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Blowup time estimates for a parabolic p-Laplacian equation with nonlinear gradient terms

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Abstract. This article studies the blowup time of weak solutions to the degenerate parabolic equation $u_t - \Delta_p u = \lambda u^m + \mu|\nabla u|^q$ with homogeneous Dirichlet boundary condition in a bounded smooth domain. We first obtain an upper bound and a lower one for the blowup time of L^∞ blowup solutions and then get the upper bound for the blowup time of gradient blowup solutions.

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1. Introduction

In this paper, we consider the blowup time of weak solutions to the following parabolic p-Laplacian equation:

$$\begin{cases} u_t - \Delta_p u = \lambda u^m + \mu|\nabla u|^q, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$ and $T \in (0,\infty]$ is the maximal existence time, that is

$$T = \sup\left\{ t^{'} > 0 : \sup_{0\leq t\leq t^{'}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \right) < \infty \right\}.$$

Δ_p is the p-Laplacian operator

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u \right), \quad p > 2,$$

∇ is the gradient operator, $m \geq 1, q \geq 1$ and $\lambda, \mu \in \mathbb{R}$. The initial value $u_0(x) \in W_0^{1,\infty}(\Omega)$ is a nonnegative and nontrivial function which satisfies the compatible condition.

If $p = 2$, the equation

$$u_t - \Delta u = \lambda u^m + \mu|\nabla u|^q \tag{1.2}$$

was introduced by Chipot and Weissler [3] to investigate the effect of a damping gradient term on existence or nonexistence of global solutions. The blowup properties of classical solutions to (1.2) have been studied

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extensively in [6,9,12–15] and the references therein. In particular, Payne and Song [12] obtained a lower bound for the blowup time of L^∞ blowup solutions to (1.2) with $\lambda > 0, \mu < 0$ in three-space dimension.

If $p > 2$, equation (1.1) often appears in the theory of non-Newtonian fluids. A lot of efforts, see [1,2,4,7,8,10,11,17–22] for examples, have been devoted to the blowup properties of solutions to (1.1). The local existence of weak solutions was established in [20]. The result also demonstrates that for $m \geq 1, q \geq 1, \lambda, \mu \in \mathbb{R}$ and $u_0(x) \in W_0^{1,\infty}(\Omega)$, if the maximal existence time T is finite, then

$$\lim_{t \to T^-} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \right) = \infty,$$

i.e., the maximal existence time T is just the blowup time. If $\lim_{t\to T^-} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$, we formally say u is a L^∞ *blowup* solution. While if $\sup_{\Omega\times[0,T)} |u| < \infty$, but $\lim_{t\to T^-} \|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$, then u is normally called a *gradient blowup* solution. If $\lambda > 0$ and $\mu = 0$, Li and Xie [10] proved existence of a L^∞ blowup solution under given conditions such as λ is large enough if $m = p - 1$, or the initial data are sufficiently large if $m > p - 1$. Furthermore, the blowup rate [22] was discussed in the radial case. If $\lambda = 0$ and $\mu > 0$, the gradient blowup solution could be obtained under certain conditions, see [1,2,7] for examples. Besides, Zhang [17] gave the gradient blowup rate in the one-dimensional case. If $\lambda\mu \neq 0$, Zhang and Li [19,20] established the complete classification of parameters λ, μ, p, m and q for global, L^∞ blowup and gradient blowup solutions to (1.1).

These known results show that the values of p, m, q, λ and μ play very vital roles in studying the blowup properties of weak solutions to (1.1). Motivated by these results, it is a natural way that we are concerned with the bounds for blowup time of L^∞ blowup and gradient blowup solutions to (1.1). In fact, we will estimate the upper bound and the lower one for the blowup time of L^∞ blowup solutions and establish the upper bound for the blowup time of gradient blowup solutions.

Firstly, we give the definition of weak solutions to (1.1).

