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On the Cauchy problem of compressible full Hall-MHD equations

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Abstract. This paper is concerned with an initial value problem of the compressible full Hall-MHD equations in three-dimensional whole space. Both the global existence and the optimal decay rates of solutions are obtained, when the smooth initial data are sufficiently close to the non-vacuum equilibrium in H^1. As a by-product of the uniform estimates, the vanishing limit of Hall coefficient is also justified.

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1. Introduction

The Hall effect was firstly discovered by Edwin Hall in 1879 (cf. [19]) and is the production of a voltage difference (the Hall voltage) across an electrical conductor, transverse to an electric current in the conductor and to an applied magnetic field perpendicular to the current. It restores the influence of the electric current in the Lorentz force occurring in Ohm's law. The motion of a conducting fluid with Hall effects in magnetic field is governed by the mathematical model of Hall magnetohydrodynamics (for short, Hall-MHD, cf. [1,23]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u + (\nabla \times b) \times b, \\ \mathrm{c_V}\left[(\rho\theta)_t + \operatorname{div}(\rho u \theta)\right] + p \operatorname{div} u = \kappa \Delta \theta + \dfrac{\mu}{2}\left|\nabla u + (\nabla u)^\top\right|^2 + \lambda (\operatorname{div} u)^2 + \nu |\nabla \times b|^2, \\ b_t - \nabla \times (u \times b) + \varepsilon \nabla \times \left(\rho^{-1}(\nabla \times b) \times b\right) = \nu \Delta b, \\ \operatorname{div} b = 0, \end{cases} \tag{1.1}$$

where the unknown functions $\rho, u \in \mathbb{R}^3, \theta$ and $b \in \mathbb{R}^3$ are the density of the fluid, velocity, temperature and magnetic field, respectively. In this paper, we focus our interest on the polytropic ideal fluids which satisfy the equations of states $p = \mathrm{R}\rho\theta$ and $e = \mathrm{c_V}\theta$, where p and e denote the pressure and internal energy of the fluids, respectively. The constants μ and λ are the viscosity coefficients, $\kappa > 0$ is the heat conductivity, $\nu > 0$ is the resistivity coefficient acting as the diffusion coefficient of the magnetic field, $\varepsilon > 0$ is the Hall coefficient, while R and $\mathrm{c_V}$ are the perfect gas constant and the specific heat at constant volume, respectively. For simplicity, we assume throughout this paper that the physical constants $\kappa = \nu = \mathrm{R} = \mathrm{c_V} \equiv 1$. Moreover, the viscosity coefficients μ and λ are assumed to satisfy the physical conditions:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0,$$

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which ensures that (1.1)$_2$ is a Láme system.

In this paper, we consider an initial value problem of (1.1) with the far-field behavior

$$(\rho, u, \theta, b)(x,t) \to (\rho_\infty, 0, \theta_\infty, 0) \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

and the initial conditions

$$(\rho, u, \theta, b)|_{t=0} = (\rho_0, u_0, \theta_0, b_0)(x), \quad x \in \mathbb{R}^3. \tag{1.3}$$

The Hall-MHD system (1.1) can be derived from fluid mechanics with appropriate modifications to account for electrical forces and Hall effects, and is believed to be an essential feature in the problems of magnetic reconnection in space plasmas, star formulation, neutron stars and geo-dynamo (see, for example, [3,16,20,23,30,33]). From the mathematical point of view, the Hall term is quadratic in magnetic field and involves second-order derivatives. So, the mathematical analysis of Hall-MHD is more complicated than that of the conventional MHD system, where the Hall effect is small and can be neglected (e.g., in the laminar situation). We refer to [15,21,26,27,29,31] and among others for the studies of compressible MHD equations without Hall term. In the past few years, the incompressible Hall-MHD system has been studied by Chae and his coauthors (cf. [4–9] and among others), where the global weak/local strong solutions with large data, the global strong solutions with small data, the blow-up criteria and the singularity analysis results were obtained. Compared with the incompressible case, the compressible Hall-MHD system receives less attention from mathematicians. For the compressible isentropic case, the global existence and asymptotic behavior were proved in [13,18,34], when the initial data are sufficiently close to the non-vacuum equilibrium. The aim of this paper is to treat the compressible heat-conductive case and to study the large-time behavior of global classical solutions to the initial value problem (1.1)–(1.3).

Without loss of generality, it is assumed that $\rho_\infty = \theta_\infty = 1$. Then, the main result of this paper can be stated as follows.

Theorem 1.1. (I. Global Existence) *For any given positive number $M_0 > 0$ (not necessarily small), assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, b_0) \in H^3$ satisfy*

$$\|\nabla^2(\rho_0 - 1, u_0, \theta_0 - 1, b_0)\|_{H^1} \le M_0, \tag{1.4}$$

then there exists a positive constant $\eta_0 > 0$, depending on M_0, such that the problem (1.1)–(1.3) has a unique global solution (ρ, u, θ, b) on $\mathbb{R}^3 \times (0, \infty)$ satisfying $\inf_{x,t} \rho(x,t) > 0$, $\inf_{x,t} \theta(x,t) > 0$, and

$$\begin{aligned} &\|(\rho - 1, u, \theta - 1, b)(t)\|_{H^3}^2 + \int_0^t (\|\nabla \rho(\tau)\|_{H^2}^2 + \|\nabla(u, \theta, b)(\tau)\|_{H^3}^2) d\tau \\ &\le C\|(\rho_0 - 1, u_0, \theta_0 - 1, b_0)\|_{H^3}^2 \end{aligned} \tag{1.5}$$

for all $t \ge 0$, provided

$$\|(\rho_0 - 1, u_0, \theta_0 - 1, b_0)\|_{H^1} \le \eta_0, \tag{1.6}$$

where $C > 0$ is a generic positive constant independent of t.

(II. Decay Rates) *Assume further that $\Lambda \triangleq \|(\rho_0 - 1, u_0, \theta_0 - 1, b_0)\|_{L^1}$ is bounded. Then, there exists a positive constant $0 < \eta_1 \le \eta_0$, depending on M_0 and Λ, such that for any $t \ge 0$*

$$\|\nabla^k(\rho - 1, u, \theta - 1, b)(t)\|_{L^2} \le C(1 + t)^{-\left(\frac{3}{4} + \frac{k}{2}\right)}, \quad k = 0, 1, \tag{1.7}$$

and that for any $t \ge \bar{T}$ with $\bar{T} > 0$ being large enough and depending on M_0, Λ,

$$\|\nabla^2(\rho - 1, u, \theta - 1)(t)\|_{H^1} \le C(1 + t)^{-\frac{7}{4}}, \tag{1.8}$$

$$\|\nabla^k b(t)\|_{L^2} \le C(1 + t)^{-\left(\frac{3}{4} + \frac{k}{2}\right)}, \quad k = 2, 3, \tag{1.9}$$

provided $\|(\rho_0 - 1, u_0, \theta_0 - 1, b_0)\|_{H^1} \le \eta_1$.

Remark 1.1. Compared with the decay estimates of linearized system (cf. Lemma 2.1), the decay rates stated in the second part of Theorem 1.1 are optimal, except the one of $\|\nabla^3(\rho-1,u,\theta-1)\|_{L^2}$. Indeed, if the initial data are more regular, say $(\rho_0-1,u_0,\theta_0-1,b_0)\in H^k$ with $k\geq 4$, then one can obtain the optimal decay rates of the solutions up to the $(k-1)$th order derivatives of $(\rho-1,u,\theta-1)$ and the kth order derivatives of b. The lack of the optimal decay estimates of $\|\nabla^k(\rho-1,u,\theta-1)\|_{L^2}$ is mainly due to the insufficient dissipation of density and the strong coupling of fluid quantities.

As a by-product, we can study the vanishing limit of Hall coefficient (i.e., $\varepsilon\to 0$).

Theorem 1.2. *Let (ρ,u,θ,b) be the classical solution of Hall-MHD system (1.1) obtained in Theorem 1.1. Then, as $\varepsilon\to 0$, it holds that*

$$(\rho,u,\theta,b)\to(\bar\rho,\bar u,\bar\theta,\bar b) \quad \textit{strongly in} \quad C([0,T];H^2) \tag{1.10}$$

for any $0<T<\infty$, where $(\bar\rho,\bar u,\bar\theta,\bar b)$ is the classical solution of MHD system (4.1) without Hall effects (cf. Theorem 4.1). Moreover,

$$\sup_{0\leq t\leq T}\|(\rho-\bar\rho,u-\bar u,\theta-\bar\theta,b-\bar b)(t)\|_{H^2}^2\leq C(T)\varepsilon^2, \tag{1.11}$$

where $C(T)$ is a positive constant depending on T.

The proofs of Theorems 1.1 and 1.2 are similar to the ones in [34,35], based on the standard L^2-method and the origin ideas developed by Matsumura-Nishida [28]. It is worth noting that though the H^1-perturbation of initial data is small, the higher-order derivatives could be of large oscillations. Compared with the results in [11,35] where the authors only obtained the optimal decay estimates for the L^p-norm $(2\leq p\leq 6)$ of the solution and the L^2-norm of its first-order derivative, the decay rates of both $\|\nabla^2(\rho,u,\theta,b)(t)\|_{L^2}$ and $\|\nabla^3 b(t)\|_{L^2}$ for large $t>0$ are also optimal in the present paper. This will be achieved by making a full use of the H^1-decay estimates and the Sobolev interpolation inequality (see (3.74) and (3.75)). The key point here is that all the estimates are uniform in the Hall coefficient ε.

The rest of this paper is organized as follows. In Sect. 2, we first reformulate the problem and recall some known results about the linear system. Theorems 1.1 and 1.2 will be shown in Sects. 3 and 4, respectively, based on the global uniform-in-ε estimates.

2. Reformulation

In order to estimate the solutions, we set

$$\phi\triangleq\rho-1,\quad \psi\triangleq\theta-1 \quad \text{with} \quad \phi_0\triangleq\rho_0-1,\quad \psi_0\triangleq\theta_0-1.$$

Then, the system (1.1)–(1.3) can be reformulated in the form:

$$\begin{cases} \phi_t+\operatorname{div}u=S_1,\\ u_t-\mu\Delta u-(\mu+\lambda)\nabla\operatorname{div}u+\nabla\phi+\nabla\psi=S_2,\\ \psi_t-\Delta\psi+\operatorname{div}u=S_3,\\ b_t-\Delta b=S_4,\\ \operatorname{div}b=0, \end{cases} \tag{2.1}$$

with the initial data

$$(\phi,u,\psi,b)|_{t=0}=(\phi_0,u_0,\psi_0,b_0) \tag{2.2}$$

and the far-field boundary conditions

$$(\phi,u,\psi,b)\to(0,0,0,0),\quad \text{as} \quad |x|\to\infty, \tag{2.3}$$

where the nonlinear terms $S_i (i = 1, 2, 3, 4)$ on the right-hand side are defined as follows:

$$
\begin{cases}
S_1 \triangleq -\phi \operatorname{div} u - u \cdot \nabla\phi, \\
S_2 \triangleq -u \cdot \nabla u - h(\phi)\left[\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\right] + g(\phi)(\nabla \times b) \times b + \left[h(\phi) - \psi g(\phi)\right]\nabla\phi, \\
S_3 \triangleq -u \cdot \nabla\psi - \psi \operatorname{div} u - h(\phi)\Delta\psi + g(\phi)\left[\frac{\mu}{2}\left|\nabla u + (\nabla u)^\top\right|^2 + \lambda(\operatorname{div} u)^2 + |\nabla \times b|^2\right], \\
S_4 \triangleq -u \cdot \nabla b + b \cdot \nabla u - b \operatorname{div} u - \varepsilon\nabla \times \left[g(\phi)((\nabla \times b) \times b)\right],
\end{cases}
\tag{2.4}
$$

and $h(\cdot), g(\cdot)$ are the functions of ϕ given by

$$
h(\phi) = \frac{\phi}{\phi+1} \quad \text{and} \quad g(\phi) = \frac{1}{\phi+1}. \tag{2.5}
$$

It is clear that for smooth solutions, the problem (1.1)–(1.3) is equivalent to the one (2.1)–(2.3). Moreover, the left-hand side of (2.1) is indeed the linearized Navier–Stokes equations coupled with a heat equation. In fact, let

$$
A \triangleq \begin{pmatrix} 0 & \operatorname{div} & 0 \\ \nabla & -\mu\Delta - (\mu+\lambda)\nabla \operatorname{div} & \nabla \\ 0 & \operatorname{div} & -\Delta \end{pmatrix}. \tag{2.6}
$$

Then, it is easy to see that ($U \triangleq (\phi, u, \psi)$)

$$
U(t) = e^{-tA}U(0) \quad \text{and} \quad B(t) = e^{-t\Delta}B(0)
$$

solve the linearized Navier–Stokes equations $U_t + AU = 0$ (cf. [11,24,25]) and the heat equations $B_t - \Delta B = 0$ (cf. [32]), respectively. Thus, it follows from [11,24,25] and [35] that

Lemma 2.1. *Assume that $(U_0, B_0) \in L^1 \cap H^3$. Let $U \triangleq U(x,t)$ and $B(t) \triangleq B(x,t)$ be the smooth solutions of $U_t + AU = 0$ and $B_t - \Delta B = 0$, respectively. Then, for any $k \in \{0, 1, 2, 3\}$,*

$$
\|\nabla^k (U, B)(t)\|_{L^2} \le C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\left(\|(U_0, B_0)\|_{L^1} + \|\nabla^k (U_0, B_0)\|_{L^2}\right), \tag{2.7}
$$

where C is a generic positive constant depending only on μ and λ.

