument from [51, Lemma 3.4]. □

Now thanks to the a priori information on c and ∇c from Lemma 3.2 and Lemma 3.3, the right-hand side in (4.1) can be estimated in terms of a sublinear power of the dissipation rate appearing in (4.3).

Lemma 4.3. *There exists $C > 0$ such that*

$$\int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\} \cdot (c + u \cdot \nabla c) \leq C \|\nabla u\|_{L^2(\Omega)} + C \quad \text{for all } t \in (0, T_{\max}). \tag{4.4}$$

Proof. We use the elementary inequality $\ln^6 \xi \leq C_1 \xi$, valid for all $\xi > 0$ with $C_1 := (\frac{6}{e})^6$, to see that thanks to (2.13),

$$\begin{aligned} \int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\}^6 &\leq 32 \int_{\Omega} \ln^6(n+1) + 32 \int_{\Omega} \frac{1}{(n+1)^6} \\ &\leq 32C_1 \int_{\Omega} (n+1) + 32|\Omega| \\ &= C_2 := 32C_1 \int_{\Omega} (n_0 + 1) + 32|\Omega| \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

Therefore, the Hölder inequality implies that

$$\begin{aligned} \int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\} \cdot (c + u \cdot \nabla c) &\leq \left\{ \int_{\Omega} \left\{ \ln(n+1) + \frac{1}{n+1} \right\}^6 \right\}^{\frac{1}{6}} \cdot \|c + u \cdot \nabla c\|_{L^{\frac{6}{5}}(\Omega)} \\ &\leq C_2^{\frac{1}{6}} \|c + u \cdot \nabla c\|_{L^{\frac{6}{5}}(\Omega)} \\ &\leq C_2^{\frac{1}{6}} \|c\|_{L^{\frac{6}{5}}(\Omega)} + C_2^{\frac{1}{6}} \|u\|_{L^6(\Omega)} \|\nabla c\|_{L^{\frac{3}{2}}(\Omega)} \\ &\leq C_2^{\frac{1}{6}} C_3 + C_2^{\frac{1}{6}} C_4 \|u\|_{L^6(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \end{aligned}$$

with $C_3 := \sup_{t \in (0, T_{\max})} \|c(\cdot, t)\|_{L^{\frac{6}{5}}(\Omega)}$ and $C_4 := \sup_{t \in (0, T_{\max})} \|\nabla c(\cdot, t)\|_{L^{\frac{3}{2}}(\Omega)}$ being finite due to Lemma 3.2 and Lemma 3.3. Since clearly $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, this entails (4.4) with some appropriately large $C > 0$. □

Therefore, adding a suitably small multiple of the inequality (4.3) to (4.1) reveals the announced property of the functional in (1.6):

Lemma 4.4. *There exist $a > 0$ and $C > 0$ such that*

$$\begin{aligned} & \frac{d}{dt} \left\{ - \int_{\Omega} \ln(n+1) + a \int_{\Omega} |u|^2 \right\} + \frac{1}{C} \cdot \left\{ - \int_{\Omega} \ln(n+1) + a \int_{\Omega} |u|^2 \right\} + \frac{1}{2} \int_{\Omega} n \ln(n+1) + \frac{a}{2} \int_{\Omega} |\nabla u|^2 \\ & \leq C \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (4.5)$$

Proof. On the basis of Lemma 4.1, Lemma 4.3, Lemma 4.2 and (2.13), let us pick $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that

$$- \frac{d}{dt} \int_{\Omega} \ln(n+1) + \int_{\Omega} n \ln(n+1) \leq C_1 \|\nabla u\|_{L^2(\Omega)} + C_1 \quad \text{for all } t \in (0, T_{\max}) \quad (4.6)$$

and

$$\frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 \leq C_2 \int_{\Omega} n \ln n + C_3 \quad \text{for all } t \in (0, T_{\max}), \quad (4.7)$$

and thereupon, let

$$a := \frac{1}{2C_2}, \quad C_4 := \frac{C_1^2}{a} + C_1 + aC_3 \quad \text{and} \quad C_5 := \max\{4C_6, C_4\}, \quad (4.8)$$

where according to the Poincaré inequality, $C_6 > 0$ is such that

$$\int_{\Omega} |\varphi|^2 \leq C_6 \int_{\Omega} |\nabla \varphi|^2 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^2). \quad (4.9)$$