Definition 1.1. Let $s = \max\{p, m, q\}$, $Q_T = \Omega \times (0,T)$, $\partial Q_T = \{\partial\Omega \times [0,T]\} \cup \left\{\overline{\Omega} \times \{0\}\right\}$. A function $u(x,t)$ is called a weak super- (sub-) solution of the problem (1.1) if it satisfies

$$\begin{cases} u \in C\left(\overline{\Omega} \times [0,T)\right) \cap L^s\left(0,T; W_0^{1,s}(\Omega)\right), \quad \partial_t u \in L^2\left(Q_T\right), \\ u(x,0) \geq (\leq)\ u_0(x) \quad \text{in} \quad \Omega, \quad u \geq (\leq)\ 0 \quad \text{on} \quad \partial\Omega, \\ \iint\limits_{Q_T} \left(\partial_t u \psi + |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\right) \mathrm{d}x\,\mathrm{d}t \geq (\leq) \iint\limits_{Q_T} \left(\lambda u^m + \mu|\nabla u|^q\right)\psi \,\mathrm{d}x\,\mathrm{d}t. \end{cases} \tag{1.3}$$

Here, $0 \leq \psi \in C\left(\overline{Q_T}\right) \cap L^p\left(0,T; W_0^{1,p}(\Omega)\right)$. A function $u(x,t)$ is a weak solution if it is a weak super-solution and a weak sub-solution.

Remark 1.1. The local existence of weak solutions to (1.1) can be found in [20, Theorem 2.1].

The following weak comparison principle of weak solutions to (1.1) will play a crucial role in establishing the blowup time results.

Lemma 1.1. (See [10,19]) *Assume that $z_1, z_2 \in L^\infty_{\mathrm{loc}}\left(0,T; W_0^{1,\infty}(\Omega)\right)$ are weak sub- and super-solutions of (1.1), respectively, and $z_1(x,0) \leq z_2(x,0)$.*

(1) *Suppose $\mu \neq 0$. If $q \geq p/2$, then $z_1 \leq z_2$ on $\Omega \times (0,T)$;*
(2) *Suppose $\mu = 0$, then $z_1 \leq z_2$ on $\Omega \times (0,T)$.*

By Lemma 1.1, we know that the solution $u(x,t)$ of (1.1) is nonnegative in the time interval of existence under corresponding parameters conditions.

This paper is organized as follows. In Sect. 2, we will obtain an upper bound and a lower one for the blowup time of L^∞ blowup solutions. In Sect. 3, the upper bound for the blowup time of gradient blowup solutions will be derived.

2. Upper and lower bounds for blowup time of L^∞ blowup solutions

In this section, we give an upper bound and a lower one for blowup time of L^∞ blowup solutions to (1.1). At first, according to [19,20], we have the following results about L^∞ blowup solutions.

Theorem 2.1. *Let $\lambda > 0$, $\mu < 0$, $m > \max\{q, p-1\}$ and $q \geq p/2$ and assume that $u_0 = \eta\psi$ for some ψ satisfying $\psi \geq 0$, $\psi|_{\partial\Omega} = 0$ and $\psi \not\equiv 0$. Then, there exists $\eta_0(p, m, q, \lambda, \mu, \Omega) > 0$, such that for all $\eta > \eta_0$,*

(i) *if $q \leq p-1$, then L^∞ blowup occurs;*

(ii) *if $q > p-1$ and u_0 satisfies, for any $\varepsilon > 0$,*

$$\operatorname{div}\left(\left(|\nabla u_0|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla u_0\right) + \lambda u_0^m + \mu|\nabla u_0|^q > 0, \tag{2.1}$$

then L^∞ blowup occurs.

Proof. When $q \leq p-1$, the conclusion has been shown in [19, Theorem 1.3]. When $q > p-1$, by [19, Theorem 1.3], we just need to exclude the possibility of gradient blowup to occur. Suppose that the solution u is uniformly bounded and u_0 satisfies (2.1), then by [19, Proposition 2.4], interior gradient blowup cannot occur. On the other hand, we can verify that $\bar{u} = \bar{R}\operatorname{dist}(x, \partial\Omega)$ is a super-solution for suitable large $\bar{R} > 0$, which guarantees that $\partial u/\partial\nu$ is bounded on $\partial\Omega$, where ν is the unit outward normal vector on $\partial\Omega$. Hence, L^∞ blowup occurs. □

Remark 2.1. Theorem 2.1 fills one gap in [19] for the case $m > q > p-1$, where the L^∞ blowup or gradient blowup was not clarified.

