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Fractional nonlinear Schrödinger equation

Jesus A. Mendez-Navarro, Pavel I. Naumkin and Isahi Sánchez-Suárez

Abstract. We consider the Cauchy problem for the fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{2}{3}\left|\partial_x\right|^{\frac{3}{2}} u = \lambda \left|u\right|^2 u, \ t > 0, & x \in \mathbb{R}, \\ u\left(1, x\right) = u_0\left(x\right), & x \in \mathbb{R}. \end{cases}$$

We develop the factorization technique to obtain the large-time asymptotic behavior of solutions which has a logarithmic phase modifications for large time comparing with the linear problem.

Mathematics Subject Classification. 35B40, 35Q35.

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1. Introduction

We study the Cauchy problem for the fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{2}{3}\left|\partial_x\right|^{\frac{3}{2}} u = \lambda \left|u\right|^2 u, \ t > 0, & x \in \mathbb{R}, \\ u\left(0, x\right) = u_0\left(x\right), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, the fractional derivative $\left|\partial_x\right|^\alpha = \mathcal{F}^{-1}\left|\xi\right|^\alpha \mathcal{F}$, here and below $\mathcal{F}$ stands for the Fourier transform $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)\mathrm{d}x$, and $\mathcal{F}^{-1}$ is the inverse Fourier transformation $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$. Fractional nonlinear Schrödinger equation (1.1) appeared in [25,26] with applications in quantum mechanics. Later, it was derived in various areas such as plasma physics, optimization, finance, free boundary obstacle problems, population dynamics and minimal surfaces. The case of fractional derivative $\left|\partial_x\right|^{\frac{3}{2}}$ has a particular relevance to the two-dimensional water waves with surface tension (see [19,20]). Recently fractional nonlinear Schrödinger equations attracted much attention of many authors, (see [3,5,6,9–11,13,21,22,24] and references cited therein).

For the fractional nonlinear Schrödinger equations

$$i\partial_t u + \frac{1}{\alpha}\left|\partial_x\right|^\alpha u = \lambda\left|u\right|^2 u, \tag{1.2}$$

the local well posedness in $\mathbf{H}^s$ for $s \geq \frac{2-\alpha}{4}$ and ill posedness in $\mathbf{H}^s$ for $\frac{2-3\alpha}{4(\alpha+1)} < s < \frac{2-\alpha}{4}$, $1 < \alpha < 2$, were obtained in [6] through the multilinear estimates based on the Bourgain spaces. Hence, the result in [6] shows that global $\mathbf{L}^2$ well posedness fails for the cubic nonlinearity, when $1 < \alpha < 2$. In [17], the local well posedness and ill-posedness were also considered for (1.2) with $0 < \alpha < 2$, $\alpha \neq 1$ and general nonlinearity $\lambda\left|u\right|^{p-1}u$ in the scaling invariant Sobolev spaces $\mathbf{H}^{\frac{1}{2}-\frac{\alpha}{p-1}}$. In particular, the small data scattering in $\mathbf{H}^{\frac{1}{2}-\frac{\alpha}{p-1}}$ was shown for the case of $p \geq 5$. Cubic nonlinearities often require some logarithmic phase corrections in the large-time asymptotics comparing to the corresponding linear problem. Our purpose in the present paper is to show that the factorization technique originated in papers [14–16,27,28] can also be developed for the fractional nonlinear Schrödinger equation (1.1).

We introduce *Notation and Function Spaces.* $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$ is the usual Lebesgue space, and the norm is defined by $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p \, dx\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\right\},$$

$m, s \in \mathbb{R}, 1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}, \langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$.

To state our main result, we introduce the dilation operator $\mathcal{D}_t\phi(x) = t^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$, the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$ and the multiplication factor $M(t,\eta) = e^{\frac{it}{3}|\eta|^{\frac{3}{2}}}$, where the stationary point $\mu(x) = x|x|$. We are in a position to state the main result of this paper.

Theorem 1.1. *Let the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,1}$ have a small norm $\|u_0\|_{\mathbf{H}^2\cap\mathbf{H}^{0,1}}$. Then, there exists a unique global solution u of the Cauchy problem (1.1) such that $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^2 \cap \mathbf{H}^{0,1}\right)$. Also the time decay estimate $\|u(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-\frac{1}{2}}$ is true. Moreover, there exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ such that the asymptotics*

$$u(t) = \mathcal{D}_t\mathcal{B}MW_+ \exp\left(-i\lambda|W_+|^2\log t\right) + O\left(\varepsilon t^{-\delta}\right) \tag{1.3}$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$, where $\delta > 0$.

For the convenience of the reader, we now give a sketch of the proof. First, by using the factorization techniques, we change $u = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi}$ so that Eq. (1.1) takes the form $i\partial_t\widehat{\varphi} = \lambda t^{-1}\mathcal{Q}^*\left(|\mathcal{Q}\widehat{\varphi}|^2\mathcal{Q}\widehat{\varphi}\right)$, where the direct defect operator $\mathcal{Q}(t)\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi$, the conjugate defect operator $\mathcal{Q}^*(t)\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,d\eta$, the phase function $S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta)$ and the symbol $\Lambda(\xi) = \frac{2}{3}|\xi|^{\frac{3}{2}}$. Then, the most difficulty is to estimate derivatives of the defect operators $\mathcal{Q}$ and $\mathcal{Q}^*$. For this purpose, we apply the $\mathbf{L}^2$-theory of pseudodifferential operators.

The rest of the paper is organized as follows. In Sect. 2, we prove some preliminary estimates for the defect operators in the uniform metrics and $\mathbf{L}^2$-norm. Section 3 is devoted to the proof of the a priori estimates for the local solutions. We prove Theorem 1.1 in Sect. 4.

2. Preliminaries

2.1. Factorization techniques

Denote the symbol $\Lambda(\xi) = \frac{2}{3}|\xi|^{\frac{3}{2}}$, then the free evolution group has the form $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$. We have $\mathcal{U}(t)\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\left(\frac{x}{t}\xi - \Lambda(\xi)\right)}\phi(\xi)\,d\xi$. Consider the stationary point $\mu(x)$ defined by the equation $\Lambda'(\mu) = x$. Since $\Lambda''(\xi) = \frac{1}{2}|\xi|^{-\frac{1}{2}} > 0$, then $\Lambda'(\xi) = |\xi|^{-\frac{1}{2}}\xi$ is monotonous. Hence, there exists a unique stationary point $\mu(x) = x|x|$ such that $\Lambda'(\mu(x)) = x$ for all $x \in \mathbb{R}$. Then, we write

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{D}_t\sqrt{\frac{t}{2\pi}}e^{\frac{it}{3}|\eta|^{\frac{3}{2}}}\int_{\mathbb{R}} e^{-it(\Lambda(\xi)-\Lambda(\mu(x))-x(\xi-\mu(x)))}\phi(\xi)\,d\xi \\ &= \mathcal{D}_t\mathcal{B}M\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\phi,\end{aligned}$$

where the dilation operator $\mathcal{D}_t\phi(x) = t^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$, the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$, the multiplication factor $M(t,\eta) = e^{\frac{it}{3}|\eta|^{\frac{3}{2}}}$, the phase function $S(\xi,\eta) = \frac{2}{3}|\xi|^{\frac{3}{2}} - \frac{2}{3}|\eta|^{\frac{3}{2}} - |\eta|^{-\frac{1}{2}}\eta(\xi-\eta)$ and the defect operator $\mathcal{Q}(t)\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\,\mathrm{d}\xi$. Also, we define the conjugate defect operator $\mathcal{Q}^*(t)\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,\mathrm{d}\eta$. Thus, we have the representation for the free evolution group $\mathcal{U}(t)\mathcal{F}^{-1} = \mathcal{D}_t\mathcal{B}M\mathcal{Q}$ and for the inverse evolution group $\mathcal{F}\mathcal{U}(-t) = \mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}$ with the inverse scaling operator $\left(\mathcal{B}^{-1}\phi\right)(\eta) = \phi(\Lambda'(\eta))$ and the inverse dilation operator $\mathcal{D}_t^{-1}\phi(x) = t^{\frac{1}{2}}\phi(xt)$. We define the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = i\partial_t\mathcal{F}\mathcal{U}(-t)$, $\mathcal{L} = i\partial_t + \frac{2}{3}|\partial_x|^{\frac{3}{2}}$, applying the operator $\mathcal{F}\mathcal{U}(-t)$ to Eq. (1.1) and substituting $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi}$, we find the following factorization property

$$\begin{aligned} i\partial_t\widehat{\varphi} &= i\partial_t\mathcal{F}\mathcal{U}(-t)u = \mathcal{F}\mathcal{U}(-t)\mathcal{L}u = \mathcal{F}\mathcal{U}(-t)\left(\lambda|u|^2u\right) \\ &= t^{-1}\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\left(\lambda|\mathcal{B}M\mathcal{Q}\widehat{\varphi}|^2\mathcal{B}M\mathcal{Q}\widehat{\varphi}\right) = \lambda t^{-1}\mathcal{Q}^*|v|^2v, \end{aligned}$$

where $v = \mathcal{Q}\widehat{\varphi}$.

2.2. Estimates for defect operator $\mathcal{Q}$ in the uniform metrics

Define the kernel $A(t,\eta) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\langle\xi\rangle^{-1}\,\mathrm{d}\xi$, where $S(\xi,\eta) = \frac{2}{3}|\xi|^{\frac{3}{2}} - \frac{2}{3}|\eta|^{\frac{3}{2}} - |\eta|^{-\frac{1}{2}}\eta(\xi-\eta)$. We change $\xi = \eta y$, then

$$A(t,\eta) = |\eta|\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-it|\eta|^{\frac{3}{2}}\left(\frac{2}{3}|y|^{\frac{3}{2}}+\frac{1}{3}-y\right)}\langle\eta y\rangle^{-1}\,\mathrm{d}y.$$

To compute the asymptotics of the kernel $A(t,\eta)$ for large time, we apply the stationary phase method (see [12], p. 110)

$$\int_{\mathbb{R}} e^{itg(y)}f(y)\,\mathrm{d}y = e^{izg(y_0)}f(y_0)\sqrt{\frac{2\pi}{t|g''(y_0)|}}e^{i\frac{\pi}{4}\operatorname{sgn}g''(y_0)} + O\left(t^{-\frac{3}{2}}\right) \tag{2.1}$$

for $t\to\infty$, where the stationary point y_0 is defined by the equation $g'(y_0) = 0$. By virtue of formula (2.1) with $g(y) = -\left(\frac{2}{3}|y|^{\frac{3}{2}} + \frac{1}{3} - y\right)$, $f(y) = \langle\eta y\rangle^{-1}$, $y_0 = 1$, we get $A(t,\eta) = \frac{\langle\eta\rangle^{-1}}{\sqrt{i\Lambda''(\eta)}} + O\left(t^{\frac{1}{2}}|\eta|\left\langle t|\eta|^{\frac{3}{2}}\right\rangle^{-\frac{3}{2}}\right)$ for $t|\eta|^{\frac{3}{2}}\to\infty$. In the next lemma, we find the large-time asymptotics for the defect operator $\mathcal{Q}\phi$. Denote $\{\eta\} = \frac{|\eta|}{\langle\eta\rangle}$.

