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Special relativity, de Broglie waves, dark energy and quantum mechanics

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Abstract. In this paper, we connect quantum mechanics with the recent work of the author Hill (Z Angew Math Phys 69:133–145, 2018; Z Angew Math Phys 70:5–14, 2019), suggesting that dark energy arises from the conventional mechanical theory neglecting the work done in the direction of time and consequently neglecting the de Broglie wave energy. Using special relativity and validation through Lorentz invariance, Hill (2018, 2019) develops expressions for the de Broglie wave energy $\mathscr{E}$ by making a distinction between particle energy $e = mc^2$ and the total work done by the particle W, so that both momentum $\mathbf{p} = m\mathbf{u}$ and particle energy e contribute to the total work done $W = e + \mathscr{E}$. This formulation provides an extension of Newton's second law that is invariant under the Lorentz group and gives work done expressions for $\mathscr{E}$ involving the log function, indicating that large energies might be generated even for slowing mechanical systems. Although inherent in Hill (2018, 2019), here we propose explicitly that the total work done W by a single particle comprises two contributions, namely particle energy e and wave energy $\mathscr{E}$; thus, $W = e + \mathscr{E}$. Since in any experiment either particles or de Broglie waves are reported, only one of e or $\mathscr{E}$ is physically measured, which leads to the expectation that particles appear for $e < \mathscr{E}$ and de Broglie waves occur for $\mathscr{E} \leqslant e$, but in either event, both a measurable energy and an unmeasurable energy exist, the latter registering its presence in the form of dark energy. In particular, in this formulation conventional quantum mechanics operates under circumstances such that the spatial physical force $\mathbf{f}$ vanishes, and the force g in the direction of time becomes pure imaginary. If both $\mathbf{f}$ and g are generated as the gradient of a potential, then the total particle energy is necessarily conserved in a conventional manner. The present paper makes a formal connection between special relativity and quantum mechanics, linking two new invariances of the Lorentz group of special relativity with the corresponding Lorentz invariant differential operators arising in quantum mechanics and the de Broglie particle and wave duality in Hill (2018, 2019) and giving rise to the Klein–Gordon equation of relativistic quantum mechanics.

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1. Introduction

Louis de Broglie [3] first predicted light to display the dual characteristics as both a collection of particles, called photons, or in some respects as a wave. He predicted that other elementary particles such as electrons, and indeed all matter, may, under appropriate circumstances, exhibit either particle-like or wave-like behaviour. For example, he envisaged that an electron orbiting a hydrogen atom is accompanied by a mysterious pilot wave (now known as a de Broglie wave) extending the circumferential length of the orbit, and he speculated that the length of the orbit circumference comprised an integer number of wavelengths, from which he deduced that $p = h/\lambda$, where p is the particle momentum, h is the Planck constant and λ is the wave length (see, for example, [7]). In two recent papers [12,13], the present author proposes that the origins of dark energy lie in conventional mechanical thinking not taking into account the work done in the direction of time and in consequence ignoring the de Broglie wave energy. The motivation for [12,13] and the present work arises from electromagnetism and the perceived fundamental importance of the vector and scalar potentials $(\mathbf{A}, V)$ as compared to the fields $(\mathbf{E}, \mathbf{B})$ that in standard notation are related by the formulae

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V.$$

In [12,13], corresponding potentials in mechanics are sought that play a similar role. Given that momentum and mass conservation are necessarily partnered in a special relativistic four-vector sense and the requirement of Lorentz invariant energy–momentum relations, the only available option is the adoption of momentum $\mathbf{p} = m\mathbf{u}$, particle energy $e = mc^2$ and necessarily with the Einstein mass variation. This approach appears to provide a natural framework, and the choice turns out to be a fortuitous one, since the proposed work done expression automatically makes an accommodation for the de Broglie wave energy. The purpose of this paper is to complete the picture for these ideas to include quantum mechanics and two new invariants of special relativity, giving rise to two Lorentz invariant operators in quantum mechanics, involving the Klein–Gordon equation and including new formulae not reported in the author's two previously cited papers.

1.1. Waves and group velocities

Energy transmission by a stream of particles involves mass transport through a flow of matter. However, wave motion as the passage of a local disturbance in a medium may transmit energy without any accompanying flow of matter. There are numerous types of wave motion, such as ocean waves in which the disturbance is an oscillation of the sea water along the path of the wave, or sound waves in which the disturbance is a deformation of the medium (perhaps elastic or viscoelastic) along the direction of the medium, or electromagnetic waves in which the disturbance consists of oscillations of the electric and magnetic fields, and so on. Since it is generally believed that even the most complicated wave structure can be built up from a linear combination of simple harmonic waves, attention is accordingly focussed on simple harmonic waves of the form $e^{2\pi i(x/\lambda - t/T)}$, where λ is the wavelength and T is the period of oscillation. On introducing, respectively, the wave number and angular frequency $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, in standard notation the simple harmonic wave becomes $e^{i(kx-\omega(k)t)}$, noting that typically the angular frequency ω depends upon the wave number k, namely the wavelength λ (see, for example, [18], pp 13–16). In this notation, the wave velocity w as measured by the movement of one wave peak (crest) to the next, namely "wavelength/ period", becomes simply $w = \omega(k)/k$. Now in any dispersive medium, that is, one in which the wave velocity depends upon the wavelength, waves of different wavelengths are propagated through the medium as a group with a velocity $u = \mathrm{d}\omega/\mathrm{d}k$ which is generally different from $w = \omega/k$ and the relationship $u = w + k\mathrm{d}w/\mathrm{d}k$ holds (see, for example, [22], pp. 208–213).

1.2. de Broglie waves

In brief, the de Broglie [3] idea of an accompanying pilot wave originates as follows. In the Bohr theory of the hydrogen atom (see, for example, [7], pp 29–61 for a general historical account), the nth orbit of the electron has radius r given by

$$r = \left(\frac{nh}{2\pi}\right)^2 \frac{1}{me*^2}, \tag{1.1}$$

where m and $e*$ denote, respectively, the mass and charge of the electron. However, from the mechanical and electrical force balance, we have $mr\omega^2 = e*^2/r^2$ where ω denotes angular velocity, and by substitution of $e*^2$ from this equation into Eq. (1.1) and taking the square root, we may deduce de Broglie's relation $2\pi r = nh/p$ noting that the momentum $p = mu$ and the electron velocity $u = r\omega$, for which he made the critical observation that $2\pi r$ is an integer multiple of the wavelength λ if $p = h/\lambda$ which he speculated is applied to all elementary particles. For such an elementary particle with wave velocity $w = \lambda\nu$ where $\nu = 1/T$ is the frequency, the analogous result for energy is $e = h\nu$, so that from $e = hw/\lambda$, together with $p = h/\lambda$ and $e^2 - (pc)^2 = e_0^2$, where $e_0 = m_0c^2$ and m_0 is the rest mass, we may deduce that the wave velocity w is given by

$$w = c\left(1 + \left(\frac{e_0\lambda}{hc}\right)^2\right)^{1/2},$$

and from the relationship $u = w - \lambda \mathrm{d}w/\mathrm{d}\lambda$, after differentiation and simplification we may deduce $uw = c^2$ connecting the group velocity u with the wave velocity w. Further, with the usual relations $e = mc^2$ and $p = mu$, and $e/p = \lambda\nu = w$ it is not difficult to show that the group velocity u of the wave package coincides with the particle velocity as defined by that velocity occurring in the expression $p = mu$ for momentum.

Thus, the group velocity of the wave u coincides with the particle velocity, and if the particle velocity u is subluminal, then the associated wave or phase velocity c^2/u through the de Broglie relation is necessarily superluminal. This is "believed" not to contradict the fact that information cannot be carried faster than the speed of light c because "supposedly" the wave phase does not carry energy. However, the superluminal phase velocity may well be physically significant, and as suggested in [12,13], dark energy may well exist as a consequence that the associated de Broglie wave energy is neglected. If the wave energy through the superluminal wave speed c^2/u is accommodated, then it is not difficult to envisage interesting outcomes for slowing particle speeds u tending to zero.

In [12,13], it is shown that within the confines of Lorentz invariance and special relativity, alternative energy accounting procedures exist for which large energies can be generated that are well in excess of that predicted by Einstein's expression. Einstein's formulae describing the variation with velocity u for the energy and mass of a particle $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, where m_0 denotes the rest mass and c is the speed of light, have been overwhelmingly verified in our local environment, but these relations are not so successful on a cosmological scale (see, for example, [21]). Einstein's formulae are based on the assumption that the particle energy e accrues from and coincides with the work done and is derived from an energy rate equation $\mathrm{d}e/\mathrm{d}t = \mathbf{u}.(\mathrm{d}\mathbf{p}/\mathrm{d}t)$, sometimes referred to as the rate-of-working equation, where $\mathbf{p} = m\mathbf{u}$ is the momentum and $\mathbf{u}$ is the velocity vector. In [12,13], a distinction is made between particle energy e and the work done, and our purpose here is to connect the ideas of [12,13] with two new invariances of the Lorentz group in special relativity, which are linked with the de Broglie particle and wave duality which in turn give rise to two corresponding Lorentz invariant operators in quantum mechanics.