Then combining (4.6) with (4.7) and using Young's inequality along with (4.9) and our definition of C_4 show that since $\ln(n+1) \geq \max\{0, \ln n\}$,

$$\begin{aligned} & \frac{d}{dt} \left\{ - \int_{\Omega} \ln(n+1) + a \int_{\Omega} |u|^2 \right\} + \frac{1}{C_5} \cdot \left\{ - \int_{\Omega} \ln(n+1) + a \int_{\Omega} |u|^2 \right\} + \int_{\Omega} n \ln(n+1) + a \int_{\Omega} |\nabla u|^2 \\ & \leq - \frac{1}{C_5} \int_{\Omega} \ln(n+1) + \frac{a}{C_5} \int_{\Omega} |u|^2 \\ & \quad + C_1 \|\nabla u\|_{L^2(\Omega)} + C_1 + aC_2 \int_{\Omega} n \ln n + aC_3 \\ & \leq \frac{a}{C_5} \int_{\Omega} |u|^2 + \frac{a}{4} \int_{\Omega} |\nabla u|^2 + \frac{C_1^2}{a} + C_1 + aC_2 \int_{\Omega} n \ln(n+1) + aC_3 \\ & = \frac{a}{C_5} \int_{\Omega} |u|^2 + \frac{a}{4} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} n \ln(n+1) + C_4 \\ & \leq \frac{aC_6}{C_5} \int_{\Omega} |\nabla u|^2 + \frac{a}{4} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} n \ln(n+1) + C_4 \\ & = \frac{a}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} n \ln(n+1) + C_4 \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (4.10)$$

because (4.8) inter alia says that $\frac{aC_6}{C_5} \leq \frac{a}{4}$. As (4.8) furthermore implies that $C_4 \leq C_5$, (4.10) immediately leads to (4.5) with $C := C_5$. \square

By straightforward integration, the latter entails the following estimates, the second and third ones among which will be of predominant importance for our subsequent analysis.

Lemma 4.5. *There exists $C > 0$ such that*

$$\int_{\Omega} |u(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\max}) \tag{4.11}$$

and

$$\int_t^{t+\tau} \int_{\Omega} n \ln(n+1) \leq C \quad \text{for all } t \in [0, T_{\max} - \tau) \tag{4.12}$$

as well as

$$\int_t^{t+\tau} \int_{\Omega} |\nabla u|^2 \leq C \quad \text{for all } t \in [0, T_{\max} - \tau), \tag{4.13}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

Proof. From Lemma 4.4 we obtain $a > 0, C_1 > 0$ and $C_2 > 0$ such that writing

$$\mathcal{F}(t) := - \int_{\Omega} \ln(u(\cdot, t) + 1) + a \int_{\Omega} |u(\cdot, t)|^2, \quad t \in [0, T_{\max}),$$

and

$$g(t) := \frac{1}{2} \int_{\Omega} n(\cdot, t) \ln(n(\cdot, t) + 1) + \frac{a}{2} \int_{\Omega} |\nabla u(\cdot, t)|^2, \quad t \in (0, T_{\max}),$$

we have

$$\mathcal{F}'(t) + C_1 \mathcal{F}(t) + g(t) \leq C_2 \quad \text{for all } t \in (0, T_{\max}). \tag{4.14}$$

Since $g \geq 0$, through an ODE comparison this firstly implies that

$$\mathcal{F}(t) \leq C_3 := \max\left\{\mathcal{F}(0), \frac{C_2}{C_1}\right\} \quad \text{for all } t \in (0, T_{\max}) \tag{4.15}$$

and thus entails (4.11) due to the fact that

$$\begin{aligned} \mathcal{F}(t) &\geq - \int_{\Omega} (n(\cdot, t) + 1) + a \int_{\Omega} |u(\cdot, t)|^2 \\ &= - \int_{\Omega} (n_0 + 1) + a \int_{\Omega} |u(\cdot, t)|^2 \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \tag{4.16}$$

thanks to the validity of $\ln(\xi + 1) \leq \xi$ for all $\xi \geq 0$ and to (2.13).

