Z. Angew. Math. Phys. (2019) 70:94

0044-2275/19/040001-17
published online June 8, 2019
https://doi.org/10.1007/s00033-019-1142-y

Zeitschrift für angewandte Mathematik und Physik ZAMP

Incompressible inviscid limit of the viscous two-fluid model with general initial data

Young-Sam Kwon and Fucai Li

Abstract. In this paper, we study the incompressible inviscid limit of the viscous two-fluid model in the whole space $\mathbb{R}^3$ with general initial data in the framework of weak solutions. By applying the refined relative entropy method and carrying out the detailed analysis on the oscillations of the densities and the velocity, we prove rigorously that the weak solutions of the compressible two-fluid model converge to the strong solution of the incompressible Euler equations in the time interval provided that the latter exists. Moreover, thanks to the Strichartz's estimates of linear wave equations, we also obtain the convergence rates. The main ingredient of this paper is that our wave equations include the oscillations caused by the two different densities and the velocity and we also give an detailed analysis on the effect of the oscillations on the evolution of the solutions.

Mathematics Subject Classification. Primary: 76T10; Secondary: 35B40.

Keywords. Viscous two-fluid model, Incompressible inviscid limit, Incompressible Euler equations, Whole space, General initial data.

1. Introduction

In this paper, we consider a compressible model of two-phase fluids in the following form [3,26,33]:

$$\partial_t n + \operatorname{div}(n\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t \varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \tag{1.2}$$

$$\begin{aligned}&\partial_t[(n+\varrho)\mathbf{u}] + \operatorname{div}[(n+\varrho)\mathbf{u}\otimes\mathbf{u}] + \nabla P(n,\varrho)\\ &\quad = \mu\Delta\mathbf{u} + (\mu+\nu)\nabla\operatorname{div}\mathbf{u},\end{aligned} \tag{1.3}$$

where the unknowns n and ϱ denote the densities of the fluids, and $\mathbf{u}\in\mathbb{R}^3$ denotes the common velocity of the fluids. Here we assume that the two fluids obey the same velocity for simplicity. The parameters μ and λ denote the viscosity coefficients satisfying $2\mu+\nu\geq 0$, and the pressure P takes the form $P(n,\varrho) = \frac{1}{\alpha}n^\alpha + \frac{1}{\gamma}\varrho^\gamma$ with $\gamma>1$ and $\alpha>1$.

The system (1.1)–(1.3) can be derived from the general two-fluid model [17,28] or from a coupled system of the compressible Navier–Stokes equation and a Vlasov–Fokker–Planck equation by taking an asymptotic limit [3,26]. The system (1.1)–(1.3) is also related to the compressible Oldroyd-B type model with stress diffusion [2].

Compared with the classical isentropic Navier–Stokes equations, the main difference is that the pressure law $P(n,\varrho)$ depends on two variables n and ϱ.

The work of the first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2017R1D1A1B03030249) and the second author was supported in part by NSFC (Grant No. 11671193) and A project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

In a series of papers [7–10] by S. Evje and his collaborators, they obtained the global existence of weak solutions to the system (1.1)–(1.3) in one-dimensional case. Yao, et al. [35] studied the existence of asymptotic behaviour of global weak solutions to a 2D viscous liquid–gas two-phase flow model with small initial data. Recently, Vasseur, et al. [33] obtained the global weak solution to the viscous two-fluid model (1.1)–(1.3) with finite energy in the framework of DiPerna–Lions' theory. Later, their results were extended to more general case by Novotný and Pokorný [28].

Yao et al. [36] studied the incompressible limit of the system (1.1)–(1.3) in the torus $\mathbb{T}^3$ with well-prepared initial data under the framework of local classical solutions. They obtained the convergence of the local strong solutions to the two-fluid model to that of the incompressible Navier–Stokes equations with an convergence rate.

In this paper, we study the incompressible limit of the compressible models of two-phase fluids (1.1)–(1.3) in the whole space $\mathbb{R}^3$ with *general initial data* in the framework of weak solutions established in [33]. We shall derive rigorously the incompressible Euler or Navier–Stokes equations based on the refined relative entropy method and the detailed analysis on the oscillations of the densities and the velocity. To begin with, we introduce the scaling

$$x \mapsto x,\ t \mapsto \epsilon t,\ \mathbf{u} \mapsto \epsilon \mathbf{u}_\epsilon,\ \mu \mapsto \epsilon\mu_\epsilon,\ \nu \mapsto \epsilon\nu_\epsilon. \tag{1.4}$$

Then the system (1.1)–(1.3) can be rewritten as

$$\partial_t n_\epsilon + \operatorname{div}(n_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.5}$$

$$\partial_t \varrho_\epsilon + \operatorname{div}(\varrho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.6}$$

$$\begin{aligned} &\partial_t[(n_\epsilon + \varrho_\epsilon)\mathbf{u}_\epsilon] + \operatorname{div}[(n_\epsilon + \varrho_\epsilon)\mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon] + \frac{1}{\epsilon^2}\nabla\left(\frac{1}{\alpha}n_\epsilon^\alpha + \frac{1}{\gamma}\varrho_\epsilon^\gamma\right) \\ &\quad = \mu_\epsilon \Delta \mathbf{u}_\epsilon + (\mu_\epsilon + \nu_\epsilon)\nabla \operatorname{div}\mathbf{u}_\epsilon, \end{aligned} \tag{1.7}$$

where we assume the initial data at the infinity:

$$n_\epsilon \to 1,\ \varrho_\epsilon \to 1,\ \mathbf{u}_\epsilon \to 0,$$

when $|x| \to \infty$.

Formally, by taking $\epsilon \to 0$ in (1.7), we get $n_\epsilon \to n(t)$ and $\varrho_\epsilon \to \varrho(t)$. If we further assume that the initial densities $n_{0,\epsilon}$ and $\varrho_{0,\epsilon}$ are small perturbations of some positive constant, say 1, we can also expect that $n_\epsilon \to 1$, $\varrho_\epsilon \to 1$ as $\epsilon \to 0$. Moreover, if we assume that the shear and bulk viscosity coefficients satisfy

$$\mu_\epsilon \to 0, \quad \nu_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{1.8}$$

then the system (1.5)–(1.7) is reduced to the classical incompressible Euler equations (suppose that the limits $n_\epsilon \mathbf{u}_\epsilon \to \mathbf{u}$ and $\varrho_\epsilon \mathbf{u}_\epsilon \to \mathbf{u}$ exist):

$$\begin{cases} \operatorname{div}\mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\Pi = 0. \end{cases} \tag{1.9}$$

In addition, if we suppose that

$$\mu_\epsilon \to \bar{\mu} > 0, \quad \nu_\epsilon \to \bar{\nu} \quad \text{as} \quad \epsilon \to 0, \tag{1.10}$$

then the system (1.5)–(1.7) is reduced to the classical incompressible Navier–Stokes equations (suppose that the limits $n_\epsilon \mathbf{u}_\epsilon \to \mathbf{u}$ and $\varrho_\epsilon \mathbf{u}_\epsilon \to \mathbf{u}$ exist):

$$\begin{cases} \operatorname{div}\mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \bar{\mu}\Delta\mathbf{u} + \nabla\Pi = 0. \end{cases} \tag{1.11}$$

.

Our goal of this paper is to investigate the above limits rigorously in some suitable sense.

The outline of this article is as follows: In Sect. 2, we present the result of global weak solutions for the compressible models of two-phase fluids (1.5)–(1.7) and state our main results. In Sect. 3, we give the proofs of our results.

Before ending Introduction, we point out that the incompressible limits of the compressible Navier–Stokes equations and related models are very interesting topics and there are a lot of works on it. Among others, we mention [5,15,23,25,27] on the isentropic Navier–Stokes equations, [1,12–14] on the full Navier–Stokes–Fourier system, [24,34] on quantum isentropic Navier–Stokes equations, [16,18,19] on isentropic compressible magnetohydrodynamic equations, and [20,21] on the full compressible magnetohydrodynamic equations.

2. Main results

In this section, we introduce our main results of incompressible limit for the compressible model of two-phase fluids (1.5)–(1.7) in the whole space $\mathbb{R}^3$.

For any vector field $\mathbf{v}$, we use $\mathbf{P}$ and $\mathbf{Q}$ to denote the divergence-free part of $\mathbf{v}$ and the gradient part of $\mathbf{v}$, respectively, i.e. $\mathbf{P}(\mathbf{v}) = \mathbf{v} - \mathbf{Q}(\mathbf{v})$ and $\mathbf{Q}(\mathbf{v}) = \nabla\Delta^{-1}\mathrm{div}\mathbf{v}$. Below the letter C denotes a generic positive constant, independent of ϵ, and may change from line to line. And the letter C_T denotes a generic positive constant, dependent on T.