2. General formulation

Assuming all quantities are both position $\mathbf{x}$ and time t dependent and assuming the usual formulae of special relativity, namely $e = mc^2$ and $m(u) = m_0[1-(u/c)^2]^{-1/2}$, in [12,13] a distinction is made between the particle energy $e = mc^2$ and the actual work done by the particle.

2.1. Force formulae

As a formal extension of Newton's second law, the following force $\mathbf{f}$ and energy–mass production g are proposed:

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla.\mathbf{p}, \tag{2.1}$$

noting that this formulation assumes that space and time are on an equal footing so that all derivatives in (2.1) are assumed to be partial. Further, the second equation of (2.1) is the generally accepted mass continuity equation but including an energy–mass production term g, and might be viewed as Newton's second law in the direction of time. We comment that at both local length scales (conservation of mass in classical mechanics) and at atomic and molecular length scales (conservation of probability in quantum mechanics) the quantity g is generally assumed to be zero. In [12], an explicit wave-like solution is obtained on the basis that g is assumed to be nonzero. The Lorentz invariance of (2.1) hinges on the precise given signatures, and a critical calculation elaborated in [12] and purposely repeated in the following section is that both $\mathbf{f}$ and g are Lorentz invariants so that $(\mathbf{f}, gc)$ is well-defined four vectors.

Assuming the Einstein mass variation implies that energy e and momentum $\mathbf{p}$ satisfy the relation $e^2 - c^2\mathbf{p}.\mathbf{p} = e_0^2$, where $e_0 = m_0c^2$ and m_0 is the rest mass, so that the following relations are applied:

$$\frac{\partial e}{\partial t} = \mathbf{u}.\frac{\partial \mathbf{p}}{\partial t}, \qquad \nabla e = (\mathbf{u}.\nabla)\mathbf{p}, \qquad \frac{\mathrm{d}e}{\mathrm{d}t} = \mathbf{u}.\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t},$$

which we may exploit to establish the identities:

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{p} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{2.2}$$

and

$$g = \frac{1}{c^2}\left(\frac{\partial e}{\partial t} + \nabla.(e\mathbf{u})\right) = \frac{1}{c^2}\left(\frac{\mathrm{d}e}{\mathrm{d}t} + e(\nabla.\mathbf{u})\right), \tag{2.3}$$

so that in this formulation, the force $\mathbf{f}$ coincides precisely with the usual notion as the total time derivative of momentum.

Of particular interest, on adopting the usual operator relations of quantum mechanics $\mathbf{p} \longrightarrow -i\hbar\nabla$ and $e \longrightarrow i\hbar\partial/\partial t$, where as usual $\hbar = h/2\pi$ and h is Planck's constant, so that

$$\mathbf{p} = -i\hbar\nabla\psi, \qquad e = i\hbar\frac{\partial \psi}{\partial t},$$

for some function $\psi = \psi(\mathbf{x}, t)$, we see that (2.1) gives $\mathbf{f} = 0$ and

$$g = \frac{i\hbar}{c^2}\left(\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi\right), \tag{2.4}$$

for which we make three observations. Firstly, in conventional quantum mechanics, a distinction is not made between physical particle energy e and wave energy $\mathscr{E}$. In the formalism developed here, it is apparent that conventional quantum mechanics operates under circumstances for which the spatial physical force $\mathbf{f}$ vanishes. Secondly, the three spatial dimensional Klein–Gordon equation (C.1) can be seen to emerge under the linearity assumption $g(\psi) \approx -i(e_0^2/\hbar c^2)\psi$, where $e_0 = m_0c^2$. We note that the Klein–Gordon equation constitutes a fundamental equation of relativistic quantum mechanics, and the observation indicates both that g might be imaginary and that the Klein–Gordon equation might be only a first approximation in a nonlinear setting. The third observation is that Schrodinger's wave equation might well be readily deduced as arising from the wave equation operator appearing in Eq. (2.4), and this is discussed further in the final subsection of this section.

2.2. Work done formulae

Accordingly, following [12,13] to extend the conventional notion of work done, say $\mathrm{d}W$ as arising from the accepted notion of force times distance, we propose that the incremental work done $\mathrm{d}W$ arises as the scalar product of the two four vectors $(\mathbf{f}, gc)$ and $(\mathbf{dx}, c\mathrm{d}t)$, thus

$$\mathrm{d}W = \mathbf{f}.\mathrm{d}\mathbf{x} + gc^2\mathrm{d}t = \left(\frac{\partial \mathbf{p}}{\partial t} + \nabla e\right).\mathbf{dx} + \left(\frac{\partial e}{\partial t} + c^2\nabla.\mathbf{p}\right)\mathrm{d}t, \tag{2.5}$$

which on using the identities simplifies to yield

$$\mathrm{d}W = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}.\mathbf{dx} + \left(\frac{\mathrm{d}e}{\mathrm{d}t} + e(\nabla.\mathbf{u})\right)\mathrm{d}t = \frac{\mathrm{d}e}{\mathrm{d}t}\mathrm{d}t + \left(\frac{\mathrm{d}e}{\mathrm{d}t} + e(\nabla.\mathbf{u})\right)\mathrm{d}t. \tag{2.6}$$

However, as reported in [12,13], immediately from Eq. (2.5), we may deduce

$$\mathrm{d}(W-e)=\frac{\partial \mathbf{p}}{\partial t}.\mathbf{dx}+c^2(\nabla.\mathbf{p})\mathrm{d}t, \tag{2.7}$$

and we may view either (2.6) or (2.7) as generalising the conventional work done equation $\mathrm{d}e=(\mathrm{d}\mathbf{p}/\mathrm{d}t).\mathbf{dx}$, and comprising two energies: the conventional particle energy e arising from $\mathrm{d}e=(\mathrm{d}\mathbf{p}/\mathrm{d}t).\mathbf{dx}$ and the wave energy $\mathscr{E}$ which is the subject of [12,13], and arising from either of the two expressions

$$\mathrm{d}\mathscr{E}=\left(\frac{\mathrm{d}e}{\mathrm{d}t}+e(\nabla.\mathbf{u})\right)\mathrm{d}t=\frac{\partial \mathbf{p}}{\partial t}.\mathbf{dx}+c^2(\nabla.\mathbf{p})\mathrm{d}t, \tag{2.8}$$

and from this equation and Eqs. (2.2) and (2.3), we may deduce the companion formulae

$$\mathbf{f}=\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t},\qquad g=\frac{1}{c^2}\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}.$$

The equation $W=e+\mathscr{E}$ accommodates both the particle and de Broglie wave energies, and it is the contribution arising from the de Broglie wave energy $\mathscr{E}$ that is not accommodated in traditional mechanical thinking and might be identified as a formal source of dark energy. We comment that while the Einstein particle energy $e=mc^2$ can be calculated for any subluminal velocity field, the de Broglie wave energy $\mathscr{E}$ is only generated from velocity fields $\mathbf{u}$ for which the corresponding momentum $\mathbf{p}$ satisfies a wave equation of the form

$$\frac{\partial^2 \mathbf{p}}{\partial t^2}=c^2\nabla(\nabla.\mathbf{p})=c^2\nabla^2\mathbf{p}+c^2\nabla_\wedge(\nabla_\wedge \mathbf{p}), \tag{2.9}$$

arising from the compatibility of either differential relations (2.5) or (2.8), while the wave energy $\mathscr{E}$ satisfies the wave equation,

$$\frac{\partial^2 \mathscr{E}}{\partial t^2}=c^2\nabla^2\mathscr{E}.$$

In one spatial dimension, Eq. (2.9) becomes simply the conventional wave equation, and a number of wave-like solutions of the one-dimensional wave equation are examined in [12,13]. In particular, the special case of light for which $\mathscr{E}=h\nu$ and $p=h\nu/c$, where h is Planck's constant and ν denotes the frequency, is shown to arise from (2.8) and the one-dimensional wave equation.

Assuming the existence of either W or $\mathscr{E}$ imposes certain constraints, it is apparent from (2.5) that $\mathbf{f}$ and g must satisfy the compatibility condition

$$\frac{\partial \mathbf{f}}{\partial t}=c^2\nabla g, \tag{2.10}$$

in order that either (2.5) or (2.8) represents well-defined differential relations for W and $\mathscr{E}$, respectively. In the present formulation, the compatibility condition (2.10) represents a new equation and an important constraint that is not present in conventional theory. For example, if our particle is exposed to some external fields such as gravity, then (2.10) implies that we might assume that $(\mathbf{f}, gc)$ are generated as external forces from a potential $V(\mathbf{x},t)$ such that

$$\mathbf{f}=-\nabla V,\qquad gc^2=-\frac{\partial V}{\partial t},$$

so that (2.1) becomes

$$\frac{\partial \mathbf{p}}{\partial t}+\nabla(e+V)=0,\qquad \frac{1}{c^2}\frac{\partial(e+V)}{\partial t}+\nabla.\mathbf{p}=0,$$

which implies that $e+V$ satisfies the wave equation and from (2.8) we find

$$\mathrm{d}\mathscr{E}=\frac{\partial \mathbf{p}}{\partial t}.\mathbf{dx}+c^2(\nabla.\mathbf{p})\mathrm{d}t=-\nabla(e+V).\mathbf{dx}-\frac{\partial(e+V)}{\partial t}\mathrm{d}t,$$

and therefore $d(e + \mathscr{E} + V) = 0$, and a conventional conservation of energy is applied, namely $W + V = e + \mathscr{E} + V = constant$. We comment that the potential energy term $V\psi$ is an essential ingredient in the conventional Schrodinger equation. In an equivalent fluid formulation due to Madelung, there is a force term $-\nabla V$, while in field theory, an external field is coupled to the Klein–Gordon field or to the Dirac field within a standard invariant Lagrangian formulation.