Based on Lemma 2.1, it is easily deduced from the Duhamel's principle that

Lemma 2.2. *Assume that the pair of functions (ϕ, u, ψ, b) is the smooth solution of (2.1) with initial data $(\phi_0, u_0, \psi_0, b_0) \in L^1 \cap H^3$. Then, for any $k \in \{0, 1, 2, 3\}$,*

$$
\begin{aligned}
\|\nabla^k (\phi, u, \psi, b)(t)\|_{L^2} \le{}& C(1+t)^{-(\frac{3}{4}+\frac{k}{2})}\|(\phi_0, u_0, \psi_0, b_0)\|_{L^1 \cap H^k} \\
&+ C\int_0^t (1+t-\tau)^{-(\frac{3}{4}+\frac{k}{2})}\|(S_1, S_2, S_3, S_4)\|_{L^1 \cap H^k} d\tau,
\end{aligned}
\tag{2.8}
$$

where C is a generic positive constant depending only on μ and λ.

The following product and commutator estimates [22] will be used repeatedly in the derivations of the global a priori estimates.

Lemma 2.3. *Let f, g be the smooth functions in Schwartz class. Then, for any $s > 0$ and $p \in (1, +\infty)$, there exists a generic positive constant $C > 0$ such that*

$$
\|D^s (fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}) \tag{2.9}
$$

and

$$
\|D^s (fg) - fD^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|D^{s-1} g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{2.10}
$$

where $p_1, p_2 > 1$ satisfy

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}. \tag{2.11}
$$

We also need the following simple facts (see, for example, [35]), which will be used to derive the decay rates.

Lemma 2.4. *Assume that $a, b, c \in R$ satisfy $0 \le a \le b, b > 1$ and $c > 0$. Then, there exists a positive constant C, depending only on a, b, c, such that for any $t > 0$,*

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}d\tau + \int_0^t (1+\tau)^{-a}e^{-c(t-\tau)}d\tau \le C(1+t)^{-a}. \tag{2.12}$$

Finally, we recall the local existence theorem of (1.1)–(1.3) [also cf. the problem (2.1)–(2.3)], which can be proved by using the Schauder fixed-point theorem (see, e.g., [12–15]).

Lemma 2.5. *Assume that the initial data satisfy*

$$(\rho_0 - 1, u_0, \theta_0 - 1, b_0) \in H^3, \quad \inf_{x\in\mathbb{R}^3} \rho_0(x) > 0, \quad \operatorname{div} b_0 = 0. \tag{2.13}$$

Then, there exists a small positive time $T^* > 0$ such that the problem (1.1)–(1.3) has a unique classical solution (ρ, u, θ, b) on $\mathbb{R}^3 \times [0, T^*]$ satisfying

$$(\rho - 1, u, \theta - 1, b) \in C([0, T^*]; H^3) \cap L^2(0, T^*; H^4), \qquad \inf_{x\in\mathbb{R}^3, t\in[0,T^*]} \rho(x, t) > 0. \tag{2.14}$$

3. Proof of Theorem 1.1

3.1. Global existence of classical solutions

To establish the global well-posedness of classical solutions stated in Theorem 1.1, it suffices to prove Theorem 3.1.

Theorem 3.1. *Given any positive number $K > 0$ (not necessarily small), assume that*

$$(\phi_0, u_0, \psi_0, b_0) \in H^3, \quad \|\nabla^2(\phi_0, u_0, \psi_0, b_0)\|_{H^1} \le K. \tag{3.1}$$

Then, there exists a positive constant $\eta > 0$, depending only on μ, λ and K, such that the problem (2.1)–(2.3) has a unique global classical solution (ϕ, u, ψ, b) on $\mathbb{R}^3 \times [0, \infty)$ satisfying

$$\begin{cases} \|\nabla(\phi, u, \psi, b)(t)\|_{H^2}^2 + \int_0^t \left(\|\nabla^2(u, \psi, b)\|_{H^2}^2 + \|\nabla^2\phi\|_{H^1}^2\right) \mathrm{ds} \le C\|\nabla(\phi_0, u_0, \psi_0, b_0)\|_{H^2}^2, \\ \|(\phi, u, \psi, b)(t)\|_{H^1}^2 + \int_0^t \left(\|\nabla(u, \psi, b)\|_{H^1}^2 + \|\nabla\phi\|_{L^2}^2\right) \mathrm{ds} \le C\|(\phi_0, u_0, \psi_0, b_0)\|_{H^1}^2, \end{cases} \tag{3.2}$$

for any $t \ge 0$, provided

$$\|(\phi_0, u_0, \psi_0, b_0)\|_{H^1} \le \eta. \tag{3.3}$$

In order to prove Theorem 3.1, we make the following a priori assumptions. For any given $M > 1$ (not necessarily small), assume that

$$\sup_{0\le t\le T} \|\nabla^2(\phi, u, \psi, b)(t)\|_{H^1} \le M \tag{3.4}$$

and

$$\sup_{0\le t\le T} \|(\phi, u, \psi, b)(t)\|_{H^1} \le \delta, \tag{3.5}$$

where $\delta > 0$ is a positive constant, depending on M, and satisfies

$$0 < \delta \le \delta_0 \triangleq (4C^2M)^{-1}, \quad (C \text{ is the Sobolev embedding constant of (3.7)}). \tag{3.6}$$

Our main purpose is to close the a priori assumptions (3.4) and (3.5) by choosing $\|(\phi_0, u_0, \psi_0, b_0)\|_{H^1}$ suitably small. To begin, we first observe that due to (3.4)–(3.6) and Sobolev embedding inequality (cf. [2]),

$$\|\phi(t)\|_{L^\infty} \le C\|\nabla\phi\|_{L^2}^{\frac{1}{2}}\|\nabla^2\phi\|_{L^2}^{\frac{1}{2}} \le \frac{1}{2}, \quad \forall\, t \in [0, T], \tag{3.7}$$

which indicates that

$$\frac{1}{2} \le \inf_{x,t} \phi(x, t) + 1 \le \sup_{x,t} \phi(x, t) + 1 \le \frac{3}{2}, \tag{3.8}$$

and moreover,

$$|h(\phi)| \le C|\phi|, \quad |h^{(k)}(\phi)|,\ |g^{(k-1)}(\phi)| \le C, \quad \forall\, 1 \le k \in \mathbb{Z}^+. \tag{3.9}$$

First of all, we derive the elementary L^2-estimates of (ϕ, u, ψ, b).

Lemma 3.1. *Under the assumptions* (3.4) *and* (3.5)*, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\phi, u, \psi, b)\|_{L^2}^2 + \|\nabla(u, \psi, b)\|_{L^2}^2 \le C\delta^{\frac{1}{2}}M^{\frac{1}{2}}\|\nabla(\phi, u, \psi, b)\|_{L^2}^2. \tag{3.10}$$

Proof. Multiplying $(2.1)_1$, $(2.1)_2$, $(2.1)_3$, $(2.1)_4$ by ϕ, u, ψ, b in L^2, respectively, and integrating by parts, we obtain after adding them together that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\phi, u, \psi, b)\|_{L^2}^2 + \left(\mu\|\nabla u\|_{L^2}^2 + (\mu+\lambda)\|\operatorname{div} u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) \\ &\quad = \langle S_1, \phi\rangle + \langle S_2, u\rangle + \langle S_3, \psi\rangle + \langle S_4, b\rangle, \end{aligned} \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2-inner product.

We are now in a position of estimating each term on the right-hand side of (3.11). First, integrating by parts and using Sobolev inequality, by (3.5) we easily get

$$\langle S_1, \phi\rangle \le C\|\phi\|_{L^3}\|\nabla u\|_{L^2}\|\phi\|_{L^6} \le C\|\phi\|_{H^1}\|\nabla u\|_{L^2}\|\nabla\phi\|_{L^2} \le C\delta(\|\nabla u\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2). \tag{3.12}$$

Integrating by parts and using Sobolev inequality (cf. [2]), we infer from (3.4), (3.5), (3.8) and (3.9) that

$$\begin{aligned} \langle S_2, u\rangle &\le C\int \left(|u|^2|\nabla u| + |\phi||\nabla u|^2 + |u||\nabla u||\nabla\phi| + |b||\nabla b||u| + |\phi||\nabla\phi||u| + |\psi||\nabla\phi||u|\right) dx \\ &\le C\|u\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6} + C\|\nabla u\|_{L^3}\left(\|\phi\|_{L^6}\|\nabla u\|_{L^2} + \|u\|_{L^6}\|\nabla\phi\|_{L^2}\right) \\ &\quad + C\|b\|_{L^3}\|\nabla b\|_{L^2}\|u\|_{L^6} + C\left(\|\phi\|_{L^3} + \|\psi\|_{L^3}\right)\|\nabla\phi\|_{L^2}\|u\|_{L^6} \\ &\le C\left(\|(\phi, u, \psi, b)\|_{H^1} + \|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\right)\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) \\ &\le C\delta^{\frac{1}{2}}M^{\frac{1}{2}}\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right), \end{aligned} \tag{3.13}$$

since $\delta \in (0, 1)$ and $M > 1$. In a similar manner, we also have

$$\begin{aligned} &\langle S_3, \psi\rangle + \langle S_4, b\rangle \\ &\le C\int \left(|\psi||u||\nabla\psi| + |\psi|^2|\nabla u| + |\phi||\nabla\psi|^2 + |\psi||\nabla\psi||\nabla\phi| + |\nabla b|^2|\psi| + |\nabla u|^2|\psi|\right) dx \\ &\quad + C\int \left(|b||u||\nabla b| + |b|^2|\nabla u| + |\nabla b|^2|b|\right) dx \\ &\le C\left(\|(\psi, b)\|_{H^1} + \|(\nabla u, \nabla\psi, \nabla b)\|_{L^3}\right)\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) \\ &\le C\delta^{\frac{1}{2}}M^{\frac{1}{2}}\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right). \end{aligned} \tag{3.14}$$

Thus, substituting (3.12)–(3.14) into (3.11) immediately leads to (3.10). □

Lemma 3.2. *Under the assumptions* (3.4) *and* (3.5), *one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\phi,u,\psi,b)\|_{L^2}^2+\|\nabla^2(u,\psi,b)\|_{L^2}^2\le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right). \tag{3.15}$$

Proof. Operating ∇ to $(2.1)_1$, $(2.1)_2$, $(2.1)_3$ and $(2.1)_4$, multiplying them by $\nabla\phi$, ∇u, $\nabla\psi$ and ∇b in L^2, respectively, and integrating by parts, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\phi,u,\psi,b)\|_{L^2}^2+\left(\mu\|\nabla^2u\|_{L^2}^2+(\mu+\lambda)\|\nabla\operatorname{div}u\|_{L^2}^2+\|\nabla^2\psi\|_{L^2}^2+\|\nabla^2b\|_{L^2}^2\right)\\&\quad=\langle\nabla S_1,\nabla\phi\rangle-\langle S_2,\nabla^2u\rangle-\langle S_3,\nabla^2\psi\rangle-\langle S_4,\nabla^2b\rangle.\end{aligned} \tag{3.16}$$

Upon integration by parts, we infer from (3.4) and (3.5) that

$$\begin{aligned}\langle\nabla S_1,\nabla\phi\rangle\le&C\int\left(|\nabla u||\nabla\phi|^2+|\phi||\nabla^2u||\nabla\phi|\right)dx\\\le&C\|\nabla\phi\|_{L^3}\left(\|\nabla\phi\|_{L^2}\|\nabla u\|_{L^6}+\|\phi\|_{L^6}\|\nabla^2u\|_{L^2}\right)\\\le&C\|\nabla\phi\|_{L^2}^{1/2}\|\nabla^2\phi\|_{L^2}^{1/2}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\right)\\\le&C\delta^{\frac12}M^{\frac12}\|\left(\|\nabla\phi\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\right).\end{aligned} \tag{3.17}$$