Proposition 2.1. (See [20, Theorem 3.1]) *Let $\lambda > 0$, $\mu > 0$ and $m > p-1 \geq q$ and assume that u_0 is large enough, then the maximal existence time $T < \infty$ and the solution u satisfies*

$$\lim_{t \to T^-} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

Remark 2.2. If $\lambda > 0$ and $\mu = 0$, the result is still valid as long as the condition $p-1 \geq q$ is replaced by $p-1 > 1$. See Theorem 4.1 in [10] for further details.

With the aid of these results, we will derive upper and lower bounds for blowup time of L^∞ blowup solutions. In order to acquire upper and lower bounds for blowup time, we introduce the auxiliary function

$$\phi(t) = \int_\Omega u^\gamma \,\mathrm{d}x, \tag{2.3}$$

where the constant $\gamma > 0$ will be decided later. Our main results of this section read as follows.

Theorem 2.2. *Let u be a solution of (1.1). Assume that $\lambda > 0$, then the following conclusions hold.*

(1) *Suppose $\mu < 0$. If $m > \max\{q, p-1\}$, $q \geq p/2$ and the initial data are sufficiently large, then the solution u blows up in finite time in measure (2.3) with*

$$\gamma > \max\left\{1,\ \frac{p^*}{p}(m-p+1) + (1-m),\ \frac{p^*}{p}(q-p+1) + (m-2q+1),\right.$$
$$\left.\frac{\frac{p^*}{p}(p-2)}{\left(\frac{p^*}{p}\right)^2 - 3\frac{p^*}{p} + 1},\ \frac{N(m+1)}{p} - N,\ \frac{N(m-q)}{q}\right\},$$

where $p^* = \frac{Np}{N-p}$ for $p < N < \frac{\sqrt{5}+1}{2}p$ and $p^* = 3p$ for $p \geq N$, and the blowup time T satisfies

$$\frac{1}{H\left[\phi(0)\right]^{\delta-1}} \leq T \leq \frac{1}{\rho} - t_0, \tag{2.4}$$

where $H = H(p, m, q, \lambda, \mu, \Omega, N) > 0$, $\delta = \delta(p, m, q, \lambda, \mu, N) > 1$, $\rho = \rho(p, m, q, \lambda) > 0$ *and* $t_0 = t_0(p, m, q, \lambda, N) > 0$ *are constants.*

(2) *Suppose* $\mu \geq 0$. *If* $m > p - 1 \geq q \geq 1$ *and the initial data are suitably large, then the solution* u *must blow up in finite time in measure* (2.3) *with*

$$\gamma > \max\left\{2, \frac{p^{**}(2m-p) - p(m-1)}{p^{**} - p}\right\},$$

where $p^{**} = \frac{Np}{N-p}$ *for* $p < N$ *and* $p^{**} = 2p$ *for* $p \geq N$, *and the blowup time* T *satisfies*

$$\frac{1}{M(\phi(0)+1)} \leq T \leq \frac{1}{\rho} - t_0, \tag{2.5}$$

where $M = M(p, m, q, \lambda, \mu, \Omega, N)$, $\rho = \rho(p, m, q, \lambda)$ *and* $t_0 = t_0(p, m, q, \lambda, N)$ *are positive constants.*

Proof. **(1) In the case of** $\mu < 0$. We first show that the corresponding solution of (1.1) blows up in finite time in measure (2.3). The proof is based upon the construction of a self-similar sub-solution which was used in [16]. Let

$$z(x,t) = \frac{1}{(1-\rho t)^k} Z\left(\frac{|x|}{(1-\rho t)^l}\right), \quad t_0 \leq t < \frac{1}{\rho}, \tag{2.6}$$

where

$$Z(y) = 1 + Aw^{-1} - \frac{y^w}{wA^{w-1}}, \quad y \geq 0, \ w = \frac{p}{p-1}$$

with $\rho, k, l, A > 0$ and t_0 to be determined. It's easy to verify that $Z(y)$ satisfies