We first recall the global weak solutions of the system (1.5)–(1.7). We assume that the initial data $(n_\epsilon, \varrho_\epsilon, \mathbf{u}_\epsilon)|_{t=0} = (n_{0,\epsilon}, \varrho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$ satisfy

$$\int_{\mathbb{R}^3} \Big[\frac{(n_{0,\epsilon}+\varrho_{0,\epsilon})|\mathbf{u}_{0,\epsilon}|^2}{2} + G_\alpha(n_{0,\epsilon}) + H_\gamma(\varrho_{0,\epsilon})\Big]\mathrm{d}x < \infty, \tag{2.1}$$

$$\inf_{x\in\mathbb{R}^3} n_{0,\epsilon} \geq 0,\ \inf_{x\in\mathbb{R}^3} \varrho_{0,\epsilon} \geq 0,\ G_\alpha(n_{0,\epsilon}),\ H_\gamma(\varrho_{0,\epsilon}) \in L^1(\mathbb{R}^3), \tag{2.2}$$

where $n_{0,\epsilon} \to 1,\ \varrho_{0,\epsilon} \to 1,\ \mathbf{u}_{0,\epsilon} \to 0$ when $|x| \to \infty$ and $n_{0,\epsilon}-1 \in L^\alpha(\mathbb{R}^3),\ \varrho_{0,\epsilon}-1 \in L^\gamma(\mathbb{R}^3)$. Furthermore, we assume

$$\frac{M_{0,\epsilon}}{\sqrt{n_{0,\epsilon}+\varrho_{0,\epsilon}}} \in L^2(\mathbb{R}^3) \text{ if } \frac{M_{0,\epsilon}}{\sqrt{n_{0,\epsilon}+\varrho_{0,\epsilon}}} = 0 \text{ on } \{x\in\mathbb{R}^3 | n_{0,\epsilon}(x)+\varrho_{0,\epsilon}(x) = 0\}, \tag{2.3}$$

where

$$G_\alpha(n_\epsilon) = \frac{1}{\alpha(\alpha-1)\epsilon^2}(n_\epsilon^\alpha - 1 - \alpha(n_\epsilon - 1)),$$
$$H_\gamma(\varrho_\epsilon) = \frac{1}{\gamma(\gamma-1)\epsilon^2}(\varrho_\epsilon^\gamma - 1 - \gamma(\varrho_\epsilon - 1)),$$

and $M_{0,\epsilon} = n_{0,\epsilon} + \varrho_{0,\epsilon}$.

Proposition 2.1. [33] *Let $\epsilon > 0$ be a fixed number. Suppose that the initial data $(n_{0,\epsilon}, \varrho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$ satisfy (2.1)–(2.3). If*

$$\alpha \geq 1, \quad \gamma > \frac{9}{5}$$

and the initial data $n_{0,\epsilon}$ and $\varrho_{0,\epsilon}$ verify

$$\frac{1}{C_0}\varrho_{0,\epsilon} \leq n_{0,\epsilon} \leq C_0\varrho_{0,\epsilon} \quad \text{on} \quad \mathbb{R}^3 \tag{2.4}$$

for some positive constant C_0, then there exists a weak solution of the system (1.5)–(1.7) in the sense of distribution verifying the following energy inequality:

$$E_\epsilon(\tau) + \int_0^T \int_{\mathbb{R}^3} \Big(\mu_\epsilon|\nabla\mathbf{u}_\epsilon|^2 + (\mu_\epsilon+\nu_\epsilon)|\mathrm{div}\mathbf{u}_\epsilon|^2\Big)\mathrm{d}x\mathrm{d}t \leq E_{0,\epsilon}, \tag{2.5}$$

where

$$E_\epsilon(\tau) := \int_{\mathbb{R}^3} \Big(\frac{1}{2}(n_\epsilon + \varrho_\epsilon)|\mathbf{u}_\epsilon|^2 + G_\alpha(n_\epsilon) + H_\gamma(\varrho_\epsilon)\Big)\mathrm{d}x,$$
$$E_{0,\epsilon} := \int_{\mathbb{R}^3} \Big(\frac{1}{2}(n_{0,\epsilon} + \varrho_{0,\epsilon})|\mathbf{u}_{0,\epsilon}|^2 + G_\alpha(n_{0,\epsilon}) + H_\gamma(\varrho_{0,\epsilon})\Big)\mathrm{d}x.$$

Remark 2.1. Although the proof of global existence of the system (1.5)–(1.7) is given in [33] on bounded domains, we can prove a similar global weak solution of it in the whole space $\mathbb{R}^3$ based on the same spirit with the standard expanding domain techniques [29]. For the value of density at the infinity, we get: for fixed number $M \gg 1$, we define $\varrho_M : [0,\infty) \to [0,\infty)$ by

$$\varrho_M \in C^\infty,\ \varphi_M(x) = x \text{ if } |x| \le M,\ \varphi_M(x) = 1 \text{ if } |x| \ge 2M.$$

Let $\overline{\varrho}_0(x) = \varphi_M(\varrho_0(x))$. Then we construct a weak solution with the initial data $\overline{\varrho}_0(x)$ on $B_M(0) = \{x||x| \le M\}$. Then, when $M \to \infty$, obviously, we get $\rho_0(x) \to 1$ when $|x| \to \infty$. Similarly, we can also prove that $n_0(x) \to 1$ when $|x| \to \infty$.

Remark 2.2. In fact, [33] is a particular case of [28]. In particular, in [28], they have constructed solutions for $\alpha > 0$ and $\gamma \ge 9/5$. However, we here use the result of [33] for the convenient presentation such that it is more convenient to deal with the pressure terms of the relative entropy. Actually, the arguments in the proof of Theorem 2.1 still hold for all $\alpha > 1$ and $\gamma > 1$ provided that the system (1.5)–(1.7) has a global weak solution as stated in Proposition 2.1.

Remark 2.3. In [33], the authors pointed that the results in Proposition 2.1 still hold without the condition (2.4) on the initial densities if we have further restrictions on the powers of the pressure:

$$\alpha, \gamma > \frac{9}{5} \text{ and } \max\left\{\frac{3\gamma}{4}, \gamma - 1, \frac{3(\gamma+1)}{5}\right\} < \alpha < \max\left\{\frac{3\gamma}{4}, \gamma + 1, \frac{5\gamma}{3} - 1\right\}.$$

Since our main concerns here are on the incompressible limit to the system (1.5)–(1.7), it is easy to check that our assumptions in Theorem 2.1 satisfy the condition (2.4).

Next, we recall the classical results on the incompressible Euler equations (1.9).

Proposition 2.2. [22,32] *Assume that the initial datum* $\mathbf{u}(x,t)|_{t=0} = \mathbf{u}_0(x)$ *satisfies*

$$\mathbf{u}_0(x) \in H^s(\mathbb{R}^3), \quad \operatorname{div}\mathbf{u}_0 = 0, \quad s > 5/2. \tag{2.6}$$

Then there exist a $T^* \in (0,\infty)$ *and a unique solution* $\mathbf{u}$ *to the incompressible Euler equations*

$$\begin{cases} \operatorname{div}\mathbf{u} = 0, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\Pi = 0 \end{cases} \tag{2.7}$$

satisfying the following estimates

$$\sup_{0<t\le T}\big(\|\mathbf{u}\|_{H^s(\mathbb{R}^3)} + \|\partial_t\mathbf{u}\|_{H^{s-1}(\mathbb{R}^3)} + \|\nabla\Pi\|_{H^s(\mathbb{R}^3)} + \|\partial_t\nabla\Pi\|_{H^{s-1}(\mathbb{R}^3)}\big) \le C(T) \tag{2.8}$$

with $C(T) > 0$*, a constant for any* $0 < T < T^*$.

Finally, we denote $n_\epsilon = 1 + \epsilon\varphi_\epsilon$, $\varrho_\epsilon = 1 + \epsilon\psi_\epsilon$ and

$$\Phi_\epsilon = \sqrt{2G_\alpha(n_\epsilon)},\ \Phi_{0,\epsilon} = \sqrt{2G_\alpha(n_{0,\epsilon})},\ \Psi_\epsilon = \sqrt{2H_\gamma(\varrho_\epsilon)},\ \Psi_{0,\epsilon} = \sqrt{2H_\gamma(\varrho_{0,\epsilon})}.$$

Now we are in a position to state our results of this paper.

Theorem 2.1. *Let $\alpha > 1$ and $\gamma > 9/5$ and*

$$\mu_\epsilon = \nu_\epsilon = \epsilon^a, \quad a > 0. \tag{2.9}$$

Assume that the initial data $(n_{0,\epsilon}, \varrho_{0,\epsilon}, \mathbf{u}_{0,\epsilon})$ satisfy the conditions in Proposition 2.1 and $n_{0,\epsilon} = 1 + \epsilon\varphi_{0,\epsilon}$, $\varrho_{0,\epsilon} = 1 + \epsilon\psi_{0,\epsilon}$,

$$\begin{aligned} &\|\Phi_{0,\epsilon} - \varphi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} + \|\Psi_{0,\epsilon} - \psi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} \\ &\quad + \|\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \hat{\mathbf{u}}_0\|^2_{L^2(\mathbb{R}^3)} + \|\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \hat{\mathbf{u}}_0\|^2_{L^2(\mathbb{R}^3)} \le C\epsilon^b, \end{aligned} \tag{2.10}$$

$$\|\varphi_{0,\epsilon} - \psi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} \le C\epsilon^c \tag{2.11}$$

for some constants $b > 0$ and $c > 0$, where

$$\|\varphi_{0,\epsilon}\|_{L^\infty \cap L^2(\mathbb{R}^3)} + \|\psi_{0,\epsilon}\|_{L^\infty \cap L^2(\mathbb{R}^3)} \le c_0, \ \mathbf{P}(\hat{\mathbf{u}}_0) \in H^k(\mathbb{R}^3; \mathbb{R}^3), \tag{2.12}$$

for some $k > 7/2$. Let

$$\sigma = \begin{cases} \min\{\frac{1}{4}, \frac{b}{2}, a, c\}, & 1 < \alpha \le 4, \ 9/5 < \gamma \le 4, \\ \min\{\frac{1}{4}, \frac{b}{2}, \frac{\alpha-4}{2\alpha}, a, c\}, & \alpha > 4, \ 9/5 < \gamma \le 4, \\ \min\{\frac{1}{4}, \frac{b}{2}, \frac{\gamma-4}{2\gamma}, a, c\}, & 1 < \alpha \le 4, \ \gamma > 4, \\ \min\{\frac{b}{2}, \frac{\alpha-4}{2\alpha}, \frac{\gamma-4}{2\gamma}, a, c\}, & \alpha > 4, \ \gamma > 4, \end{cases} \tag{2.13}$$

and $(\mathbf{u}, \Pi)$ be the local strong solution, on the time interval $0, T^*)$, to the incompressible Euler equations (2.7) with the initial datum $\mathbf{u}(x, 0) = \mathbf{u}_0 = \mathbf{P}(\hat{\mathbf{u}}_0)$. Then, for any $T < T_*$, the weak solution $(n_\epsilon, \varrho_\epsilon, \mathbf{u}_\epsilon)$ of the system (1.5)–(1.7) established in Proposition 2.1 satisfies:

$$\|\sqrt{n_\epsilon} - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\sqrt{\varrho^\epsilon} - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C_T\epsilon, \tag{2.14}$$

$$\|\mathbf{P}(\sqrt{n_\epsilon}\mathbf{u}_\epsilon) - \mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\mathbf{P}(\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon) - \mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C_T\epsilon^\sigma, \tag{2.15}$$

$$\|\sqrt{n_\epsilon}\mathbf{u}_\epsilon - \mathbf{u}\|_{L^r(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))} + \|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \mathbf{u}\|_{L^r(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))} \le C_T\epsilon^d \tag{2.16}$$

for all $2 < r < \infty$. Here $d = \min\left\{\frac{\sigma}{2}, \frac{1}{r}\right\}$.