Lemma 2.1. *The estimate*

$$\|\mathcal{Q}\phi - A(t,\eta)\langle\eta\rangle\phi(\eta)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{4}}\left(t^{\gamma-\frac{1}{4}} + \{\eta\}^{\frac{3}{8}}\right)\left(\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{L}^2}\right)$$

is valid for all $t\ge 1$, where $\gamma > 0$ is small.

Proof. We integrate by parts via the identity $e^{-itS(\xi,\eta)} = H_1\partial_\xi\left((\xi-\eta)e^{-itS(\xi,\eta)}\right)$ with $H_1 = (1 - it(\xi-\eta)\partial_\xi S(\xi,\eta))^{-1}$

$$\begin{aligned} \mathcal{Q}\phi - A(t,\eta)\langle\eta\rangle\phi(\eta) = &-\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\frac{\langle\xi\rangle\phi(\xi) - \langle\eta\rangle\phi(\eta)}{\xi-\eta}(\xi-\eta)^2\partial_\xi\left(H_1\langle\xi\rangle^{-1}\right)\mathrm{d}\xi \\ &-\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}(\xi-\eta)H_1\langle\xi\rangle^{-1}\partial_\xi\left(\langle\xi\rangle\phi(\xi)\right)\mathrm{d}\xi. \end{aligned}$$

Since $\partial_{\xi} S(\xi,\eta) = |\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta$, we get the estimates

$$\left|(\xi-\eta) H_1 \langle\xi\rangle^{-1}\right| + \left|(\xi-\eta)^2 \partial_{\xi}\left(H_1 \langle\xi\rangle^{-1}\right)\right| \leq \frac{C|\xi-\eta|\langle\xi\rangle^{-1}}{1+t|\xi-\eta|\left||\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta\right|}.$$

Hence, by the Hardy inequality, we obtain

$$|\mathcal{Q}\phi - A(t,\eta)\langle\eta\rangle\phi(\eta)| \leq Ct^{\frac{1}{2}}\left(\|\langle\xi\rangle\phi_{\xi}\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{L}^2}\right) I^{\frac{1}{2}},$$

where

$$I = \int_{\mathbb{R}} \frac{(\xi-\eta)^2\langle\xi\rangle^{-2}\mathrm{d}\xi}{\left(1+t|\xi-\eta|\left||\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta\right|\right)^2}.$$

We have $\left||\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta\right| = \frac{|\xi-\eta|}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\theta(\xi\eta) + \left(|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}\right)\theta(-\xi\eta)$, where $\theta(x)$ is the Heaviside function. Hence, we find $I = I_1 + I_2$, where

$$I_1 = \int_{\mathbb{R}} \frac{(\xi-\eta)^2\langle\xi\rangle^{-2}\theta(\xi\eta)\mathrm{d}\xi}{\left(1+t\frac{(\xi-\eta)^2}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\right)^2}, I_2 = \int_{\mathbb{R}} \frac{(\xi-\eta)^2\langle\xi\rangle^{-2}\theta(-\xi\eta)\mathrm{d}\xi}{\left(1+t|\xi-\eta|\left||\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}\right|\right)^2}.$$

For $\eta > 0$, we have

$$\begin{aligned} I_1 &\leq \frac{C\eta^2}{1+t^2\eta^3}\int_0^{\frac{\eta}{2}}\mathrm{d}\xi + C\langle\eta\rangle^{-2}\int_{\frac{\eta}{2}}^{2\eta}\frac{(\xi-\eta)^2\mathrm{d}\xi}{\left(1+t\eta^{-\frac{1}{2}}(\xi-\eta)^2\right)^2} \\ &\quad + Ct^{2\gamma-2}\int_{2\eta}^{\infty}\frac{\mathrm{d}\xi}{\xi^{1-3\gamma}\langle\xi\rangle^2} \leq Ct^{2\gamma-2} + \frac{C\eta^3\langle\eta\rangle^{-2}}{\left\langle t\eta^{\frac{3}{2}}\right\rangle^{\frac{3}{2}}} \leq Ct^{2\gamma-2} + Ct^{-\frac{3}{2}}\{\eta\}^{\frac{3}{4}} \end{aligned}$$

and

$$I_2 \leq Ct^{2\gamma-2}\int_0^1\frac{\mathrm{d}\xi}{(|\xi|+|\eta|)^{1-3\gamma}} + Ct^{-2}\int_1^{\infty}\xi^{-3}\mathrm{d}\xi \leq Ct^{2\gamma-2}.$$

Thus, we get

$$|\mathcal{Q}\phi - A(t,\eta)\langle\eta\rangle\phi(\eta)| \leq C\left(t^{\gamma-\frac{1}{2}} + t^{-\frac{1}{4}}\{\eta\}^{\frac{3}{8}}\right)\left(\|\langle\xi\rangle\phi_{\xi}\|_{\mathbf{L}^2} + \|\phi\|_{\mathbf{L}^2}\right).$$

Lemma 2.1 is proved. □

2.3. Estimates for conjugate defect operator $\mathcal{Q}^*$ in the uniform metrics

Define $\chi_1(x) \in \mathbf{C}^4(\mathbb{R})$ such that $\chi_1(x) = 1$ for $|x| \leq \frac{1}{3}$ and $\chi_1(x) = 0$ for $|x| \geq \frac{2}{3}$, $\chi_2(x) = 1 - \chi_1(x)$. Denote the conjugate kernel

$$A^*(t,\xi) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\chi_2(\eta t)\chi_2\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\mathrm{d}\eta.$$

By virtue of formula (2.1) with $g(y) = S(\xi,y)$, $f(y) = \chi_2(yt)\chi_2\left(\frac{y}{\xi}\right)\Lambda''(y)$, $y_0 = \xi$, we obtain the asymptotics $A^*(t,\xi) = \chi_2(\xi t)\sqrt{i\Lambda''(\xi)}\left(1+O\left(t^{-1}\right)\right)$ for $t \to \infty$.

In the next lemma, we estimate the conjugate defect operator $\mathcal{Q}^*$.

Lemma 2.2. *The estimate*

$$\left\|\langle\xi\rangle^{-\frac{1}{8}}\left(\mathcal{Q}^*\phi - A^*\phi\right)\right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{4}}\left\||\eta|^{-\frac{1}{4}}\partial_\eta\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3}t^{-1}\right)} + Ct^{-\frac{1}{4}}\left\||\eta|^{-\frac{5}{4}}\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3}t^{-1}\right)} + C\|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}$$

is true for all $t \geq 1$.

Proof. We write

$$\begin{aligned}\mathcal{Q}^*\phi - A^*\phi &= \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\chi_1(\eta t)\Lambda''(\eta)\,\mathrm{d}\eta \\ &+ \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\chi_2(\eta t)\chi_1\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\,\mathrm{d}\eta \\ &+ \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\left(\phi(\eta)-\phi(\xi)\right)\chi_2(\eta t)\chi_2\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\,\mathrm{d}\eta = I_1 + I_2 + I_3.\end{aligned}$$

For the first integral, we have $|I_1| \leq Ct^{\frac{1}{2}}\|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}\int_{|\eta|\leq t^{-1}}|\eta|^{-\frac{1}{2}}\,\mathrm{d}\eta \leq C\|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}$. In the second integral, we integrate by parts via the identity $e^{itS(\xi,\eta)} = H_2\partial_\eta\left(\eta e^{itS(\xi,\eta)}\right)$ with $H_2 = \left(1 + it\eta\partial_\eta S(\xi,\eta)\right)^{-1}$, $\partial_\eta S(\xi,\eta) = \frac{1}{2}|\eta|^{-\frac{1}{2}}(\eta-\xi)$, then we get

$$\begin{aligned}I_2 &= -\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\eta\partial_\eta\left(H_2\chi_2(\eta t)\chi_1\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\right)\mathrm{d}\eta \\ &- \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\eta H_2\chi_2(\eta t)\chi_1\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\partial_\eta\phi(\eta)\,\mathrm{d}\eta.\end{aligned}$$

We have the estimate

$$\left|\eta H_2\chi_2(\eta t)\chi_1\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\right| + \left|\eta^2\partial_\eta\left(H_2\chi_2(\eta t)\chi_1\left(\frac{\eta}{\xi}\right)\Lambda''(\eta)\right)\right| \leq \frac{C|\eta|^{\frac{1}{2}}}{1+t|\eta|^{\frac{1}{2}}|\xi|}$$

in the domain $\frac{1}{3}t^{-1} \leq |\eta| \leq |\xi|$. Hence, we obtain

$$\begin{aligned}|I_2| &\leq Ct^{\frac{1}{2}}\int_{\frac{1}{3}t^{-1}\leq|\eta|\leq|\xi|}\frac{\left(|\phi(\eta)||\eta|^{-1} + |\partial_\eta\phi(\eta)|\right)|\eta|^{\frac{1}{2}}\,\mathrm{d}\eta}{1+t|\eta|^{\frac{1}{2}}|\xi|} \\ &\leq Ct^{\frac{1}{2}}\left(\left\||\eta|^{-\frac{5}{4}}\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} + \left\||\eta|^{-\frac{1}{4}}\partial_\eta\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)}\right)I_4^{\frac{1}{2}} \\ &\leq Ct^{-\frac{1}{3}}\left(\left\||\eta|^{-\frac{5}{4}}\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} + \left\||\eta|^{-\frac{1}{4}}\partial_\eta\phi\right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)}\right),\end{aligned}$$

since for $I_4 = \int_{\frac{1}{3}t^{-1}\leq|\eta|\leq|\xi|}\frac{|\eta|^{\frac{3}{2}}\,\mathrm{d}\eta}{\left(1+t|\eta|^{\frac{1}{2}}|\xi|\right)^2}$, we have the estimate

$$I_4 \leq C|\xi|^{\frac{5}{2}}\int_{|z|\leq 1}\frac{|z|^{\frac{3}{2}}\,\mathrm{d}z}{1+t^2|\xi|^3|z|} \leq Ct^{-\frac{5}{3}}\int_{|z|\leq 1}|z|^{\frac{2}{3}}\,\mathrm{d}z \leq Ct^{-\frac{5}{3}}.$$

Finally, in I_3, we integrate by parts via the identity $e^{itS(\xi,\eta)} = H_3 \partial_\eta \left((\eta - \xi) e^{itS(\xi,\eta)} \right)$ with $H_3 = (1 + it(\eta - \xi) \partial_\eta S(\xi, \eta))^{-1}$, then we get

$$I_3 = -\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \frac{\phi(\eta) - \phi(\xi)}{\eta - \xi} (\eta - \xi)^2 \partial_\eta \left(H_3 \chi_2(\eta t) \chi_2\left(\frac{\eta}{\xi}\right) \Lambda''(\eta) \right) \mathrm{d}\eta$$
$$- \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} (\eta - \xi) H_3 \chi_2(\eta t) \chi_2\left(\frac{\eta}{\xi}\right) \Lambda''(\eta) \partial_\eta \phi(\eta) \mathrm{d}\eta.$$