By direct computations, we easily get that

$$\begin{aligned}\|(S_2,S_3,S_4)\|_{L^2}\le&\, C\left(\|(u,\psi,b)\|_{L^3}\|\nabla(u,\psi,b)\|_{L^6}+\|(\phi,b)\|_{L^\infty}\|\nabla^2(u,\psi,b)\|_{L^2}+\|\nabla(u,b)\|_{L^4}^2\right)\\&+C\left(\|(\phi,\psi)\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\nabla\phi\|_{L^2}\right)\\\le&\, C\left(\|(u,\psi,b)\|_{H^1}+\|\nabla(\phi,u,b)\|_{L^2}^{\frac12}\|\nabla^2(\phi,u,b)\|_{L^2}^{\frac12}\right)\|\nabla^2(u,\psi,b)\|_{L^2}\\&+C\left(\|\nabla(\phi,\psi)\|_{L^2}^{\frac12}\|\nabla^2(\phi,\psi)\|_{L^2}^{\frac12}+\|\nabla b\|_{L^2}^{\frac12}\|\nabla^2b\|_{L^2}\|\nabla^3b\|_{L^2}^{\frac12}\right)\|\nabla\phi\|_{L^2}\\\le&\, C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{L^2}+\|\nabla\phi\|_{L^2}\right),\end{aligned} \tag{3.18}$$

and consequently,

$$\left|\langle S_2,\nabla^2u\rangle\right|+\left|\langle S_3,\nabla^2\psi\rangle\right|+\left|\langle S_4,\nabla^2b\rangle\right|\le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right). \tag{3.19}$$

Now, putting (3.17) and (3.19) into (3.16), we arrive at the desired result of (3.15). □

Remark 3.1. In fact, it can be also deduced from (3.17) that

$$\begin{aligned}\langle\nabla S_1,\nabla\phi\rangle\le&C\int\left(|\nabla u||\nabla\phi|^2+|\phi||\nabla^2u||\nabla\phi|\right)dx\\&\le C\left(\|\nabla u\|_{L^3}\|\nabla\phi\|_{L^3}^2+\|\phi\|_{L^3}\|\nabla^2u\|_{L^2}\|\nabla\phi\|_{L^6}\right)\\&\le C\left(\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2u\|_{L^2}^{\frac12}\|\nabla\phi\|_{L^2}\|\nabla^2\phi\|_{L^2}+\|\phi\|_{H^1}\|\nabla^2u\|_{L^2}\|\nabla^2\phi\|_{L^2}\right)\\&\le C\left(\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2u\|_{L^2}^{\frac12}\|\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac32}+\|\phi\|_{H^1}\|\nabla^2u\|_{L^2}\|\nabla^2\phi\|_{L^2}\right)\\&\le C\delta\left(\|\nabla^2\phi\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\right),\end{aligned} \tag{3.20}$$

where we have used the Sobolev interpolation inequality $\|\nabla\phi\|_{L^2}^2\le C\|\phi\|_{L^2}\|\nabla^2\phi\|_{L^2}$. Similarly,

$$\|(S_2,S_3,S_4)\|_{L^2}\le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{L^2}+\|\nabla^2\phi\|_{L^2}\right).$$

This, combined with (3.16) and (3.20), shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\phi,u,\psi,b)\|_{L^2}^2+\|\nabla^2(u,\psi,b)\|_{L^2}^2\le C\delta^{\frac12}M^{\frac32}\|\nabla^2(\phi,u,\psi,b)\|_{L^2}^2, \tag{3.21}$$

which will be used to close the estimate of $\|\nabla(\phi,u,\psi,b)\|_{L^2}^2$.

Clearly, we need to estimate $\|\nabla\phi\|_{L^2(0,T;L^2)}$.

Lemma 3.3. *Under the assumptions* (3.4) *and* (3.5), *one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u,\nabla\phi\rangle+\|\nabla\phi\|_{L^2}^2\le C\|\nabla(u,\psi)\|_{H^1}^2+C\delta^{\frac12}M^{\frac32}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla(u,\psi,b)\|_{H^1}^2\right). \tag{3.22}$$

Proof. Multiplying $(2.1)_2$ by $\nabla\phi$ in L^2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u,\nabla\phi\rangle+\|\nabla\phi\|_{L^2}^2=\langle u,\nabla\phi_t\rangle+\langle\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u-\nabla\psi,\nabla\phi\rangle+\langle S_2,\nabla\phi\rangle. \tag{3.23}$$

Upon integration by parts, one infers from (2.1) that

$$\begin{aligned}\langle u,\nabla\phi_t\rangle&=-\langle u,\nabla(\operatorname{div}u+\operatorname{div}(\phi u))\rangle=\langle\operatorname{div}u,\operatorname{div}u+\operatorname{div}(\phi u)\rangle\\&\le C\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^3}\left(\|\nabla\phi\|_{L^2}\|u\|_{L^6}+\|\phi\|_{L^6}\|\nabla u\|_{L^2}\right)\\&\le C\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2u\|_{L^2}^{\frac12}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\&\le C\|\nabla u\|_{L^2}^2+C\delta^{\frac12}M^{\frac12}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right).\end{aligned} \tag{3.24}$$

It is easy to see that

$$\langle\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u-\nabla\psi,\nabla\phi\rangle\le\frac14\|\nabla\phi\|_{L^2}^2+C\left(\|\nabla^2u\|_{L^2}^2+\|\nabla\psi\|_{L^2}^2\right), \tag{3.25}$$

and moreover, it follows from (3.18) that

$$\langle S_2,\nabla\phi\rangle\le C\|S_2\|_{L^2}\|\nabla\phi\|_{L^2}\le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right) \tag{3.26}$$

Hence, substituting (3.24)–(3.26) into (3.23), we obtain (3.22).

We proceed to prove the L^2-estimates of the second-order derivatives.

Lemma 3.4. *Under the assumptions* (3.4) *and* (3.5), *one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2(\phi,u,\psi,b)\|_{L^2}^2+\|\nabla^3(u,\psi,b)\|_{L^2}^2\le C\delta^{\frac14}M^{\frac32}\left(\|\nabla^3(u,\psi,b)\|_{L^2}^2+\|\nabla^2\phi\|_{L^2}^2\right). \tag{3.27}$$

Proof. Similarly to the proof of Lemma 3.2, we deduce from (2.1) that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2(\phi,u,\psi,b)\|_{L^2}^2+\left(\mu\|\nabla^3u\|_{L^2}^2+(\mu+\lambda)\|\nabla^2\operatorname{div}u\|_{L^2}^2+\|\nabla^3\psi\|_{L^2}^2+\|\nabla^3b\|_{L^2}^2\right)\\&\quad=\langle\nabla^2S_1,\nabla^2\phi\rangle+\langle\nabla^2S_2,\nabla^2u\rangle+\langle\nabla^2S_3,\nabla^2\psi\rangle+\langle\nabla^2S_4,\nabla^2b\rangle.\end{aligned} \tag{3.28}$$

Using Lemma 2.3 and Gagliardo–Nirenberg inequality (cf. [2]), we find (keeping in mind that $M>1$)

$$\begin{aligned}\langle\nabla^2S_1,\nabla^2\phi\rangle&=-\langle\nabla^2(\phi\operatorname{div}u),\nabla^2\phi\rangle-\langle\nabla^2(u\cdot\nabla\phi)-u\cdot\nabla^2(\nabla\phi),\nabla^2\phi\rangle+\frac12\langle\operatorname{div}u,|\nabla^2\phi|^2\rangle\\&\le C\left(\|\nabla u\|_{L^\infty}\|\nabla^2\phi\|_{L^2}+\|\phi\|_{L^\infty}\|\nabla^3u\|_{L^2}+\|\nabla^2u\|_{L^6}\|\nabla\phi\|_{L^3}\right)\|\nabla^2\phi\|_{L^2}\\&\le C\left(\|\nabla u\|_{L^2}^{\frac14}\|\nabla^3u\|_{L^2}^{\frac34}\|\nabla^2\phi\|_{L^2}+\|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^3u\|_{L^2}\right)\|\nabla^2\phi\|_{L^2}\\&\le C\delta^{\frac14}M^{\frac34}\|\left(\|\nabla^2\phi\|_{L^2}^2+\|\nabla^3u\|_{L^2}^2\right).\end{aligned} \tag{3.29}$$

Next, we estimate some terms involving in ∇S_i for $i=2,3,4$. Indeed, by making use of the Sobolev interpolation inequality (cf. [10,17])

$$\|D^jf\|_{L^2}\le C\|f\|_{L^2}^{\frac{m-j}{m}}\|D^mf\|_{L^2}^{\frac{j}{m}},\quad\forall\ 0\le j\le m,$$

we get that

$$
\begin{aligned}
\|\nabla(u\cdot\nabla u)\|_{L^2} \le& \|u\|_{L^3}\|\nabla^2 u\|_{L^6} + \|\nabla u\|_{L^3}\|\nabla u\|_{L^6}\\
&\le C\left(\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac32}\right)\\
&\le C\left(\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}\right)\\
&\le C\delta\|\nabla^3 u\|_{L^2}.
\end{aligned}
\tag{3.30}
$$

Similarly to the derivation of (3.30), we also have

$$
\begin{aligned}
&\|\nabla(u\cdot\nabla\psi)\|_{L^2} + \|\nabla(u\cdot\nabla b)\|_{L^2} + \|\nabla(b\cdot\nabla u)\|_{L^2} + \|\nabla(\psi\operatorname{div} u)\|_{L^2} + \|\nabla(b\operatorname{div} u)\|_{L^2}\\
&\quad\le C\|(u,\psi,b)\|_{L^2}^{\frac12}\|\nabla(u,\psi,b)\|_{L^2}^{\frac12}\|\nabla^3(u,\psi,b)\|_{L^2} \le C\delta\|\nabla^3(u,\psi,b)\|_{L^2}.
\end{aligned}
\tag{3.31}
$$

It is easy to deduce from (3.7)–(3.9) and the interpolation inequality that

$$
\begin{aligned}
&\|\nabla(h(\phi)\nabla\phi)\|_{L^2} + \|\nabla(\psi g(\phi)\nabla\phi)\|_{L^2} + \|\nabla(g(\phi)|\nabla(u,b)|^2)\|_{L^2}\\
&\quad\le C\left(\|(\phi,\psi)\|_{L^\infty}\|\nabla^2\phi\|_{L^2} + \|\nabla(\phi,\psi)\|_{L^3}\|\nabla\phi\|_{L^6} + \|\psi\|_{L^\infty}\|\nabla\phi\|_{L^3}\|\nabla\phi\|_{L^6}\right)\\
&\qquad+C\left(\|\nabla\phi\|_{L^6}\|\nabla(u,b)\|_{L^6}^2 + \|\nabla(u,b)\|_{L^3}\|\nabla^2(u,b)\|_{L^6}\right)\\
&\quad\le C\left(\|\nabla(\phi,\psi)\|_{L^2}^{\frac12}\|\nabla^2(\phi,\psi)\|_{L^2}^{\frac12} + \|\nabla\psi\|_{L^2}^{\frac12}\|\nabla^2\psi\|_{L^2}^{\frac12}\|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac12}\right)\|\nabla^2\phi\|_{L^2}\\
&\qquad+C\left(\|\nabla^2\phi\|_{L^2}\|\nabla(u,b)\|_{L^2} + \|\nabla(u,b)\|_{L^2}^{\frac12}\|\nabla^2(u,b)\|_{L^2}^{\frac12}\right)\|\nabla^3(u,b)\|_{L^2}\\
&\quad\le C\delta^{\frac12}M\left(\|\nabla^2\phi\|_{L^2} + \|\nabla^3(u,b)\|_{L^2}\right),
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&\|\nabla(h(\phi)(\Delta u,\nabla\operatorname{div} u,\Delta\psi))\|_{L^2} \le C\left(\|\phi\|_{L^\infty}\|\nabla^3(u,\psi)\|_{L^2} + \|\nabla\phi\|_{L^3}\|\nabla^2(u,\psi)\|_{L^6}\right)\\
&\quad\le C\|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^3(u,\psi)\|_{L^2} \le C\delta^{\frac12}M^{\frac12}\|\nabla^3(u,\psi)\|_{L^2},
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
&\|\nabla(g(\phi)((\nabla\times b)\times b))\|_{L^2} \le C\left(\|\nabla\phi\|_{L^6}\|b\|_{L^6}\|\nabla b\|_{L^6} + \|b\|_{L^3}\|\nabla^2 b\|_{L^6} + \|\nabla b\|_{L^3}\|\nabla b\|_{L^6}\right)\\
&\quad\le C\left(\|\nabla^2\phi\|_{L^2}\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2} + \|b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\nabla^3 b\|_{L^2}\right)\\
&\quad\le C\delta^{\frac12}M\left(\|\nabla^2\phi\|_{L^2} + \|\nabla^3 b\|_{L^2}\right)
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
\|\nabla^2(g(\phi)((\nabla\times b)\times b))\|_{L^2} \le& C\left(\|\nabla^2\phi\|_{L^2}\|b\|_{L^\infty} + \|\nabla\phi\|_{L^6}\|\nabla\phi\|_{L^\infty}\|b\|_{L^3}\right)\|\nabla b\|_{L^\infty}\\
&+C\left(\|b\|_{L^\infty}\|\nabla^3 b\|_{L^2} + \|\nabla b\|_{L^3}\|\nabla^2 b\|_{L^6}\right)\\
&+C\left(\|\nabla\phi\|_{L^6}\|b\|_{L^\infty}\|\nabla^2 b\|_{L^3} + \|\nabla\phi\|_{L^6}\|\nabla b\|_{L^6}^2\right)\\
\le& C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2\phi\|_{L^2} + \|\nabla^3 b\|_{L^2}\right).
\end{aligned}
\tag{3.35}
$$

Collecting (3.30)–(3.35) together, we arrive at

$$
\|\nabla(S_2,S_3,S_4)\|_{L^2} \le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^3(u,\psi,b)\|_{L^2} + \|\nabla^2\phi\|_{L^2}\right),
\tag{3.36}
$$

from which and (3.28), (3.29), we obtain (3.27) after integrating by parts and using Cauchy–Schwarz inequality. □

The next lemma is concerned with the estimate of $\|\nabla^2\phi\|_{L^2(0,T;L^2)}$, which can be achieved in a similar manner as that used in the proof of Lemma 3.3.