2.3. Some comments

For this general formulation, we make the following observations. The term $c^2(\nabla.\mathbf{p})\mathrm{d}t$ does not appear in conventional special relativistic mechanics, and it is this term which automatically takes into account the de Broglie wave energy. Secondly, we might anticipate that e is applied for subluminal particle motion, while $\mathscr{E}$ is applied for superluminal de Broglie waves. Thirdly, that since in any experiment either particles or de Broglie waves are reported, only one of e or $\mathscr{E}$ is physically measured, which leads to the expectation that particles appear for $e < \mathscr{E}$ and de Broglie waves occur for $\mathscr{E} \leqslant e$. However, in either event, both a measurable energy and an unmeasurable energy exist, the latter registering its presence in the form of dark energy. Fourthly, it is clear that $e < \mathscr{E}$ or $\mathscr{E} \leqslant e$ hinges on the sign of the divergence of the velocity field $\nabla.\mathbf{u}$, and the particle-wave transition occurs for precisely volume preserving or incompressible velocity fields $\mathbf{u}$ for which $\nabla.\mathbf{u} = 0$.

2.4. Schrodinger wave equation

We further comment that in conventional quantum mechanics, the Schrodinger equation is usually motivated as arising from the classical wave equation. The usual requirements are that the equation is linear so that different solutions may be superimposed and that it involves only fundamental constants, rather than parameters associated with a particular motion of the particle such as momentum, energy, frequency or propagation number.

It is therefore important to emphasise that within the theory proposed here, the classical wave equation is not a matter of speculation, but rather a consequence, and it is not difficult to envisage the Schrodinger wave equation arising in the present context as a formal consequence, following the numerous ad hoc derivations of the Schrodinger wave equation presented in several texts (such as [18], pp 18–19 or [22], pp 218–220). So, for example, for a single non-relativistic particle, Semat [22] starts with the wave equation

$$\frac{\partial^2 \Psi}{\partial t^2} = w^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right), \tag{2.11}$$

where w denotes the wave speed, assumed constant, and $\Psi(x, y, z, t)$ denotes a wave function. On using the three relations $p = h/\lambda$, $w = \lambda\nu$ and $E = h\nu$, [22] makes use of conservation of energy $E = mv^2/2 + V$ where V denotes potential energy, to deduce $p = mv = (2m(E - V))^{1/2}$ and therefore $w = h\nu/(2m(E - V))^{1/2}$. On looking for solutions of (2.11) of the form $\Psi(x, y, z, t) = \psi(x, y, z)\exp(2\pi i\nu t)$, we might readily deduce Schrodinger's wave equation for a single particle, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{8\pi^2 m}{h^2}(E - V)\psi, \tag{2.12}$$

and the question arises as to whether or not there is a more immediate derivation of Schrodinger's equation arising from an alternative formulation.

For completeness, in the following section we detail the major formulae of special relativity that are required to establish the Lorentz invariance of the fundamental force relations (2.1) and the two further invariances $\xi(x, t)$ and $\eta(x, t)$ of special relativity given by (4.1), which are the subject of the subsequent section.

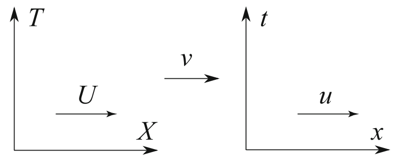

FIG. 1. Inertial frames moving along x-axis with relative velocity v

3. Special relativity

We consider a rectangular Cartesian frame $\mathbf{X} = (X, Y, Z)$ and another frame $\mathbf{x} = (x, y, z)$ moving with constant velocity v relative to the first frame, and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 1. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest.

3.1. Lorentz transformations

For $0 \leqslant v < c$, the standard Lorentz transformations are

$$x = \frac{X - vT}{[1-(v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1-(v/c)^2]^{1/2}}, \tag{3.1}$$

and various derivations of these equations can be found in many standard textbooks, such as Feynmann et al. [6] and Landau and Lifshitz [15], and other novel derivations are given by Lee and Kalotas [16] and Levy-Leblond [17]. We also observe that viewing the relative frame velocity v as a parameter, the envelope parameter values $v = \pm c$ (see, for example, [8]) emerge from solving the two equations

$$\frac{\partial x}{\partial v} = \frac{\left(Xv - Tc^2\right)}{c^2\left(1-(v/c)^2\right)^{3/2}} = 0, \qquad \frac{\partial t}{\partial v} = \frac{(Tv - X)}{c^2\left(1-(v/c)^2\right)^{3/2}} = 0,$$

as might be expected.

3.2. Velocity addition formulae

With velocities $U = \mathrm{d}X/\mathrm{d}T$ and $u = \mathrm{d}x/\mathrm{d}t$, (3.1) yields the addition of velocity law

$$u = \frac{U - v}{(1 - Uv/c^2)}, \tag{3.2}$$

which is well known and due to Einstein, and an immediate consequence is the identity

$$[1-(u/c)^2](1-Uv/c^2)^2 = [1-(v/c)^2][1-(U/c)^2]. \tag{3.3}$$

Another formula arising from (3.2) is

$$\left(\frac{1+U/c}{1-U/c}\right) = \left(\frac{1+u/c}{1-u/c}\right)\left(\frac{1+v/c}{1-v/c}\right), \tag{3.4}$$

so that on introducing velocity variables $(\Theta, \theta, \epsilon)$ defined by

$$\Theta = \tanh^{-1}(U/c), \quad \theta = \tanh^{-1}(u/c), \quad \epsilon = \tanh^{-1}(v/c), \tag{3.5}$$

Eq. (3.4) becomes simply the translation $\Theta = \theta + \epsilon$ noting again that within the context of special relativity, v and therefore ϵ are both assumed to be constants. Subsequently, we find that the angle θ assumes an important role, so for completeness we note the elementary relations

$$\theta = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \tanh^{-1}(u/c), \quad \left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e^{\theta}, \tag{3.6}$$

and emphasise that θ is the angle in which Lorentz invariance appears through a translational invariance. We comment that the formulation of Lorentz transformations as a one-parameter group of geometric transformations is due to Minkowski [19].

We observe for (3.2) that an outcome predicted in [11] is that Einstein's velocity addition formula (3.2) remains valid for the proposed extension of special relativity beyond the speed of light, and in particular, in the limit $v \to \infty$ the de Broglie relation $uU = c^2$ emerges. It also formally emerges from the envelope [8], namely by simultaneously solving (3.2) and

$$\frac{\partial u}{\partial v} = \frac{-1}{(1-Uv/c^2)} + \frac{U(U-v)}{c^2\left(1-Uv/c^2\right)^2} = \frac{-\left(1-(U/c)^2\right)}{\left(1-Uv/c^2\right)^2} = 0,$$

which can only vanish in the limit $v \to \infty$. The relation $uU = c^2$ formally arises from the underlying transformation $x = cT, t = X/c$. With a primed notation, the space–time transformation $x' = ct$ and $t' = x/c$, for which $u' = \mathrm{d}x'/\mathrm{d}t' = c^2\mathrm{d}t/\mathrm{d}x = c^2/u$ and has been widely used to connect the Galilean and Carroll transformations as significant limits of Lorentz invariant theories, for example, in electromagnetism. The Carrollian transformations $x' = ct$ and $t' = x/c$ were originally introduced by Jean-Marc Levy-Leblond, and their origin and development are fully detailed by Rousseaux [20] and Houlrik and Rousseaux [14].

3.3. Lorentz invariant energy–momentum relations

For $v, u, U < c$, assuming the Einstein mass variation in both frames

$$m(u) = \frac{m_0}{[1-(u/c)^2]^{1/2}}, \quad M(U) = \frac{m_0}{[1-(U/c)^2]^{1/2}},$$

and with momenta $P = MU$ and $p = mu$, we have on multiplication of (3.2) by $m_0\left[1-(u/c)^2\right]^{-1/2}$ and using the square root identity from (3.3), we may readily deduce the Lorentz invariant energy–momentum relations in consideration of the formulae $e = mc^2$ and $E = Mc^2$, thus

$$p = \frac{P - Ev/c^2}{[1-(v/c)^2]^{1/2}}, \quad e = \frac{E - Pv}{[1-(v/c)^2]^{1/2}}, \tag{3.7}$$

and noting that (3.1) and (3.7) give rise to the two Lorentz invariants

$$x^2 - (ct)^2 = X^2 - (cT)^2, \qquad e^2 - (pc)^2 = E^2 - (Pc)^2 = e_0^2, \tag{3.8}$$

where $e_0 = m_0c^2$ denotes the rest mass energy. Subsequently, we see that the derived form of the latter relation, namely

$$\frac{e}{p}\frac{\mathrm{d}e}{\mathrm{d}p} = c^2, \tag{3.9}$$

emerges as an alternative formulation of the de Broglie relation $uw = c^2$ connecting the group velocity u with the wave velocity w. Further, as a consequence of (3.1) we have the following important relations applying for the characteristic variables $x + ct$ and $x - ct$, as given by [11] thus

$$x + ct = \left(\frac{1-v/c}{1+v/c}\right)^{1/2}(X + cT), \qquad x - ct = \left(\frac{1+v/c}{1-v/c}\right)^{1/2}(X - cT),$$

from which the first relation of (3.8) is apparent. In the following sections, we shall make use of the notation

$$\alpha = x + ct, \beta = x - ct, \zeta = (x^2 - (ct)^2)^{1/2}, \rho = \frac{1}{2}\log\left(\frac{x+ct}{x-ct}\right), \tag{3.10}$$

noting that only ζ is a full invariant of the Lorentz group (3.1) while the other three might be viewed as partial invariants.