We find the estimates

$$\left| (\eta - \xi) H_3 \chi_2(\eta t) \chi_2\left(\frac{\eta}{\xi}\right) \Lambda''(\eta) \right| + \left| (\eta - \xi)^2 \partial_\eta \left(H_3 \chi_2(\eta t) \chi_2\left(\frac{\eta}{\xi}\right) \Lambda''(\eta) \right) \right| \leq \frac{C|\eta - \xi| |\eta|^{-\frac{1}{2}}}{1 + t|\eta|^{-\frac{1}{2}} (\eta - \xi)^2}$$

in the domain $\frac{1}{3} \max\left(t^{-1}, |\xi|\right) \leq |\eta|$. Then, we obtain

$$|I_3| \leq Ct^{\frac{1}{2}} \int_{\frac{1}{3}\max(t^{-1},|\xi|) \leq |\eta|} \left(\left| \frac{\phi(\eta) - \phi(\xi)}{\eta - \xi} \right| + |\partial_\eta \phi(\eta)| \right) \frac{|\eta - \xi| |\eta|^{-\frac{1}{2}} \mathrm{d}\eta}{1 + t|\eta|^{-\frac{1}{2}} (\eta - \xi)^2}$$
$$\leq Ct^{\frac{1}{2}} \left(\left\| |\eta|^{-\frac{1}{4}} \partial_\eta \phi \right\|_{\mathbf{L}^2\left(|\eta| \geq \frac{1}{3t}\right)} + \left\| |\eta|^{-\frac{5}{4}} \phi \right\|_{\mathbf{L}^2\left(|\eta| \geq \frac{1}{3t}\right)} \right) I_5^{\frac{1}{2}}$$
$$\leq C\langle \xi \rangle^{\frac{1}{8}} t^{-\frac{1}{4}} \left(\left\| |\eta|^{-\frac{1}{4}} \partial_\eta \phi \right\|_{\mathbf{L}^2\left(|\eta| \geq \frac{1}{3t}\right)} + \left\| |\eta|^{-\frac{5}{4}} \phi \right\|_{\mathbf{L}^2\left(|\eta| \geq \frac{1}{3t}\right)} \right),$$

since for $I_5 = \int_{\frac{1}{3}\max(t^{-1},|\xi|) \leq |\eta|} \frac{(\eta - \xi)^2 |\eta|^{-\frac{1}{2}} \mathrm{d}\eta}{\left(1 + t|\eta|^{-\frac{1}{2}} (\eta - \xi)^2\right)^2}$, we have

$$I_5 \leq C|\xi|^{\frac{5}{2}} \int_{\frac{1}{3} \leq |z| \leq 2} \frac{(z - 1)^2 \mathrm{d}z}{\left(1 + t|\xi|^{\frac{3}{2}} (z - 1)^2\right)^2} + C|\xi|^{\frac{5}{2}} \int_{|z| \geq 2} \frac{|z|^{\frac{3}{2}} \mathrm{d}z}{1 + t^2 |\xi|^3 |z|^3}$$
$$\leq Ct^{-\frac{3}{2}} |\xi|^{\frac{1}{4}} + Ct^{-\frac{5}{3}}.$$

Lemma 2.2 is proved. □

2.4. Estimates of pseudodifferential operators

There are many papers devoted to the $\mathbf{L}^2$-estimates of pseudodifferential operators (see, e.g., [2,7,8,18]). Below, we will need the following result on the $\mathbf{L}^2$-boundedness of pseudodifferential operator $\mathbf{a}(t, x, \mathbf{D}) \phi \equiv \int_{\mathbb{R}} e^{ix\xi} \mathbf{a}(t, x, \xi) \widehat{\phi}(\xi) \mathrm{d}\xi$. See [1] for the proof.

Lemma 2.3. *Let the symbol* $\mathbf{a}(t, x, \xi)$ *be such that* $\sup_{x,\xi \in \mathbb{R}, t \geq 1} \left| \frac{\langle \xi \rangle^\nu}{\{\xi\}^\nu} (\xi \partial_\xi)^k \mathbf{a}(t, x, \xi) \right| \leq C$ *for* $k = 0, 1, 2,$ *where* $\nu \in (0, 1)$*. Then* $\|\mathbf{a}(t, x, \mathbf{D}) \phi\|_{\mathbf{L}^2_x} \leq C \|\phi\|_{\mathbf{L}^2}$ *for all* $t \geq 1$*.*

Similarly, by considering the conjugate operator, we have.

Lemma 2.4. *Let the symbol* $\mathbf{a}(t, x, \xi)$ *be such that* $\sup_{x,\xi \in \mathbb{R}, t \geq 1} \left| \{x\}^{-\nu} \langle x \rangle^\nu (x \partial_x)^k \mathbf{a}(t, x, \xi) \right| \leq C$ *for* $k = 0, 1, 2,$ *where* $\nu \in (0, 1)$*. Then* $\|\mathbf{a}(t, x, \mathbf{D}) \phi\|_{\mathbf{L}^2_x} \leq C \|\phi\|_{\mathbf{L}^2}$ *for all* $t \geq 1$*.*

2.5. Estimate for derivative of $\mathcal{Q}$

Next, we consider $\mathbf{L}^2$-estimate for the derivative $\partial_\eta \mathcal{Q}$.

Lemma 2.5. *The estimate*

$$\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} \partial_\eta \mathcal{Q}\phi \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} \leq Ct^{2\nu} \|\xi\phi_\xi\|_{\mathbf{L}^2} + C\|\phi_\xi\|_{\mathbf{L}^2} + Ct^\nu \|\phi\|_{\mathbf{L}^2} + Ct^{\frac{2\nu}{1-4\nu}} \|\phi\|_{\mathbf{L}^\infty}$$

is true for all $t \geq 1$, where $\nu > 0$ is small.

Proof. Integrating by parts, we obtain

$$\partial_\eta \mathcal{Q}\phi = -\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)} \phi_\xi(\xi)\, \mathrm{d}\xi - \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \phi(\xi)\, \partial_\xi \left(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right) \mathrm{d}\xi.$$

Since $\partial_\xi S(\xi,\eta) = |\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta$ and $\partial_\eta S(\xi,\eta) = \frac{1}{2}|\eta|^{-\frac{1}{2}}(\eta-\xi)$, we get $-\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)} = \frac{1}{2}|\eta|^{-\frac{1}{2}}|\xi|^{\frac{1}{2}} + \frac{1}{2} + b(\xi,\eta)$, where $b(\xi,\eta) = -\frac{|\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\theta(-\xi\eta)$, $\theta(x)$ is the Heaviside function. Hence,

$$\partial_\eta \mathcal{Q}\phi = \frac{1}{2}|\eta|^{-\frac{1}{2}} \mathcal{Q}|\xi|^{\frac{1}{2}}\phi_\xi + \frac{1}{2}\mathcal{Q}\phi_\xi + \frac{1}{4}|\eta|^{-\frac{1}{2}}\mathcal{Q}|\xi|^{\frac{1}{2}}\xi^{-1}\phi + \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \partial_\xi (b(\xi,\eta)\phi(\xi))\, \mathrm{d}\xi.$$

Then, we represent

$$\begin{aligned}\partial_\eta \mathcal{Q}\phi &= \frac{1}{2}|\eta|^{-\frac{1}{2}} \mathcal{Q}|\xi|^{\frac{1}{2}}\phi_\xi + \frac{1}{2}\mathcal{Q}\phi_\xi \\ &+ \frac{1}{4}|\eta|^{-\frac{1}{2}}\mathcal{Q}|\xi|^{\frac{1}{2}}\xi^{-1}\left(\phi - \langle\xi\rangle^{-1}\phi(0)\right) + \frac{1}{4}\phi(0)|\eta|^{-\frac{1}{2}}\mathcal{Q}|\xi|^{\frac{1}{2}}\xi^{-1}\langle\xi\rangle^{-1} \\ &+ \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} b(\xi,\eta)\phi_\xi(\xi)\, \mathrm{d}\xi \\ &+ \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \langle\xi\rangle^{-\frac{1}{2}} \xi \partial_\xi b(\xi,\eta) \frac{\langle\xi\rangle^{\frac{1}{2}}\phi(\xi)-\phi(0)}{\xi} \mathrm{d}\xi \\ &+ \phi(0)\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)} \langle\xi\rangle^{-\frac{1}{2}} \partial_\xi b(\xi,\eta)\, \mathrm{d}\xi = \sum_{j=1}^{7} I_j.\end{aligned}$$

Note that $\left\| |\Lambda''|^{\frac{1}{2}} \mathcal{Q}\phi \right\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}$. Hence using inequality $t^\nu |\xi|^{\frac{1}{2}} \leq 1 + t^{2\nu}|\xi|$, we get

$$\begin{aligned}\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} I_1 \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} &\leq Ct^\nu \left\| |\Lambda''|^{\frac{1}{2}} \mathcal{Q}|\xi|^{\frac{1}{2}}\phi_\xi \right\|_{\mathbf{L}^2} \\ &= Ct^\nu \left\| |\xi|^{\frac{1}{2}}\phi_\xi \right\|_{\mathbf{L}^2} \leq C\|\phi_\xi\|_{\mathbf{L}^2} + Ct^{2\nu}\|\xi\phi_\xi\|_{\mathbf{L}^2}.\end{aligned}$$

Similarly $\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} I_2 \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} \leq C\left\| |\Lambda''|^{\frac{1}{2}} \mathcal{Q}\phi_\xi \right\|_{\mathbf{L}^2} \leq C\|\phi_\xi\|_{\mathbf{L}^2}$. Also we find

$$\begin{aligned}\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} I_3 \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} &\leq Ct^\nu \left\| |\Lambda''|^{\frac{1}{2}} \mathcal{Q}|\xi|^{\frac{1}{2}}\xi^{-1}\left(\phi - \langle\xi\rangle^{-1}\phi(0)\right) \right\|_{\mathbf{L}^2} \\ &\leq Ct^\nu \left\| |\xi|^{-\frac{1}{2}}\left(\phi - \langle\xi\rangle^{-1}\phi(0)\right) \right\|_{\mathbf{L}^2} \\ &\leq C\|\partial_\xi\phi\|_{\mathbf{L}^2} + Ct^\nu\|\phi\|_{\mathbf{L}^2} + Ct^\nu \log t\, \|\phi\|_{\mathbf{L}^\infty},\end{aligned}$$

since

$$\begin{aligned}\left\| |\xi|^{-\frac{1}{2}} \left(\phi - \langle \xi \rangle^{-1} \phi(0) \right) \right\|_{\mathbf{L}^2}^2 &\leq C \int_0^{t^{-2\nu}} \left(\phi(\xi) - \langle \xi \rangle^{-1} \phi(0) \right)^2 \frac{\mathrm{d}\xi}{\xi} + C \|\phi\|_{\mathbf{L}^\infty} \int_{t^{-2\nu}}^1 \frac{\mathrm{d}\xi}{\xi} \\ &\quad + C \int_1^\infty \left(|\phi(\xi)|^2 + \langle \xi \rangle^{-2} |\phi(0)|^2 \right) \frac{\mathrm{d}\xi}{\xi} \\ &\leq C t^{-2\nu} \|\partial_\xi \phi\|_{\mathbf{L}^2}^2 + C \log t \|\phi\|_{\mathbf{L}^\infty}^2 + C \|\phi\|_{\mathbf{L}^2}^2 .\end{aligned}$$