Lemma 3.5. *Under the assumptions* (3.4) *and* (3.5), *one has*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle\operatorname{div} u,\Delta\phi\rangle + \|\nabla^2\phi\|_{L^2}^2 \le C\|\nabla^2(u,\psi)\|_{H^1}^2 + C\delta^{\frac12}M^{\frac32}\left(\|\nabla^2(u,\psi,b)\|_{H^1}^2 + \|\nabla^2\phi\|_{L^2}^2\right).
\tag{3.37}
$$

Proof. Operating div to $(2.1)_2$ and multiplying it by $\Delta\phi$ in L^2, we obtain after integrating by parts that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \operatorname{div} u, \Delta\phi\rangle + \|\nabla^2\phi\|_{L^2}^2 = &-\langle \nabla \operatorname{div} u, \nabla\phi_t\rangle - \langle \Delta\psi, \Delta\phi\rangle + \langle \operatorname{div} S_2, \Delta\phi\rangle \\ &+ \langle \operatorname{div}[\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u], \Delta\phi\rangle.\end{aligned} \tag{3.38}$$

In view of $(2.1)_1$, we have

$$\begin{aligned}|\langle \nabla \operatorname{div} u, \nabla\phi_t\rangle| &\le C\left(\|\nabla^2 u\|_{L^2} + \|(\phi,u)\|_{L^\infty}\|\nabla^2(\phi,u)\|_{L^2} + \|\nabla\phi\|_{L^3}\|\nabla u\|_{L^6}\right)\|\nabla^2 u\|_{L^2} \\ &\le C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla(\phi,u)\|_{L^2}^{\frac12}\|\nabla^2(\phi,u)\|_{L^2}^{\frac12}\|\nabla^2(\phi,u)\|_{L^2}^2 \\ &\le C\|\nabla^2 u\|_{L^2}^2 + C\delta^{\frac12}M^{\frac12}\|\nabla^2(\phi,u)\|_{L^2}^2,\end{aligned} \tag{3.39}$$

and by virtue of (3.36) we see that

$$\begin{aligned}&|\langle \Delta\psi, \Delta\phi\rangle| + |\langle \operatorname{div} S_2, \Delta\phi\rangle| + |\langle \operatorname{div}[\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u], \Delta\phi\rangle| \\ &\le \frac12\|\nabla^2\phi\|_{L^2}^2 + C\left(\|\nabla^3 u\|_{L^2}^2 + \|\nabla^2\psi\|_{L^2}^2\right) + C\delta^{\frac12}M^{\frac32}\left(\|\nabla^3(u,\psi,b)\|_{L^2}^2 + \|\nabla^2\phi\|_{L^2}^2\right)\end{aligned}$$

which, combined with (3.38) and (3.39), yields (3.37). □

Next, it remains to deal with H^3-estimates of the solutions.

Lemma 3.6. *Under the assumptions* (3.4) *and* (3.5)*, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3(\phi,u,\psi,b)\|_{L^2}^2 + \|\nabla^4(u,\psi,b)\|_{L^2}^2 \le C\delta^{\frac14}M^{\frac52}\left(\|\nabla^3\phi\|_{L^2}^2 + \|\nabla^4(u,\psi,b)\|_{L^2}^2\right). \tag{3.40}$$

Proof. It follows from (2.1) that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3(\phi,u,\psi,b)\|_{L^2}^2 + \left(\mu\|\nabla^4 u\|_{L^2}^2 + (\mu+\lambda)\|\nabla^3 \operatorname{div} u\|_{L^2}^2 + \|\nabla^4\psi\|_{L^2}^2 + \|\nabla^4 b\|_{L^2}^2\right) \\ &= \langle \nabla^3 S_1, \nabla^3\phi\rangle + \langle \nabla^3 S_2, \nabla^3 u\rangle + \langle \nabla^3 S_3, \nabla^3\psi\rangle + \langle \nabla^3 S_4, \nabla^3 b\rangle.\end{aligned} \tag{3.41}$$

Based on Lemma 2.3 and Sobolev inequality, it is easily seen that

$$\begin{aligned}\langle \nabla^3 S_1, \nabla^3\phi\rangle = &-\langle \nabla^3(\phi \operatorname{div} u), \nabla^3\phi\rangle - \langle \nabla^3(u\cdot\nabla\phi) - u\cdot\nabla^3(\nabla\phi), \nabla^3\phi\rangle - \frac12\langle \operatorname{div} u, |\nabla^3\phi|^2\rangle \\ &\le C\left(\|\nabla u\|_{L^\infty}\|\nabla^3\phi\|_{L^2} + \|\phi\|_{L^\infty}\|\nabla^4 u\|_{L^2} + \|\nabla^3 u\|_{L^6}\|\nabla\phi\|_{L^3}\right)\|\nabla^3\phi\|_{L^2} \\ &\le C\left(\|\nabla u\|_{L^2}^{\frac14}\|\nabla^3 u\|_{L^2}^{\frac34}\|\nabla^3\phi\|_{L^2} + \|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^4 u\|_{L^2}\right)\|\nabla^3\phi\|_{L^2} \\ &\le C\delta^{\frac14}M^{\frac34}\left(\|\nabla^3\phi\|_{L^2}^2 + \|\nabla^4 u\|_{L^2}^2\right).\end{aligned} \tag{3.42}$$

In the following, similarly to the proofs of Lemma 3.4, we estimate the terms involved in $\|\nabla^2 S_i\|_{L^2}$ for $i = 2,3,4$. First, by virtue of the interpolation inequality we have

$$\begin{aligned}\|\nabla^2(u\cdot\nabla u)\|_{L^2} &\le C\left(\|\nabla u\|_{L^3}\|\nabla^2 u\|_{L^6} + \|u\|_{L^3}\|\nabla^3 u\|_{L^6}\right) \\ &\le C\left(\|\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2} + \|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^4 u\|_{L^2}\right) \\ &\le C\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\nabla^4 u\|_{L^2} \le C\delta\|\nabla^4 u\|_{L^2}\end{aligned} \tag{3.43}$$

and similarly,

$$\begin{aligned}&\|\nabla^2(u\cdot\nabla\psi)\|_{L^2} + \|\nabla^2(u\cdot\nabla b)\|_{L^2} + \|\nabla^2(b\cdot\nabla u)\|_{L^2} + \|\nabla^2(\psi \operatorname{div} u)\|_{L^2} + \|\nabla^2(b \operatorname{div} u)\|_{L^2} \\ &\le C\|(u,\psi,b)\|_{L^2}^{\frac12}\|\nabla(u,\psi,b)\|_{L^2}^{\frac12}\|\nabla^4(u,\psi,b)\|_{L^2} \le C\delta\|\nabla^4(u,\psi,b)\|_{L^2}.\end{aligned} \tag{3.44}$$

By (3.8) and (3.9), we obtain

$$\begin{aligned}\|\nabla^2(h(\phi)\nabla\phi)\|_{L^2} &\le C\left(\|\phi\|_{L^\infty}\|\nabla^3\phi\|_{L^2}+\|\nabla^2\phi\|_{L^6}\|\nabla\phi\|_{L^3}+\|\nabla^2\phi\|_{L^2}^3\right)\\ &\le C\left(\|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^2\phi\|_{L^2}^{\frac12}+\|\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\right)\|\nabla^3\phi\|_{L^2}\\ &\le C\delta^{\frac12}M\|\nabla^3\phi\|_{L^2}.\end{aligned}\tag{3.45}$$

In a similar manner,

$$\begin{aligned}&\|\nabla^2(\psi g(\phi)\nabla\phi)\|_{L^2}\le C\left(\|\psi\|_{L^\infty}\|\nabla^2(g(\phi)\nabla\phi)\|_{L^2}+\|\nabla^2\psi\|_{L^3}\|\nabla\phi\|_{L^6}\right)\\ &\quad\le C\delta^{\frac12}M^{\frac12}\|\nabla^3\phi\|_{L^2}\left(1+\delta^{\frac12}M\right)+C\|\nabla^2\phi\|_{L^2}\|\nabla^2\psi\|_{L^2}^{\frac12}\|\nabla^3\psi\|_{L^2}^{\frac12}\\ &\quad\le C\delta^{\frac12}M^{\frac32}\|\nabla^3\phi\|_{L^2}+C\|\nabla\phi\|_{L^2}^{\frac12}\|\nabla^3\phi\|_{L^2}^{\frac12}\|\psi\|_{L^2}^{\frac14}\|\nabla^2\psi\|_{L^2}^{\frac14}\|\nabla^4\psi\|_{L^2}^{\frac12}\\ &\quad\le C\delta^{\frac12}M^{\frac32}\|\nabla^3\phi\|_{L^2}+C\delta^{\frac34}M^{\frac14}\left(\|\nabla^3\phi\|_{L^2}+\|\nabla^4\psi\|_{L^2}\right)\\ &\quad\le C\delta^{\frac12}M^{\frac32}\left(\|\nabla^3\phi\|_{L^2}+\|\nabla^4\psi\|_{L^2}\right),\end{aligned}\tag{3.46}$$

$$\begin{aligned}&\|\nabla^2(g(\phi)|\nabla(u,b)|^2)\|_{L^2}\le C\left(\|\nabla^2\phi\|_{L^6}+\|\nabla\phi\|_{L^\infty}\|\nabla\phi\|_{L^6}\right)\|\nabla(u,b)|\|_{L^6}^2\\ &\qquad+C\|\nabla^2(u,b)\|_{L^3}\|\nabla^2(u,b)\|_{L^6}\\ &\qquad+C\left(\|\nabla\phi\|_{L^\infty}\|\nabla^2(u,b)\|_{L^6}\|\nabla(u,b)\|_{L^3}+\|\nabla(u,b)\|_{L^3}\|\nabla^3(u,b)\|_{L^6}\right)\\ &\quad\le C\left(\|\nabla^3\phi\|_{L^2}+\|\phi\|_{L^2}^{\frac12}\|\nabla^3\phi\|_{L^2}^{\frac32}\right)\|(u,b)\|_{L^2}\|\nabla^4(u,b)\|_{L^2}\\ &\qquad+C\left(\|\nabla^2\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\|(u,b)\|_{H^1}+\|\nabla(u,b)\|_{L^2}^{\frac12}\|\nabla^2(u,b)\|_{L^2}^{\frac12}\right)\|\nabla^4(u,b)\|_{L^2}\\ &\quad\le C\delta^{\frac12}M^{\frac32}\|\nabla^4(u,b)\|_{L^2}.\end{aligned}\tag{3.47}$$

In terms of Lemma 2.3 and the interpolation inequality again, we see that

$$\begin{aligned}&\|\nabla^2(h(\phi)\Delta u)\|_{L^2}\le C\left[\|\phi\|_{L^\infty}\|\nabla^4u\|_{L^2}+\left(\|\nabla^2\phi\|_{L^3}+\|\nabla\phi\|_{L^6}^2\right)\|\nabla^2u\|_{L^6}\right]\\ &\quad\le C\delta^{\frac12}M^{\frac12}\|\nabla^4u\|_{L^2}+C\left(\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^3\phi\|_{L^2}^{\frac12}+\|\nabla^2\phi\|_{L^2}^2\right)\|\nabla^3u\|_{L^2}\\ &\quad\le C\delta^{\frac12}M^{\frac12}\|\nabla^4u\|_{L^2}+C\left(\|\nabla\phi\|_{L^2}^{\frac14}\|\nabla^3\phi\|_{L^2}^{\frac34}+\|\nabla\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\right)\|\nabla^2u\|_{L^2}^{\frac12}\|\nabla^4u\|_{L^2}^{\frac12}\\ &\quad\le C\delta^{\frac14}M\left(\|\nabla^4u\|_{L^2}+\|\nabla^3\phi\|_{L^2}\right),\end{aligned}\tag{3.48}$$

and analogously,

$$\|\nabla^2(h(\phi)(\nabla\operatorname{div}u,\Delta\psi))\|_{L^2}\le C\delta^{\frac14}M\left(\|\nabla^4(u,\psi)\|_{L^2}+\|\nabla^3\phi\|_{L^2}\right).\tag{3.49}$$