As (4.16) moreover means that $-\mathcal{F}(t) \leq C_4 := \int_{\Omega} (n_0 + 1)$ for all $t \in (0, T_{\max})$, an integration of (4.14) finally shows that

$$\begin{aligned} \int_t^{t+\tau} g(s) ds &\leq \mathcal{F}(t) - \mathcal{F}(t + \tau) - C_1 \int_t^{t+\tau} \mathcal{F}(s) ds + C_2 \tau \\ &\leq C_3 + C_4 + C_1 C_4 \tau + C_2 \tau \quad \text{for all } t \in [0, T_{\max} - \tau) \end{aligned}$$

and that thus also (4.13) and (4.12) are valid, because $\tau \leq 1$. □

5. Higher regularity information. Proof of Theorem 1.1

With the information from Lemma 4.5 in hand, we can now proceed to derive improved regularity properties on the basis of more standard testing procedures. Our first step in this respect is concerned with a functional of classical logarithmic entropy type.

Lemma 5.1. *We have*

$$\frac{d}{dt} \int_{\Omega} n \ln \frac{n}{\bar{n}_0} + \int_{\Omega} \frac{|\nabla n|^2}{n} + \frac{1}{2} \int_{\Omega} n^2 \leq \int_{\Omega} c^2 + \int_{\Omega} |u \cdot \nabla c|^2 \quad \text{for all } t \in (0, T_{\max}). \tag{5.1}$$

Proof. Once more using that $\nabla \cdot u = 0$, from (2.13) and the first two equations in (1.3) we obtain that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} n \ln \frac{n}{\bar{n}_0} + \frac{|\nabla n|^2}{n} &= - \int_{\Omega} \nabla n \cdot \nabla c - \int_{\Omega} u \cdot \nabla(n \ln n) \\ &= \int_{\Omega} n \Delta c \\ &= - \int_{\Omega} n^2 + \int_{\Omega} nc - \int_{\Omega} n(u \cdot \nabla c) \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

This implies (5.1) upon two applications of Young’s inequality, which namely show that

$$\int_{\Omega} nc \leq \frac{1}{4} \int_{\Omega} n^2 + \int_{\Omega} c^2 \quad \text{and} \quad - \int_{\Omega} n(u \cdot \nabla c) \leq \frac{1}{4} \int_{\Omega} n^2 + \int_{\Omega} |u \cdot \nabla c|^2$$

for all $t \in (0, T_{\max})$. □

In order to appropriately absorb the rightmost contribution therein, we shall linearly combine (5.1) with the following outcome of a standard testing procedure performed to the Navier–Stokes subsystem of (1.3).

Lemma 5.2. *There exists $C > 0$ such that*

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |Au|^2 \leq C \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + C \int_{\Omega} n^2 \quad \text{for all } t \in (0, T_{\max}). \tag{5.2}$$

Proof. As a proof of this can be built on (4.11) through a rather standard argument, we may confine ourselves to referring to e.g. [42, Proof of Theorem 1.1] and [51, Lemma 4.2] for details in two closely related situations. □

In fact, the crucial term on the right of (5.1), even when augmented by some second-order expression containing c , can be estimated in terms of a quadratic power of the functional on the left of (5.2) and of an integral involving an essentially arbitrary subquadratic power of n :

Lemma 5.3. *For all $p \in (1, 2)$, there exists $C(p) > 0$ such that*

$$\int_{\Omega} |u \cdot \nabla c|^2 + \|c\|_{W^{2,p}(\Omega)}^p \leq C(p) \int_{\Omega} n^p + C(p) \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + C(p) \quad \text{for all } t \in (0, T_{\max}). \tag{5.3}$$