3.4. Lorentz invariance of force relations

Fundamental to this investigation is the invariance of the given expressions for $(\mathbf{f}, gc)$ under Lorentz transformation established in [12]. For completeness and given the crucial nature of this result, the major details are repeated here. In one spatial dimension x, the proposed Eqs. (2.1) become simply

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x},$$

and it is not difficult to show that these equations remain invariant under the Lorentz group (3.1) and (3.7); in other words, the following relations hold

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial P}{\partial T} + \frac{\partial E}{\partial X}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X},$$

which we establish as follows. From Eq. (3.1), we have the differential relations

$$\frac{\partial}{\partial x} = \frac{1}{(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \quad \frac{\partial}{\partial t} = \frac{1}{(1-(v/c)^2)^{1/2}}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\}, \tag{3.11}$$

so that on using (3.7) and the subscript notation for partial derivatives we have

$$\begin{aligned} &p_t + e_x \\ &\quad = \frac{1}{(1-(v/c)^2)}\left\{\left(P_T - \frac{v}{c^2}E_T\right) + v\left(P_X - \frac{v}{c^2}E_X\right) + (E_X - vP_X) + \frac{v}{c^2}(E_T - vP_T)\right\} \\ &\quad = P_T + E_X, \end{aligned}$$

and similarly

$$\begin{aligned} &\frac{1}{c^2}e_t + p_x \\ &\quad = \frac{1}{(1-(v/c)^2)}\left\{\frac{1}{c^2}(E_T - vP_T) + \frac{v}{c^2}(E_X - vP_X) + \left(P_X - \frac{v}{c^2}E_X\right) + \frac{v}{c^2}\left(P_T - \frac{v}{c^2}E_T\right)\right\} \\ &\quad = \frac{1}{c^2}E_T + P_X. \end{aligned}$$

This key outcome indicates that Eqs. (2.1) are well formulated. In the following section, we examine two related invariances of special relativity.

4. Two Lorentz invariants of special relativity

By direct substitution, we may establish, using Eqs. (3.1) and (3.7), that $\xi(x,t)$ and $\eta(x,t)$ as defined by the equations

$$\xi = ex - c^2pt, \qquad \eta = px - et, \tag{4.1}$$

constitute two Lorentz invariances of special relativity, which are readily verified as follows. On evaluating $\xi = ex - c^2pt$ and $\eta = px - et$ using (3.1) and (3.7), we have

$$\frac{\big((E-Pv)(X-vT)-c^2(P-Ev/c^2)(T-vX/c^2)\big)}{(1-(v/c)^2)} = EX - c^2PT,$$

and

$$\frac{\big((P-Ev/c^2)(X-vT)-(E-Pv)(T-vX/c^2)\big)}{(1-(v/c)^2)} = PX - ET,$$

as required. The invariant $\eta = px - et$ has the usual dimensions of action and arises directly for simple harmonic waves of the form $e^{2\pi i(x/\lambda - t/T)} = e^{i(kx-\omega(k)t)}$, where λ is the wavelength, T is the period of oscillation, $k = 2\pi/\lambda$ is the wave number and $\omega = 2\pi/T$ is the angular frequency, assuming the de Broglie relations. On the other hand, the invariant $\xi = ex - c^2pt$ is more surprising and as noted in Sect. 5.1 associates with the wave or superluminal world, in contrast to η which associates with the particle or subluminal world. As demonstrated explicitly above, both constitute bona fide Lorentz invariants of conventional special relativity.

4.1. Algebraic relations

For the space–time transformation $x' = ct$ and $t' = x/c$, for which $u' = \mathrm{d}x'/\mathrm{d}t' = c^2\mathrm{d}t/\mathrm{d}x = c^2/u$, and assuming the Einstein expressions $e' = m'c^2$ and $m' = m_0/((u'/c)^2 - 1)^{1/2}$ for $u' > c$ as proposed in [11], by direct calculation we may readily establish the skew-symmetric relations $\xi(x,t) = -\xi(x',t')$ and $\eta(x,t) = -\eta(x',t')$. Quantities that change sign under this reflection are referred to as "pseudo-scalars" in the particle physics literature.

We comment that Guemez et al. [10] have provided the special relativistic four-vector extension of the de Broglie relation valid for three spatial dimensions as simply the scalar product $\mathbf{u}.\mathbf{u}' = c^2$, using an obvious abbreviated formalism. One possible coordinate decomposition of this formula might be $\mathbf{r}' = ct\mathbf{r}/r$ and $t' = r/c$, where $\mathbf{r}$ and $\mathbf{r}'$ denote the position vectors and $r = (x^2 + y^2 + z^2)^{1/2}$. Note especially the identical inverse transformations $\mathbf{r} = ct'\mathbf{r}'/r'$ and $t = r'/c$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$ and that this coordinate decomposition although apparently a natural extension of the one-dimensional coordinate transformation $x' = ct$ and $t' = x/c$ for $uu' = c^2$ may not be unique. Indeed, the one-dimensional transformation itself may not provide a unique decomposition of the equation $uu' = c^2$. The coordinate transformation $\mathbf{r}' = ct\mathbf{r}/r$ is essentially one-dimensional and might well apply in a predominantly Friedmann–LeMaitre universe in which there is a preferred reference frame in which the cosmic microwave background is isotropic, since the transformation is spatially spherically symmetric and polar angles remain unchanged and $r' = ct$ and $t' = r/c$.

Further, from (4.1) it is not difficult to establish the algebraic relations

$$\xi + c\eta = (e+cp)(x-ct), \qquad \xi - c\eta = (e-cp)(x+ct),$$

and therefore, we have

$$\xi^2 - (c\eta)^2 = (e^2-(cp)^2)(x^2-(ct)^2) = e_0^2(x^2-(ct)^2), \tag{4.2}$$

on using $e^2 - (cp)^2 = e_0^2$ where $e_0 = m_0c^2$. Now on using the relations

$$e + cp = e_0\left(\frac{1+u/c}{1-u/c}\right)^{1/2} = e_0e^{\theta}, \qquad e - cp = e_0\left(\frac{1-u/c}{1+u/c}\right)^{1/2} = e_0e^{-\theta},$$

where θ is the angle defined by (3.5) and satisfying relations (3.6), we may deduce

$$\xi + c\eta = e_0(x-ct)e^{\theta}, \qquad \xi - c\eta = e_0(x+ct)e^{-\theta},$$

from which we may readily obtain

$$\xi = e_0 x\cosh\theta - e_0 ct\sinh\theta, \qquad \eta = e_0 x\sinh\theta - e_0 ct\cosh\theta. \tag{4.3}$$

Further, relations (4.1) may be inverted to yield

$$e = \left(\frac{x\xi + c^2 t\eta}{x^2-(ct)^2}\right), \qquad p = \left(\frac{t\xi + x\eta}{x^2-(ct)^2}\right), \tag{4.4}$$

and from these expressions and (2.1), we might deduce the following for the force f and mass energy production g

$$f = \left(\frac{x(\eta_t+\xi_x) + t(c^2\eta_x+\xi_t)}{x^2-(ct)^2}\right), \quad g = \left(\frac{t(\eta_t+\xi_x) + x(c^2\eta_x+\xi_t)/c^2}{x^2-(ct)^2}\right),$$

where subscripts denote partial derivatives.

4.2. Differential relations

On taking the total time derivative $\mathrm{d}/\mathrm{d}t$ of the two invariants ξ and η given by (4.1) and making use of $\mathrm{d}e/\mathrm{d}t = u\mathrm{d}p/\mathrm{d}t$, we obtain

$$e_0\frac{\mathrm{d}\xi}{\mathrm{d}s} = fc\eta, \qquad e_0 c\frac{\mathrm{d}\eta}{\mathrm{d}s} = f\xi - e_0^2, \tag{4.5}$$

where $f = \mathrm{d}p/\mathrm{d}t$ is the force and $\mathrm{d}s$ denotes the line element $\mathrm{d}s = c\left(1-(u/c)^2\right)^{1/2}\mathrm{d}t$ arising from $(\mathrm{d}s)^2 = (c\mathrm{d}t)^2 - (\mathrm{d}x)^2$. We may confirm these relations since on differentiating Eq. (4.2) totally with respect to time, we may deduce

$$\xi\frac{\mathrm{d}\xi}{\mathrm{d}t} - c^2\eta\frac{\mathrm{d}\eta}{\mathrm{d}t} = e_0^2(xu - c^2 t),$$

which simplifies to give

$$\xi\frac{\mathrm{d}\xi}{\mathrm{d}s} - c^2\eta\frac{\mathrm{d}\eta}{\mathrm{d}s} = e_0 c\eta,$$

and Eqs. (4.5) can be readily seen to be consistent with this result.