Since

$$|\nabla^2g(\phi)|\le C\left(|\nabla^2\phi|+|\nabla\phi|^2\right),\quad |\nabla^3g(\phi)|\le C\left(|\nabla^3\phi|+|\nabla\phi||\nabla^2\phi|+|\nabla\phi|^3\right).$$

we deduce from Lemma 2.3 and the interpolation inequality that

$$\begin{aligned}&\|\nabla^2(g(\phi)((\nabla\times b)\times b))\|_{L^2}\\ &\quad\le C\left(\|\nabla^2\phi\|_{L^3}\|b\|_{L^\infty}\|\nabla b\|_{L^6}+\|\nabla\phi\|_{L^6}^2\|b\|_{L^\infty}\|\nabla b\|_{L^6}+\|b\|_{L^3}\|\nabla^3b\|_{L^6}+\|\nabla b\|_{L^3}\|\nabla^2b\|_{L^6}\right)\\ &\quad\le C\left(\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^3\phi\|_{L^2}^{\frac12}+\|\nabla\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\right)\|b\|_{H^1}^{\frac12}\|\nabla^2b\|_{L^2}^{\frac32}+C\|b\|_{H^1}\|\nabla^4b\|_{L^2}\\ &\quad\le C\left(\|\nabla^2\phi\|_{L^2}^{\frac12}\|\nabla^3\phi\|_{L^2}^{\frac12}+\|\nabla\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\right)\|b\|_{H^1}^{\frac54}\|\nabla^4b\|_{L^2}^{\frac34}+C\|b\|_{H^1}\|\nabla^4b\|_{L^2}\\ &\quad\le C\delta M^{\frac34}\left(\|\nabla^3\phi\|_{L^2}+\|\nabla^4b\|_{L^2}\right)\end{aligned}\tag{3.50}$$

and

$$
\begin{aligned}
&\|\nabla^3(g(\phi)((\nabla\times b)\times b))\|_{L^2}\\
&\quad\le C\left(\|\nabla^3\phi\|_{L^2}+\|\nabla\phi\|_{L^3}\|\nabla^2\phi\|_{L^6}+\|\nabla\phi\|_{L^6}^3\right)\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}+C\|\nabla b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\nabla^4 b\|_{L^2}\\
&\quad\le C\delta^{\frac12}M^{\frac52}\left(\|\nabla^3\phi\|_{L^2}+\|\nabla^4 b\|_{L^2}\right).
\end{aligned}\tag{3.51}
$$

Collecting (3.43)–(3.51) together, we find

$$
\|\nabla^2(S_2,S_3,S_4)\|_{L^2}\le C\delta^{\frac14}M^{\frac52}\left(\|\nabla^3\phi\|_{L^2}+\|\nabla^4(u,\psi,b)\|_{L^2}\right),\tag{3.52}
$$

which, together (3.42) and (3.41), immediately leads to (3.40), using integration by parts and Cauchy–Schwarz inequality. □

Finally, to close the estimate, we still need to deal with $\|\nabla^3\phi\|_{L^2(0,T;L^2)}$.

Lemma 3.7. *Under the assumptions* (3.4) *and* (3.5), *one has*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla\operatorname{div}u,\nabla\Delta\phi\rangle+\|\nabla^3\phi\|_{L^2}^2\le C\|\nabla^3(u,\psi)\|_{H^1}^2+C\delta^{\frac14}M^{\frac52}\left(\|\nabla^3\phi\|_{L^2}^2+\|\nabla^3(u,\psi,b)\|_{H^1}^2\right).\tag{3.53}
$$

Proof. Operating $\nabla\operatorname{div}$ to both sides of $(2.1)_2$ and multiplying it by $\nabla\Delta\phi$ in L^2, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla\operatorname{div}u,\nabla\Delta\phi\rangle+\|\nabla\Delta\phi\|_{L^2}^2&=\langle\nabla\operatorname{div}u,\nabla\Delta\phi_t\rangle+\langle\nabla\operatorname{div}S_2,\nabla\Delta\phi\rangle\\
&\quad+\langle\nabla\operatorname{div}(\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u)-\nabla\Delta\psi,\nabla\Delta\phi\rangle.
\end{aligned}\tag{3.54}
$$

In view of $(2.1)_1$, we have from Lemma 2.3 that

$$
\begin{aligned}
\langle\nabla\operatorname{div}u,\nabla\Delta\phi_t\rangle&=-\langle\Delta\operatorname{div}u,\Delta\phi_t\rangle\\
&\le C\|\nabla^3u\|_{L^2}\left(\|\nabla^3u\|_{L^2}+\|\nabla^3(\phi u)\|_{L^2}\right)\\
&\le C\|\nabla^3u\|_{L^2}\left(\|\nabla^3u\|_{L^2}+\|\nabla^3\phi\|_{L^2}\|u\|_{L^\infty}+\|\phi\|_{L^\infty}\|\nabla^3u\|_{L^2}\right)\\
&\le C\|\nabla^3u\|_{L^2}^2+C\delta^{\frac12}M^{\frac12}\left(\|\nabla^3u\|_{L^2}^2+\|\nabla^3\phi\|_{L^2}^2\right).
\end{aligned}\tag{3.55}
$$

For the other terms, we have

$$
\begin{aligned}
&|\langle\nabla\operatorname{div}S_2,\nabla\Delta\phi\rangle|+|\langle\nabla\operatorname{div}(\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u)-\nabla\Delta\psi,\nabla\Delta\phi\rangle|\\
&\quad\le\frac12\|\nabla^3\phi\|_{L^2}^2+C\|\nabla^3(u,\psi)\|_{H^1}^2+C\delta^{\frac14}M^{\frac52}\left(\|\nabla^3\phi\|_{L^2}^2+\|\nabla^4(u,\psi,b)\|_{L^2}^2\right).
\end{aligned}\tag{3.56}
$$

Thus, substituting (3.55) and (3.56) into (3.54), we obtain (3.53). □

With Lemmas 3.1–3.7 at hand, we are now ready to prove Theorem 3.1.

Proof of Global Existence. On the one hand, in view of (3.10), (3.15) and (3.22), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|(\phi,u,\psi,b)\|_{H^1}^2+\|\nabla(u,\psi,b)\|_{H^1}^2&\le C_1\delta^{\frac12}M^{\frac32}\left(\|\nabla(u,\psi,b)\|_{H^1}^2+\|\nabla\phi\|_{L^2}^2\right)\\
&\le\frac12\|\nabla(u,\psi,b)\|_{H^1}^2+C_1\delta^{\frac12}M^{\frac32}\|\nabla\phi\|_{L^2}^2
\end{aligned}\tag{3.57}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle u,\nabla\phi\rangle+\|\nabla\phi\|_{L^2}^2&\le C_2\|\nabla(u,\psi)\|_{H^1}^2+C_2\delta^{\frac12}M^{\frac32}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla(u,\psi,b)\|_{H^1}^2\right)\\
&\le 2C_2\|\nabla(u,\psi,b)\|_{H^1}^2+C_2\delta^{\frac12}M^{\frac32}\|\nabla\phi\|_{L^2}^2,
\end{aligned}\tag{3.58}
$$

provided $\delta>0$ is chosen to be small enough such that

$$
0<\delta\le\delta_1\triangleq\min\left\{\delta_0,\left(2C_1M^{\frac32}\right)^{-2},M^{-3}\right\}.
$$

It follows from (3.57) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\phi,u,\psi,b)\|_{H^1}^2+\frac{1}{2}\|\nabla(u,\psi,b)\|_{H^1}^2\le C_1\delta^{\frac12}M^{\frac32}\|\nabla\phi\|_{L^2}^2,$$

which, multiplied by a suitably large number $K_1=\max\{4,8C_2\}$ and added to (3.58), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(K_1\|(\phi,u,\psi,b)\|_{H^1}^2+\langle u,\nabla\phi\rangle\right)+2C_2\|\nabla(u,\psi,b)\|_{H^1}^2+\|\nabla\phi\|_{L^2}^2\le C_3(K_1)\delta^{\frac12}M^{\frac32}\|\nabla\phi\|_{L^2}^2,$$

where $C_3(K_1)$ is a positive constant depending on K_1. So, if $\delta>0$ is chosen to be such that

$$0<\delta\le\delta_2\triangleq\min\left\{\delta_1,\left(2C_3(K_1)M^{\frac32}\right)^{-2}\right\},$$

then one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(K_1\|(\phi,u,\psi,b)\|_{H^1}^2+\langle u,\nabla\phi\rangle\right)+2C_2\|\nabla(u,\psi,b)\|_{H^1}^2+\frac{1}{2}\|\nabla\phi\|_{L^2}^2\le 0,$$

which, integrated over $(0,T)$, immediately gives

$$\sup_{0\le t\le T}\|(\phi,u,\psi,b)(t)\|_{H^1}^2+\int_0^T\left(\|\nabla(u,\psi,b)\|_{H^1}^2+\|\nabla\phi\|_{L^2}^2\right)\mathrm{d}t\le\bar{C}\|(\phi_0,u_0,\psi_0,b_0)\|_{H^1}^2, \tag{3.59}$$

since the fact $K_1=\max\{4,8C_2\}$ implies

$$|\langle u,\nabla\phi\rangle|\le\frac{K_1}{2}\|(\phi,u)\|_{H^1}^2.$$

On the other hand, one easily concludes from (3.21), Lemmas 3.4 and 3.6 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\phi,u,\psi,b)(t)\|_{H^2}^2+\|\nabla^2(u,\psi,b)\|_{H^2}^2&\le C_4\delta^{\frac14}M^{\frac52}\left(\|\nabla^2(u,\psi,b)\|_{H^2}^2+\|\nabla^2\phi\|_{H^1}^2\right)\\&\le\frac{1}{2}\|\nabla^2(u,\psi,b)\|_{H^2}^2+C_4\delta^{\frac14}M^{\frac52}\|\nabla^2\phi\|_{H^1}^2,\end{aligned}$$

and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\phi,u,\psi,b)(t)\|_{H^2}^2+\frac{1}{2}\|\nabla^2(u,\psi,b)\|_{H^2}^2\le C_4\delta^{\frac14}M^{\frac52}\|\nabla^2\phi\|_{H^1}^2, \tag{3.60}$$

provided $\delta>0$ is chosen to be such that

$$0<\delta\le\delta_3\triangleq\min\left\{\delta_2,\left(2C_4M^{\frac52}\right)^{-4}\right\}.$$

It follows from (3.37) and (3.53) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\operatorname{div}u,\Delta\phi\rangle+\langle\nabla\operatorname{div}u,\nabla\Delta\phi\rangle\right)+\|\nabla^2\phi\|_{H^1}^2\\&\quad\le C_5\|\nabla^2(u,\psi)\|_{H^2}^2+C_5\delta^{\frac14}M^{\frac52}\left(\|\nabla^2\phi\|_{H^1}^2+\|\nabla^2(u,\psi,b)\|_{H^2}^2\right)\\&\quad\le 2C_5\|\nabla^2(u,\psi,b)\|_{H^2}^2+\frac{1}{2}\|\nabla^2\phi\|_{H^1}^2,\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle\operatorname{div}u,\Delta\phi\rangle+\langle\nabla\operatorname{div}u,\nabla\Delta\phi\rangle\right)+\frac{1}{2}\|\nabla^2\phi\|_{H^1}^2\le 2C_5\|\nabla^2(u,\psi,b)\|_{H^2}^2, \tag{3.61}$$

provided

$$0<\delta\le\delta_4\triangleq\min\left\{\delta_3,\left(2C_5M^{\frac52}\right)^{-4},\left(M^{\frac52}\right)^{-4}\right\}.$$

Similarly to the proof of (3.59), multiplying (3.60) by a suitably large number $K_2 = \max\{8, 8C_5\}$, adding it to (3.61), and integrating the resulting relation over $(0, T)$, we obtain

$$\sup_{0\leq t\leq T} \|\nabla(\phi, u, \psi, b)(t)\|_{H^2}^2 + \int_0^T \left(\|\nabla^2(u, \psi, b)\|_{H^2}^2 + \|\nabla^2\phi\|_{H^1}^2\right) \mathrm{d}t \leq \tilde{C}\|\nabla(\phi_0, u_0, \psi_0, b_0)\|_{H^2}^2. \tag{3.62}$$

With the help of (3.59) and (3.62), by bootstrap arguments we easily close the a priori assumptions (3.4) and (3.5) by taking $M^2 = 2\tilde{C}\|\nabla(\phi_0, u_0, \psi_0, b_0)\|_{H^2}^2$ and choosing $\eta > 0$ sufficiently small to be such that $2\bar{C}\|(\phi_0, u_0, \psi_0, b_0)\|_{H^1}^2 \leq \delta^2$. This, together with the local existence result (cf. Lemma 2.5), finishes the proof of Theorem 3.1, and thus, the proof of the first part of Theorem 1.1 is complete. □

3.2. Decay rates

This subsection is devoted to the derivations of the decay rates of the solutions. As a first step, we derive the decay estimates of $\|\nabla(\phi, u, \psi, b)\|_{H^2}$.