Proof. Given $p \in (1, 2)$, using that then $4 > 4 - p + 2 - p$ we can pick $r \in (1, 2)$ suitably close to 2 such that $4 > \frac{8}{r} - p + 2 - \frac{2p}{r}$. This in turn enables us to fix $s > 2$ close enough to 2 such that

$$\frac{8}{s} > \frac{8}{r} - p + 2 - \frac{2p}{r}, \tag{5.4}$$

and we thereupon let

$$\lambda := \frac{\frac{2}{r} - \frac{2}{s}}{1 - \frac{2}{p} + \frac{2}{r}}.$$

Then $\lambda > 0$ since $r < 2 < s$, and (5.4) asserts that moreover

$$\left(1 - \frac{2}{p} + \frac{2}{r}\right) \cdot (4\lambda - p) = \frac{8}{r} - \frac{8}{s} - p + 2 - \frac{2p}{r} < 0,$$

and thus, $4\lambda < p$. In consequence, after employing the Gagliardo–Nirenberg inequality and relying on the fact that $r < 2$ in applying Lemma 3.3, we may use Young’s inequality to see that with some positive constants $C_1 = C_1(p), C_2 = C_2(p)$ and $C_3 = C_3(p)$ we have

$$\begin{aligned} \|\nabla c\|_{L^s(\Omega)}^4 &\leq C_1 \|c\|_{W^{2,p}(\Omega)}^{4\lambda} \|\nabla c\|_{L^r(\Omega)}^{4(1-\lambda)} \\ &\leq C_2 \|c\|_{W^{2,p}(\Omega)}^{4\lambda} \\ &\leq \frac{1}{2} \|c\|_{W^{2,p}(\Omega)}^p + C_3 \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

Thanks to the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2s}{s-2}}(\Omega)$, by once more utilizing Young’s inequality we therefore obtain $C_4 = C_4(p) > 0$ such that

$$\begin{aligned} \int_{\Omega} |u \cdot \nabla c|^2 &\leq \|u\|_{L^{\frac{2s}{s-2}}(\Omega)}^4 + \|\nabla c\|_{L^s(\Omega)}^4 \\ &\leq C_4 \|\nabla u\|_{L^2(\Omega)}^4 + \frac{1}{2} \|c\|_{W^{2,p}(\Omega)}^p + C_4 \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \tag{5.5}$$

Independently, we next employ standard elliptic regularity theory to the second equation in (1.3) to find $C_5 = C_5(p) > 0$ such that due to Young’s inequality and the restriction that $p < 2$,

$$\begin{aligned} 2\|c\|_{W^{2,p}(\Omega)}^p &\leq C_5 \|n - u \cdot \nabla c\|_{L^p(\Omega)}^p \\ &\leq 2^{p-1} C_5 \|n\|_{L^p(\Omega)}^p + 2^{p-1} C_5 \|u \cdot \nabla c\|_{L^p(\Omega)}^p \\ &\leq 2^{p-1} C_5 \|n\|_{L^p(\Omega)}^p + \|u \cdot \nabla c\|_{L^2(\Omega)}^2 + C_6 \quad \text{for all } t \in (0, T_{\max}) \end{aligned}$$

with some $C_6 = C_6(p) > 0$. When added to (5.5), this shows that for all $t \in (0, T_{\max})$,

$$\begin{aligned} \int_{\Omega} |u \cdot \nabla c|^2 + 2\|c\|_{W^{2,p}(\Omega)}^p &\leq 2 \int_{\Omega} |u \cdot \nabla c|^2 + 2^{p-1} C_5 \int_{\Omega} n^p + C_6 \\ &\leq 2C_4 \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + \|c\|_{W^{2,p}(\Omega)}^p + 2C_3 + 2^{p-1} C_5 \int_{\Omega} n^p + C_6, \end{aligned}$$

and therefore implies (5.3) with $C(p) := \max\{2^{p-1} C_5(p), 2C_4(p), 2C_3(p) + C_6(p)\}$. □

Thanks to the spatio-temporal L^2 bound for ∇u available due to Lemma 4.5, in view of Lemma 5.3 a suitable combination of the inequalities from Lemma 5.1 and Lemma 5.2 yields the following improvement on our knowledge about the regularity of the fluid flow.