On introducing the function $\sigma(x,t)$ defined by the differential relation $\mathrm{d}\sigma = f\mathrm{d}s = fc\left(1-(u/c)^2\right)^{1/2}\mathrm{d}t$, from $f = \mathrm{d}p/\mathrm{d}t$ we may readily deduce $\mathrm{d}\sigma = m_0 c\mathrm{d}u/\left(1-(u/c)^2\right)$ so that the function σ is given by

$$\sigma = \frac{m_0 c^2}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = e_0\tanh^{-1}\left(\frac{u}{c}\right). \tag{4.6}$$

Accordingly, in terms of the angle θ defined by (3.5), we have simply $\sigma = e_0\theta$, and Eqs. (4.5) become

$$\frac{\mathrm{d}\xi}{\mathrm{d}\theta} = c\eta, \qquad c\frac{\mathrm{d}\eta}{\mathrm{d}\theta} = \xi - \frac{e_0^2}{f}, \tag{4.7}$$

from which we may readily deduce

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} - \xi = -\frac{e_0^2}{f}, \qquad \frac{\mathrm{d}^2\eta}{\mathrm{d}\theta^2} - \eta = -\frac{1}{c}\frac{\mathrm{d}(e_0^2/f)}{\mathrm{d}\theta}. \tag{4.8}$$

Equations (4.7) and (4.8) can appear in a variety of forms. For example, on noting the relations for the force and the line element, namely $f = \mathrm{d}p/\mathrm{d}t$ and $\mathrm{d}s = c\left(1-(u/c)^2\right)^{1/2}\mathrm{d}t$, we may deduce the equation $e_0/f = \mathrm{d}s/\mathrm{d}\theta$, and in place of Eqs. (4.7) and (4.8) we have

$$\frac{\mathrm{d}\xi}{\mathrm{d}\theta} = c\eta, \qquad c\frac{\mathrm{d}\eta}{\mathrm{d}\theta} = \xi - e_0\frac{\mathrm{d}s}{\mathrm{d}\theta}, \tag{4.9}$$

from which we may readily deduce

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} - \xi = -e_0\frac{\mathrm{d}s}{\mathrm{d}\theta}, \qquad c\frac{\mathrm{d}^2\eta}{\mathrm{d}\theta^2} - c\eta = -e_0\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2}, \tag{4.10}$$

and an independent derivation of Eqs. (4.9) and (4.10) is provided in "Appendix A". Further, we comment that by direct differentiation of (4.3) totally with respect to time and using $u = \mathrm{d}x/\mathrm{d}t = c\tanh\theta$ we have

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = e_0\left(x\sinh\theta - ct\cosh\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}, \quad c\frac{\mathrm{d}\eta}{\mathrm{d}t} = e_0\left(x\cosh\theta - ct\sinh\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} - e_0\,\mathrm{sech}\,\theta, \tag{4.11}$$

which coincide with (4.7) and (4.9) since on using $u = c\tanh\theta$, we have $\mathrm{d}s/\mathrm{d}t = c\,\mathrm{sech}\,\theta$ and (4.9) follows immediately from (4.11). Some further differential relations are presented in "Appendix B".

5. Quantum mechanics and the Lorentz invariants ξ and η

The determination of a formal connection between special relativity and quantum mechanics has long attracted the interest of many eminent researchers, including Einstein himself, and culminating in the highly successful Dirac and Klein–Gordon equations and the literature leading to these developments and their numerous consequences is now extensive, see, for example, Bjorken and Drell [1], Dirac [4], Dirac [5] and Gross [9] to name only four of many substantial texts on this topic. This formal connection has always alluded researchers since in conventional quantum mechanics, there is no distinction made between energy arising from particle energy e or from wave energy $\mathscr{E}$. Once this distinction is made clear and the two invariants ξ and η of special relativity are identified, the connection between the two topics becomes apparent.

The purpose of this section is to formulate the role of the invariants $\xi(x,t)$ and $\eta(x,t)$ defined by Eq. (4.1) relating to de Broglie waves and quantum mechanics. In the following subsection, we show that the simple harmonic waves $e^{2\pi i\eta/h}$ and $e^{2\pi i\xi/hc}$ are connected to particles moving with velocities u and c^2/u, respectively, and moreover with corresponding de Broglie wave velocities c^2/u and u, respectively. Subsequently, following the usual replacement with operators in quantum mechanics $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, where as usual $\hbar = h/2\pi$, we show that ξ and η give rise to two commuting Lorentz invariant operators.

5.1. Wave and particle velocities

With the relations $p = h/\lambda$ and $e = h\nu$ in mind, on comparison of the expressions $e^{2\pi i(ex-c^2pt)/hc}$ and $e^{2\pi i(px-et)/h}$ with the previously noted standard simple harmonic wave $e^{i(kx-\omega(k)t)} = e^{2\pi i(x/\lambda-\nu t)}$, we see an immediate correspondence in the latter case, while in the former case for the invariant ξ we have $e^{2\pi i(ex-c^2pt)/hc} = e^{2\pi i(wx/\lambda-\nu ut)/hc}$ involving both the wave and group velocities w and u, respectively. It is apparent that the invariant η is intimately connected to a particle moving with velocity u with an accompanying de Broglie wave speed c^2/u, since waves of the form $e^{2\pi i\eta/h} = e^{2\pi i(px-et)/h}$ have a particle velocity $\mathrm{d}e/\mathrm{d}p = u$ and a wave velocity $e/p = c^2/u$. On the other hand, it is apparent that the invariant ξ is intimately connected to a particle moving with velocity c^2/u with an accompanying de Broglie wave speed u, since waves of the form $e^{2\pi i\xi/hc} = e^{2\pi i(ex-c^2pt)/hc}$ have a particle velocity $c^2\mathrm{d}p/\mathrm{d}e = c^2/u$ and a wave velocity $c^2p/e = u$. We comment that in each case the de Broglie relation $uw = c^2$ arises from (3.9) and that we might anticipate that the invariant η associates with the subluminal or particle world, while the invariant ξ associates with the superluminal or wave world.

5.2. Preliminary quantum mechanical observations

In quantum mechanics, it is well established that the variables become operators and wave functions involve a probability density. In the present formulation, for a single spatial dimension, the following equations are applied for the momentum p and the wave energy $\mathscr{E}$,

$$\frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t},$$

so that there certainly exists a function $\psi(x,t)$ such that

$$p = \frac{\partial \psi}{\partial x}, \qquad \mathscr{E} = \frac{\partial \psi}{\partial t}. \tag{5.1}$$

and satisfying the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{5.2}$$

from which, as previously described (see Eq. (2.12)), we may readily deduce a Schrodinger equation, which perhaps indicates that there are alternative interpretations for $\psi(x,t)$ other than the probabilistic approach. The probabilistic interpretation in quantum mechanics arises primarily in consequence of identifying variables as complex (pure imaginary) operators in order to adopt analogous energy integrals from classical mechanics. Strictly speaking, we may derive Schrodinger's equation from the present formulation, and in particular for a single spatial dimension from the wave Eq. (5.2), without the need to introduce complex variables.

We also observe that the one-dimensional version of (2.8), namely

$$\mathrm{d}\mathscr{E} = u\mathrm{d}p + eu_x \mathrm{d}t,$$

on using a subscript notation for partial derivatives, can be shown to become

$$\psi_{tx}\mathrm{d}x + \psi_{tt}\mathrm{d}t = u(\psi_{xx}\mathrm{d}x + \psi_{xt}\mathrm{d}t) + eu_x\mathrm{d}t.$$

On using $u = \mathrm{d}x/\mathrm{d}t$, together with the wave Eq. (5.2) to eliminate ψ_{tt}, the resulting equation for ψ_{xx} can be seen to be merely the partial derivative with respect to x of the relation $p = mu = \partial\psi/\partial x$.

Also as previously noted, relations arising from the quantum mechanical operators $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, namely

$$p = -i\hbar \frac{\partial \Psi}{\partial x}, \qquad e = i\hbar \frac{\partial \Psi}{\partial t}, \tag{5.3}$$

for some function $\Psi(x,t)$, are such that the physical force f vanishes. If for relations (5.1) and (5.2) we make the additional assumption that $f = p_t + e_x = 0$, then in addition to $p = \partial\psi/\partial x$ we have the further relation $e = -\partial\psi/\partial t$, so that in this particular case we have $g = 0$ and therefore $W = e + \mathscr{E} = 0$ as might be anticipated. Further, within the present theory it is a simple matter to show that there are no non-trivial solutions for $\psi(x,t)$ for which $f = 0$. In terms of the characteristic coordinates $\alpha = x + ct$ and $\beta = x - ct$ and $e^2 - (pc)^2 = e_0^2$, it is not difficult to show that the only solutions for e and p arising from $\psi_{\alpha\beta} = 0$ and $\psi_\alpha \psi_\beta = -(e_0/2c)^2$ are constant state solutions emanating from $\psi(x,t) = C_1\alpha - C_2\beta$ where C_1 and C_2 denote constants such $C_1C_2 = (e_0/2c)^2$, and for which we have

$$e = -c(C_1 + C_2), \qquad p = (C_1 - C_2), \qquad \mathscr{E} = c(C_1 + C_2),$$

corresponding to the constant velocity $u/c = (C_1 - C_2)/(C_1 + C_2)$. However, since e is a constant, we have $(C_1 + C_2) = -e_0/c$ and together with $C_1C_2 = (e_0/2c)^2$ we might readily conclude $C_1 = C_2 = -e_0/2c$ and therefore only the trivial solution $p = u = 0$ is applied.