Since $\|\mathcal{Q}\phi\|_{\mathbf{L}^\infty} \leq C|t|^{\frac{1}{2}} \|\phi\|_{\mathbf{L}^1}$, then by the Riesz interpolation theorem (see [29], p. 52), we have $\left\| |\Lambda''|^{\frac{1}{p}} \mathcal{Q}(t) \phi \right\|_{\mathbf{L}^p} \leq C |t|^{\frac{1}{2}-\frac{1}{p}} \|\phi\|_{\mathbf{L}^{\frac{p}{p-1}}}$ for $2 \leq p \leq \infty$. Hence taking $p = 2 + \frac{8\nu}{1-8\nu}$, we find

$$\begin{aligned}&\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} I_4 \right\|_{\mathbf{L}^2\left(|\eta| \geq \frac{1}{3t}\right)} \\ &\leq C|\phi(0)| \left\| |\Lambda''|^{\frac{1}{2}-\frac{1}{p}} \{\eta\}^{-\nu} \langle \eta \rangle^{-\frac{1}{2}} \right\|_{\mathbf{L}^{\frac{2p}{p-2}}\left(|\eta| \geq \frac{1}{3t}\right)} \left\| |\Lambda''|^{\frac{1}{p}} \mathcal{Q} |\xi|^{\frac{1}{2}} \xi^{-1} \langle \xi \rangle^{-1} \right\|_{\mathbf{L}^p} \\ &\leq C|\phi(0)| |t|^{\frac{1}{2}-\frac{1}{p}} \left\| |\xi|^{\frac{1}{2}} \xi^{-1} \langle \xi \rangle^{-1} \right\|_{\mathbf{L}^{\frac{p}{p-1}}} \leq C|t|^{\frac{2\nu}{1-4\nu}} \|\phi\|_{\mathbf{L}^\infty}\end{aligned}$$

since $\left(\frac{1}{2} \left(\frac{1}{2} - \frac{1}{p} \right) + \nu \right) \frac{2p}{p-2} < 1$, in view of $p > 2 + \frac{8\nu}{1-4\nu}$. Next, we change $\eta = \mu(x)$, then we get

$$\{\eta\}^{\frac{1}{2}-\nu} I_5 = \sqrt{\frac{1}{2\pi}} \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1} \int_{\mathbb{R}} e^{ix\xi} b_1\left(\mu\left(xt^{-1} \right), \xi \right) e^{-it\Lambda(\xi)} \langle \xi \rangle^{\frac{1}{2}} \phi_\xi(\xi) \mathrm{d}\xi,$$

where $b_1(\xi, \eta) = \{\eta\}^{\frac{1}{2}-\nu} \langle \xi \rangle^{-\frac{1}{2}} b(\xi, \eta)$, and similarly

$$\{\eta\}^{\frac{1}{2}-\nu} I_6 = \sqrt{\frac{1}{2\pi}} \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1} \int_{\mathbb{R}} e^{ix\xi} b_2\left(\mu\left(xt^{-1} \right), \xi \right) e^{-it\Lambda(\xi)} \frac{\langle \xi \rangle^{\frac{1}{2}} \phi(\xi) - \phi(0)}{\xi} \mathrm{d}\xi,$$

where $b_2(\xi, \eta) = \{\eta\}^{\frac{1}{2}-\nu} \langle \xi \rangle^{-\frac{1}{2}} \xi \partial_\xi b(\xi, \eta)$. Define the pseudodifferential operators $\mathbf{a}_k(t, x, D) \phi \equiv \int_{\mathbb{R}} e^{ix\xi} \mathbf{a}_k (t, x, \xi) \widehat{\phi}(\xi) \mathrm{d}\xi$ with symbols $\mathbf{a}_k(t, x, \xi) = \sqrt{\frac{1}{2\pi}} b_k\left(\mu\left(xt^{-1} \right), \xi \right)$. Then, we get

$$\begin{aligned}\{\eta\}^{\frac{1}{2}-\nu} I_5 &= \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1} \mathbf{a}_1(t, x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \langle \xi \rangle^{\frac{1}{2}} \phi_\xi, \\ \{\eta\}^{\frac{1}{2}-\nu} I_6 &= \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1} \mathbf{a}_2(t, x, \mathbf{D}) \mathcal{F}^{-1} e^{-it\Lambda} \frac{\langle \xi \rangle^{\frac{1}{2}} \phi(\xi) - \phi(0)}{\xi}.\end{aligned}$$

Let us prove the $\mathbf{L}^2$-boundedness of the pseudodifferential operators $\mathbf{a}_k(t, x, D)$. We have

$$\mathbf{a}_k(t, x, \xi) = -\sqrt{\frac{1}{2\pi}} \langle \xi \rangle^{-\frac{1}{2}} \left(\xi \partial_\xi \right)^{k-1} \left(\frac{\{\eta\}^{\frac{1}{2}-\nu} |\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}}} \theta(-\xi\eta) \right) \Bigg|_{\eta = \mu(xt^{-1})}.$$

Hence, we obtain the estimates

$$\begin{aligned}&\left| \frac{\langle \xi \rangle^\nu}{\{\xi\}^\nu} \left(\xi \partial_\xi \right)^l \mathbf{a}_k(t, x, \xi) \right| \\ &= O\left(\frac{\langle \xi \rangle^\nu}{\{\xi\}^\nu} \left(\xi \partial_\xi \right)^l \langle \xi \rangle^{-\frac{1}{2}} \left(\xi \partial_\xi \right)^{k-1} \left(\frac{\{\eta\}^{\frac{1}{2}-\nu} |\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}} + |\eta|^{\frac{1}{2}}} \theta(-\xi\eta) \right) \Bigg|_{\eta = \mu(xt^{-1})} \right) \leq C\end{aligned}$$

for all $x,\xi\in\mathbb{R}, t\geq 1,\ k=0,1,2, k=1,2$, with $\nu>0$. Therefore, by Lemma 2.3, we find $\left\|\mathbf{a}_k\left(t,x,\mathbf{D}\right)\phi\right\|_{\mathbf{L}^2_x}\leq C\left\|\phi\right\|_{\mathbf{L}^2}, k=1,2$. Then, we obtain

$$\left\|\left|\Lambda''\right|^{\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}I_5\right\|_{\mathbf{L}^2_\eta}\leq\left\|\mathbf{a}_1\left(t,x,\mathbf{D}\right)\mathcal{F}^{-1}e^{-it\Lambda}\left\langle\xi\right\rangle^{\frac{1}{2}}\phi_\xi\right\|_{\mathbf{L}^2_x}$$
$$\leq C\left\|\mathcal{F}^{-1}e^{-it\Lambda}\left\langle\xi\right\rangle^{\frac{1}{2}}\phi_\xi\right\|_{\mathbf{L}^2}\leq C\left\|\left\langle\xi\right\rangle^{\frac{1}{2}}\phi_\xi\right\|_{\mathbf{L}^2}$$

and

$$\left\|\left|\Lambda''\right|^{\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}I_6\right\|_{\mathbf{L}^2_\eta}\leq\left\|\mathbf{a}_2\left(t,x,\mathbf{D}\right)\mathcal{F}^{-1}e^{-it\Lambda}\frac{\left\langle\xi\right\rangle^{\frac{1}{2}}\phi\left(\xi\right)-\phi\left(0\right)}{\xi}\right\|_{\mathbf{L}^2_x}$$
$$\leq C\left\|\frac{\left\langle\xi\right\rangle^{\frac{1}{2}}\phi\left(\xi\right)-\phi\left(0\right)}{\xi}\right\|_{\mathbf{L}^2}\leq C\left\|\left\langle\xi\right\rangle^{\frac{1}{2}}\phi_\xi\right\|_{\mathbf{L}^2}+C\left\|\phi\right\|_{\mathbf{L}^2}.$$

Finally, we need to estimate the integral I_7. Consider $\eta>0$, then we get

$$\left\{\eta\right\}^{\frac{1}{2}-\nu}I_7=-\phi\left(0\right)\sqrt{\frac{t}{2\pi}}\int_{-\infty}^0e^{-itS(\xi,\eta)}\left\langle\xi\right\rangle^{-\frac{1}{2}}\partial_\xi\frac{\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\xi\right|^{\frac{1}{2}}}{\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}}\mathrm{d}\xi.$$

We integrate by parts via the identity $e^{-itS(\xi,\eta)}=H_4\partial_\xi\left(\xi e^{-itS(\xi,\eta)}\right)$ with $H_4=\left(1-it\xi\partial_\xi S\left(\xi,\eta\right)\right)^{-1}$ with $\partial_\xi S\left(\xi,\eta\right)=\left|\xi\right|^{-\frac{1}{2}}\xi-\left|\eta\right|^{-\frac{1}{2}}\eta=-\left(\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}\right)$ for $\eta>0$, $\xi<0$. Therefore,

$$\left\{\eta\right\}^{\frac{1}{2}-\nu}I_7=Ct^{\frac{1}{2}}\phi\left(0\right)\int_{-\infty}^0e^{-itS(\xi,\eta)}\xi\partial_\xi\left(H_4\left\langle\xi\right\rangle^{-\frac{1}{2}}\partial_\xi\frac{\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\xi\right|^{\frac{1}{2}}}{\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}}\right)\mathrm{d}\xi.$$

We have

$$\left|\xi\partial_\xi\left(H_4\left\langle\xi\right\rangle^{-\frac{1}{2}}\partial_\xi\frac{\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\xi\right|^{\frac{1}{2}}}{\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}}\right)\right|\leq\frac{C\left\langle\xi\right\rangle^{-\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\xi\right|^{-\frac{1}{2}}}{\left(\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}\right)\left(1+t\left|\xi\right|\left(\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}\right)\right)}.$$

Hence changing $\xi=\eta z$, we get

$$\left\{\eta\right\}^{\frac{1}{2}-\nu}I_7\leq Ct^{\frac{1}{2}}\left|\phi\left(0\right)\right|\int_{-\infty}^0\frac{\left\langle\xi\right\rangle^{-\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\xi\right|^{-\frac{1}{2}}\mathrm{d}\xi}{\left(\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}\right)\left(1+t\left|\xi\right|\left(\left|\xi\right|^{\frac{1}{2}}+\left|\eta\right|^{\frac{1}{2}}\right)\right)}$$
$$\leq Ct^{\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\phi\left(0\right)\right|\int_{-\infty}^0\frac{\mathrm{d}z}{\left|z\right|^{\frac{1}{2}}\left(1+t\left|\eta\right|^{\frac{3}{2}}\left|z\right|\right)}$$
$$\leq Ct^{\frac{1}{2}}\left\{\eta\right\}^{\frac{1}{2}-\nu}\left|\phi\left(0\right)\right|\left\langle t\left|\eta\right|^{\frac{3}{2}}\right\rangle^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}-\nu} I_7 \right\|_{\mathbf{L}^2_\eta} &\leq Ct^{\frac{1}{2}} |\phi(0)| \left\| \frac{\{\eta\}^{\frac{1}{4}-\nu}}{\left\langle t|\eta|^{\frac{3}{2}} \right\rangle^{\frac{1}{2}}} \right\|_{\mathbf{L}^2_\eta} \\ &\leq Ct^{\frac{1}{2}} |\phi(0)| \left(\int_0^1 \frac{\eta^{\frac{1}{2}-2\nu}}{1+t|\eta|^{\frac{3}{2}}} \mathrm{d}\eta \right)^{\frac{1}{2}} + Ct^{\frac{1}{2}} |\phi(0)| \left(\int_1^\infty \frac{\mathrm{d}\eta}{t|\eta|^{\frac{3}{2}}} \right)^{\frac{1}{2}} \\ &\leq Ct^{\frac{2}{3}\nu} |\phi(0)| \left(\int_0^{t^{\frac{2}{3}}} \frac{\eta^{\frac{1}{2}-2\nu} \mathrm{d}\eta}{1+|\eta|^{\frac{3}{2}}} \right)^{\frac{1}{2}} + C|\phi(0)| \leq Ct^{\frac{2}{3}\nu} |\phi(0)|.\end{aligned}$$

Lemma 2.5 is proved. □

2.6. Estimate for derivative of $\mathcal{Q}^*$

In the next lemma, we estimate the derivative $\partial_\xi \mathcal{Q}^*$ in the domain $|\xi| \leq 1$.