$$\begin{cases} 1 \leq Z(y) \leq 1 + Aw^{-1}, \quad -1 \leq Z'(y) \leq 0, & \text{if } 0 \leq y \leq A, \\ 0 \leq Z(y) \leq 1, \quad -\left(RA^{-1}\right)^{w-1} \leq Z'(y) \leq -1, & \text{if } A \leq y \leq R, \\ \left(|Z'|^{p-2}Z'\right)' + \dfrac{N-1}{y}|Z'|^{p-2}Z' = -NA^{-1}, & \text{if } 0 \leq y \leq R, \end{cases} \tag{2.7}$$

where $R = \left(A^{w-1}(w+A)\right)^{1/w}$ is the zero of $Z(y)$. Let

$$D = \left\{(x,t) \middle|\ t_0 \leq t < 1/\rho, \ |x| < R(1-\rho t)^l\right\},$$

then $z(x,t) > 0$ if and only if $(x,t) \in D$, and $z(x,t)$ is smooth in D. Define

$$\mathcal{L}_p^1 z = z_t - \Delta_p z - \lambda z^m - \mu|\nabla z|^q, \tag{2.8}$$

and let $y = |x|/(1-\rho t)^l$, then we have

$$\mathcal{L}_p^1 z = \frac{\rho(kZ + lyZ')}{(1-\rho t)^{k+1}} - \frac{\left(|Z'|^{p-2}Z'\right)' + \frac{N-1}{y}|Z'|^{p-2}Z'}{(1-\rho t)^{(k+l)(p-1)+l}} - \frac{\lambda Z^m}{(1-\rho t)^{km}} - \frac{\mu|Z'|^q}{(1-\rho t)^{q(k+l)}}. \tag{2.9}$$

We first choose

$$\rho < \frac{\lambda}{k(1+Aw^{-1})}, \ k = \frac{1}{m-1}, \ 0 < l < \min\left\{\frac{m-p+1}{p(m-1)}, \frac{m-q}{q(m-1)}\right\}, \ A > \frac{k}{l},$$

and next we choose $t_0(p, m, q, \rho, N, A)$ sufficiently close to $1/\rho$, then

$$\begin{aligned} \mathcal{L}_p^1 z \leq \frac{1}{(1-\rho t)^{k+1}} \Big(&\rho k\left(1 + Aw^{-1}\right) - \lambda + NA^{-1}(1-\rho t_0)^{1+k-l-(p-1)(k+l)} \\ &- \mu(1-\rho t_0)^{k+1-q(k+l)}\Big) \leq 0 \end{aligned} \tag{2.10}$$

when $0 \le y \le A$, and

$$\mathcal{L}_p^1 z \le \frac{1}{(1-\rho t)^{k+1}}\Big(\rho(k-lA) + NA^{-1}(1-\rho t_0)^{1+k-l-(p-1)(k+l)} \\ - \mu\left(RA^{-1}\right)^{q(w-1)}(1-\rho t_0)^{k+1-q(k+l)}\Big) \le 0 \tag{2.11}$$

when $A \le y \le R$. Combining (2.10) with (2.11), we know $\mathcal{L}_p^1 z \le 0$ in D. By translation, we may assume without loss of generality that $0 \in \Omega$. Choosing t_0 still closer to $1/\rho$ if necessary, we have $B\left(0, R(1-\rho t)^l\right) \subset \Omega$. Thus, by the definition of $z(x,t)$, we have $u_0(x) \ge z(x,t_0)$ in $\overline{\Omega}$ for sufficiently large initial data, and then, z is a sub-solution of (1.1). By Lemma 1.1(1), it follows that

$$u(x, t-t_0) \ge z(x,t), \quad x \in \Omega,\ t_0 \le t < 1/\rho.$$

Therefore,

$$\begin{aligned} \phi(t-t_0) &= \int\limits_\Omega u^\gamma(x,t-t_0)\,\mathrm{d}x \\ &\ge \int\limits_{B(0,R(1-\rho t)^l)} u^\gamma(x,t-t_0)\,\mathrm{d}x \\ &\ge \int\limits_{B(0,R(1-\rho t)^l)} z^\gamma(x,t)\,\mathrm{d}x \\ &\ge \int\limits_{B(0,A(1-\rho t)^l)} z^\gamma(x,t)\,\mathrm{d}x \\ &\ge \frac{\Upsilon(N)\left(A(1-\rho t)^l\right)^N}{(1-\rho t)^{k\gamma}}, \end{aligned} \tag{2.12}$$