Remark 2.4. If we replace the assumption (2.9) by (1.10), we can obtain a similar result to Theorem 2.1. Here the target equations are the incompressible Navier–Stokes equations (1.11). Since the proof is similar to that of Theorem 2.1, we omit the details here for brevity.

Remark 2.5. When the initial data are well prepared, it is much easier to show that the weak solutions of the compressible two-fluid model converge to the local strong solution of the incompressible Euler or Navier–Stokes equations in the time interval provided that the latter exists. Moreover, we can obtain the convergence rates. In fact, in this case, because there are no oscillations, we do not need to use the Strichartz's estimates of linear wave equations. Thus, the relative entropy to the system (1.5)–(1.7) becomes

$$\mathcal{E}_\epsilon(\tau) = \frac{1}{2}\int_{\mathbb{R}^3} \Big((n_\epsilon + \varrho_\epsilon)|\mathbf{u}_\epsilon - \mathbf{u}|^2 + |\Phi_\epsilon|^2 + |\Psi_\epsilon|^2\Big)\mathrm{d}x, \tag{2.17}$$

where $\mathbf{u}$ is the local strong solution of the incompressible Euler or Navier–Stokes equations. Since the proof is much simpler than that in Theorem 2.1, we will omit the details here. The readers can refer [15] on the discussion of isentropic Navier–Stokes equations.

Remark 2.6. For well-prepared initial data, we can also study the incompressible limit to the system (1.5)–(1.7) in the torus $\mathbb{T}^3$ or bounded domain and obtain a similar convergence result stated in the above remark. In fact, no Poincaré inequality is needed in our arguments. Thus, our results can be regarded as an extension and improvement in that in [36], where the incompressible limit is studied only for local strong solutions with well-prepared initial data in the $\mathbb{T}^3$. However, if we consider the general initial data case in the torus, the oscillations will survive forever and satisfy a parabolic equations; the readers can refer [27,30] on the discussion of isentropic Navier–Stokes equations in the torus $\mathbb{T}^3$. See also [5] on the isentropic Navier–Stokes equations in the bounded domain case.

We give some comments on the proof of Theorem 2.1. We will make full use of the energy inequality, compact arguments, the refined relative entropy method (see [11,13,27]), and the Strichartz's estimates of linear wave equations (see [6]). Thanks to the dispersive effects of the linear wave equations in the whole space $\mathbb{R}^3$, we can further obtain the convergence rate of the incompressible limit. Compared with the previous results on the isentropic Navier–Stokes equations (see [4,15,25,27]), the main ingredient of this paper is that our wave equation includes the oscillations caused by the two different densities and the velocity and we also give an detailed analysis on the effect of the oscillations on the evolutions of the solutions. In fact, we have used the oscillation of $\frac{\varrho_\epsilon-1+n_\epsilon-1}{2}$ for the density to figure out the pressure term in the relative entropy.

3. Proof of Theorem 2.1

In this section, we are going to give a rigorous proof of Theorem 2.1. We will make full use of the energy inequality, the refined relative entropy inequality, and the Strichartz's estimates of linear wave equations to obtain the convergence rate of the solutions. From now on, we work on any time $T < T_*$, where T_* is the maximal existing time of solutions to the incompressible Euler equations (2.7).

3.1. Uniform bounds

In this subsection, we are going to derive some estimates on the sequence $\{(n_\epsilon, \varrho_\epsilon, \mathbf{u}_\epsilon)\}_{\epsilon>0}$.

From the energy inequality (2.5), we obtain that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{n_\epsilon}\mathbf{u}_\epsilon(t)\|_{L^2(\mathbb{R}^3)} \le C_T, \tag{3.1}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon(t)\|_{L^2(\mathbb{R}^3)} \le C_T, \tag{3.2}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|n_\epsilon^\alpha - 1 - \alpha(n_\epsilon - 1)\|_{L^1(\mathbb{R}^3)} \le C_T\epsilon^2, \tag{3.3}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\epsilon^\gamma - 1 - \gamma(\varrho_\epsilon - 1)\|_{L^1(\mathbb{R}^3)} \le C_T\epsilon^2. \tag{3.4}$$

We consider the properties of convex function

$$\begin{cases} s^\gamma - 1 - \gamma(s-1) \ge C|s-1|^2 & \text{if} \quad \gamma \ge 2, \\ s^\gamma - 1 - \gamma(s-1) \ge C|s-1|^2 & \text{if} \quad 1 < \gamma < 2 \quad \text{and} \quad 0 < s \le R, \\ s^\gamma - 1 - \gamma(s-1) \ge C|s-1|^\gamma & \text{if} \quad 1 < \gamma < 2 \quad \text{and} \quad s \ge R \end{cases} \tag{3.5}$$

for some constant $R > 0$.

Let us introduce the set of the essential and residual values

$$g = [g]_{\text{ess}} + [g]_{\text{res}},$$

where $[g]_{\text{ess}} = \chi(\varrho_\epsilon)g$, $[g]_{\text{res}} = (1 - \chi(\varrho_\epsilon))g$, and χ is defined as

$$\chi(r) = \begin{cases} 1, & r \in [1/2, 2], \\ 0, & \text{otherwise}. \end{cases}$$

Following the estimates (3.3), (3.4), and the convexity (3.5), we get

$$\operatorname{ess}\sup_{t\in(0,T)} \Big\| \Big[\frac{n_\epsilon - 1}{\epsilon}\Big]_{\text{ess}} \Big\|_{L^2(\mathbb{R}^3)} \le C_T, \tag{3.6}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \Big\| \Big[\frac{\varrho_\epsilon - 1}{\epsilon}\Big]_{\text{ess}} \Big\|_{L^2(\mathbb{R}^3)} \le C_T, \tag{3.7}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\mathbb{R}^3} [1 + n_\epsilon^\alpha]_{\text{res}} \mathrm{d}x \le C_T \epsilon^2 \tag{3.8}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\mathbb{R}^3} [1 + \varrho_\epsilon^\gamma]_{\text{res}} \mathrm{d}x \le C_T \epsilon^2. \tag{3.9}$$

Noting that the following two elementary inequalities

$$|\sqrt{x} - 1|^2 \le M|x-1|^\gamma, \quad |x-1| \ge \delta, \ \ \gamma \ge 1, \tag{3.10}$$

$$|\sqrt{x} - 1|^2 \le M|x-1|^2, \quad x \ge 0, \tag{3.11}$$

for some positive constant M and $0 < \delta < 1$, it is easy to obtain that

$$\begin{aligned} \int_{\mathbb{R}^3} |\sqrt{n_\epsilon} - 1|^2 &= \int_{\mathbb{R}^3} |\sqrt{n_\epsilon} - 1|^2 1_{\{|n_\epsilon - 1| \le 1/2\}} + \int_{\mathbb{R}^3} |\sqrt{n_\epsilon} - 1|^2 1_{\{|n_\epsilon - 1| > 1/2\}} \\ &\le M \int_{\mathbb{R}^3} |\sqrt{n_\epsilon} - 1|^2 1_{\{|n_\epsilon - 1| \le 1/2\}} + M \int_{\mathbb{R}^3} |\sqrt{n_\epsilon} - 1|^\gamma 1_{\{|n_\epsilon - 1| > 1/2\}} \\ &\le CM\epsilon^2. \end{aligned} \tag{3.12}$$

It then follows that

$$\|\sqrt{n_\epsilon} - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C_T \epsilon.$$

Similarly, we have

$$\|\sqrt{\varrho^\epsilon} - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C_T \epsilon.$$

Hence, (2.14) holds.