Lemma 5.4. *There exists $C > 0$ such that*

$$\int_{\Omega} |\nabla u(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{5.6}$$

Proof. We apply Lemma 5.2 and Lemma 5.3 to fix $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{d}{dt} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |Au|^2 \leq C_1 \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + C_1 \int_{\Omega} n^2 \quad \text{for all } t \in (0, T_{\max}) \tag{5.7}$$

and

$$\int_{\Omega} |u \cdot \nabla c|^2 \leq C_2 \int_{\Omega} n^{\frac{3}{2}} + C_2 \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + C_2 \quad \text{for all } t \in (0, T_{\max}), \tag{5.8}$$

and let $b := \frac{1}{4C_1}$. Then taking an appropriate linear combination of the inequality from Lemma 5.1 with (5.7) shows that

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_{\Omega} n \ln \frac{n}{\bar{n}_0} + b \int_{\Omega} |\nabla u|^2 + 1 \right\} + \frac{1}{2} \int_{\Omega} n^2 \\ & \leq \int_{\Omega} c^2 + \int_{\Omega} |u \cdot \nabla c|^2 + bC_1 \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + bC_1 \int_{\Omega} n^2 \\ & \leq \int_{\Omega} c^2 + (C_2 + bC_1) \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\}^2 + C_2 \int_{\Omega} n^{\frac{3}{2}} + C_2 + \frac{1}{4} \int_{\Omega} n^2 \quad \text{for all } t \in (0, T_{\max}), \end{aligned}$$

and that thus $y(t) := \int_{\Omega} n(\cdot, t) \ln \frac{n(\cdot, t)}{\bar{n}_0} + b \int_{\Omega} |\nabla u(\cdot, t)|^2 + 1$, $t \in [0, T_{\max})$, satisfies

$$y'(t) + \frac{1}{4} \int_{\Omega} n^2 \leq (C_2 + bC_1) \cdot \left\{ \int_{\Omega} |\nabla u|^2 \right\} \cdot y(t) + C_2 \int_{\Omega} n^{\frac{3}{2}} + \int_{\Omega} c^2 + C_2 \quad \text{for all } t \in (0, T_{\max}).$$

Since here Young's inequality provides $C_4 > 0$ such that

$$C_2 \int_{\Omega} n^{\frac{3}{2}} \leq \frac{1}{4} \int_{\Omega} n^2 + C_4 \quad \text{for all } t \in (0, T_{\max}),$$

and since $C_5 := \sup_{t \in (0, T_{\max})} \int_{\Omega} c^2(\cdot, t)$ is finite by Lemma 3.2, this entails that

$$y'(t) \leq h(t)y(t) \quad \text{for all } t \in (0, T_{\max}) \tag{5.9}$$

with $h(t) := (C_2 + bC_1) \int_{\Omega} |\nabla u(\cdot, t)|^2 + C_2 + C_4 + C_5$, $t \in (0, T_{\max})$, because $y \geq 1$.

Now Lemma 4.5 ensures the existence of $C_6 > 0$ and $C_7 > 0$ such that again writing $\tau := \min\{1, \frac{1}{2}T_{\max}\}$ we have

$$\int_{(t-\tau)_+}^t h(s)ds \leq C_6 \quad \text{for all } t \in [0, T_{\max}),$$

and that for any choice of $t \in (0, T_{\max})$ we can fix $t_0(t) \in (t - \tau, t) \cap [0, T_{\max})$ fulfilling

$$y(t_0(t)) \leq C_7.$$

Integrating (5.9) hence shows that

$$\begin{aligned} y(t) & \leq y(t_0(t)) \cdot e^{\int_{t_0(t)}^t h(s)ds} \\ & \leq C_7 e^{C_6} \quad \text{for all } t \in (0, T_{\max}) \end{aligned}$$

and therefore establishes (5.6), because $t \in (0, T_{\max})$ was arbitrary. □

The latter in turn provides bounds sufficient for estimating the contribution of u to the forcing terms arising in the course of a standard L^2 testing procedure applied to the first equation in (1.3):

Lemma 5.5. *There exists $C > 0$ such that*

$$\int_{\Omega} n^2(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{5.10}$$