Lemma 2.6. *The estimate*

$$\begin{aligned}\|\partial_\xi \mathcal{Q}^* \phi\|_{\mathbf{L}^2(|\xi|\leq 1)} &\leq C \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{-\nu} \partial_\eta \phi \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} + C \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{-1-\nu} \phi \right\|_{\mathbf{L}^2\left(|\eta|\geq \frac{1}{3t}\right)} \\ &\quad + Ct^{\frac{3}{4}} \|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}\end{aligned}$$

is true for all $t \geq 1$, where $\nu > 0$ is small.

Proof. Using $\chi_1(x) \in \mathbf{C}^4(\mathbb{R})$ such that $\chi_1(x) = 1$ for $|x| \leq \frac{1}{3}$ and $\chi_1(x) = 0$ for $|x| \geq \frac{2}{3}$, $\chi_2(x) = 1 - \chi_1(x)$, we write

$$\begin{aligned}\partial_\xi \mathcal{Q}^* \phi &= it\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \partial_\xi S(\xi,\eta) \phi(\eta) \chi_1(\eta t) \Lambda''(\eta) \mathrm{d}\eta \\ &\quad + it\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \partial_\xi S(\xi,\eta) \chi_2(\eta t) \phi(\eta) \Lambda''(\eta) \mathrm{d}\eta = I_1 + I_2.\end{aligned}$$

Note that $\|\mathcal{Q}^*\phi\|_{\mathbf{L}^2} = \left\| |\Lambda''|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^2}$. Then, the first integral, we can estimate as follows

$$\begin{aligned}\|I_1\|_{\mathbf{L}^2(|\xi|\leq 1)} &\leq Ct \|\Lambda' \mathcal{Q}^* \chi_1(\eta t) \phi\|_{\mathbf{L}^2(|\xi|\leq 1)} + Ct \|\mathcal{Q}^* \Lambda' \chi_1(\eta t) \phi\|_{\mathbf{L}^2(|\xi|\leq 1)} \\ &\leq Ct \left\| |\Lambda''|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^2(|\eta|\leq t^{-1})} \leq Ct \|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})} \left\| |\eta|^{-\frac{1}{4}} \right\|_{\mathbf{L}^2(|\eta|\leq t^{-1})} \\ &\leq Ct^{\frac{3}{4}} \|\phi\|_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}.\end{aligned}$$

Integrating by parts, we obtain for the second integral

$$\begin{aligned}I_2 &= -\sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} t\chi_2'(\eta t) \phi(\eta) \Lambda''(\eta) \mathrm{d}\eta \\ &\quad - \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} \chi_2(\eta t) \partial_\eta \phi(\eta) \Lambda''(\eta) \mathrm{d}\eta \\ &\quad - \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)} \chi_2(\eta t) \phi(\eta) \partial_\eta \left(\frac{\Lambda''(\eta) \partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} \right) \mathrm{d}\eta = \sum_{j=3}^{5} I_j.\end{aligned}$$

Since $\partial_\xi S(\xi,\eta) = |\xi|^{-\frac{1}{2}}\xi - |\eta|^{-\frac{1}{2}}\eta$ and $\partial_\eta S(\xi,\eta) = \Lambda''(\eta)(\eta-\xi)$, we get $-\frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} = 2 + B(\xi,\eta)$, where $B(\xi,\eta) = -\frac{2|\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\theta(\xi\eta) + \frac{2|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{1}{2}}-|\xi|^{\frac{1}{2}}\right)}{|\xi|+|\eta|}\theta(-\xi\eta)$. Also, we find $-\frac{1}{\Lambda''(\eta)}\partial_\eta \frac{\Lambda''(\eta)\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} = -\frac{1}{4} + B_1(\xi,\eta)$, where $B_1(\xi,\eta) = \frac{\frac{1}{4}|\xi|^{\frac{1}{2}}\left(2|\eta|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}}\right)}{\left(|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}\right)^2}\theta(\xi\eta) - \frac{\frac{1}{2}|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{3}{2}}-\frac{1}{2}|\xi|^{\frac{3}{2}}-\frac{3}{2}|\xi|^{\frac{1}{2}}|\eta|\right)}{(|\xi|+|\eta|)^2}\theta(-\xi\eta)$. Hence, we represent $I_4 = 2\mathcal{Q}^*\chi_2(\eta t)\partial_\eta\phi(\eta) + I_6$ and $I_5 = -\frac{1}{4}\mathcal{Q}^*\chi_2(\eta t)\frac{\phi(\eta)}{\eta} + I_7$, where

$$I_6 = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}B(\xi,\eta)\chi_2(\eta t)\partial_\eta\phi(\eta)\Lambda''(\eta)\,\mathrm{d}\eta,$$
$$I_7 = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}B_1(\xi,\eta)\chi_2(\eta t)\frac{\phi(\eta)}{\eta}\Lambda''(\eta)\,\mathrm{d}\eta.$$

The integral I_3 is estimated as I_1 above

$$\|I_3\|_{\mathbf{L}^2(|\xi|\le 1)} \le Ct\|\phi\|_{\mathbf{L}^\infty(|\eta|\le t^{-1})}\left\||\eta|^{-\frac{1}{4}}\right\|_{\mathbf{L}^2(|\eta|\le t^{-1})} \le Ct^{\frac{3}{4}}\|\phi\|_{\mathbf{L}^\infty(|\eta|\le t^{-1})}.$$

Also, we find

$$\|\mathcal{Q}^*\chi_2(\eta t)\partial_\eta\phi(\eta)\|_{\mathbf{L}^2(|\xi|\le 1)} \le \left\||\Lambda''|^{\frac{1}{2}}\partial_\eta\phi\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)}$$

and

$$\left\|\mathcal{Q}^*\chi_2(\eta t)\frac{\phi(\eta)}{\eta}\right\|_{\mathbf{L}^2} \le \left\||\Lambda''|^{\frac{1}{2}}\frac{\phi(\eta)}{\eta}\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)}.$$

Then changing the variable of integration $\eta = \mu(x)$, we get

$$\chi_1(3\xi)I_6 = e^{it\Lambda(\xi)}\int_{\mathbb{R}} e^{-ix\xi}b_1^*\left(\xi,\mu\left(xt^{-1}\right)\right)\left(\mathcal{D}_t\mathcal{B}M\{\eta\}^{-\nu}\chi_2(\eta t)\partial_\eta\phi\right)\mathrm{d}x,$$
$$\chi_1(3\xi)I_7 = e^{it\Lambda(\xi)}\int_{\mathbb{R}} e^{-ix\xi}b_2^*\left(\xi,\mu\left(xt^{-1}\right)\right)\left(\mathcal{D}_t\mathcal{B}M\{\eta\}^{-\nu}\chi_2(\eta t)\frac{\phi(\eta)}{\eta}\right)\mathrm{d}x,$$

where $b_1^*(\xi,\eta) = \sqrt{\frac{t}{2\pi}}\chi_1(3\xi)\{\eta\}^\nu B(\xi,\eta)$ and $b_2^*(\xi,\eta) = \sqrt{\frac{t}{2\pi}}\chi_1(3\xi)\{\eta\}^\nu B_1(\xi,\eta)$. We define the pseudodifferential operators $\mathbf{a}_k^*(t,\xi,\mathbf{D})\phi = \int_{\mathbb{R}} e^{-ix\xi}\mathbf{a}_k(t,x,\xi)\widehat{\phi}(x)\,\mathrm{d}x$, with symbols $\mathbf{a}_k(t,x,\xi) = b_k^*\left(\xi,\mu\left(xt^{-1}\right)\right)$, and then we get

$$\chi_1(3\xi)I_6 = e^{it\Lambda(\xi)}\mathbf{a}_1^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\{\eta\}^{-\nu}\chi_2(\eta t)\partial_\eta\phi(\eta),$$
$$\chi_1(3\xi)I_7 = e^{it\Lambda(\xi)}\mathbf{a}_2^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\chi_2(\eta t)\frac{\phi(\eta)}{\eta\{\eta\}^\nu}.$$

We prove the $\mathbf{L}^2$-boundedness of the pseudodifferential operators $\mathbf{a}_k^*(t,\xi,\mathbf{D})$, $k=1,2$. We have

$$\mathbf{a}_1(t,x,\xi) = -\sqrt{\frac{t}{2\pi}}\frac{2\chi_1(3\xi)|\xi|^{\frac{1}{2}}\{\eta\}^\nu}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\theta(\xi\eta)\bigg|_{\eta=\mu(xt^{-1})}$$
$$= \sqrt{\frac{t}{2\pi}}\frac{2\chi_1(3\xi)\{\eta\}^\nu|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{1}{2}}-|\xi|^{\frac{1}{2}}\right)}{|\xi|+|\eta|}\theta(-\xi\eta)\bigg|_{\eta=\mu(xt^{-1})}$$

and

$$\mathbf{a}_2(t,x,\xi) = \frac{1}{4}\sqrt{\frac{t}{2\pi}}\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}\left(2|\eta|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}}\right)}{\left(|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}\right)^2}\theta(\xi\eta)\Big|_{\eta=\mu(xt^{-1})} - \frac{1}{2}\sqrt{\frac{t}{2\pi}}\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{3}{2}}-\frac{1}{2}|\xi|^{\frac{3}{2}}-\frac{3}{2}|\xi|^{\frac{1}{2}}|\eta|\right)}{(|\xi|+|\eta|)^2}\theta(-\xi\eta)\Big|_{\eta=\mu(xt^{-1})}.$$