In accordance with (3.6) and (3.8), we obtain that

$$n_\epsilon \to 1 \ \text{ in } \ L^\infty(0,T;(L^2 + L^p)(\mathbb{R}^3)), \quad p = \min\{\alpha, 2\}. \tag{3.13}$$

Similarly,

$$\varrho_\epsilon \to 1 \ \text{ in } \ L^\infty(0,T;(L^2 + L^p)(\mathbb{R}^3)), \quad p = \min\{\gamma, 2\}. \tag{3.14}$$

(3.13) and (3.14), together with (3.1) and (3.2), imply that, up to extraction an subsequence, still denoted by $n_\epsilon \mathbf{u}_\epsilon$ and $\varrho_\epsilon \mathbf{u}_\epsilon$,

$$n_\epsilon \mathbf{u}_\epsilon \to \mathbf{u} \ \text{ weakly} - * \ \text{ in } \ L^\infty(0,T;(L^2 + L^{2\alpha/(\alpha+1)})(\mathbb{R}^3;\mathbb{R}^3)), \tag{3.15}$$

$$\varrho_\epsilon \mathbf{u}_\epsilon \to \mathbf{u} \ \text{ weakly} - * \ \text{ in } \ L^\infty(0,T;(L^2 + L^{2\gamma/(\gamma+1)})(\mathbb{R}^3;\mathbb{R}^3)). \tag{3.16}$$

Combining this with continuity Eqs. (1.5) or (1.6), we deduce that, up to extraction an subsequence,

$$\operatorname{div}\mathbf{u}_\epsilon \ \text{ converges weakly to0in} H^{-1}((0,T) \times \mathbb{R}^3). \tag{3.17}$$

3.2. Stricharz's estimates

In this subsection, we introduce the Strichartz's estimate of linear wave equations to handle the oscillations of the densities and the velocity. To do this, we consider the following acoustic system:

$$\begin{cases} \epsilon\partial_t s_\epsilon + \Delta q_\epsilon = 0, \quad s_\epsilon(0,\cdot) = s_{0,\epsilon} = \frac{1}{2}(\Phi_{0,\epsilon} + \Psi_{0,\epsilon}), \\ \epsilon\partial_t \nabla q_\epsilon + \nabla s_\epsilon = 0, \nabla q_\epsilon(0,\cdot) = \nabla q_{0,\epsilon} = \frac{1}{2}(\mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon}) + \mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon})). \end{cases} \tag{3.18}$$

Here we have used $\mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon})$ and $\mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon})$ as an approximation of $\mathbf{Q}(\varrho_{0,\epsilon}\mathbf{u}_{0,\epsilon})$ and $\mathbf{Q}(n_{0,\epsilon}\mathbf{u}_{0,\epsilon})$, respectively, since

$$\|\mathbf{Q}(\varrho_{0,\epsilon}\mathbf{u}_{0,\epsilon}) - \mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon})\|_{L^1(\mathbb{R}^3)} \le C\epsilon, \tag{3.19}$$

$$\|\mathbf{Q}(n_{0,\epsilon}\mathbf{u}_{0,\epsilon}) - \mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon})\|_{L^1(\mathbb{R}^3)} \le C\epsilon. \tag{3.20}$$

We shall regularize the initial data $(\frac{1}{2}(\Psi_{0,\epsilon} + \Psi_{0,\epsilon}), \frac{1}{2}(\mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon}) + \mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon})))$ to remove the interruption of computation of convergence. Let us choose the following initial data:

$$\begin{cases} s_{0,\epsilon} = s_{0,\epsilon,\delta} = \chi_\delta * [\frac{1}{2}(\Phi_{0,\epsilon} + \Psi_{0,\epsilon})], \\ \nabla q_{0,\epsilon} = \nabla q_{0,\epsilon,\delta} = \chi_\delta * \left[\frac{1}{2}(\mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon}) + \mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon}))\right]. \end{cases} \tag{3.21}$$

Here $\chi^\delta(x) = (1/\delta^3)\chi(x/\delta)$ and $\chi \in C_0^\infty(\mathbb{R}^3)$ is the Friedrich's mollifier, i.e. $\int_{\mathbb{R}^3} \chi \mathrm{d}x = 1$. From now on, we remove δ to proceed the convenient presentation. Then we have the Strichartz's estimates:

Proposition 3.1. [6] *Let* $(s_\epsilon, \nabla q_\epsilon)$ *be the solution of system* (3.18) *with initial data* $(s^j_{\epsilon,0}, q^j_{\epsilon,0})$ *given in* (3.21). *Then, one has*

$$\begin{aligned} &\|s_\epsilon(\cdot,t)\|^2_{H^k(\mathbb{R}^3)} + \|\nabla q_\epsilon(\cdot,t)\|^2_{H^k(\mathbb{R}^3;\mathbb{R}^3)} \\ &\quad \le \|s_{0,\epsilon}\|^2_{H^k(\mathbb{R}^3)} + \|\nabla q_{0,\epsilon}\|^2_{H^k(\mathbb{R}^3;\mathbb{R}^3)}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\|s_\epsilon\|_{L^l(\mathbb{R};W^{k,p}(\mathbb{R}^3;\mathbb{R}^3))} + \|\nabla q_\epsilon\|_{L^l(\mathbb{R};W^{k,p}(\mathbb{R}^3;\mathbb{R}^3))} \\ &\quad \le C\epsilon^{1/l}\big(\|s_{0,\epsilon}\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)} + \|\nabla q_{0,\epsilon}\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)}\big) \end{aligned} \tag{3.23}$$

with

$$2 < p,\, l \le \infty, \ \frac{1}{p} + \frac{1}{l} = \frac{1}{2}, \ k = 0, 1, 2, \dots.$$

3.3. Relative entropy inequality

Recall that we have assumed that

$$\mu_\epsilon = \nu_\epsilon = \epsilon^a, \quad a > 0.$$

In order to prove the convergence of Theorem 2.1, we will introduce the relative entropy to the two-fluid system (1.5)–(1.7):

$$\mathcal{E}_\epsilon(\tau) = \frac{1}{2}\int_{\mathbb{R}^3} \Big((n_\epsilon + \varrho_\epsilon)|\mathbf{u}_\epsilon - \mathbf{U}|^2 + |\Phi_\epsilon - s_\epsilon|^2 + |\Psi_\epsilon - s_\epsilon|^2\Big)\mathrm{d}x,$$

where $\mathbf{U} = \mathbf{u} + \nabla q_\epsilon$.

Let first recall the energy inequality to the system (1.5)–(1.7):

$$E_\epsilon(\tau) + \int_0^T \int_{\mathbb{R}^3} \Big(\mu_\epsilon|\nabla \mathbf{u}_\epsilon|^2 + (\mu_\epsilon + \nu_\epsilon)|\mathrm{div}\mathbf{u}_\epsilon|^2\Big)\mathrm{d}x\mathrm{d}t \le E_{0,\epsilon}, \tag{3.24}$$

where

$$E_\epsilon(\tau) := \int_{\mathbb{R}^3} \Big(\frac{1}{2}(n_\epsilon+\varrho_\epsilon)|\mathbf{u}_\epsilon|^2 + G_\alpha(n_\epsilon) + H_\gamma(\varrho_\epsilon)\Big)\mathrm{d}x,$$
$$E_{0,\epsilon} := \int_{\mathbb{R}^3} \Big(\frac{1}{2}(n_{0,\epsilon}+\varrho_{\epsilon,0})|\mathbf{u}_{0,\epsilon}|^2 + G_\alpha(n_{0,\epsilon}) + H_\gamma(\varrho_{0,\epsilon})\Big)\mathrm{d}x.$$

The conservation of energy for the incompressible Euler equations (2.7) reads

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(t)|^2\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3}|u_0|^2\mathrm{d}x. \tag{3.25}$$

From the system (3.18), we get

$$\int_{\mathbb{R}^3} (|s_\epsilon|^2(\tau) + |\nabla q_\epsilon|^2(\tau))\mathrm{d}x = \int_{\mathbb{R}^3}(|s_{0,\epsilon}|^2 + |\nabla q_{0,\epsilon}|^2)\mathrm{d}x. \tag{3.26}$$

We adapt $\mathbf{U}$ as test functions to the moment equation (1.7):

$$\begin{aligned}
-\int_{\mathbb{R}^3}((n_\epsilon+\varrho_\epsilon)\mathbf{u}_\epsilon\cdot\mathbf{U})(\tau)\mathrm{d}x &= -\int_{\mathbb{R}^3}((n_{0,\epsilon}+\varrho_{0,\epsilon})\mathbf{U}_{0,\epsilon})\cdot\mathbf{u}_{0,\epsilon}\mathrm{d}x\\
&-\int_0^\tau\int_{\mathbb{R}^3}\big[((n_\epsilon+\varrho_\epsilon)\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon : \nabla\mathbf{U} - \mu_\epsilon\nabla\mathbf{u}_\epsilon:\nabla\mathbf{U}\big]\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\mathbb{R}^3}(\mu_\epsilon+\nu_\epsilon)\mathrm{div}\mathbf{u}_\epsilon\ \mathrm{div}\mathbf{U}\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_{\mathbb{R}^3}(n_\epsilon+\varrho_\epsilon)\mathbf{u}_\epsilon\cdot\partial_t\mathbf{U}\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{\epsilon}\int_0^\tau\int_{\mathbb{R}^3}(\varphi_\epsilon+\psi_\epsilon)\mathrm{div}\mathbf{U}\mathrm{d}x\mathrm{d}t - \frac{\alpha-1}{2}\int_0^\tau\int_{\mathbb{R}^3}|\Phi_\epsilon|^2\mathrm{div}\mathbf{U}\mathrm{d}x\mathrm{d}t,\\
&-\frac{\gamma-1}{2}\int_0^\tau\int_\Omega|\Psi_\epsilon|^2\mathrm{div}\mathbf{U}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.27}$$

We also use $\frac{1}{2}|\mathbf{U}|^2$ as a test function to continuity Eqs. (1.5) and (1.6), respectively, to deduce that

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^3}(n_\epsilon+\varrho_\epsilon)|\mathbf{U}|^2\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^3}(n_{0,\epsilon}+\varrho_{0,\epsilon})|\mathbf{U}_0|^2\mathrm{d}x\\
&+\int_0^\tau\int_{\mathbb{R}^3}(n_\epsilon+\varrho_\epsilon)\partial_t\mathbf{U}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\mathbb{R}^3}(n_\epsilon+\varrho_\epsilon)\mathbf{u}_\epsilon\cdot\nabla\mathbf{U}\cdot\mathbf{U}\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.28}$$