Proof. We integrate by parts in the first equation from (1.3) and again rely on the solenoidality of u to obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} n^2 + \int_{\Omega} |\nabla n|^2 &= - \int_{\Omega} n \nabla n \cdot \nabla c \\ &= \frac{1}{2} \int_{\Omega} n^2 \Delta c \\ &= \frac{1}{2} \int_{\Omega} n^2 c - \frac{1}{2} \int_{\Omega} n^3 + \frac{1}{2} \int_{\Omega} n^2 (u \cdot \nabla c) \\ &= \frac{1}{2} \int_{\Omega} n^2 c - \frac{1}{2} \int_{\Omega} n^3 - \int_{\Omega} n c (u \cdot \nabla n) \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \tag{5.11}$$

and using Young’s inequality, we find $C_1 > 0, C_2 > 0$ and $C_3 > 0$ such that

$$\int_{\Omega} n^2 \leq \frac{1}{6} \int_{\Omega} n^3 + C_1 \quad \text{for all } t \in (0, T_{\max})$$

and

$$\frac{1}{2} \int_{\Omega} n^2 c \leq \frac{1}{6} \int_{\Omega} n^3 + C_2 \int_{\Omega} c^3 \quad \text{for all } t \in (0, T_{\max})$$

as well as

$$\begin{aligned} - \int_{\Omega} n c (u \cdot \nabla n) &\leq \int_{\Omega} |\nabla n|^2 + \frac{1}{4} \int_{\Omega} n^2 + \frac{1}{4} \int_{\Omega} n^2 c^2 |u|^2 \\ &\leq \int_{\Omega} |\nabla n|^2 + \frac{1}{6} \int_{\Omega} n^3 + C_3 \int_{\Omega} c^6 |u|^6 \\ &\leq \int_{\Omega} |\nabla n|^2 + \frac{1}{6} \int_{\Omega} n^3 + \frac{C_3}{2} \int_{\Omega} c^{12} + \frac{C_3}{2} \int_{\Omega} |u|^{12} \quad \text{for all } t \in (0, T_{\max}). \end{aligned}$$

Observing that the constants $C_4 := \sup_{t \in (0, T_{\max})} \int_{\Omega} c^3(\cdot, t)$, $C_5 := \sup_{t \in (0, T_{\max})} \int_{\Omega} c^{12}(\cdot, t)$ and $C_6 := \sup_{t \in (0, T_{\max})} \int_{\Omega} |u(\cdot, t)|^{12}$ are all finite thanks to Lemma 3.2, Lemma 5.4 and the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{12}(\Omega)$, from (5.11) we thus infer that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} n^2 + \int_{\Omega} n^2 \leq C_1 + C_2 C_4 + \frac{C_3 C_5}{2} + \frac{C_3 C_6}{2} \quad \text{for all } t \in (0, T_{\max})$$

and that therefore (5.10) follows by means of an ODE comparison argument. □

We can therefore easily improve our knowledge about regularity of the signal gradient:

Lemma 5.6. *Let $r > 2$. Then there exists $C(r) > 0$ fulfilling*

$$\|\nabla c(\cdot, t)\|_{L^r(\Omega)} \leq C(r) \quad \text{for all } t \in (0, T_{\max}). \quad (5.12)$$

Proof. Using that $p := \frac{2r}{r+2}$ satisfies $p \in (1, 2)$ by assumption on r , we see that Lemma 5.3 in conjunction with Lemma 5.5 and Lemma 5.4 ensures the existence of $C_1 = C_1(r) > 0$ such that $\|c\|_{W^{2,p}(\Omega)} \leq C_1$ for all $t \in (0, T_{\max})$. Since $W^{2,p}(\Omega) \hookrightarrow W^{1,r}(\Omega)$, this implies (5.12). \square

The ultimate step within our bootstrap process yields bounds not only sufficient to assert global extensibility via Lemma 2.2, but moreover also to cover the intended boundedness feature in (1.7):

Lemma 5.7. *Given any $\alpha \in (\frac{1}{2}, 1)$, one can find $C(\alpha) > 0$ such that*

$$\|A^\alpha u(\cdot, t)\|_{L^2(\Omega)}^2 \leq C(\alpha) \quad \text{for all } t \in (0, T_{\max}). \quad (5.13)$$