Although relations (5.1) are clearly different from the quantum mechanical relations (5.3), nevertheless they enjoy some common structure. In both cases, we might generate some insight as to what is happening by evaluating the respective ratios $p/\mathscr{E}$ for (5.1) and p/e for (5.3), thus:

$$\frac{p}{\mathscr{E}} \approx \frac{p}{e} = \frac{u}{c^2} = \frac{\Delta\psi}{\Delta x}\frac{\Delta t}{\Delta\psi} = \frac{\Delta t}{\Delta x},$$

noting that here we are assuming the approximate relation $\mathscr{E} \approx e$. Since we are dealing with complex variables, the corresponding approximate calculation in quantum mechanics for p/e produces

$$\left(\frac{pp*}{ee*}\right)^{1/2} = \frac{u}{c^2} = \frac{\Delta\Psi}{\Delta x}\frac{\Delta t}{\Delta\Psi} = \frac{\Delta t}{\Delta x}.$$

Both cases indicate that the underlying formality is a de Broglie relationship $uw = c^2$ produced in the form $u/c^2 \approx 1/w$ where w denotes a certain velocity identified as $\Delta x/\Delta t$.

We also make the observation that from a special relativistic perspective, the conventional signatures of these operators are meaningful in the sense of being precisely what is required to produce the correct Lorentz invariances. Firstly, if we adopt $P \longrightarrow -i\hbar\partial/\partial X$ and $E \longrightarrow i\hbar\partial/\partial T$, and we apply these operator relations to the Lorentz invariant energy–momentum relations (3.7), then from $p \longrightarrow -i\hbar\partial/\partial x$ and $e \longrightarrow i\hbar\partial/\partial t$, we obtain precisely the correct differential transformation formulae (3.11). Furthermore, the usual signatures of the quantum mechanical operators are precisely that required to ensure the Lorentz invariances of the operators arising from the Lorentz invariants ξ and η defined by (4.1) and established below.

5.3. Lorentz invariant quantum mechanical operators

Here, it proves convenient to adopt the convention that the operator corresponding to a given variable is subscripted, so that for example in this notation, the two standard operators arising from momentum p and energy e become

$$L_p = -i\hbar\frac{\partial}{\partial x}, \qquad L_e = i\hbar\frac{\partial}{\partial t},$$

so that directly from Eq. (4.1), namely $\xi = ex - c^2pt$ and $\eta = px - et$, we might introduce operators L_ξ and L_η that are defined by

$$L_\xi = i\hbar\left(x\frac{\partial}{\partial t} + c^2t\frac{\partial}{\partial x}\right), \qquad L_\eta = -i\hbar\left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right), \tag{5.4}$$

both of which are fully Lorentz invariant operators, as can be verified from Eqs. (3.1) and (3.11), thus

$$L_\xi = \frac{i\hbar(X - vT)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\} + \frac{i\hbar c^2(T - vX/c^2)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\},$$

and

$$L_\eta = -\frac{i\hbar(X - vT)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\} - \frac{i\hbar(T - vX/c^2)}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\},$$

from which we may deduce

$$L_\xi = i\hbar\left(X\frac{\partial}{\partial T} + c^2T\frac{\partial}{\partial X}\right), \qquad L_\eta = -i\hbar\left(X\frac{\partial}{\partial X} + T\frac{\partial}{\partial T}\right),$$

and therefore the operators L_ξ and L_η are Lorentz invariant. Furthermore, by direct computation, it is not difficult to show that the two operators commute, namely $L_\xi L_\eta = L_\eta L_\xi$. In conventional quantum mechanical thinking, this means that the corresponding observables ξ and η are simultaneously measurable (compatible) and the two operators share the same eigenfunctions (see, for example, [2], p 101). We further observe that the operator L_ξ is essentially the Lorentz operator L_v arising from the one-parameter group of Lorentz transformations (3.1) which is given by

$$L_v = -\left(T\frac{\partial}{\partial X} + \frac{X}{c^2}\frac{\partial}{\partial T}\right),$$

which therefore also commutes with L_η. The full implications of this intriguing correspondence are not immediately apparent, but further underscore the formal connection established here between special relativity and quantum mechanics. Starting with the full invariant $\xi = ex - c^2pt$ of the Lorentz group, we formulate the quantum mechanical operator L_ξ given by (5.4), which turns out to be the Lorentz operator formed from the one-parameter group of Lorentz transformations (3.1).

The various properties of the two operators L_ξ and L_η are most apparent in terms of the characteristic coordinates $\alpha = x + ct$ and $\beta = x - ct$ and the variables ζ and ρ defined by (3.10). Using the differential formulae

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}, \qquad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \alpha} - c\frac{\partial}{\partial \beta},$$

we may deduce

$$L_\xi = i\hbar c\left\{\alpha\frac{\partial}{\partial \alpha} - \beta\frac{\partial}{\partial \beta}\right\}, \qquad L_\eta = -i\hbar\left\{\alpha\frac{\partial}{\partial \alpha} + \beta\frac{\partial}{\partial \beta}\right\}, \tag{5.5}$$

from which there arises the intriguingly simple formulae

$$L_{\xi-c\eta} = 2i\hbar c\alpha\frac{\partial}{\partial \alpha}, \qquad L_{\xi+c\eta} = -2i\hbar c\beta\frac{\partial}{\partial \beta}.$$

From these two expressions, the formal operator equation corresponding to the identity (4.2), namely $\xi^2 - (c\eta)^2 = e_0^2(x^2 - (ct)^2)$, gives rise immediately to the Klein–Gordon equation (see, for example, [2], p 312)

$$L_{\xi-c\eta}L_{\xi+c\eta}\Psi = 4(\hbar c)^2\alpha\beta\frac{\partial^2\Psi}{\partial\alpha\partial\beta} = e_0^2\alpha\beta\Psi,$$

for some function $\Psi(x,t)$ from which we have

$$4\frac{\partial^2\Psi}{\partial\alpha\partial\beta} = \left(\frac{e_0}{\hbar c}\right)^2\Psi,$$

or alternatively, in terms of the conventional (x,t) wave equation operator, we have the more usual form of the Klein–Gordon equation ([2], p 313)

$$\frac{\partial^2\Psi}{\partial t^2} - c^2\frac{\partial^2\Psi}{\partial x^2} = -\left(\frac{e_0}{\hbar}\right)^2\Psi.$$

The Klein–Gordon equation is second order in both space and time coordinates, and it is a fundamental equation of relativistic quantum mechanics. Dirac's Lorentz invariant relativistic equation, which is the first order in both space and time coordinates, is purposely constructed so that the probability density remains non-negative, which is not a feature of the Klein–Gordon equation (see, for example, [2], p 317). For completeness, some details for the time-dependent Dirac equation for a free particle are summarised in "Appendix C", and the equation itself for three spatial dimensions (x, y, z) becomes

$$i\hbar\frac{\partial\mathbf{\Psi}}{\partial t} + ic\hbar(\mathbf{A}.\mathbf{\nabla})\mathbf{\Psi} = e_0\mathbf{B}\mathbf{\Psi},$$

where $\mathbf{\Psi}$ is given by

$$\mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

where the four matrices $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{A_z}$ and $\mathbf{B}$ are as defined in "Appendix C". For three spatial dimensions, each component of $\mathbf{\Psi}$, namely $\psi_j = \psi_j(x, y, z, t)$ for $j = 1, 2, 3, 4$, may be shown to satisfy the three spatial dimensional Klein–Gordon equation (C.1).

In terms of the characteristic coordinates $\alpha = x + ct$ and $\beta = x - ct$, the variables ζ and ρ defined by (3.10) become

$$\zeta = (x^2 - (ct)^2)^{1/2} = (\alpha\beta)^{1/2}, \qquad \rho = \frac{1}{2}\log\left(\frac{x+ct}{x-ct}\right) = \frac{1}{2}\log\left(\frac{\alpha}{\beta}\right),$$

so that from (5.5) we may readily deduce the formulae

$$L_\xi(\zeta) = 0, \qquad L_\eta(\zeta) = -i\hbar\zeta, \qquad L_\xi(\rho) = i\hbar c, \qquad L_\eta(\rho) = 0,$$

along with

$$L_\xi L_\eta \Psi = \hbar^2 c\left\{\alpha^2\frac{\partial^2\Psi}{\partial\alpha^2} + \alpha\frac{\partial\Psi}{\partial\alpha} - \beta^2\frac{\partial^2\Psi}{\partial\beta^2} - \beta\frac{\partial\Psi}{\partial\beta}\right\},$$

so that on making the Euler transformations $\gamma = \log\alpha$ and $\delta = \log\beta$ we obtain

$$L_\xi L_\eta \Psi = \hbar^2 c\left(\frac{\partial^2\Psi}{\partial\gamma^2} - \frac{\partial^2\Psi}{\partial\delta^2}\right).$$

We also observe that for the variables ζ and ρ, the Jacobian and wave equation for the momentum become

$$\frac{\partial(\zeta,\rho)}{\partial(x,t)} = \frac{c}{\zeta}, \qquad \frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 4c^2\frac{\partial^2 p}{\partial\alpha\partial\beta} = 0, \tag{5.6}$$

and the latter equation becomes

$$\zeta^2\frac{\partial^2 p}{\partial\zeta^2} + \zeta\frac{\partial p}{\partial\zeta} - \frac{\partial^2 p}{\partial\rho^2} = 0, \qquad \frac{\partial^2 p}{\partial\tau^2} - \frac{\partial^2 p}{\partial\rho^2} = 0, \tag{5.7}$$

where we have made another Euler transformation $\tau = \log\zeta$. On noting the relations $\gamma = \log\alpha = \tau + \rho$ and $\delta = \log\beta = \tau - \rho$, the well-known general solution of the wave equation $p = F(\alpha) + G(\beta)$ is apparent from the latter equations of either (5.6) or (5.7).