Using s_ϵ as a test function to continuity Eqs. (1.5) and (1.6), respectively, we get

$$-\int_{\mathbb{R}^3}\varphi_\epsilon s_\epsilon\mathrm{d}x = -\int_{\mathbb{R}^3}\varphi_{0,\epsilon}\varphi_0\mathrm{d}x - \int_0^\tau\int_{\mathbb{R}^3}\varphi_\epsilon\partial_t s_\epsilon\mathrm{d}x\mathrm{d}t - \frac{1}{\epsilon}\int_0^\tau\int_{\mathbb{R}^3}n_\epsilon\mathbf{u}_\epsilon\cdot\nabla s_\epsilon\mathrm{d}x\mathrm{d}t. \tag{3.29}$$
$$-\int_{\mathbb{R}^3}\psi_\epsilon s_\epsilon\mathrm{d}x = -\int_{\mathbb{R}^3}\psi_{0,\epsilon}\varphi_0\mathrm{d}x - \int_0^\tau\int_{\mathbb{R}^3}\psi_\epsilon\partial_t s_\epsilon\mathrm{d}x\mathrm{d}t - \frac{1}{\epsilon}\int_0^\tau\int_\Omega\varrho_\epsilon\mathbf{u}_\epsilon\cdot\nabla s_\epsilon\mathrm{d}x\mathrm{d}t. \tag{3.30}$$

Thus, we deduce, after adding up (3.24)–(3.30), the following inequality:

$$\mathcal{E}_\epsilon(\tau) + \int_0^\tau\int_{\mathbb{R}^3}\Big(\mu_\epsilon|\nabla\mathbf{u}_\epsilon|^2 + (\mu_\epsilon+\nu_\epsilon)|\mathrm{div}\mathbf{u}_\epsilon|^2\Big)\mathrm{d}x\mathrm{d}t \le \Sigma_{j=1}^7 A_j^\epsilon, \tag{3.31}$$

where

$$
\begin{aligned}
A_1^\epsilon &= \mathcal{E}_\epsilon(0) - \Big[\int_{\mathbb{R}^3} |\nabla q_\epsilon|^2 \mathrm{d}x\Big]_0^\tau,\\
A_2^\epsilon &= -\int_{\mathbb{R}^3} (\Phi_\epsilon(\tau) - \varphi_\epsilon(\tau)) s_\epsilon \mathrm{d}x + \int_{\mathbb{R}^3} (\Phi_\epsilon(0) - \varphi_\epsilon(0)) s_{0,\epsilon} \mathrm{d}x,\\
A_3^\epsilon &= -\int_{\mathbb{R}^3} (\Psi_\epsilon(\tau) - \psi_\epsilon(\tau)) s_\epsilon \mathrm{d}x + \int_{\mathbb{R}^3} (\Psi_\epsilon(0) - \psi_\epsilon(0)) s_{0,\epsilon} \mathrm{d}x,\\
A_4^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} (n_\epsilon + \varrho_\epsilon)\Big(\partial_t \mathbf{U} + \mathbf{u}_\epsilon \cdot \nabla \mathbf{U}\Big) \cdot (\mathbf{U} - \mathbf{u}_\epsilon) \mathrm{d}x\mathrm{d}t,\\
A_5^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} \mu_\epsilon \nabla \mathbf{u}_\epsilon : \nabla \mathbf{U} \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} (\mu_\epsilon + \nu_\epsilon) \mathrm{div}\mathbf{u}_\epsilon \,\mathrm{div}\mathbf{U} \mathrm{d}x\mathrm{d}t,\\
A_6^\epsilon &= -\frac{1}{\epsilon} \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon \mathbf{u}_\epsilon \cdot \nabla s_\epsilon \mathrm{d}x\mathrm{d}t - \frac{\alpha-1}{2} \int_0^\tau \int_{\mathbb{R}^3} |\Phi_\epsilon|^2 \mathrm{div}\mathbf{U} \mathrm{d}x\mathrm{d}t,\\
A_7^\epsilon &= -\frac{1}{\epsilon} \int_0^\tau \int_{\mathbb{R}^3} \varrho_\epsilon \mathbf{u}_\epsilon \cdot \nabla s_\epsilon \mathrm{d}x\mathrm{d}t - \frac{\gamma-1}{2} \int_0^\tau \int_{\mathbb{R}^3} |\Psi_\epsilon|^2 \mathrm{div}\mathbf{U} \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Note that thanks to $\mathrm{div}\mathbf{u} = 0$ and the wave equation (3.18), the terms $-\int_0^\tau \int_{\mathbb{R}^3} \varphi_\epsilon \partial_t s_\epsilon \mathrm{d}x\mathrm{d}t$ in (3.29), $-\int_0^\tau \int_{\mathbb{R}^3} \psi_\epsilon \partial_t s_\epsilon \mathrm{d}x\mathrm{d}t$ in (3.30), and the term $-\frac{1}{\epsilon}\int_0^\tau \int_{\mathbb{R}^3} (\varphi_\epsilon + \psi_\epsilon) \mathrm{div}\mathbf{U} \mathrm{d}x\mathrm{d}t$ in (3.27) are cancelled.

3.4. Computation of relative entropy

In this subsection, we are going to compute the estimates of relative entropy. Let us now carry out on the estimates of $\{A_j^\epsilon\}_{j=1}^7$. Thanks to (2.10), (3.19), (3.20), the regularity of $\mathbf{u}_0$ and $\hat{\mathbf{u}}_0$, and Hausdorff–Young's inequality, we first deal with the initial data part of A_1^ϵ. We have

$$
\begin{aligned}
&\|\sqrt{n_{0,\epsilon}}(\mathbf{u}_{0,\epsilon} - \mathbf{u}_0 - \nabla q_{0,\epsilon})\|_{L^2(\mathbb{R}^3)}^2\\
&= \|\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \hat{\mathbf{u}}_0 + \hat{\mathbf{u}}_0 - \sqrt{n_{0,\epsilon}}\hat{\mathbf{u}}_0 + \sqrt{n_{0,\epsilon}}(\hat{\mathbf{u}}_0 - \mathbf{u}_0 - \nabla q_{0,\epsilon})\|_{L^2(\mathbb{R}^3)}^2\\
&\le C\|\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon} - \hat{\mathbf{u}}_0\|_{L^2(\mathbb{R}^3)}^2 + C\|(1 - \sqrt{n_{0,\epsilon}})\hat{\mathbf{u}}_0\|_{L^2(\mathbb{R}^3)}^2\\
&\quad + C\|\sqrt{n_{0,\epsilon}}(\hat{\mathbf{u}}_0 - \mathbf{u}_0 - \nabla q_{0,\epsilon})\|_{L^2(\mathbb{R}^3)}^2\\
&\le C\epsilon^b + C\epsilon^2 + C\|\sqrt{n_{0,\epsilon}}(\hat{\mathbf{u}}_0 - \mathbf{u}_0 - \nabla q_{0,\epsilon})\|_{L^2(\mathbb{R}^3)}^2\\
&\le C\epsilon^b + C\epsilon^2 + C\|\sqrt{n_{0,\epsilon}}(\hat{\mathbf{u}}_0 - \mathbf{u}_0 - \mathbf{Q}(\hat{\mathbf{u}}_0) * \chi^\delta)\|_{L^2(\mathbb{R}^3)}^2\\
&\quad + C\|\sqrt{n_{0,\epsilon}}(\mathbf{Q}(\hat{\mathbf{u}}_0) * \chi^\delta - \mathbf{Q}(\sqrt{n_{0,\epsilon}}\mathbf{u}_{0,\epsilon}) * \chi^\delta)\|_{L^2(\mathbb{R}^3)}^2\\
&\quad + C\|\sqrt{n_{0,\epsilon}}(\mathbf{Q}(\hat{\mathbf{u}}_0) * \chi^\delta - \mathbf{Q}(\sqrt{\varrho_{0,\epsilon}}\mathbf{u}_{0,\epsilon}) * \chi^\delta)\|_{L^2(\mathbb{R}^3)}^2\\
&\le C\epsilon^b + C\epsilon^2 + \chi(\delta).
\end{aligned} \tag{3.32}
$$

Here and below we use $\chi(\delta)$ to denote a generic function of δ satisfying

$$
\lim_{\delta\to 0} \chi(\delta) = 0. \tag{3.33}
$$

Similarly,

$$
\|\sqrt{\varrho_{0,\epsilon}}(\mathbf{u}_{0,\epsilon} - \mathbf{u}_0 - \nabla q_{0,\epsilon})\|_{L^2(\mathbb{R}^3)}^2 \le C\epsilon^b + C\epsilon^2 + \chi(\delta). \tag{3.34}
$$

For the term $\|\Phi_{0,\epsilon}-s_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}$, it can be treated as

$$\begin{aligned}\|\Phi_{0,\epsilon}-s_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} &= \|\Phi_{0,\epsilon}-\varphi_{0,\epsilon}+\varphi_{0,\epsilon}-\varphi_{0,\epsilon}*\chi^\delta+\varphi_{0,\epsilon}*\chi^\delta-s_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}\\ &\le C\|\Phi_{0,\epsilon}-\varphi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}+C\|\varphi_{0,\epsilon}-\varphi_{0,\epsilon}*\chi^\delta\|^2_{L^2(\mathbb{R}^3)}\\ &\quad +C\|\varphi_{0,\epsilon}*\chi^\delta-\Phi_{0,\epsilon}*\chi^\delta\|^2_{L^2(\mathbb{R}^3)}\\ &\quad +C\|\psi_{0,\epsilon}*\chi^\delta-\Psi_{0,\epsilon}*\chi^\delta\|^2_{L^2(\mathbb{R}^3)}\\ &\quad +C\|\varphi_{0,\epsilon}*\chi^\delta-\psi_{0,\epsilon}*\chi^\delta\|^2_{L^2(\mathbb{R}^3)}\\ &\le C\|\Phi_{0,\epsilon}-\varphi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}+C\|\varphi_{0,\epsilon}-\Phi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}\\ &\quad +C\|\psi_{0,\epsilon}-\Psi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}+C\|\varphi_{0,\epsilon}-\psi_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)}+\chi(\delta)\\ &\le C\epsilon^b+C\epsilon^c+\chi(\delta).\end{aligned} \tag{3.35}$$

Similarly, we have

$$\|\Psi_{0,\epsilon}-s_{0,\epsilon}\|^2_{L^2(\mathbb{R}^3)} \le C\epsilon^b+C\epsilon^c+\chi(\delta). \tag{3.36}$$

Thus, we have

$$A_1^\epsilon \le -\left[\int_{\mathbb{R}^3}|\nabla q_\epsilon|^2\mathrm{d}x\right]_0^\tau + C\epsilon^b+C\epsilon^c+\chi(\delta)+C\epsilon^2+\chi(\delta). \tag{3.37}$$

We remark that the first term on the right-hand side of (3.37) will be cancelled later.