Then, we obtain the estimates

$$\left|\langle\xi\rangle^{\nu}\{\xi\}^{-\nu}(\xi\partial_\xi)^l\mathbf{a}_1(t,x,\xi)\right| = O\left(\{\xi\}^{-\nu}(\xi\partial_\xi)^l\left(\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}}\theta(\xi\eta)\right)\Big|_{\eta=\mu(xt^{-1})}\right) + O\left(\{\xi\}^{-\nu}(\xi\partial_\xi)^l\left(\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{1}{2}}-|\xi|^{\frac{1}{2}}\right)}{|\xi|+|\eta|}\theta(-\xi\eta)\right)\Big|_{\eta=\mu(xt^{-1})}\right) \le C$$

and

$$\left|\langle\xi\rangle^{\nu}\{\xi\}^{-\nu}(\xi\partial_\xi)^l\mathbf{a}_2(t,x,\xi)\right| = O\left(\{\xi\}^{-\nu}(\xi\partial_\xi)^l\left(\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}\left(2|\eta|^{\frac{1}{2}}+|\xi|^{\frac{1}{2}}\right)}{\left(|\xi|^{\frac{1}{2}}+|\eta|^{\frac{1}{2}}\right)^2}\theta(\xi\eta)\right)\Big|_{\eta=\mu(xt^{-1})}\right) + O\left(\{\xi\}^{-\nu}(\xi\partial_\xi)^l\left(\frac{\chi_1(3\xi)\{\eta\}^{\nu}|\xi|^{\frac{1}{2}}\left(|\eta|^{\frac{3}{2}}-\frac{1}{2}|\xi|^{\frac{3}{2}}-\frac{3}{2}|\xi|^{\frac{1}{2}}|\eta|\right)}{(|\xi|+|\eta|)^2}\theta(-\xi\eta)\right)\Big|_{\eta=\mu(xt^{-1})}\right) \le C$$

for all $x,\xi\in\mathbb{R}$, $t\ge 1$, $l=0,1,2$, with some small $\nu>0$. Therefore, applying Lemma 2.4, we find $\|\mathbf{a}_k(t,\xi,\mathbf{D})\phi\|_{\mathbf{L}^2_\xi} \le C\|\phi\|_{\mathbf{L}^2}$. Thus, we get

$$\|I_6\|_{\mathbf{L}^2(|\xi|\le 1)} \le \|\chi_1(3\xi)I_6\|_{\mathbf{L}^2} \le C\left\|\mathbf{a}_1^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\{\eta\}^{-\nu}\chi_2(\eta t)\partial_\eta\phi(\eta)\right\|_{\mathbf{L}^2} \le C\left\|\mathcal{D}_t\mathcal{B}M\{\eta\}^{-\nu}\chi_2(\eta t)\partial_\eta\phi(\eta)\right\|_{\mathbf{L}^2} \le C\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{-\nu}\partial_\eta\phi\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)}$$

and

$$\|I_7\|_{\mathbf{L}^2(|\xi|\le 1)} \le \|\chi_1(3\xi)I_7\|_{\mathbf{L}^2} \le \left\|\mathbf{a}_2^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\chi_2(\eta t)\frac{\phi(\eta)}{\eta\{\eta\}^{\nu}}\right\|_{\mathbf{L}^2} \le C\left\|\mathcal{D}_t\mathcal{B}M\chi_2(\eta t)\frac{\phi(\eta)}{\eta\{\eta\}^{\nu}}\right\|_{\mathbf{L}^2} \le C\left\||\Lambda''|^{\frac{1}{2}}\{\eta\}^{-\nu-1}\phi\right\|_{\mathbf{L}^2(|\eta|\ge t^{-1})}.$$

Lemma 2.6 is proved. □

3. A priori estimates

Define the norms $\|u\|_{\mathbf{Z}_T} = \sup_{t\in[0,T]} \|\langle\xi\rangle \widehat{\varphi}\|_{\mathbf{L}^\infty}$ and

$$\|u\|_{\mathbf{Y}_T} = \sup_{t\in[1,T]} \left(t^{-\gamma} \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + t^{-4\gamma} \|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + t^{-\gamma} \left\|\langle\xi\rangle^2 \widehat{\varphi}\right\|_{\mathbf{L}^2} \right),$$

where $\gamma > 0$ is small, $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)\,u(t)$. Also define $\|u\|_{\mathbf{X}_T} = \|u\|_{\mathbf{Z}_T} + \|u\|_{\mathbf{Y}_T}$.

We first state the local existence of solutions to the Cauchy problem (1.1) (see [4,23]).

Theorem 3.1. *Assume that the initial data $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{0,1}$ and the norm $\varepsilon = \|u_0\|_{\mathbf{H}^2\cap\mathbf{H}^{0,1}}$ is sufficiently small. Then, there exists a time $T > 1$ such that the Cauchy problem (1.1) has a unique solution $u \in \mathbf{C}\left([0,T];\mathbf{H}^2\cap\mathbf{H}^{0,1}\right)$ such that $\|u\|_{\mathbf{X}_T} \le C\varepsilon$.*

3.1. $\mathbf{L}^2$-norm

The next lemma gives us the estimate of the derivative $\partial_\xi\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)$ in the domain $|\xi| \le 1$.

Lemma 3.1. *The estimate $\left\|\partial_\xi\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)\right\|_{\mathbf{L}^2(|\xi|\le1)} \le Ct^{4\gamma-1}\|u\|^3_{\mathbf{X}_T}$ is true for all $t \ge 1$.*

Proof. By the factorization techniques, we have $\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right) = t^{-1}\mathcal{Q}^*|v|^2 v$, $v = \mathcal{Q}\widehat{\varphi}$. Then applying Lemma 2.6 with $\nu = \gamma$, we find

$$\begin{aligned}\left\|\partial_\xi\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)\right\|_{\mathbf{L}^2(|\xi|\le1)} &\le Ct^{-1}\left\|\{\eta\}^{-\frac14} v\right\|^2_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)} \left\||\Lambda''|^{\frac12}\{\eta\}^{\frac12-\gamma}\partial_\eta v\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)} \\ &\quad + Ct^{-1}\left\||\Lambda''|^{\frac12}\{\eta\}^{-1-\gamma}|v|^2 v\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)} \\ &\quad + Ct^{\gamma-\frac14}\|v\|^3_{\mathbf{L}^\infty(|\eta|\le t^{-1})}.\end{aligned}$$

Using Lemma 2.5 with $\nu = \gamma$, we get

$$\begin{aligned}\left\||\Lambda''|^{\frac12}\{\eta\}^{\frac12-\gamma}\partial_\eta v\right\|_{\mathbf{L}^2\left(|\eta|\ge\frac{1}{3t}\right)} &\le Ct^{2\gamma}\|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + C\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \\ &\quad + Ct^{\gamma}\|\widehat{\varphi}\|_{\mathbf{L}^2} + Ct^{3\gamma}\|\widehat{\varphi}\|_{\mathbf{L}^\infty} \le Ct^{4\gamma}\|u\|_{\mathbf{X}_T}.\end{aligned}$$

Next, we apply Lemma 2.1, to find

$$\left\|\{\eta\}^{-\frac14} v\right\|_{\mathbf{L}^\infty\left(|\eta|\ge\frac{1}{3t}\right)} \le C\|u\|_{\mathbf{Z}_T} + Ct^{4\gamma-\frac14}\|u\|_{\mathbf{Y}_T} \le C\|u\|_{\mathbf{X}_T}$$

and

$$\|v\|_{\mathbf{L}^\infty(|\eta|\le t^{-1})} \le Ct^{-\frac14}\|u\|_{\mathbf{Z}_T} + Ct^{5\gamma-\frac12}\|u\|_{\mathbf{Y}_T} \le Ct^{-\frac14}\|u\|_{\mathbf{X}_T}.$$

Hence,

$$\begin{aligned}\left\||\Lambda''|^{\frac12}\{\eta\}^{-1-\gamma}|v|^2 v\right\|_{\mathbf{L}^2(|\eta|\ge t^{-1})} &\le C\left\|\{\eta\}^{-\frac14} v\right\|^3_{\mathbf{L}^\infty\left(\frac{1}{3t}\le|\eta|\le1\right)} \left\|\{\eta\}^{-\frac12-\gamma}\right\|_{\mathbf{L}^2\left(\frac{1}{3t}\le|\eta|\le1\right)} \\ &\quad + C\|v\|^2_{\mathbf{L}^\infty(|\eta|>1)}\left\||\Lambda''|^{\frac12} v\right\|_{\mathbf{L}^2(|\eta|>1)} \le Ct^{\gamma}\|u\|^3_{\mathbf{X}_T}.\end{aligned}$$

Thus, we obtain $\left\|\partial_\xi\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)\right\|_{\mathbf{L}^2(|\xi|\le1)} \le Ct^{4\gamma-1}\|u\|^3_{\mathbf{X}_T}$. Lemma 3.1 is proved. □

We prove the a priori estimate in $\mathbf{Y}_T$ norm under the condition that the local solution is bounded in $\mathbf{Z}_T$.

Lemma 3.2. *Let u be the solution stated in Theorem 3.1 and $\|u\|_{\mathbf{Z}_T} \leq \varepsilon$. Then, the estimate $\|u\|_{\mathbf{Y}_T} < 6\varepsilon$ is true.*

Proof. We prove estimate of the lemma by a contradiction. By the continuity, we can find a time $T_1 < T$ such that $\|u\|_{\mathbf{Y}_{T_1}} = 6\varepsilon$. Thus, we have $\|u\|_{\mathbf{X}_{T_1}} \leq C\varepsilon$. Applying Lemma 2.1, we get

$$\|u\|_{\mathbf{L}^\infty} = t^{-\frac{1}{2}} \|\mathcal{Q}\widehat{\varphi}\|_{\mathbf{L}^\infty} \leq t^{-\frac{1}{2}} C \|u\|_{\mathbf{Z}_{T_1}} + Ct^{4\gamma-\frac{3}{4}} \|u\|_{\mathbf{Y}_{T_1}} \leq C\varepsilon t^{-\frac{1}{2}}.$$

for all $t \in [1, T_1]$. By the classical energy method, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{\mathbf{H}^2} \leq C \|u\|_{\mathbf{L}^\infty}^2 \|u\|_{\mathbf{H}^2} \leq C\varepsilon^2 \langle t\rangle^{-1} \|u\|_{\mathbf{H}^2}.$$