6. Numerical results and conclusions

In order to provide a numerical illustration that total particle work done W is given by $W = e + \mathscr{E}$ where e is the particle energy and $\mathscr{E}$ is the wave energy, we exploit the explicit wave-like solution and formulae for $\mathscr{E}$ that are derived in [12] and characterised by an arbitrary parameter λ. For this solution, the particle velocity $u(x, t)$ is given explicitly by

$$u(x,t) = c\left\{\frac{\lambda x + ct}{((e_0/f_0)^2 + (\lambda x + ct)^2)^{1/2}}\right\},$$

and explicit formulae are presented in [12] for the wave energy $\mathscr{E}$ for the two cases $\lambda^2 < 1$ and $\lambda^2 > 1$. Using these expressions, we may illustrate the relative importance of the wave energy by evaluating the ratio defined by R = Einstein energy/(Einstein energy + Wave energy), namely $R = e/(e + \mathscr{E})$, and adopting the same datum energy levels as those used in [12]. With Q precisely as defined in [12], we have simply $R = (2+Q)^{-1}$. In Figs. 2, 3 and Table 1, the angle ϕ denotes $\sin^{-1}(u/c)$. In [12], numerical values

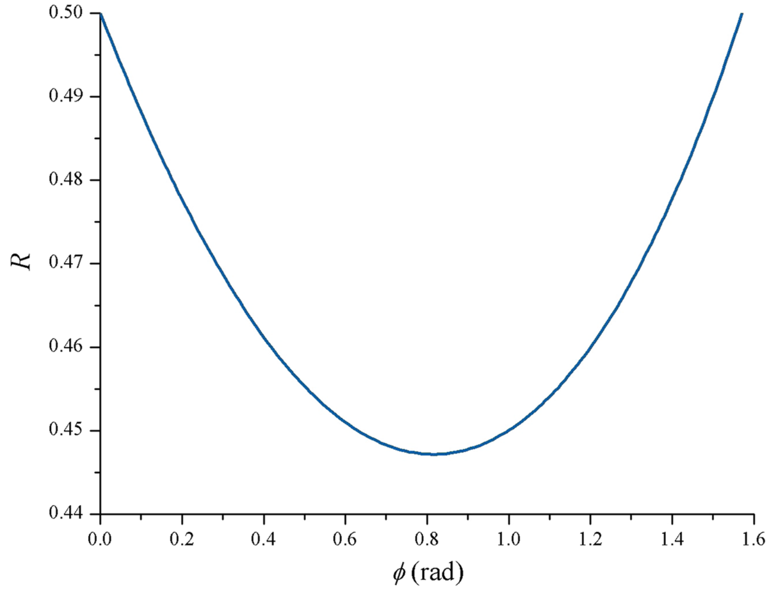

Fig. 2. Typical variation of $R = e/(e + \mathscr{E})$ with $\phi = \sin^{-1}(u/c)$ for $\lambda = 1/2$

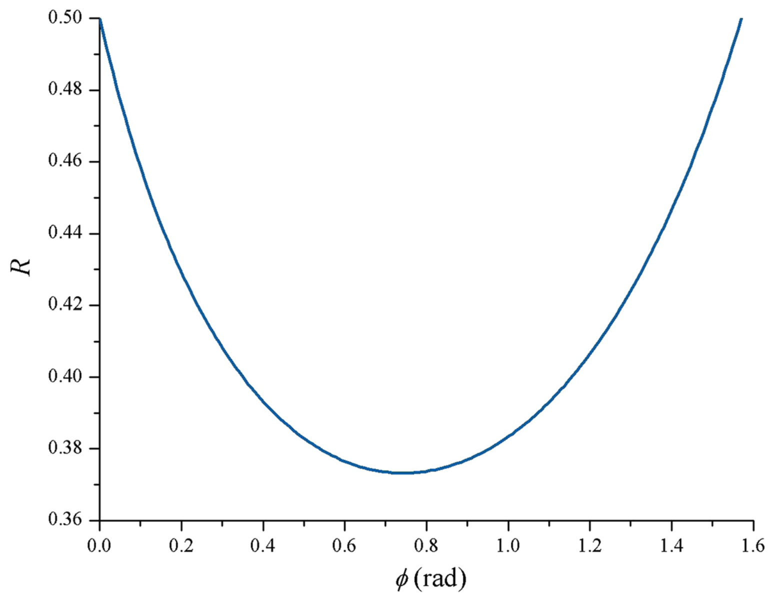

Fig. 3. Typical variation of $R = e/(e + \mathscr{E})$ with $\phi = \sin^{-1}(u/c)$ for $\lambda = 2$

are given for the ratio $e/\mathscr{E}$ as an indicator of the ratio of Einstein energy to dark energy, but with the present interpretation that total particle work done W is given by $W = e + \mathscr{E}$, the ratio $e/(e + \mathscr{E})$ more accurately reflects the ratio of the Einstein energy to the total energy, and indeed, the numbers presented here are closer to the commonly accepted estimates, which are often as low as 5 or 6%.

Table 1. *Numerical values for* $R = e/(e + \mathscr{E})$ *for two values of* ϕ *and various values of* λ

	$\phi = \pi/6$	$\phi = \pi/3$
$\lambda = 1/4$	0.4747	0.4723
$\lambda = 1/2$	0.4542	0.4517
$\lambda = 3/4$	0.4371	0.4357
$\lambda = 2$	0.3811	0.3876
$\lambda = 5$	0.3204	0.3398
$\lambda = 10$	0.2780	0.3072

In special relativity, the formulae for the energy and mass of a particle $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ are based on the assumption that the particle energy e accrues from and coincides with the work done e, and formally arises as a consequence of the rate-of-working equation which in three spatial dimensions becomes $\mathrm{d}e/\mathrm{d}t = \mathbf{u}.(\mathrm{d}\mathbf{p}/\mathrm{d}t)$ where $\mathbf{p} = m\mathbf{u}$ is the momentum and $\mathbf{u}$ is the velocity vector. In [12,13], the usual formulae of special relativity are adopted, except that a distinction is made between particle energy $e = mc^2$ and work done W, and Eq. (2.1) is proposed as a fully Lorentz invariant alternative of Newton's second law. Although inherent in [12,13], here we propose explicitly that the total work done W by a single particle comprises two contributions, namely particle energy e and wave energy $\mathscr{E}$, thus $W = e + \mathscr{E}$.

In summary for a single spatial dimension, we propose that every particle moving with velocity u acquires a conventional momentum $p = mu$ and energy $e = mc^2$ where $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ such that p satisfies the wave equation. We propose that associated with this particle motion is a de Broglie wave moving with a wave velocity c^2/u for which there is an associated wave energy $\mathscr{E}$ such that the total energy of the particle W is given by $W = e + \mathscr{E}$. For a given momentum $p(x, t)$, we have

$$u(x,t) = \frac{pc^2}{(e_0^2 + (pc)^2)^{1/2}}, \qquad e(x,t) = (e_0^2 + (pc)^2)^{1/2},$$

and arising from $\mathrm{d}\mathscr{E} = (\partial p/\partial t)\mathrm{d}x + c^2(\partial p/\partial x)\mathrm{d}t$, we have

$$\frac{\partial \mathscr{E}}{\partial t} = c^2 \frac{\partial p}{\partial x}, \qquad \frac{\partial \mathscr{E}}{\partial x} = \frac{\partial p}{\partial t},$$

so that if $p(x, t) = F(x + ct) + G(x - ct)$ for arbitrary functions F and G, then the wave energy $\mathscr{E}(x, t)$ is given by $\mathscr{E}(x, t) = c\,(F(x + ct) - G(x - ct))$, and by total differentiation of this expression with respect to time, it is not difficult to show that the three expressions

$$\begin{aligned} \mathscr{E}(x,t) &= c\,(F(x+ct) - G(x-ct))\,, \\ u(x,t) &= \frac{c^2\,(F(x+ct) + G(x-ct))}{(e_0^2 + c^2\,(F(x+ct) + G(x-ct))^2)^{1/2}}, \\ e(x,t) &= (e_0^2 + c^2\,(F(x+ct) + G(x-ct))^2)^{1/2}, \end{aligned}$$

automatically satisfy the equation

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{\mathrm{d}e}{\mathrm{d}t} + e\frac{\partial u}{\partial x},$$

for all functions $F(x + ct)$ and $G(x - ct)$, which constitute the formal general solution in a single spatial dimension.