For the terms A_2^ϵ and A_3^ϵ, by the arguments of ([31], pp. 13–14), we have

$$A_2^\epsilon + A_3^\epsilon \le \begin{cases} C_T\epsilon^{\min\{1,\frac{b}{2}\}}, & 1<\alpha\le 4,\ 9/5<\gamma\le 4,\\ C_T\epsilon^{\min\{1,\frac{b}{2},\frac{\alpha-4}{2\alpha}\}}, & \alpha>4,\ 9/5<\gamma\le 4,\\ C_T\epsilon^{\min\{1,\frac{b}{2},\frac{\gamma-4}{2\gamma}\}}, & 1<\alpha\le 4,\ \gamma>4,\\ C_T\epsilon^{\min\{\frac{b}{2},\frac{\alpha-4}{2\alpha},\frac{\gamma-4}{2\gamma}\}}, & \alpha>4,\ \gamma>4. \end{cases} \tag{3.38}$$

We next control A_4^ϵ and denote it by $A_4^\epsilon = B_1^\epsilon + B_2^\epsilon$ where

$$\begin{aligned} B_1^\epsilon &= \int_0^\tau\int_{\mathbb{R}^3} n_\epsilon(\partial_t\mathbf{U}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{U})(\mathbf{U}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t,\\ B_2^\epsilon &= \int_0^\tau\int_{\mathbb{R}^3} \varrho_\epsilon(\partial_t\mathbf{U}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{U})(\mathbf{U}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

For the term B_1^ϵ, we have

$$\begin{aligned} B_1^\epsilon &= \int_0^\tau\int_{\mathbb{R}^3} n_\epsilon(\partial_t\mathbf{U}+\mathbf{u}_\epsilon\cdot\nabla\mathbf{U})(\mathbf{U}-\mathbf{u}_\epsilon)\mathrm{d}x\mathrm{d}t\\ &= \int_0^\tau\int_{\mathbb{R}^3} n_\epsilon(\mathbf{U}-\mathbf{u}_\epsilon)\otimes(\mathbf{U}-\mathbf{u}_\epsilon):\nabla\mathbf{U}\ \mathrm{d}x\mathrm{d}t\\ &\quad + \int_0^\tau\int_{\mathbb{R}^3}\Big(n_\epsilon(\mathbf{U}-\mathbf{u}_\epsilon)\cdot\partial_t\mathbf{U}+\varrho_\epsilon(\mathbf{U}-\mathbf{u}_\epsilon):\nabla\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t\\ &\le C_T\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t+\sum_{k=1}^5 J_k^\epsilon, \end{aligned}$$

where

$$
\begin{aligned}
J_1^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \cdot (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \mathrm{d}x \mathrm{d}t, \\
J_2^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \cdot \partial_t \nabla q_\epsilon \mathrm{d}x \mathrm{d}t, \\
J_3^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \otimes \nabla q_\epsilon : \nabla \mathbf{v} \mathrm{d}x \mathrm{d}t, \\
J_4^\epsilon &= \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \otimes \mathbf{v} : \nabla^2 q_\epsilon \mathrm{d}x \mathrm{d}t, \\
J_5^\epsilon &= \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \cdot \nabla |\nabla q_\epsilon|^2 \mathrm{d}x \mathrm{d}t.
\end{aligned}
$$

For J_1^ϵ, recalling that $\mathbf{U} = \mathbf{u} + \nabla q_\epsilon$ and using the incompressible Euler equations (1.9) and $\mathrm{div}\mathbf{u} = 0$, we have

$$
\begin{aligned}
|J_1^\epsilon| &\le \Big| \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon (\mathbf{U} - \mathbf{u}_\epsilon) \cdot (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \mathrm{d}x \mathrm{d}t \Big| \\
&\le \Big| \int_0^\tau \int_{\mathbb{R}^3} (n_\epsilon - 1) \mathbf{U} \cdot \nabla \Pi \mathrm{d}x \mathrm{d}t \Big| + \Big| \int_0^\tau \int_{\mathbb{R}^3} \nabla q_\epsilon \cdot \nabla \Pi \mathrm{d}x \mathrm{d}t \Big| \\
&\quad + \Big| \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon \mathbf{u}_\epsilon \cdot \nabla \Pi \mathrm{d}x \mathrm{d}t \Big|.
\end{aligned} \tag{3.39}
$$

For the first term on right-hand side of (3.39), we use the estimates (3.6) and (3.8) to obtain that

$$
\begin{aligned}
&\Big| \int_0^\tau \int_{\mathbb{R}^3} (n_\epsilon - 1) \mathbf{U} \cdot \nabla \Pi \mathrm{d}x \mathrm{d}t \Big| \\
&\le C \|(n_\epsilon - 1) 1_{\{\{|n_\epsilon - 1| \le 1/2\}}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\mathbf{U}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \|\nabla \Pi\|_{L^2(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|(n_\epsilon - 1) 1_{\{|n_\epsilon - 1| \ge 1/2\}}\|_{L^\alpha(0,T;L^\alpha(\mathbb{R}^3))} \\
&\quad \times \|\mathbf{U}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \|\nabla \Pi\|_{L^{\frac{\alpha}{\alpha-1}}(0,T;L^{\frac{\alpha}{\alpha-1}}(\mathbb{R}^3))} \\
&\le C_T (\epsilon + \epsilon^\kappa) \le C_T \epsilon,
\end{aligned}
$$

where $\kappa := \min\{2, \alpha\}$ and we have used the condition $k > 7/2$ and $\alpha > 1$.

For the second term on the right-hand side of (3.39), together with using the equation (3.18) and the dispersive regularity (3.1), it can be estimated as follows:

$$
\begin{aligned}
\Big| \int_0^\tau \int_{\mathbb{R}^3} \nabla q_\epsilon \cdot \nabla \Pi \mathrm{d}x \mathrm{d}t \Big| &\le \epsilon \Big[\int_{\mathbb{R}^3} |s_\epsilon| |\Pi| \mathrm{d}x \Big]_0^\tau + \epsilon \int_0^\tau \int_{\mathbb{R}^3} |s_\epsilon| |\partial_t \Pi| \mathrm{d}x \mathrm{d}t \\
&\le \epsilon \big(\|s_\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \|\Pi\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|s_{0,\epsilon}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \|\Pi_0\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|s_\epsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))} \|\partial_t \Pi\|_{L^2(0,T;L^2(\mathbb{R}^3))} \big) \\
&\le C_T \epsilon.
\end{aligned}
$$

For the third term on the right-hand side of (3.39), together with using continuity Eqs. (1.5), (3.6), and (3.8), it can be bounded as follows:

$$
\begin{aligned}
&\Big|\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\mathbf{u}_\epsilon\cdot\nabla\Pi\mathrm{d}x\mathrm{d}t\Big|\\
&\le\Big|\int_{\mathbb{R}^3}\big\{((n_\epsilon-1)\Pi)(t)-((n_\epsilon-1)\Pi)(0)\big\}\mathrm{d}x+\Big|\int_0^\tau\int_{\mathbb{R}^3}(n_\epsilon-1)\cdot\partial_t\Pi\mathrm{d}x\mathrm{d}t\Big|\\
&\le\|(n_\epsilon-1)1_{\{|n_\epsilon-1|\le1/2\}}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}+\|(n_\epsilon-1)1_{\{|n_\epsilon-1|\ge1/2\}}\|_{L^\infty(0,T;L^\alpha(\mathbb{R}^3))}\\
&\quad+\|(n_\epsilon-1)1_{\{|n_\epsilon-1|\le1/2\}}\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|\partial_t\Pi\|_{L^2(0,T;L^2(\mathbb{R}^3))}\\
&\quad+\|(n_\epsilon-1)1_{\{|n_\epsilon-1|\ge1/2\}}\|_{L^\alpha(0,T;L^\alpha(\mathbb{R}^3))}\|\mathbf{U}\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|\partial_t\Pi\|_{L^{\frac{\alpha}{\alpha-1}}(0,T;L^{\frac{\alpha}{\alpha-1}}(\mathbb{R}^3))}\\
&\quad+C_T\epsilon\\
&\le C_T(\epsilon+\epsilon^\kappa)\le C_T\epsilon,
\end{aligned}
$$

where we have here used (2.12). Thus, we get

$$|J_1^\epsilon|\le C_T\epsilon.$$

Next, we have

$$
\begin{aligned}
J_2^\epsilon=&-\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\mathbf{u}_\epsilon\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\mathbf{u}\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t\\
&+\frac12\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\partial_t|\nabla q_\epsilon|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.40}
$$