Lemma 3.8. *Assume that the conditions of Theorem 3.1 are satisfied. If $\eta > 0$ is small enough and $\|(\phi_0, u_0, \psi_0, b_0)\|_{L^1}$ is bounded, then for any $t \geq 0$,*

$$\|\nabla(\phi, u, \psi, b)(t)\|_{H^2}^2 \leq C(1+t)^{-\frac{5}{2}} \quad \text{and} \quad \|(\phi, u, \psi, b)(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{3}{2}}. \tag{3.63}$$

Proof. Similarly to the derivation of (3.59), it is easily deduced from (3.21) and Lemmas 3.4–3.7 that there exist positive numbers $\bar{K} > 0$ and k such that if $\eta > 0$ is small enough, then

$$E'(t) + k\left(\|\nabla^2(u, \psi, b)\|_{H^2}^2 + \|\nabla^2\phi\|_{H^1}^2\right) \leq 0,$$

where

$$E(t) \triangleq \bar{K}\|\nabla(\phi, u, \psi, b)\|_{H^2}^2 + \langle \operatorname{div} u, \Delta\phi\rangle + \langle \nabla \operatorname{div} u, \nabla\Delta\phi\rangle.$$

Adding $\|\nabla(\phi, u, \psi, b)\|_{L^2}^2$ to both sides of the last inequality, we have

$$E'(t) + k\|\nabla(\phi, u, \psi, b)\|_{H^2}^2 \leq C\|\nabla(\phi, u, \psi, b)\|_{L^2}^2. \tag{3.64}$$

Noting that for suitably large $\bar{K} > 0$,

$$E(t) \sim \|\nabla(\phi, u, \psi, b)\|_{H^2}^2,$$

and thus, it follows from (3.64) that there exists some positive number $c > 0$ such that

$$E'(t) + cE(t) \leq C\|\nabla(\phi, u, \psi, b)\|_{L^2}^2,$$

so that

$$E(t) \leq E(0)e^{-ct} + C\int_0^t e^{-c(t-\tau)}\|\nabla(\phi, u, \psi, b)(\tau)\|_{L^2}^2 d\tau. \tag{3.65}$$

In order to deal with $\|\nabla(\phi, u, \psi, b)\|_{L^2}^2$, we first make use of Lemma 2.2 to see that

$$\begin{aligned}\|\nabla(\phi, u, \psi, b)(t)\|_{L^2} &\leq C(1+t)^{-\frac{5}{4}} + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|(S_1, S_2, S_3, S_4)\|_{L^1\cap H^1} d\tau \\ &\leq C(1+t)^{-\frac{5}{4}} + C\eta^{\frac{1}{2}}\int_0^t (1+t-\tau)^{-\frac{5}{4}}\sqrt{E(\tau)} d\tau,\end{aligned} \tag{3.66}$$

where Theorem 3.1, (3.18), (3.36) and the following simple facts were also used.

$$\begin{aligned}&\|(S_1,S_2,S_3,S_4)\|_{L^1}\le C\left(\|(\phi,u,\psi,b)\|_{L^2}\|\nabla(\phi,u,\psi,b)\|_{H^1}+\|\nabla(u,b)\|_{L^2}^2+\|b\|_{L^2}\|\nabla\phi\|_{L^3}\|\nabla b\|_{L^6}\right)\\&\quad\le \eta C(K)\|\nabla(\phi,u,\psi,b)\|_{H^1}\le \eta C(K)\sqrt{E(t)},\end{aligned}\tag{3.67}$$

$$\begin{aligned}&\|S_1\|_{H^1}+\|(S_2,S_3,S_4)\|_{H^1}\le C\|(\phi,u)\|_{H^1}\|\nabla(\phi,u)\|_{H^2}+\eta^{\frac12}C(K)\|\nabla(\phi,u,\psi,b)\|_{H^2}\\&\quad\le \eta^{\frac12}C(K)\sqrt{E(t)}.\end{aligned}\tag{3.68}$$

Define

$$\mathcal{E}(t)\triangleq\sup_{0\le\tau\le t}(1+\tau)^{\frac52}E(\tau).$$

Then, it follows from Lemma 2.4 that

$$\begin{aligned}\|\nabla(\phi,u,\psi,b)(t)\|_{L^2}&\le C(1+t)^{-\frac54}+C\eta^{\frac12}\sqrt{\mathcal{E}(t)}\int_0^t(1+t-\tau)^{-\frac54}(1+\tau)^{-\frac54}d\tau\\&\le C(1+t)^{-\frac54}\left(1+\eta^{\frac12}\sqrt{\mathcal{E}(t)}\right)\end{aligned}$$

which, inserted into (3.65), shows that for any $t\ge0$ (noting that $\mathcal{E}(t)$ is non-decreasing)

$$\begin{aligned}(1+t)^{\frac52}E(t)&\le C(1+t)^{\frac52}e^{-ct}+C\left(1+\eta\mathcal{E}(t)\right)(1+t)^{\frac52}\int_0^te^{-c(t-\tau)}(1+\tau)^{-\frac52}d\tau\\&\le C+C\eta\mathcal{E}(t).\end{aligned}$$

As a result, if $\eta>0$ is chosen to be small enough, then one has $\mathcal{E}(t)\le C$. This, together with the fact that $E(t)\sim\|\nabla(\phi,u,\psi,b)\|_{H^2}^2$, proves $(3.63)_1$.

Moreover, using $(3.63)_1$, (3.67) and (3.68), we have from Lemmas 2.2 and 2.4 that

$$\begin{aligned}\|(\phi,u,\psi,b)(t)\|_{L^2}&\le C(1+t)^{-\frac34}+C\int_0^t(1+t-\tau)^{-\frac34}\|(S_1,S_2,S_3,S_4)\|_{L^1\cap L^2}d\tau\\&\le C(1+t)^{-\frac34}+C\int_0^t(1+t-\tau)^{-\frac34}\sqrt{E(\tau)}d\tau\\&\le C(1+t)^{-\frac34}+C\mathcal{E}(t)\int_0^t(1+t-\tau)^{-\frac34}(1+\tau)^{-\frac54}d\tau\\&\le C(1+t)^{-\frac34},\end{aligned}\tag{3.69}$$

which immediately yields the desired estimate of $(3.63)_2$. The proof of (3.63) is thus finished. □

Compared with Lemma 2.1, it is easily seen that the decay rates of H^1-norm of solutions stated in (3.63) are optimal. In the next lemma, we aim to improve the decay estimates of higher derivatives, which will be achieved by using Lemma 3.8.

Lemma 3.9. *There exists a positive time $T_1>0$ such that if $\eta>0$ is small enough, then*

$$\|\nabla^2(\phi,u,\psi,b)(t)\|_{H^1}^2\le C(1+t)^{-\frac72},\quad\forall\ t\ge T_1.\tag{3.70}$$

Proof. In view of Lemma 2.3, we have

$$\begin{aligned}\|\nabla S_1\|_{L^2}&\le\|\nabla^2(\phi u)\|_{L^2}\le C\left(\|\phi\|_{L^3}\|\nabla^2u\|_{L^6}+\|u\|_{L^3}\|\nabla^2\phi\|_{L^6}\right)\\&\le C\|(\phi,u)\|_{H^1}\|\nabla^3(\phi,u)\|_{L^2}\le C\eta\|\nabla^3(\phi,u)\|_{L^2}.\end{aligned}\tag{3.71}$$

From the derivations of (3.30)–(3.35), by Theorem 3.1 and Lemma 3.8 we obtain

$$\begin{aligned}\|\nabla(S_2,S_3,S_4)\|_{L^2} &\le C\eta\|\nabla^3(u,\psi,b)\|_{L^2}+C(K)\|\nabla(\phi,u,\psi,b)\|_{H^2}^2\\ &\le C\eta\|\nabla^3(u,\psi,b)\|_{L^2}+C(K)(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{3.72}$$

Thus, substituting (3.71) and (3.72) into (3.28) and integrating by parts, by Young inequality we see that if $\eta>0$ is small enough, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2(\phi,u,\psi,b)\|_{L^2}^2+\|\nabla^3(u,\psi,b)\|_{L^2}^2\le C\eta\|\nabla^3\phi\|_{L^2}^2+C(1+t)^{-5},$$

which, combined with (3.40) and (3.54), yields that there exist positive numbers $\tilde{K}$ (suitably large) and k_1 (suitably small) such that if $\eta>0$ is small enough, then

$$E_1'(t)+k_1\left(\|\nabla^3(u,\psi,b)\|_{H^1}^2+\|\nabla^3\phi\|_{L^2}^2\right)\le C(1+t)^{-5},\tag{3.73}$$

where

$$E_1(t)\triangleq\tilde{K}\|\nabla^2(\phi,u,\psi,b)\|_{H^1}^2+\langle\nabla\operatorname{div}u,\nabla\Delta\phi\rangle\sim\|\nabla^2(\phi,u,\psi,b)\|_{H^1}^2.$$

Noting that the Sobolev interpolation inequality, together with the Young inequality, gives

$$\|\nabla^2\phi\|_{L^2}^2\le C\|\nabla\phi\|_{L^2}\|\nabla^3\phi\|_{L^2}\le\alpha^{-1}(1+t)\|\nabla^3\phi\|_{L^2}^2+C(\alpha)(1+t)^{-1}\|\nabla\phi\|_{L^2}^2$$

where $\alpha>0$ is a positive constant to be chosen later. This particularly implies that

$$\|\nabla^3\phi\|_{L^2}^2\ge\alpha(1+t)^{-1}\|\nabla^2\phi\|_{L^2}^2-C(\alpha)(1+t)^{-2}\|\nabla\phi\|_{L^2}^2.\tag{3.74}$$

In a similar manner,

$$\|\nabla^3(u,\psi,b)\|_{H^1}^2\ge\alpha(1+t)^{-1}\|\nabla^2(u,\psi,b)\|_{H^1}^2-C(\alpha)(1+t)^{-2}\|\nabla(u,\psi,b)\|_{H^1}^2.\tag{3.75}$$

As a result of (3.74) and (3.75), we deduce from (3.73) that

$$\begin{aligned}&E_1'(t)+\frac{\alpha k_1}{2}(1+t)^{-1}\|\nabla^2(u,\psi,b)\|_{H^1}^2+\frac{k_1}{2}\left(\alpha(1+t)^{-1}\|\nabla^2\phi\|_{L^2}^2+\|\nabla^3\phi\|_{L^2}^2\right)\\ &\le C(1+t)^{-5}+C(1+t)^{-2}\left(\|\nabla\phi\|_{L^2}^2+\|\nabla(u,\psi,b)\|_{H^1}^2\right)\le C(\alpha)(1+t)^{-\frac{9}{2}}.\end{aligned}\tag{3.76}$$

It is clear that if $t\ge\alpha>0$, then $\|\nabla^3\phi\|_{L^2}\ge\alpha(1+t)^{-1}\|\nabla^3\phi\|_{L^2}$. So, we have from (3.76) that

$$E_1'(t)+\frac{\alpha k_1}{2}(1+t)^{-1}\|\nabla^2(\phi,u,\psi,b)\|_{H^1}^2\le C(\alpha)(1+t)^{-\frac{9}{2}}.\tag{3.77}$$

Moreover, since $E_1(t)\sim\|\nabla^2(\phi,u,\psi,b)\|_{H^1}^2$ for suitably large $\tilde{K}>0$, there exists a positive constant c_1, depending only on $\tilde{K}$ and k_1, such that

$$E_1'(t)+\alpha c_1(1+t)^{-1}E_1(t)\le C(\alpha)(1+t)^{-\frac{9}{2}}.\tag{3.78}$$

Thus, if $\alpha>0$ is chosen to be $\alpha=4c_1^{-1}$, then one easily concludes from (3.78) that

$$E_1'(t)+4(1+t)^{-1}E_1(t)\le C(1+t)^{-\frac{9}{2}},$$

and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^4E_1(t)\right]=(1+t)^4\left(E_1'(t)+4(1+t)^{-1}E_1(t)\right)\le C(1+t)^{-\frac{1}{2}},$$

which, integrated over $(0,t)$, results in

$$(1+t)^4E_1(t)\le E_1(0)+C_1(1+t)^{\frac{1}{2}}\le 2C_1(1+t)^{\frac{1}{2}},\tag{3.79}$$

provided $t>0$ is large enough such that

$$t\ge T_1\triangleq\max\left\{\alpha,\left(\frac{E_1(0)}{C_1}\right)^2-1\right\}.$$

Now, (3.70) readily follows from (3.79) and the fact that $E_1(t) \sim \|\nabla^2(\phi, u, \psi, b)\|_{H^1}^2$. The proof of Lemma 3.9 is therefore finished. □

Finally, we improve the decay rates of $\|\nabla^3 b\|_{L^2}$, based on a full use of Lemmas 3.8 and 3.9.