Hence integrating in time, we obtain $\|u\|_{\mathbf{H}^2} < 2\varepsilon \langle t\rangle^\gamma$. We mention some important identities. The operator $\mathcal{J} = \mathcal{U}(t) x \mathcal{U}(-t) = x + it|\partial_x|^{-\frac{1}{2}} \partial_x$ commutes with $\mathcal{L} = i\partial_t + \frac{2}{3}|\partial_x|^{\frac{3}{2}}$, i.e., $[\mathcal{J}, \mathcal{L}] = 0$. Since the symbol $\Lambda(\xi) = \frac{2}{3}|\xi|^{\frac{3}{2}}$ is homogeneous, we can use the operator $\mathcal{P} = x\partial_x + \frac{3}{2}t\partial_t$, which is related with $\mathcal{J}$ by the identity $\mathcal{P} = \mathcal{J}\partial_x - \frac{3}{2}it\mathcal{L}$. So we consider the estimate of $\|\mathcal{P}u\|_{\mathbf{L}^2}$. We have the commutator $[\mathcal{L}, \mathcal{P}] = \frac{3}{2}\mathcal{L}$, where $\mathcal{L} = i\partial_t + \frac{2}{3}|\partial_x|^{\frac{3}{2}}$. Applying $\mathcal{P}$ to Eq. (1.1) $\mathcal{L}u = \lambda|u|^2 u$ we get $\mathcal{L}\mathcal{P}u = \left(\mathcal{P} + \frac{3}{2}\right)\mathcal{L}u = \left(\mathcal{P} + \frac{3}{2}\right)\lambda|u|^2 u$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathcal{P}u\|_{\mathbf{L}^2} \leq C \|u\|_{\mathbf{L}^\infty}^2 \left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right) \leq C\varepsilon^2 t^{-1} \|\mathcal{P}u\|_{\mathbf{L}^2} + C\varepsilon^3 t^{\gamma-1}$$

for $t \in [1, T_1]$. Integration with respect to time yields $\|\mathcal{P}u\|_{\mathbf{L}^2} \leq \varepsilon + C\varepsilon^3 \langle t\rangle^\gamma$. Then by the identity $\mathcal{P} = \mathcal{J}\partial_x - \frac{3}{2}it\mathcal{L}$, we obtain

$$\|\partial_x \mathcal{J}u\|_{\mathbf{L}^2} \leq \|\mathcal{P}u\|_{\mathbf{L}^2} + Ct \|u\|_{\mathbf{L}^\infty}^2 \|u\|_{\mathbf{L}^2} < 2\varepsilon \langle t\rangle^\gamma.$$

Finally, let us estimate $\|\mathcal{J}u\|_{\mathbf{L}^2}$. Since

$$\|\mathcal{J}u\|_{\mathbf{L}^2} = \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\leq 1)} + \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\geq 1)} \leq \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\leq 1)} + \|\partial_x \mathcal{J}u\|_{\mathbf{L}^2},$$

then it is sufficient to estimate the norm $\|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\leq 1)}$. Multiplying equation (1.1) by $\mathcal{F}\mathcal{U}(-t)$, we obtain $i\partial_t \widehat{\varphi} = \lambda \mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)$ for the function $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. Differentiating we get $i\partial_t \partial_\xi \widehat{\varphi} = \lambda \partial_\xi \mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)$. Applying Lemma 3.1, we find $\left\|\partial_\xi \mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)\right\|_{\mathbf{L}^2(|\xi|\leq 1)} \leq C\varepsilon^3 t^{4\gamma-1}$ for $t \in [1, T_1]$. Then denoting the norm $y = \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\leq 1)}$ we get $\frac{\mathrm{d}y}{\mathrm{d}t} \leq C\varepsilon^3 t^{4\gamma-1}$. Integrating in time, we find $y = \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\leq 1)} < 2\varepsilon t^{4\gamma}$ for $t \in [1, T_1]$. Thus, we obtain $\|u\|_{\mathbf{Y}_{T_1}} < 6\varepsilon$, which yields a desired contradiction. Lemma 3.2 is proved. □

3.2. $\mathbf{L}^\infty$-norm

In the next lemma, we calculate the asymptotic representation for $\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right)$.

Lemma 3.3. *The asymptotic representation*

$$\mathcal{F}\mathcal{U}(-t)\left(|u|^2 u\right) = \frac{1}{t\Lambda''(\xi)} |\widehat{\varphi}(\xi)|^2 \widehat{\varphi}(\xi) + O\left(\langle\xi\rangle^{\frac{1}{8}} t^{12\gamma-\frac{5}{4}} \|u\|_{\mathbf{X}_T}^3\right)$$

is true for all $t \geq 1$, $\xi \in \mathbb{R}$, where $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$.

Proof. By the factorization property, we obtain $\mathcal{FU}(-t)\left(|u|^2 u\right) = t^{-1}\mathcal{Q}^* |v|^2 v$, where $v = \mathcal{Q}\widehat{\varphi}$. By virtue of Lemma 2.2, we get

$$\langle\xi\rangle^{-\frac{1}{8}} \mathcal{Q}^* |v|^2 v = \langle\xi\rangle^{-\frac{1}{8}} A^* |v|^2 v + O\left(t^{-\frac{1}{4}} \left\| |\eta|^{-\frac{1}{4}} \partial_\eta |v|^2 v \right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)}\right)$$
$$+ O\left(t^{-\frac{1}{4}} \left\| |\eta|^{-\frac{5}{4}} |v|^2 v \right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)}\right) + O\left(\|v\|^3_{\mathbf{L}^\infty(|\eta|\leq t^{-1})}\right).$$

As in the proof of Lemma 3.1, we get

$$\left\| |\eta|^{-\frac{1}{4}} \partial_\eta |v|^2 v \right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} \leq C \left\| \{\eta\}^{-\frac{1}{4}} v \right\|^2_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} \left\| |\Lambda''|^{\frac{1}{2}} \{\eta\}^{\frac{1}{2}} \partial_\eta v \right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} \leq Ct^{4\gamma} \|u\|^3_{\mathbf{X}_T},$$

$$\left\| |\eta|^{-\frac{5}{4}} |v|^2 v \right\|_{\mathbf{L}^2\left(|\eta|\geq\frac{1}{3t}\right)} \leq Ct^{\gamma} \|u\|^3_{\mathbf{X}_T}$$

and $\|v\|^3_{\mathbf{L}^\infty(|\eta|\leq t^{-1})} \leq Ct^{-\frac{3}{4}} \|u\|_{\mathbf{X}_T}$. Thus, we get in view of the asymptotics of the kernel A^*

$$\mathcal{Q}^* |v|^2 v = \sqrt{i\Lambda''(\xi)}\, |v|^2 v + O\left(t^{4\gamma-\frac{1}{4}} \langle\xi\rangle^{\frac{1}{8}} \|u\|^3_{\mathbf{X}_T}\right).$$

Then, we apply Lemma 2.1 to represent the first summand on the right-hand side of the above formula in the form

$$\sqrt{i\Lambda''(\xi)}\, |v|^2 v = \frac{1}{\Lambda''(\xi)} |\widehat{\varphi}(\xi)|^2 \widehat{\varphi}(\xi) + O\left(t^{12\gamma-\frac{1}{4}} \|u\|^3_{\mathbf{X}_T}\right).$$

Hence, we get

$$\mathcal{Q}^* |v|^2 v = \frac{1}{\Lambda''(\xi)} |\widehat{\varphi}(\xi)|^2 \widehat{\varphi}(\xi) + O\left(\langle\xi\rangle^{\frac{1}{8}} t^{12\gamma-\frac{1}{4}} \|u\|^3_{\mathbf{X}_T}\right).$$

Lemma 3.3 is proved. □

We next prove a priori estimate of the local solutions in $\mathbf{Z}_T$ norm under the boundedness condition in $\mathbf{Y}_T$.

Lemma 3.4. *Let u be the solution stated in Theorem 3.1 and $\|u\|_{\mathbf{Y}_T} \leq \varepsilon$. Then, the estimate $\|u\|_{\mathbf{Z}_T} < 2\varepsilon$ is true.*

Proof. In the domain $|\xi| \geq \langle t\rangle^\nu$, we get by the Sobolev embedding inequality

$$\|\langle\xi\rangle \widehat{\varphi}\|_{\mathbf{L}^\infty(|\xi|\geq\langle t\rangle^\nu)} \leq \sqrt{2}\, \|\langle\xi\rangle \widehat{\varphi}\|^{\frac{1}{2}}_{\mathbf{L}^2(|\xi|\geq\langle t\rangle^\nu)} \|\partial_\xi \langle\xi\rangle \widehat{\varphi}\|^{\frac{1}{2}}_{\mathbf{L}^2(|\xi|\geq\langle t\rangle^\nu)}$$
$$\leq \sqrt{2}\, \langle t\rangle^{-\frac{\nu}{2}} \left\| \langle\xi\rangle^2 \widehat{\varphi} \right\|^{\frac{1}{2}}_{\mathbf{L}^2(|\xi|\geq\langle t\rangle^\nu)} \|\partial_\xi \langle\xi\rangle \widehat{\varphi}\|^{\frac{1}{2}}_{\mathbf{L}^2(|\xi|\geq\langle t\rangle^\nu)} < 2\varepsilon \langle t\rangle^{-\frac{\nu}{2}+\frac{3}{2}\gamma} < 2\varepsilon$$

if $\frac{\nu}{2} > \frac{3}{2}\gamma$. Therefore, we need to estimate the function $\langle\xi\rangle \widehat{\varphi}(t,\xi)$ in the domain $|\xi| \leq \langle t\rangle^\nu$. Applying the operator $\mathcal{FU}(-t)$ to Eq. (1.1) $\mathcal{L}u = \lambda |u|^2 u$, we get $i\widehat{\varphi}_t(t,\xi) = \lambda \mathcal{FU}(-t)\left(|u|^2 u\right)$. By virtue of Lemma 3.3, we obtain

$$i\langle\xi\rangle \widehat{\varphi}_t(t,\xi) = \frac{\lambda}{\Lambda''(\xi)} |\widehat{\varphi}(\xi)|^2 \langle\xi\rangle \widehat{\varphi}(\xi) + O\left(\langle\xi\rangle^{\frac{5}{4}} t^{12\gamma-\frac{5}{4}} \|u\|^3_{\mathbf{X}_T}\right).$$

Multiplying by $\langle\xi\rangle \overline{\widehat{\varphi}_t(t,\xi)}$, and taking the imaginary part, we get $\frac{\mathrm{d}}{\mathrm{d}t} |\langle\xi\rangle \widehat{\varphi}(t,\xi)|^2 \leq C\varepsilon^4 t^{\frac{5}{4}\nu+12\gamma-\frac{5}{4}}$ in the domain $|\xi| \leq \langle t\rangle^\nu$. Define t_1 such that $\langle t_1\rangle^\nu = |\xi|$, then integrating in time from t_1 to t, we obtain

$$|\langle\xi\rangle \widehat{\varphi}(t,\xi)|^2 \leq |\langle\xi\rangle \widehat{\varphi}(t_1,\xi)|^2 + C\varepsilon^4 \int_{t_1}^t \tau^{\frac{5}{4}\nu+12\gamma-\frac{5}{4}} \mathrm{d}\tau < 2\varepsilon^2.$$

Lemma 3.4 is proved. □

4. Proof of Theorem 1.1

By Lemma 3.2, we see that a priori estimate of $\|u\|_{\mathbf{Z}_T}$ implies a priori estimate of $\|u\|_{\mathbf{Y}_T}$. On the other hand, by Lemma 3.4, a priori estimate of $\|u\|_{\mathbf{Y}_T}$ yields a priori estimate of $\|u\|_{\mathbf{Z}_T}$. Therefore, global existence of solutions of the Cauchy problem (1.1) satisfying estimates $\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$ follow by a standard continuation argument via the local existence Theorem 3.1. Thus, we have the global in time existence of solutions to the Cauchy problem (1.1).