In virtue of $\mathrm{div}\mathbf{u}=0$, (3.1), (3.6), and (3.8), we get

$$
\begin{aligned}
&\Big|\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\mathbf{u}\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t\Big|\\
&=\Big|\int_0^\tau\int_{\mathbb{R}^3}(n_\epsilon-1)\mathbf{u}\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t\Big|\\
&\le\Big\|\frac{n_\epsilon-1}{\epsilon}1_{\{|n_\epsilon-1|\le1/2\}}\Big\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|\mathbf{u}\|_{L^{4/3}(0,T;L^4(\mathbb{R}^3))}\|\nabla s_\epsilon\|_{L^4(0,T;L^4(\mathbb{R}^3))}\\
&\quad+\frac{C}{\epsilon}\|(n_\epsilon^\alpha+1)1_{\{|n_\epsilon-1|>1/2\}}\|_{L^\infty(0,T;L^1(\mathbb{R}^3))}\\
&\le C_T(\epsilon^{1/4}+\epsilon),
\end{aligned}
$$

where we have used the facts that

$$\|\mathbf{u}\|_{L^\infty((0,T)\times\mathbb{R}^3)}\le C_T,\quad \|\nabla s_\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^3)}\le C_T.$$

Similarly, we have

$$\frac12\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\partial_t|\nabla q_\epsilon|^2\mathrm{d}x\mathrm{d}t\le\frac12\Big[\int_{\mathbb{R}^3}|\nabla q_\epsilon|^2\mathrm{d}x\Big]_0^\tau+C_T(\epsilon^{1/4}+\epsilon).$$

Thus, the term J_2^ϵ is bounded by:

$$
\begin{aligned}
J_2^\epsilon&=\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon(\mathbf{U}-\mathbf{u}_\epsilon)\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t\\
&\le\frac12\Big[\int_{\mathbb{R}^3}|\nabla q_\epsilon|^2\mathrm{d}x\Big]_0^\tau-\int_0^\tau\int_{\mathbb{R}^3} n_\epsilon\mathbf{u}_\epsilon\cdot\partial_t\nabla q_\epsilon\mathrm{d}x\mathrm{d}t+C_T(\epsilon^{1/4}+\epsilon).
\end{aligned}
$$

Using the regularity (2.8), the dispersive property (3.23), (3.1), and (3.12), the term J_3^ϵ can be estimated as:

$$J_3^\epsilon = \int_0^\tau \int_{\mathbb{R}^3} (n_\epsilon - 1)\mathbf{U} \otimes \nabla q_\epsilon : \nabla \mathbf{u} \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \mathbf{U} \otimes \nabla q_\epsilon : \nabla \mathbf{u} \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{n_\epsilon} - 1)\sqrt{n_\epsilon}\mathbf{u}_\epsilon \otimes \nabla q_\epsilon : \nabla \mathbf{u} \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\mathbb{R}^3} \sqrt{n_\epsilon}\mathbf{u}_\epsilon \otimes \nabla q_\epsilon : \nabla \mathbf{u} \mathrm{d}x\mathrm{d}t$$
$$\leq C_T(\epsilon^{1/4} + \epsilon).$$

Similarly, we get

$$J_4^\epsilon + J_5^\epsilon \leq C(\epsilon^{1/4} + \epsilon).$$

So, the term B_1^ϵ can be estimated as follows:

$$B_1^\epsilon \leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + \frac{1}{2}\Big[\int_{\mathbb{R}^3} |\nabla q_\epsilon|^2 \mathrm{d}x\Big]_0^\tau - \int_0^\tau \int_{\mathbb{R}^3} n_\epsilon \mathbf{u}_\epsilon \cdot \partial_t \nabla q_\epsilon \mathrm{d}x\mathrm{d}t$$
$$+ C(\epsilon^{1/4} + \epsilon),$$

and with the same arguments, B_2^ϵ can be handled as follows:

$$B_2^\epsilon \leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + \frac{1}{2}\Big[\int_{\mathbb{R}^3} |\nabla q_\epsilon|^2 \mathrm{d}x\Big]_0^\tau - \int_0^\tau \int_{\mathbb{R}^3} \varrho_\epsilon \mathbf{u}_\epsilon \cdot \partial_t \nabla q_\epsilon \mathrm{d}x\mathrm{d}t$$
$$+ C_T(\epsilon^{1/4} + \epsilon).$$

The above inequalities imply that

$$A_4^\epsilon \leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + \Big[\int_{\mathbb{R}^3} |\nabla q_\epsilon|^2 \mathrm{d}x\Big]_0^\tau - \int_0^\tau \int_{\mathbb{R}^3} (\varrho_\epsilon + n_\epsilon) \mathbf{u}_\epsilon \cdot \partial_t \nabla q_\epsilon \mathrm{d}x\mathrm{d}t$$
$$+ C_T(\epsilon^{1/4} + \epsilon).$$

Thanks to $\mathrm{div}\mathbf{u} = 0$, the estimate (3.22) and the viscosity term $A_5^\epsilon(\tau)$ can be bounded by the following:

$$A_5^\epsilon(\tau) = \int_0^\tau \int_{\mathbb{R}^3} \Big(\sqrt{\mu_\epsilon}\nabla \mathbf{u}_\epsilon : \sqrt{\mu_\epsilon}\nabla \mathbf{U} + \sqrt{\mu_\epsilon + \nu_\epsilon}\mathrm{div}\mathbf{u}_\epsilon \ \sqrt{\mu_\epsilon + \nu_\epsilon}\mathrm{div}\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t$$
$$\leq \frac{\mu_\epsilon}{2}\|\nabla \mathbf{u}_\epsilon\|^2_{L^2((0,T)\times\mathbb{R}^3)} + \frac{\mu_\epsilon + \nu_\epsilon}{2}\|\mathrm{div}\mathbf{u}_\epsilon\|^2_{L^2((0,T)\times\mathbb{R}^3)} + C_T(\mu_\epsilon + \nu_\epsilon). \tag{3.41}$$

Noticing that after adding the second term on the right-hand side of J_2^ϵ in (3.40) to A_6^ϵ, we get that the first term of A_6^ϵ vanishes. The second term of A_6^ϵ can be estimated as follows:

$$\Big|\int_0^\tau \int_{\mathbb{R}^3} |\Phi_\epsilon|^2 \Delta q_\epsilon \mathrm{d}x\mathrm{d}t\Big|$$
$$\leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\Big|\int_0^\tau \int_{\mathbb{R}^3} |s_\epsilon|^2 \Delta q_\epsilon \mathrm{d}x\mathrm{d}t\Big|$$
$$\leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C\|s_\epsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))}\|s_\epsilon\|_{L^4(0,T;L^4(\mathbb{R}^3))}\|\nabla q_\epsilon\|_{L^4(0,T;W^{1,4}(\mathbb{R}^3))}$$
$$\leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C_T\epsilon^{1/4}. \tag{3.42}$$

Here we have used the regularity of s_ϵ and q_ϵ in (3.22), the dispersive regularity (3.23), and the computation in [12] together with the Strichartz's estimate (3.23).

Similarly, the second term of A_7^ϵ can be estimated as follows:

$$\Big|\int_0^\tau \int_{\mathbb{R}^3} |\Psi_\epsilon|^2 \Delta q_\epsilon \mathrm{d}x\mathrm{d}t\Big| \leq C \int_0^\tau \mathcal{E}_\epsilon(t)\mathrm{d}t + C_T\epsilon^{1/4}. \tag{3.43}$$

Consequently, putting all the above estimates related to $\{A_j^\epsilon\}_{j=1}^7$ into the inequality (3.31), we obtain that

$$\mathcal{E}_\epsilon(\tau)+\frac{\mu_\epsilon}{2}\int_0^\tau\int_{\mathbb{R}^3}|\nabla\mathbf{u}_\epsilon|^2\mathrm{d}x\mathrm{d}t+\frac{\mu_\epsilon+\nu_\epsilon}{2}\int_0^\tau\int_{\mathbb{R}^3}|\mathrm{div}\mathbf{u}_\epsilon|^2\mathrm{d}x\mathrm{d}t \\ \leq C_T\int_0^\tau\mathcal{E}_\epsilon(t)\mathrm{d}t+C_T\epsilon^\sigma+\chi(\delta), \tag{3.44}$$

where the number ω is defined as follows:

$$\sigma=\begin{cases}\min\{\frac14,\frac b2,a,c\}, & 1<\alpha\leq 4,\ 9/5<\gamma\leq 4,\\ \min\{\frac14,\frac b2,\frac{\alpha-4}{2\alpha},a,c\}, & \alpha>4,\ 9/5<\gamma\leq 4,\\ \min\{\frac14,\frac b2,\frac{\gamma-4}{2\gamma},a,c\}, & 1<\alpha\leq 4,\ \gamma>4,\\ \min\{\frac b2,\frac{\alpha-4}{2\alpha},\frac{\gamma-4}{2\gamma},a,c\}, & \alpha>4,\ \gamma>4,\end{cases} \tag{3.45}$$

and we have used the condition (2.9) and the facts that:

$$\epsilon^{1/4}\geq\epsilon^{1/3}\geq\epsilon$$

for sufficiently small $0<\epsilon<1$.

Applying the Gronwall's inequality to (3.44) gives

$$\mathcal{E}_\epsilon(\tau)\leq C_T\epsilon^\sigma+\chi(\delta) \tag{3.46}$$

for any $\tau\in(0,T]$.