Moreover,

$$\sup_{t \in (0, T_{\max})} \|n(\cdot, t)\|_{L^\infty(\Omega)} < \infty. \quad (5.14)$$

Proof. The inequality in (5.13) can be derived in quite a standard manner from the regularity properties obtained in Lemma 5.4 and Lemma 5.5 (see, e.g., [42, Proof of Theorem 1.1] or also [51, Lemma 4.3] for closely related precedents). As $D(A^\alpha) \hookrightarrow L^{12}(\Omega; \mathbb{R}^2)$ for any such α , this particularly entails the existence of $C_1 > 0$ such that $\|u\|_{L^{12}(\Omega)} \leq C_1$ for all $t \in (0, T_{\max})$, so that according to Lemma 5.6 we can find $C_2 > 0$ such that $h := \nabla c - u$ satisfies $\|h\|_{L^{12}(\Omega)} \leq C_2$ for all $t \in (0, T_{\max})$. Therefore, using the Hölder inequality along with (2.13) we can estimate

$$\|nh\|_{L^3(\Omega)} \leq \|n\|_{L^4(\Omega)} \|h\|_{L^{12}(\Omega)} \leq \|n\|_{L^\infty(\Omega)}^{\frac{3}{4}} \|n\|_{L^1(\Omega)}^{\frac{1}{4}} \|h\|_{L^{12}(\Omega)} \leq C_3 \|n\|_{L^\infty(\Omega)}^{\frac{3}{4}} \quad \text{for all } t \in (0, T_{\max})$$

with $C_3 := C_2 \|n_0\|_{L^1(\Omega)}^{\frac{1}{4}}$, so that employing known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ on Ω [13], we see that for some $C_4 > 0$ and all $t \in (0, T_{\max})$ we have

$$\begin{aligned} \|n(\cdot, t)\|_{L^\infty(\Omega)} &= \left\| e^{t(\Delta-1)} n_0 + \int_0^t e^{(t-s)(\Delta-1)} \nabla \cdot \left(n(\cdot, s) h(\cdot, s) \right) ds + \int_0^t e^{(t-s)(\Delta-1)} n(\cdot, s) ds \right\|_{L^\infty(\Omega)} \\ &\leq e^{-t} \|n_0\|_{L^\infty(\Omega)} + C_4 \int_0^t \left(1 + (t-s)^{-\frac{5}{6}} \right) e^{-(t-s)} \|n(\cdot, s) h(\cdot, s)\|_{L^3(\Omega)} ds \\ &\quad + C_4 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \right) e^{-(t-s)} \|n(\cdot, s)\|_{L^2(\Omega)} ds \\ &\leq \|n_0\|_{L^\infty(\Omega)} + C_3 C_4 C_5 \sup_{s \in (0, t)} \|n(\cdot, s)\|_{L^\infty(\Omega)}^{\frac{3}{4}} + C_4 C_6 \sup_{s \in (0, t)} \|n(\cdot, s)\|_{L^2(\Omega)} \end{aligned}$$

with $C_5 := \int_0^\infty (1 + \sigma^{-\frac{5}{6}}) e^{-\sigma} d\sigma$ and $C_6 := \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-\sigma} d\sigma$. Now since $C_7 := \sup_{t \in (0, T_{\max})} \|n(\cdot, t)\|_{L^2(\Omega)}$ is finite due to Lemma 5.5, this shows that writing $C_8 := \max\{\|n_0\|_{L^\infty(\Omega)} + C_4 C_6 C_7, C_3 C_4 C_5\}$ and $M(T) := \sup_{t \in (0, T)} \|n(\cdot, t)\|_{L^\infty(\Omega)}$, $T \in (0, T_{\max})$, we have

$$M(T) \leq C_8 + C_8 M^{\frac{3}{4}}(T) \quad \text{for all } T \in (0, T_{\max}).$$

This implies that $M(T) \leq \max\{1, (2C_8)^4\}$ for all $T \in (0, T_{\max})$ and hence completes the proof. \square

Our main goal has therefore been accomplished:

Proof of Theorem 1.1. The claim directly results from Lemma 5.7 and Lemma 5.6 upon an application of Lemma 2.2. \square

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