For three spatial dimensions, the momentum $\mathbf{p}(\mathbf{x}, \mathbf{t})$ satisfies the wave-like Eq. (2.9) for which the wave energy $\mathscr{E}(\mathbf{x}, \mathbf{t})$ is obtained from

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \frac{\mathrm{d}e}{\mathrm{d}t} + e(\nabla.\mathbf{u}) = \mathbf{u}.\frac{\partial \mathbf{p}}{\partial t} + c^2(\nabla.\mathbf{p}), \tag{6.1}$$

where $\mathrm{d}/\mathrm{d}t$ denotes the total time derivative, and the Newtonian force $\mathbf{f}$ and the "force" in the direction of time g are linked to momentum $\mathbf{p}$ and de Broglie energy $\mathscr{E}$ through the simple formulae

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \qquad g = \frac{1}{c^2}\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}.$$

In the presence of an applied external field with potential $V(\mathbf{x},t)$, the two energies $\mathscr{E}$ and $e+V$ individually satisfy the wave equation, and the total particle energy $W+V = e+\mathscr{E}+V$ is necessarily conserved in a conventional manner.

The conventional operator relations of quantum mechanics $\mathbf{p} \longrightarrow -i\hbar\nabla$ and $e \longrightarrow i\hbar\partial/\partial t$, where $\hbar = h/2\pi$ and h is Planck's constant, indicate that quantum mechanics operates under circumstances for which the spatial physical force $\mathbf{f}$ as formulated here vanishes, and the force g in the direction of time becomes pure imaginary. Further, two Lorentz invariants ξ and η of special relativity that are defined by (4.1) provide the 'long searched for' formal connection between special relativity and quantum mechanics, such that $\eta = px - et$ associates with the particle or subluminal world, while $\xi = ex - c^2pt$ associates with the wave or superluminal world.

Since in any experiment either particles or de Broglie waves are reported, only one of e or $\mathscr{E}$ is physically measured, which leads to the expectation that particles appear for $e < \mathscr{E}$ and de Broglie waves occur for $\mathscr{E} \leqslant e$, but in either event, both a measurable energy and an unmeasurable energy exist, the latter registering its presence in the form of dark energy. Further, it is clear from Eq. (6.1) that the critical transition occurs for precisely volume preserving or incompressible motions for which $\nabla.\mathbf{u} = 0$.

The theory presented here is applied at a single particle level, and a future aim is to undertake an experiment which might measure any missing energy after work is done. If the only measurable consequence is the amount of cosmological dark energy, then another future aim might be to undertake a calculation to explain why the Einstein constant Λ is so large and why it appears to be constant, that is, why the density of dark energy remains constant as obtained from the Friedmann equation, as the universe expands. These are future aspirations, but it is presently clear that any realistic incorporation of the de Broglie wave energy will inevitably lead to a singularity for slowing systems and display characteristics of unstoppable mechanical systems, which might well provide the key idea to understanding the accelerating expansion of the universe.

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Appendices

Appendix A: Alternative derivation of Eqs. (4.9) and (4.10)

On taking the total time derivative $\mathrm{d}/\mathrm{d}t$ of the two invariants ξ and η given by (4.1) and making use of $\mathrm{d}e/\mathrm{d}t = u\mathrm{d}p/\mathrm{d}t$, we obtain

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\eta}{(1-(u/c)^2)}\frac{\mathrm{d}u}{\mathrm{d}t}, \qquad c\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\xi}{c\,(1-(u/c)^2)}\frac{\mathrm{d}u}{\mathrm{d}t} - e_0 c\left(1-(u/c)^2\right)^{1/2},$$

on using $\mathrm{d}p/\mathrm{d}t = m_0\left(1-(u/c)^2\right)^{-3/2}\mathrm{d}u/\mathrm{d}t$. These two equations simplify to give

$$\frac{\mathrm{d}\xi}{\mathrm{d}u} = \frac{\eta}{(1-(u/c)^2)}, \qquad c\frac{\mathrm{d}\eta}{\mathrm{d}u} = \frac{\xi}{c\,(1-(u/c)^2)} - e_0\frac{\mathrm{d}s}{\mathrm{d}u},$$

so that on introducing the substitution $u = c\sin\phi$ we have

$$\cos\phi\frac{\mathrm{d}\xi}{\mathrm{d}\phi} = c\eta, \qquad c\cos\phi\frac{\mathrm{d}\eta}{\mathrm{d}\phi} = \xi - e_0\cos\phi\frac{\mathrm{d}s}{\mathrm{d}\phi}.$$

Now on introducing χ defined by

$$\mathrm{d}\chi = \frac{\mathrm{d}\phi}{\cos\phi} = \frac{\mathrm{d}\phi}{\left(\cos^2(\phi/2)-\sin^2(\phi/2)\right)} = \frac{\sec^2(\phi/2)\mathrm{d}\phi}{\left(1-\tan^2(\phi/2)\right)},$$

we may readily deduce that

$$\chi = \log\left(\frac{1+\tan(\phi/2)}{1-\tan(\phi/2)}\right) = \frac{1}{2}\log\left(\frac{1+\sin\phi}{1-\sin\phi}\right) = \frac{1}{2}\log\left(\frac{1+u/c}{1-u/c}\right) = \theta,$$

where θ is as defined in Eq. (3.5), and Eqs. (4.9) and (4.10) follow immediately.

Appendix B: Further differential relations in terms of angles

From the two relations $\xi^2-(c\eta)^2 = e_0^2(x^2-(ct)^2)$ and $x^2-(ct)^2=\zeta^2$, we may formally introduce angles Φ and Ψ such that

$$\xi = e_0\zeta\cosh\Phi, \quad c\eta = e_0\zeta\sinh\Phi, \quad x = \zeta\cosh\Psi, \quad ct = \zeta\sinh\Psi, \tag{B.1}$$

so that from relations (4.4) and using Eq. (4.6) we may deduce

$$e = e_0\cosh(\Phi+\Psi) = e_0\cosh\theta, \qquad pc = e_0\sinh(\Phi+\Psi) = e_0\sinh\theta,$$

and therefore, we may conclude that simply $\theta = \Phi+\Psi$. On totally differentiating $\zeta^2 = x^2-(ct)^2$ with respect to time, we find that $\zeta\mathrm{d}\zeta/\mathrm{d}t = xu - c^2t = (xp-c^2et)/m$ from which we may deduce the equation

$$\zeta\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{c\eta}{f}, \tag{B.2}$$

and the three relations Eqs. (4.7) and (B.2) constitute the three basic equations connecting the three variables ξ,η and ζ as functions of $\theta=\Phi+\Psi$, where the angle θ relates to the velocity $u = c\tanh\theta$, and Φ and Ψ connect, respectively, with $(\xi, c\eta)$ and (x, ct) through (B.1), noting that only one of the two relations (4.7) is independent. Indeed, we may show from Eqs. (4.7) and the basic definition $u = \mathrm{d}x/\mathrm{d}t = c\tanh\theta$ that all three relations give rise to the single condition

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\theta} + \frac{e_0}{f\zeta}\cosh\Phi = 1, \tag{B.3}$$

while (B.2) yields

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\theta} = \frac{e_0\sinh\Phi}{f}, \tag{B.4}$$

further noting that the physical force f needs to be specified (say gravitational or electrical) before these two key Eqs. ((B.3) and (B.4)) can be fully solved as two equations in the two unknowns ζ and Φ. Again on using $e_0/f = \mathrm{d}s/\mathrm{d}\theta$, Eqs. (B.3) and (B.4)

$$\zeta\frac{\mathrm{d}\Phi}{\mathrm{d}s} + \cosh\Phi = \zeta\frac{\mathrm{d}\theta}{\mathrm{d}s}, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}s} = \sinh\Phi,$$

so that on using $\theta = \Phi + \Psi$ we have simply

$$\zeta \frac{\mathrm{d}\Psi}{\mathrm{d}s} = \cosh \Phi, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}s} = \sinh \Phi,$$

and therefore by division, we obtain the deceptively simple result

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\Psi} = \zeta \tanh \Phi,$$

connecting the three variables $\zeta = (x^2 - (ct)^2)^{1/2}$, $\Phi = \tanh^{-1}(c\eta/\xi)$ and $\Psi = \tanh^{-1}(ct/x)$.

Appendix C: Time-dependent Dirac equation for a free particle

In this appendix, for ease of reference, we state the details for the time-dependent Dirac equation for a free particle. The matrices $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{A_z}$ and $\mathbf{B}$ appearing in the time-dependent Dirac equation for a free particle are given, respectively, by

$$\mathbf{A_x} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A_y} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A_z} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and the Dirac equation becomes

$$\begin{pmatrix} \frac{\partial}{\partial t} & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial t} & \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} & \frac{\partial}{\partial t} & 0 \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} -if_0\psi_1 \\ -if_0\psi_2 \\ if_0\psi_3 \\ if_0\psi_4 \end{pmatrix},$$

where $f_0 = e_0/\hbar$. Further, it can be shown that each component $\psi_j = \psi_j(x, y, z, t)$ for $j = 1, 2, 3, 4$ satisfies the three spatial dimensions Klein–Gordon equation, thus

$$\frac{\partial^2 \psi_j}{\partial t^2} - c^2 \nabla^2 \psi_j = -\left(\frac{e_0}{\hbar}\right)^2 \psi_j. \tag{C.1}$$

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