We are now able to prove the convergence of $\sqrt{n_\epsilon}\mathbf{u}_\epsilon$ and $\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon$. Note that the projection $\mathbf{P}$ is a bounded linear mapping from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Hence, we have

$$\sup_{0\leq t\leq T}\|\mathbf{P}(\sqrt{n_\epsilon}\mathbf{u}_\epsilon)-\mathbf{u}\|_{L^2(\mathbb{R}^3)}=\sup_{0\leq t\leq T}\left\|\mathbf{P}\left(\sqrt{n_\epsilon}\mathbf{u}_\epsilon-\mathbf{u}-\nabla q_\epsilon\right)\right\|_{L^2(\mathbb{R}^3)} \\ \leq C\sup_{0\leq t\leq T}\|\sqrt{n_\epsilon}\mathbf{u}_\epsilon-\mathbf{u}-\nabla q_\epsilon\|_{L^2(\mathbb{R}^3)}. \tag{3.47}$$

Thus, combining (2.10) and (3.46) and letting $\delta\to 0$, we further derive that

$$\|\mathbf{P}(\sqrt{n_\epsilon}\mathbf{u}_\epsilon)-\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\leq C_T(\epsilon^{\frac b2}+\epsilon^{\frac\sigma2})\leq C_T\epsilon^{\frac\sigma2}.$$

Similarly, we have

$$\|\mathbf{P}(\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon)-\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\leq C_T\epsilon^{\frac\sigma2}.$$

It is not difficult to show the local strong convergence of $\sqrt{n_\epsilon}\mathbf{u}^\epsilon$ and $\sqrt{\varrho_\epsilon}\mathbf{u}^\epsilon$ to $\mathbf{u}$ in $L^r(0,T;L^2(\Omega))$ for all $2<r<+\infty$ on any bounded domain $K\subset\mathbb{R}^3$. In fact, for any $t\in(0,T]$ and any compact subset $K\subset\mathbb{R}^3$, we have

$$\int_K|\sqrt{n_\epsilon}\mathbf{u}_\epsilon-\mathbf{u}|^2\mathrm{d}x \\ \leq\int_K|\sqrt{n_\epsilon}(\mathbf{u}_\epsilon-\mathbf{u}-\nabla q_\epsilon)-(1-\sqrt{n_\epsilon})(\mathbf{v}+\nabla q_\epsilon)+\nabla q_\epsilon|^2\mathrm{d}x \\ \leq\mathcal{E}_\epsilon(\tau)+C(K)\left(\int_K|\nabla q_\epsilon|^{\frac{2r}{r-2}}\mathrm{d}x\right)^{\frac{r-2}{2r}}+C_T\epsilon^\sigma+\chi(\delta) \tag{3.48}$$

for any $r>2$, where we have here used (3.12) and (3.23). Similarly, we have

$$\int_K|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon-\mathbf{u}|^2\mathrm{d}x\leq\mathcal{E}_\epsilon(\tau)+C(K)\left(\int_K|\nabla q_\epsilon|^{\frac{2r}{r-2}}\mathrm{d}x\right)^{\frac{r-2}{2r}}+C\epsilon^\sigma+\chi(\delta). \tag{3.49}$$

Consequently, using (3.46), (3.48), and (3.49) together with (3.23) and passing to the limit for $\delta \to 0$, we get

$$\|\sqrt{n_\epsilon}\mathbf{u}_\epsilon - \mathbf{v}\|_{L^r(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))} + \|\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon - \mathbf{u}\|_{L^r(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))} \le C(\epsilon^{\frac{\sigma}{2}} + \epsilon^{\frac{1}{r}}) \le C_T \epsilon^d$$

with $d = \min\left\{\frac{\sigma}{2}, \frac{1}{r}\right\}$. Thus, we prove (2.15) and (2.16) where the terms of constant depend on

$$\|\nabla q_{0,\epsilon}\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)} + \|s_{0,\epsilon}\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)},$$

and it is uniformly bounded by a constant number when $\delta \to 0$. This completes the proof of Theorem 2.1.

References

[1] Alazard, T.: Low Mach number limit of the full Navier–Stokes equations. Arch. Ration. Mech. Anal. **180**, 1–73 (2006)
[2] Barrett, J.W., Lu, Y., Süli, E.: Existence of large-data finite-energy global weak solutions to a compressible Oldroyd-B model. Commun. Math. Sci. **15**(5), 1265–1323 (2017)
[3] Carrillo, J.A., Goudon, T.: Stability and asymptotic analysis of a fluid–particle interaction model. Comm. Partial Differ. Eqs. **31**, 1349–1379 (2006)
[4] Caggio, M., Nečasová, S.: Inviscid incompressible limits for rotating fluids. Nonlinear Anal. **163**, 1–18 (2017)
[5] Desjardins, B., Grenier, E., Lions, P.-L., Masmoudi, N.: Incompressible limit for solutions of the isentropic Navier–Stokes equations with Dirichlet boundary conditions. J. Math. Pures Appl. **78**, 461–471 (1999)
[6] Desjardins, B., Grenier, E.: Low Mach number limit of viscous compressible flows in the whole space. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455**, 2271–2279 (1999)
[7] Evje, S.: Weak solution for a gas-liquid model relevant for describing gas–kick oil wells. SIAM J. Math. Anal. **43**, 1887–1922 (2011)
[8] Evje, S., Karlsen, K.H.: Global existence of weak solutions for a viscous two-fluid model. J. Differ. Equ. **245**(9), 2660–2703 (2008)
[9] Evje, S., Karlsen, K.H.: Global weak solutions for a viscous liquid-gas model with singular pressure law. Commun. Pure Appl. Anal. **8**, 1867–1894 (2009)
[10] Evje, S., Wen, H., Zhu, C.: On global solutions to the viscous liquid-gas model with unconstrained transition to single-phase flow. Math. Models Methods Appl. Sci. **27**(2), 323–346 (2017)
[11] Feireisl, E., Jin, B., Novotný, A.: Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier–Stokes system. J. Math. Fluid Mech. **14**(4), 717–730 (2012)
[12] Feireisl, E., Novotný, A.: Inviscid incompressible limits of the full Navier–Stokes–Fourier system. Commun. Math. Phys. **321**(3), 605–628 (2013)
[13] Feireisl, E., Novotný, A.: The low Mach number limit for the full Navier–Stokes–Fourier system. Arch. Ration. Mech. Anal. **186**(1), 77–107 (2007)
[14] Feireisl, E., Novotný, A.: Singular Limits in Thermodynamics of Viscous Fluids. Springer, Cham (2017)
[15] Hsiao, H., Ju, Q., Li, F.: The incompressible limits of compressible Navier–Stokes equations in the whole space with general initial data. Chin. Ann. Math. Ser. B **30**(1), 17–26 (2009)
[16] Hu, X.P., Wang, D.H.: Low Mach number limit of viscous compressible magnetohydrodynamic flows. SIAM J. Math. Anal. **41**, 1272–1294 (2009)
[17] Ishii, M., Hibiki, T.: Thermo-Fluid Dynamics of Two-phase Flow. Springer, Berlin (2006)
[18] Jiang, S., Ju, Q.C., Li, F.C.: Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. Commun. Math. Phys. **297**, 371–400 (2010)
[19] Jiang, S., Ju, Q.C., Li, F.C.: Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients. SIAM J. Math. Anal. **42**, 2539–2553 (2010)
[20] Jiang, S., Ju, Q.C., Li, F.C., Xin, Z.-P.: Low Mach number limit for the full compressible magnetohydrodynamic equations with general initial data. Adv. Math. **259**, 384–420 (2014)
[21] Ju, Q.C., Li, F.C., Li, Y.: Asymptotic limits of the full compressible magnetohydrodynamic equations. SIAM J. Math. Anal. **45**(5), 2597–2624 (2013)
[22] Kato, T.: Nonstationary flow of viscous and ideal fluids in R^3. J. Funct. Anal. **9**, 296–305 (1972)
[23] Klainerman, S., Majda, A.: Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids Commun. Pure Appl. Math. **34**, 481–524 (1981)

[24] Kwon, Y.-S., Li, F.-C.: Incompressible limit of the degenerate quantum compressible Navier–Stokes equations with general initial data. J. Differ. Equ. **264**(5), 3253–3284 (2018)
[25] Lions, P.-L., Masmoudi, N.: Incompressible limit for a viscous compressible fluid. J. Math. Pures Appl. **77**, 585–627 (1998)
[26] Mellet, A., Vasseur, A.: Asymptotic analysis for a Vlasov–Fokker–Planck/compressible Navier–Stokes system of equations. Commun. Math. Phys. **281**(3), 573–596 (2008)
[27] Masmoudi, N.: Incompressible, inviscid limit of the compressible Navier–Stokes system. Ann. Inst. H. Poincaré Anal. Non Linéaire **18**(2), 199–224 (2001)
[28] Novotný, A., Pokorný, M.: Weak solutions for some compressible multicomponent fluid models, arXiv:1802.00798v2
[29] Novotný, A., Straškraba, I.: Introduction to the Mathematical Theory of Compressible Flow. Oxford University Press, Oxford (2004)
[30] Schochet, S.: Fast singular limits of hyperbolic PDEs. J. Differ. Equ. **114**, 476–512 (1994)
[31] Su, J.: Incompressible limit of a compressible micropolar fluid model with general initial data. Nonlinear Anal. **132**, 1–24 (2016)
[32] Swann, H.S.G.: The convergence with vanishing viscosity of nonstationary Navier–Stokes flow to ideal flow in R^3. Trans. Am. Math. Soc. **157**, 373–397 (1971)
[33] Vasseur, A., Wen, H., Yu, C.: Global weak solution to the viscous two-fluid model with finite energy. J. Math. Pures Appl. **125**, 247–282 (2019)
[34] Yang, J., Ju, Q., Yang, Y.: Asymptotic limits of Navier–Stokes equations with quantum effects. Z. Angew. Math. Phys. **66**, 2271–2283 (2015)
[35] Yao, L., Zhang, T., Zhu, C.: Existence of asymptotic behavior of global weak solutions to a 2D viscous liquid-gas two-phase flow model. SIAM J. Math. Anal. **42**(4), 1874–1897 (2010)
[36] Yao, L., Zhu, Z., Zi, R.: Incompressible limit of viscous liquid-gas two-phase flow model. SIAM J. Math. Anal. **44**(5), 3324–3345 (2012)

Young-Sam Kwon
Department of Mathematics
Dong-A University
Busan 604-714
Korea
e-mail: ykwon@dau.ac.kr

Fucai Li
Department of Mathematics
Nanjing University
Nanjing 210093
People's Republic of China
e-mail: fli@nju.edu.cn

(Received: August 31, 2018; revised: February 28, 2019)