Now, we turn to the proof of the asymptotic formula (1.3) for the solutions u of the Cauchy problem (1.1). By the factorization formula $u(t) = \mathcal{D}_t \mathcal{B} M \mathcal{Q} \widehat{\varphi}$ and Lemma 2.1, we find $u(t) = \mathcal{D}_t \mathcal{B} M \frac{1}{\sqrt{i\Lambda''}} \widehat{\varphi} + C\varepsilon t^{-\frac{1}{4}+3\gamma}$. As in the proof of Lemma 3.4, we get $\|\langle\xi\rangle \widehat{\varphi}\|_{\mathbf{L}^\infty(|\xi|\geq\langle t\rangle^\nu)} \leq C\varepsilon \langle t\rangle^{-\frac{\nu}{2}+\frac{3}{2}\gamma}$. So we need to compute the asymptotics of the function $\frac{1}{\sqrt{i\Lambda''}}\widehat{\varphi}$ in the domain $|\xi| \leq \langle t\rangle^\nu$. As in the proof of Lemma 3.4, we get

$$\frac{1}{\sqrt{i\Lambda''}} \partial_t \widehat{\varphi}(t,\xi) = -\frac{i\lambda}{\Lambda''(\xi)} |\widehat{\varphi}(\xi)|^2 \frac{1}{\sqrt{i\Lambda''}} \widehat{\varphi}(\xi) + O\left(\varepsilon^3 t^{\frac{1}{2}\nu+12\gamma-\frac{5}{4}}\right).$$

Then, we change the dependent variable $\frac{1}{\sqrt{i\Lambda''}}\widehat{\varphi}(t,\xi) = y(t,\xi)\Psi(t,\xi)$ with $\Psi(t,\xi) = \exp\left(-\frac{i\lambda}{\Lambda''(\xi)}\int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau}\right)$, to get $\partial_t y(t,\xi) = O\left(\varepsilon^3 t^{\frac{1}{2}\nu+12\gamma-\frac{5}{4}}\right)$. Integration in time yields $\|y(t)-y(s)\|_{\mathbf{L}^\infty} \leq C\int_s^t \left(\varepsilon^3 \tau^{\frac{1}{2}\nu+12\gamma-\frac{5}{4}}\right) d\tau \leq C\varepsilon s^{-\delta_1}$ for all $t > s > 0$, with $\delta_1 = \frac{1}{4} - \frac{1}{2}\nu - 12\gamma > 0$. Therefore, there exists a unique final state $y_+ \in \mathbf{L}^\infty$ such that $\|y(t)-y_+\|_{\mathbf{L}^\infty} \leq C\varepsilon t^{-\delta_1}$ for all $t > 0$. Since $\left|\frac{1}{\sqrt{i\Lambda''}}\widehat{\varphi}(\tau,\xi)\right|^2 = |y(t,\xi)|^2$, we have $\frac{1}{\Lambda''(\xi)}\int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau} = \int_1^t |y(\tau,\xi)|^2 \frac{d\tau}{\tau}$. Denote $\Phi(t) = \int_1^t |y(\tau)|^2 \frac{d\tau}{\tau} - |y_+|^2 \log t$. We study the asymptotics in time of the remainder term $\Phi(t)$. We have

$$\Phi(t) - \Phi(s) = \int_s^t \left(|y(\tau)|^2 - |y(t)|^2\right)\frac{d\tau}{\tau} + \left(|y(t)|^2 - |y_+|^2\right)\log\frac{t}{s}$$

and $\|\Phi(t)-\Phi(s)\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 s^{-\delta_1}$ for all $t > s > 0$. Hence, there exists a unique real-valued function Φ_+ such that $\Phi_+ \in \mathbf{L}^\infty$ and $\|\Phi(t)-\Phi_+\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 t^{-\delta_1}$. Therefore, we obtain

$$\frac{1}{\Lambda''(\xi)}\int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau} = \int_1^t |y(\tau,\xi)|^2 \frac{d\tau}{\tau} = \Phi_+ + |y_+|^2 \log t + O\left(\varepsilon^2 t^{-\delta_1}\right)$$

for all $t > 0$. Then, we obtain $\left\|\Psi(t,\xi) - \exp\left(-i\lambda |y_+|^2 \log t - i\lambda\Phi_+ + O\left(\varepsilon^2 t^{-\delta_1}\right)\right)\right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 t^{-\delta_1}$ for all $t > 0$. Thus, we get the large-time asymptotics

$$\begin{aligned}\frac{1}{\sqrt{i\Lambda''}}\widehat{\varphi}(t,\xi) &= y(t,\xi)\Psi(t,\xi) = y_+\Psi(t,\xi) + O\left(\varepsilon t^{-\delta_1}\right)\\ &= W_+ \exp\left(-i\lambda |W_+|^2 \log t\right) + O\left(\varepsilon t^{-\delta_1}\right),\end{aligned}$$

where $W_+ = y_+ \exp(-i\lambda\Phi_+)$. Note that $W_+ \in \mathbf{L}^\infty$. Using the factorization of $\mathcal{U}(t)$, we have

$$u(t) = \mathcal{D}_t \mathcal{B} M \frac{1}{\sqrt{i\Lambda''}}\widehat{\varphi} + C\varepsilon t^{-\frac{1}{4}+3\gamma} = \mathcal{D}_t \mathcal{B} M W_+ \exp\left(-i\lambda |W_+|^2 \log t\right) + O\left(\varepsilon t^{-\delta_1}\right).$$

This completes the proof of the asymptotics (1.3). Theorem 1.1 is proved.

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References

[1] Bernal-Vílchis, F., Naumkin, P.I.: Self-similar asymptotics for solutions to the intermediate long-wave equation. J. Evol. Equ. **19**, 729–770 (2019)
[2] Calderon, A.P., Vaillancourt, R.: A class of bounded pseudo-differential operators. Proc. Nat. Acad. Sci. USA **69**, 1185–1187 (1972)
[3] Cai, D., Majda, A.J., McLaughlin, D.W., Tabak, E.G.: Dispersive wave turbulence in one dimension. Phys. D **152**, 551–572 (2001)
[4] Cazenave, T.: Semilinear Schrödinger Equations. Courant Institute of Mathematical Sciences. American Mathematical Society, New York (2003)
[5] Cho, Y., Hajaiej, H., Hwang, G., Ozawa, T.: On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity. Funkc. Ekvac. **56**(2), 193–224 (2013)
[6] Cho, Y., Hwang, G., Kwon, S., Lee, S.: Profile decompositions and blowup phenomena of mass critical fractional Schrödinger equations. Nonlinear Anal. **86**, 12–29 (2013)
[7] Coifman, R.R., Meyer, Y.: Au dela des operateurs pseudo-differentiels, p. 185. Societe Mathematique de France, Paris (1978)
[8] Cordes, H.O.: On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators. J. Funct. Anal. **18**, 115–131 (1975)
[9] Dinh, V.D.: Blow-up criteria for fractional nonlinear Schrödinger equations. Nonlinear Anal. Real World Appl. **48**, 117–140 (2019)
[10] Esquivel, L., Kaikina, E.I.: A forced fractional Schrödinger equation with a Neumann boundary condition. Nonlinearity **29**(7), 2082–2111 (2016)
[11] Esquivel, L., Kaikina, E.I.: Robin initial-boundary value problem for nonlinear Schrodinger equation with potential. J. Evol. Equ. **18**(2), 583–613 (2018)
[12] Fedoryuk, M.V.: Asymptotics: Integrals and Series. Mathematical Reference Library, p. 544. Nauka, Moscow (1987)
[13] Guo, B., Han, Z.: Global well-posedness for the fractional nonlinear Schrödinger equation. Commun. Partial Differ. Equ. **36**(2), 247–255 (2010)
[14] Hayashi, N., Naumkin, P.I.: The initial value problem for the cubic nonlinear Klein–Gordon equation. Z. Angew. Math. Phys. **59**(6), 1002–1028 (2008)
[15] Hayashi, N., Naumkin, P.I.: Large time asymptotics for the fractional order cubic nonlinear Schrödinger equations. Ann. Henri Poincaré **18**(3), 1025–1054 (2017)
[16] Hayashi, N., Ozawa, T.: Scattering theory in the weighted $L^2(R^n)$ spaces for some Schrödinger equations. Ann. I.H.P. (Phys. Théor.) **48**, 17–37 (1988)
[17] Hong, Y., Sire, Y.: On fractional Schrödinger equations in Sobolev spaces. Commun. Pure Appl. Anal. **14**(6), 2265–2282 (2015)
[18] Hwang, I.L.: The L^2-boundedness of pseudodifferential operators. Trans. Am. Math. Soc. **302**(1), 55–76 (1987)
[19] Ionescu, A., Pusateri, F.: Global analysis of a model for capillary water waves in two dimensions. Commun. Pure Appl. Math. **69**(11), 2015–2071 (2016)
[20] Ionescu, A., Pusateri, F.: Global regularity for 2D water waves with surface tension. Memory Am. Math. Soc. **256**(1227), v+124 (2018)
[21] Kaikina, E.I.: Nonlinear fractional Schrödinger equation on a half-line. J. Math. Phys. **56**(9), 091511 (2015)
[22] Kenig, C.E., Ponce, G., Vega, L.: Oscillatory integrals and regularity of dispersive equations. Indiana Univ. Math. J. **40**, 33–69 (1991)
[23] Kenig, C.E., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de-Vries equation via the contraction principle. Commun. Pure App. Math. **46**(4), 527–620 (1993)
[24] Krieger, J., Lenzmann, E., Raphael, P.: Nondispersive solutions to the L^2-critical half-wave equation. Arch. Ration. Mech. Anal. **209**(1), 61–129 (2013)
[25] Laskin, N.: Fractional quantum mechanics and Levy path integrals. Phys. Lett. A **268**, 298–305 (2000)
[26] Laskin, N.: Fractional Schrödinger equation. Phys. Rev. E **66**(5) Article ID 056108 (2002)

[27] Naumkin, I.P.: Sharp asymptotic behavior of solutions for cubic nonlinear Schrödinger equations with a potential. J. Math. Phys. **57**, 051501 (2016). https://doi.org/10.1063/1.4948743
[28] Naumkin, I.P.: Nonlinear Schrödinger equations with exceptional potentials. J. Differ. Equ. **265**(9), 4575–4631 (2018)
[29] Stein, E.M., Shakarchi, R.: Functional Analysis: Introduction to Further Topics in Analysis. Princeton Lectures in Analysis. Princeton University Press, Princeton (2011)

Jesus A. Mendez-Navarro and Pavel I. Naumkin
Centro de Ciencias Matemáticas
UNAM Campus Morelia
AP 61-3 (Xangari)
Morelia CP 58089 Michoacán
Mexico
e-mail: pavelni@matmor.unam.mx

Jesus A. Mendez-Navarro
e-mail: jam_arturo@hotmail.com

Isahi Sánchez-Suárez
Universidad Politécnica de Uruapan
Uruapan CP 60210 Michoacán
Mexico
e-mail: isahi_ss@hotmail.com

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