Lemma 3.10. *There exists a positive time $T_2 > 0$ such that if $\eta > 0$ is small enough, then*

$$\|\nabla^3 b(t)\|_{H^1}^2 \le C(1+t)^{-\frac{9}{2}}, \quad \forall\, t \ge T_2. \tag{3.80}$$

Proof. Indeed, similarly to the derivation of (3.41), we obtain after integrating by parts and using Cauchy–Schwarz inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3 b\|_{L^2}^2 + \|\nabla^4 b\|_{L^2}^2 \le C\|\nabla^2 S_4\|_{L^2}^2. \tag{3.81}$$

We are now ready to deal with $\|\nabla^2 S_4\|_{L^2}$. First, using Lemmas 3.8 and 3.9, we have

$$\begin{aligned}
&\|\nabla^2(u\cdot\nabla b)\|_{L^2} + \|\nabla^2(b\cdot\nabla u)\|_{L^2} + \|\nabla^2(b\,\mathrm{div}\,u)\|_{L^2}\\
&\quad\le C\left(\|(u,b)\|_{L^\infty}\|\nabla^3(u,b)\|_{L^2} + \|\nabla(u,b)\|_{L^3}\|\nabla^2(u,b)\|_{L^6}\right)\\
&\quad\le C\|\nabla(u,b)\|_{L^2}^{\frac{1}{2}}\|\nabla^2(u,b)\|_{L^2}^{\frac{1}{2}}\|\nabla^3(u,b)\|_{L^2} \le C(1+t)^{-\frac{26}{8}}
\end{aligned} \tag{3.82}$$

and moreover, by Lemma 2.3 we obtain in a manner similarly to the derivation of (3.51) that

$$\begin{aligned}
\|\nabla^3(g(\phi)(\nabla\times b)\times b)\|_{L^2} &\le C(K)(1+t)^{-\frac{26}{8}} + C\|b\|_{L^\infty}\|\nabla^4 b\|_{L^2}\\
&\le C(K)(1+t)^{-\frac{26}{8}} + \eta^{\frac{1}{2}}C(K)\|\nabla^4 b\|_{L^2}.
\end{aligned} \tag{3.83}$$

Taking (3.82) and (3.83) into account using Cauchy–Schwarz inequality and choosing $\eta > 0$ small enough, we infer from (3.81) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3 b\|_{L^2}^2 + \|\nabla^4 b\|_{L^2}^2 \le C(1+t)^{-\frac{26}{4}}. \tag{3.84}$$

Similarly to (3.74), by Lemma 3.9 we have

$$\begin{aligned}
\|\nabla^4 b\|_{L^2}^2 &\ge 5(1+t)^{-1}\|\nabla^3 b\|_{L^2}^2 - C(1+t)^{-2}\|\nabla^2 b\|_{L^2}^2\\
&\ge 5(1+t)^{-1}\|\nabla^3 b\|_{L^2}^2 - C(1+t)^{-\frac{11}{2}},
\end{aligned}$$

which, inserted into (3.84), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3 b\|_{L^2}^2 + 5(1+t)^{-1}\|\nabla^3 b\|_{L^2}^2 \le C(1+t)^{-\frac{11}{2}}. \tag{3.85}$$

Thus, we conclude from (3.85) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^5\|\nabla^3 b\|_{L^2}^2\right) = (1+t)^5\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^3 b\|_{L^2}^2 + 5(1+t)^{-1}\|\nabla^3 b\|_{L^2}^2\right) \le C(1+t)^{-\frac{1}{2}}, \tag{3.86}$$

which immediately shows

$$(1+t)^5\|\nabla^3 b(t)\|_{L^2}^2 \le \|\nabla^3 b_0\|_{L^2}^2 + C_2(1+t)^{\frac{1}{2}} \le 2C_2(1+t)^{\frac{1}{2}},$$

provided

$$t \ge T_2 \triangleq \max\left\{T_1, \left(\frac{\|\nabla^3 b_0\|_{L^2}^2}{C_2}\right)^2 - 1\right\}.$$

As a result of (3.86), we arrive at the desired decay rate stated in (3.81). □

Proof of Decay Rates. Collecting Lemmas 3.8–3.10 together, one immediately obtains the desired decay rates stated in the second part of Theorem 1.1. □

4. Proof of Theorem 1.2

To prove Theorem 1.2, we first observe that all the global estimates and the decay rates established in Sect. 3 also hold for the standard MHD equations without Hall effects (i.e., $\varepsilon \equiv 0$):

$$\begin{cases} \bar\rho_t + \operatorname{div}(\bar\rho\bar u) = 0, \\ (\bar\rho\bar u)_t + \operatorname{div}(\bar\rho\bar u\otimes\bar u) + \nabla\bar p = \mu\Delta\bar u + (\mu+\lambda)\nabla\operatorname{div}\bar u + (\nabla\times\bar b)\times\bar b, \\ \mathrm{c_V}\left[(\bar\rho\bar\theta)_t + \operatorname{div}(\bar\rho\bar u\bar\theta)\right] + \bar p\operatorname{div}\bar u = \kappa\Delta\bar\theta + \frac{\mu}{2}\left|\nabla\bar u + (\nabla\bar u)^\top\right|^2 + \lambda(\operatorname{div}\bar u)^2 + \nu|\nabla\times\bar b|^2, \\ \bar b_t - \nabla\times(\bar u\times\bar b) = \nu\Delta\bar b, \\ \operatorname{div}\bar b = 0, \end{cases} \tag{4.1}$$

which are equipped with the far-field boundary conditions and the initial conditions:

$$\begin{cases} (\bar\rho,\bar u,\bar\theta,\bar b)(x,t) \to (1,0,1,0) \quad \text{as} \quad |x|\to\infty, \\ (\bar\rho,\bar u,\bar\theta,\bar b)|_{t=0} = (\rho_0,u_0,\theta_0,b_0)(x), \quad x\in\mathbb{R}^3. \end{cases} \tag{4.2}$$

Here, the pressure $\bar p = R\bar\rho\bar\theta$.

With the help of the global a priori estimates, we can show the following global existence theorem for the Cauchy problem (4.1)–(4.2).

Theorem 4.1. *For any given positive number $\bar M_0 > 0$ (not necessarily small), assume that the initial data $(\rho_0-1,u_0,\theta_0-1,b_0)\in H^3$ satisfy*

$$\|\nabla^2(\rho_0-1,u_0,\theta_0-1,b_0)\|_{H^1} \le \bar M_0, \tag{4.3}$$

then there exists a positive constant $\bar\eta_0>0$, depending on $\bar M_0$, such that the problem (4.1)–(4.2) has a unique global classical solution $(\bar\rho,\bar u,\bar\theta,\bar b)$ on $\mathbb{R}^3\times(0,\infty)$ satisfying

$$\begin{aligned} &\|(\bar\rho-1,\bar u,\bar\theta-1,\bar b)(t)\|_{H^3}^2 + \int_0^t (\|\nabla\bar\rho(\tau)\|_{H^2}^2 + \|\nabla(\bar u,\bar\theta,\bar b)(\tau)\|_{H^3}^2)d\tau \\ &\le C\|(\rho_0-1,u_0,\theta_0-1,b_0)\|_{H^3}^2 \end{aligned} \tag{4.4}$$

for all $t\ge 0$, provided

$$\|(\rho_0-1,u_0,\theta_0-1,b_0)\|_{H^1} \le \bar\eta_0, \tag{4.5}$$

where $C>0$ is a generic positive constant independent of t.

Let (ρ,u,θ,b) be the global solutions of the problem (1.1)–(1.3) obtained in Theorem 1.1. Indeed, based on the global uniform-in-ε estimates derived in Sect. 3, it is easily checked that as $\varepsilon\to 0$, there exists a subsequence of (ρ,u,θ,b) (still denoted by (ρ,u,θ,b)) such that

$$(\rho,u,\theta,b)\to(\bar\rho,\bar u,\bar\theta,\bar b) \quad \text{strongly in} \quad C([0,T];H^2), \quad \forall\, T\in(0,\infty).$$

To prove the convergence rates, set

$$\Phi = \rho-\bar\rho, \quad U = u-\bar u, \quad \Psi = \theta-\bar\theta, \quad B = b-\bar b.$$

Then, it can be easily derived from (1.1) and (4.1) that

$$\Phi_t + U\cdot\nabla\rho + \bar u\cdot\nabla\Phi + \rho\operatorname{div}U + \Phi\operatorname{div}\bar u = 0, \tag{4.6}$$

$$\begin{aligned} &\rho U_t + \rho u\cdot\nabla U - \mu\Delta U - (\mu+\lambda)\nabla\operatorname{div}U = -\Phi\bar u_t - \rho U\cdot\nabla\bar u - \Phi\bar u\cdot\nabla\bar u - \nabla(p-\bar p) \\ &\quad -\frac{1}{2}\nabla\left(|b|^2-|\bar b|^2\right) + b\cdot\nabla B + B\cdot\nabla\bar b, \end{aligned} \tag{4.7}$$

$$\rho\Psi_t + \rho u\cdot\nabla\Psi - \Delta\Psi = -\Phi\bar\theta_t - \rho U\cdot\nabla\bar\theta - \Phi\bar u\cdot\nabla\bar\theta - p\operatorname{div}U - (p-\bar p)\operatorname{div}\bar u$$

$$+\frac{\mu}{2}\left(|\nabla u+(\nabla u)^\top|^2-|\nabla\bar{u}+(\nabla\bar{u})^\top|^2\right)$$
$$+\lambda\left((\operatorname{div}u)^2-(\operatorname{div}\bar{u})^2\right)+\left(|\nabla\times b|^2-|\nabla\times\bar{b}|^2\right) \tag{4.8}$$

and

$$\begin{aligned}B_t-\Delta B=&-u\cdot\nabla B-U\cdot\nabla\bar{b}+b\cdot\nabla U+B\cdot\nabla\bar{u}-b\operatorname{div}U-B\operatorname{div}\bar{u}\\&-\varepsilon\nabla\times\left(\rho^{-1}(\nabla\times b)\times b\right).\end{aligned} \tag{4.9}$$

Moreover, the pair of functions (Φ,U,Ψ,B) satisfies the vanishing far-field and initial conditions.

We first prove the L^2-convergence rate of the vanishing limit of Hall coefficient.

Lemma 4.1. *For any $T\in(0,\infty)$, there exists a positive constant $C=C(T)$, independent of ε, such that*

$$\sup_{0\le t\le T}\|(\Phi,U,\Psi,B)(t)\|_{L^2}^2+\int_0^T\|\nabla(U,\Psi,B)\|_{L^2}^2\mathrm{d}t\le C\varepsilon^2. \tag{4.10}$$

Proof. The proofs will be carried out by the standard L^2-method, based on the global estimates stated in Theorems 1.1 and 4.1. For completeness, we sketch the proofs.

First, multiplying (4.6) by Φ in L^2, integrating parts, using Theorems 1.1, 4.1 and Cauchy–Schwarz inequality, we easily get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_{L^2}^2\le C\|(\nabla\rho,\nabla\bar{u})\|_{H^2}\left(\|(\Phi,U)\|_{L^2}^2+\|\nabla U\|_{L^2}^2\right)\le C\left(\|(\Phi,U)\|_{L^2}^2+\|\nabla U\|_{L^2}^2\right). \tag{4.11}$$

Note that

$$|p-\bar{p}|+|b|^2-|\bar{b}|^2=|\Phi\theta+\bar{\rho}\Psi|+(b+\bar{b})\cdot B\le C\left(|\Phi|+|\Psi|+|B|\right). \tag{4.12}$$

Thus, multiplying (4.7) by U in L^2 and integrating by parts, by Theorems 1.1 and 4.1 we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2\le C\|(\Phi,U,\Psi,B)\|_{L^2}^2 \tag{4.13}$$

where we have used the following estimates:

$$\|(u_t,\bar{u}_t,\theta_t,\bar{\theta}_t,b_t,\bar{b}_t)\|_{H^1}^2+\int_0^T\|(u_t,\bar{u}_t,\theta_t,\bar{\theta}_t,b_t,\bar{b}_t)\|_{H^2}^2\mathrm{d}t\le C, \tag{4.14}$$

due to (1.1), (4.1), Theorems 1.1 and 4.1. Similarly, since

$$\begin{aligned}&\left(|\nabla u+(\nabla u)^\top|^2-|\nabla\bar{u}+(\nabla\bar{u})^\top|^2\right)+\left((\operatorname{div}u)^2-(\operatorname{div}\bar{u})^2\right)+\left(|\nabla\times b|^2-|\nabla\times\bar{b}|^2\right)\\&\quad\le C\|\nabla(u,\bar{u},b,\bar{b})\|_{L^\infty}\left(|\nabla U|+|\nabla B|\right)\le C\left(|\nabla U|+|\nabla B|\right),\end{aligned} \tag{4.15}$$

we obtain after multiplying (4.8) by Ψ in L^2 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}\Psi\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2\le C\|(\Phi,U,\Psi)\|_{L^2}^2+C\|\nabla(U,B)\|_{L^2}^2. \tag{4.16}$$

In a similar manner,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|B\|_{L^2}^2+\|\nabla B\|_{L^2}^2\le C\|(U,B)\|_{L^2}^2+C\varepsilon^2, \tag{4.17}$$

since Theorem 1.1 implies that

$$\varepsilon\|\rho^{-1}(\nabla\times b)\times b\|_{H^2}\le C\varepsilon. \tag{4.18}$$

Multiplying (4.13) and (4.17) by a suitable large constant, then adding them with (4.11) and (4.16) together, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\Phi,\sqrt{\rho}U,\sqrt{\rho}\Psi,B)(t)\|_{L^2}^2+\|\nabla(U,\Psi,B)\|_{L^2}^2\le C\|(\Phi,U,\Psi,B)\|_{L^2}^2+C\varepsilon^2,$$

which, combined with Gronwall inequality and the fact that ρ is strictly positive, leads to the desired result of (4.10). □

The next lemma is concerned with the H^1-convergence rate of the vanishing limit of Hall coefficient.

Lemma 4.2. *There exists a positive constant $C = C(T)$, independent of ε, such that for any $T > 0$,*

$$\sup_{0\le t\le T} \|\nabla(\Phi, U, \Psi, B)(t)\|_{L^2}^2 + \int_0^T \|\nabla^2(U, \Psi, B)\|_{L^2}^2 \mathrm{d}t \le C\varepsilon^2. \tag{4.19}$$

Proof. Operating ∇ to both sides of (4.6) and multiplying it by $\nabla\Phi$ in L^2, by Lemma 4.1 and Young inequality we obtain after integrating by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi\|_{L^2}^2 \le C(\delta)\|(\Phi, U)\|_{H^1}^2 + \delta\|\nabla^2 U\|_{L^2}^2 \le C\varepsilon^2 + C(\delta)\|\nabla(\Phi, U)\|_{L^2}^2 + \delta\|\nabla^2 U\|_{L^2}^2. \tag{4.20}$$

Multiplying (4.7) by U_t in L^2, integrating by parts, using (4.10), (4.14), Theorems 1.1, 4.1, Sobolev inequalities and Cauchy–Schwarz inequality, we find (noting that ρ is strictly lower-bounded)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla U\|_{L^2}^2 + \|\sqrt{\rho}U_t\|_{L^2}^2 &\le C\|(\Phi, U, \Psi, B)\|_{L^2}^2 + C\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2 \\ &\le C\varepsilon^2 + C\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2.\end{aligned} \tag{4.21}$$

Similarly, using (4.10), (4.14), (4.15), (4.18), Theorems 1.1 and 4.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Psi\|_{L^2}^2 + \|\sqrt{\rho}\Psi_t\|_{L^2}^2 \le C\varepsilon^2 + C\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2, \tag{4.22}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla B\|_{L^2}^2 + \|B_t\|_{L^2}^2 \le C\varepsilon^2 + C\|\nabla(U, B)\|_{L^2}^2. \tag{4.23}$$

Noting that ρ is strictly lower-bounded due to (3.8), it follows from (4.20)–(4.23) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2 + \frac{1}{2}\left(\|U_t\|_{L^2}^2 + \|\Psi_t\|_{L^2}^2 + \|B_t\|_{L^2}^2\right) \\ &\le C\varepsilon^2 + C(\delta)\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2 + \delta\|\nabla^2 U\|_{L^2}^2.\end{aligned} \tag{4.24}$$

In view of (4.10), (4.14), (4.15), (4.18), Theorems 1.1 and 4.1, we easily have from (4.7)–(4.9) that

$$\begin{aligned}\|\nabla^2(U, \Psi, B)\|_{L^2} &\le C\|(U_t, \Psi_t, B_t)\|_{L^2} + C\|(\Phi, U, \Psi, B)\|_{H^1} + C\varepsilon \\ &\le C\|(U_t, \Psi_t, B_t)\|_{L^2} + C\|\nabla(\Phi, U, \Psi, B)\|_{L^2} + C\varepsilon,\end{aligned} \tag{4.25}$$

Choosing δ to be suitable small in (4.24), then (4.24) together with (4.25) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2 + \tilde{C}\|\nabla^2(U, \Psi, B)\|_{L^2} \le C\varepsilon^2 + C\|\nabla(\Phi, U, \Psi, B)\|_{L^2}^2$$

for some constant $\tilde{C} > 0$. Consequently, Lemma 4.1 and Gronwall inequality give the desired result of Lemma 4.2. □

Finally, we derive the H^2-convergence rate of the vanishing limit of Hall coefficient.

Lemma 4.3. *There exists a positive constant $C = C(T)$, independent of ε, such that for any $T > 0$,*

$$\sup_{0\le t\le T} \|\nabla^2(\Phi, U, \Psi, B)(t)\|_{L^2}^2 + \int_0^T \|\nabla^3(U, \Psi, B)\|_{L^2}^2 \mathrm{d}t \le C\varepsilon^2. \tag{4.26}$$

Proof. Using Lemma 2.3, Theorems 1.1, 4.1, (4.10), (4.19), Young inequality and Sobolev inequalities, by direct calculations we have from (4.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\Phi\|_{L^2}^2 \le C(\delta)\|(\Phi,U)\|_{H^2}^2 + \delta\|\nabla^3 U\|_{L^2}^2 \le C\varepsilon^2 + C(\delta)\|\nabla^2(\Phi,U)\|_{L^2}^2 + \delta\|\nabla^3 U\|_{L^2}^2. \tag{4.27}$$

Rewriting (4.7) in the form:

$$U_t + u\cdot\nabla U - \rho^{-1}\left(\mu\Delta U + (\mu+\lambda)\nabla\operatorname{div} U\right) = -\rho^{-1}\Big[\Phi\bar{u}_t + \rho U\cdot\nabla\bar{u} + \Phi\bar{u}\cdot\nabla\bar{u} + \nabla(p-\bar{p}) + \frac{1}{2}\nabla\left(|b|^2 - |\bar{b}|^2\right) - b\cdot\nabla B - B\cdot\nabla\bar{b}\Big].$$

Operating ∇^2 to the above equation and multiplying it by $\nabla^2 U$ in L^2, by tedious but straightforward computations we obtain after integrating by parts and using Lemma 2.3 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 U\|_{L^2}^2 + \|\nabla^3 U\|_{L^2}^2 \le C\|(\Phi,U,\Psi,B)\|_{H^2}^2 \le C\varepsilon^2 + C\|\nabla^2(\Phi,U,\Psi,B)\|_{L^2}^2, \tag{4.28}$$

where we have used (4.10), (4.14), (4.19), Theorems 1.1 and 4.1. Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2\Psi\|_{L^2}^2 + \|\nabla^3\Psi\|_{L^2}^2 \le C\varepsilon^2 + C\|\nabla^2(\Phi,U,\Psi,B)\|_{L^2}^2. \tag{4.29}$$

Operating ∇^2 to both sides of (4.9) and multiplying the resulting equation by $\nabla^2 B$ in L^2, by virtue of (4.10), (4.18) and (4.19) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 B\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2 \le C\varepsilon^2 + C\|\nabla^2(\Phi,U,\Psi,B)\|_{L^2}^2, \tag{4.30}$$

choosing δ to be suitable small in (4.27), then (4.27)–(4.30) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2(\Phi,U,\Psi,B)(t)\|_{L^2}^2 + \|\nabla^3(U,\Psi,B)\|_{L^2}^2 \le C\varepsilon^2 + C\|\nabla^2(\Phi,U,\Psi,B)\|_{L^2}^2, \tag{4.31}$$

and thus, an application of Gronwall inequality yields (4.26). □

Proof of Theorem 1.2. Now, the convergence rates of vanishing limit of Hall coefficient stated in Theorem 1.2 readily follow from Lemmas 4.1–4.3. □

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References

[1] Acheritogaray, M., Degond, P., Frouvelle, A., Liu, J.G.: Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system. Kinet. Relat. Models **4**, 901–918 (2011)
[2] Adams, R.A.: Sobolev Space. Academic Press, New York (1975)
[3] Balbus, S.A., Terquem, C.: Linear analysis of the Hall effect in protostellar disks. Astrophys. J. **552**, 235–247 (2001)
[4] Chae, D.H., Degond, P., Liu, J.G.: Well-posedness for Hall-magnetohydrodynamics. Ann. Inst. H. Poincar Anal. Non Linéaire **31**, 555–565 (2014)
[5] Chae, D.H., Lee, J.H.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. J. Differ. Equ. **256**, 3835–3858 (2014)
[6] Chae, D.H., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equatioins. J. Differ. Equ. **255**(11), 3971–3982 (2013)
[7] Chae, D.H., Wan, R.H., Wu, J.H.: Local well-posedness for the Hall-MHD equations with fractional magnetic diffusion. J. Math. Fluid Mech. **17**, 627–638 (2015)
[8] Chae, D.H., Weng, S.K.: Singularity formation for the incompressible Hall-MHD equations without resistivity. Ann. Inst. H. Poincaré Anal. Non Linéaire **4**, 1009–1022 (2016)
[9] Chae, D.H., Wolf, J.: On partial regularity for the 3D nonstationary Hall magnetohydrodynamics equations on the plane. SIAM J. Math. Anal. **48**, 443–469 (2016)

[10] Crispo, F., Maremonti, P.: An interpolation inequality in exterior domains. Rend. Sem. Mat. Univ. Padova **112**, 11–39 (2004)

[11] Duan, R.J., Liu, H.X., Ukai, S.J., Yang, T.: Optimal L^p-L^q convergence rates for the compressible Navier–Stokes equations with potential force. J. Differ. Equ. **238**, 220–233 (2007)

[12] Fan, J.S., Ahmad, B., Hayat, T., Zhou, Y.: On well-posedness and blow-up for the full compressible Hall-MHD system. Nonlinear Anal. Real World Appl. **31**, 569–579 (2016)

[13] Fan, J.S., Alsaedi, A., Hayat, T., Nakamura, G., Zhou, Y.: On strong solutions to the compressible Hall-magnetohydrodynamic system. Nonlinear Anal. Real World Appl. **22**, 423–434 (2015)

[14] Fan, J.S., Jia, X.J., Nakamura, G., Zhou, Y.: On well-posedness and blow-up criteria for the magnetohydrodynamics with the Hall and ion-slip effects. Z. Angew. Math. Phys. **66**, 1695–1706 (2015)

[15] Fan, J.S., Yu, W.H.: Strong solution to the compressible magnetohydrodynamic equations with vacuum. Nonlinear Anal. Real World Appl. **10**, 392–409 (2009)

[16] Forbes, T.G.: Magnetic reconnection in solar flares. Geophys. Astrophys. Fluid Dyn. **62**, 15–36 (1991)

[17] Gagliardo, E.: Ulteriori proprietà di alcune classi di funzioni in più variabili. Ricerche Mat. Univ. Napoli **8**, 24–51 (1959)

[18] Gao, J.C., Yao, Z.A.: Global existence and optimal decay rates of solutions for compressible Hall-MHD equations. Discrete Contin. Dyn. Syst. **36**, 3077–3106 (2016)

[19] Hall, E.: On a new action of the magnet on electric currents. Am. J. Math. **2**, 287–92 (1879)

[20] Homann, H., Grauer, R.: Bifurcation analysis of magnetic reconnection in Hall-MHD systems. Phys. D **208**, 59–72 (2005)

[21] Hu, X.P., Wang, D.H.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)

[22] Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Springer, New York (1984)

[23] Mininni, P.D., Gòmez, D.O., Mahajan, S.M.: Dynamo action in magnetohydrodynamics and Hall magnetohydrodynamics. Astrophys. J. **587**, 472–481 (2003)

[24] Kobayashi, T.: Some estimates of solutions for the equations of motion of compressible viscous fluid in the three-dimensional exterior domain. J. Differ. Equ. **184**, 587–619 (2002)

[25] Kobayashi, T., Shibata, Y.: Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$. Commun. Math. Phys. **251**, 365–376 (2004)

[26] Li, H.L., Xu, X.Y., Zhang, J.W.: Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vaccum. SIAM J. Math. Anal. **45**, 1356–1387 (2013)

[27] Lv, B.Q., Shi, X.D., Xu, X.Y.: Global existence and large-time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum. Indiana Univ. Math. J. **65**, 925–975 (2016)

[28] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. **20**, 67–104 (1980)

[29] Pu, X.K., Guo, B.L.: Global existence and convergence rates of smooth solutions for the full compressible MHD equations. Z. Angew. Math. Phys. **64**, 519–538 (2013)

[30] Shalybkov, D.A., Urpin, V.A.: The Hall effect and the decay of magnetic fields. Astron. Astrophys. **321**, 685–690 (1997)

[31] Tan, Z., Wang, H.Q.: Optimal decay rates of the compressible magnetohydrodynamic equations. Nonlinear Anal. Real World Appl. **14**, 188–201 (2013)

[32] Treves, F.: Basic Linear Partial Differential Equations. Academic Press, New York (1975)

[33] Wardle, M.: Star formation and the Hall effect. Astrophys. Space Sci. **292**, 317–323 (2004)

[34] Xiang, Z.Y.: On the Cauchy problem for the compressible Hall-magneto-hydrodynamic equatioins. J. Evol. Equ. **17**, 685–715 (2017)

[35] Zhang, J.W., Zhao, J.N.: Some decay estimates of solutions for the 3-D compressible isentropic magnetohydrodynamics. Commun. Math. Sci. **8**, 835–850 (2010)

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