where $\Upsilon(N) = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ is the volume of unit ball in $\mathbb{R}^N$. Since $k\gamma - lN > 0$, we note that $\phi(t-t_0) \to \infty$ as $t \to 1/\rho$. Hence, u blows up in finite time in measure (2.3) and the blowup time $T \le 1/\rho - t_0$.

Next, we estimate the lower bound for the blowup time T. Directly calculating to (2.3) shows that

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}t} &= \gamma \int\limits_\Omega u^{\gamma-1}\left(\Delta_p u + \lambda u^m + \mu|\nabla u|^q\right)\mathrm{d}x \\ &= -\gamma(\gamma-1)\int\limits_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x + \lambda\gamma\int\limits_\Omega u^{m+\gamma-1}\,\mathrm{d}x + \mu\gamma\int\limits_\Omega u^{\gamma-1}|\nabla u|^q\,\mathrm{d}x. \end{aligned} \tag{2.13}$$

Letting $a = m-1$ and $\gamma = ra$, we rewrite (2.13) as

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -ra(ra-1)\int\limits_\Omega u^{ra-2}|\nabla u|^p\,\mathrm{d}x + \lambda ra\int\limits_\Omega u^{ra+a}\,\mathrm{d}x + \mu ra\int\limits_\Omega u^{ra-1}|\nabla u|^q\,\mathrm{d}x. \tag{2.14}$$

We notice that

$$\begin{aligned} \int\limits_\Omega u^{ra-1}|\nabla u|^q\,\mathrm{d}x &= \Big(\frac{q}{ra+q-1}\Big)^q \int\limits_\Omega \left|\nabla u^{\frac{ra+q-1}{q}}\right|^q \mathrm{d}x \\ &\ge C_q\Big(\frac{q}{ra+q-1}\Big)^q \int\limits_\Omega u^{ra+q-1}\,\mathrm{d}x \end{aligned} \tag{2.15}$$

with $C_q = (\Upsilon(N))^{\frac{q}{N}} |\Omega|^{-\frac{q}{N}}$, where we have used the inequality (7.44) in [5]. For convenience, we set

$$v = u^a,\ b = \frac{p-2}{a},\ d = \frac{q-1}{a};$$

then, we have $d, b < 1$. Combining (2.14) with (2.15), we obtain

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t} &\le -ra(ra-1)\Big(\frac{p}{ra+p-2}\Big)^p \int_\Omega \Big|\nabla u^{\frac{ra+p-2}{p}}\Big|^p \mathrm{d}x + \lambda ra \int_\Omega u^{ra+a}\,\mathrm{d}x \\ &\quad + C_q \mu ra\Big(\frac{q}{ra+q-1}\Big)^q \int_\Omega u^{ra+q-1}\,\mathrm{d}x \\ &= -ra(ra-1)\Big(\frac{p}{ra+p-2}\Big)^p \int_\Omega \Big|\nabla v^{\frac{r+b}{p}}\Big|^p \mathrm{d}x + \lambda ra \int_\Omega v^{r+1}\,\mathrm{d}x \\ &\quad + C_q \mu ra\Big(\frac{q}{ra+q-1}\Big)^q \int_\Omega v^{r+d}\,\mathrm{d}x.\end{aligned} \tag{2.16}$$

We now seek a bound for $\int_\Omega v^{r+1}\,\mathrm{d}x$ in terms of $\phi(t)$, the first and third terms on the right-hand side of (2.16). Using Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned}\int_\Omega v^{r+1}\,\mathrm{d}x &\le \left(\int_\Omega v^{r+d}\,\mathrm{d}x\right)^{\frac{r+1+\frac{p^*}{p}(b-1)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}} \left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{(1-d)\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]}} \\ &\quad \cdot \left(\int_\Omega v^{\left(\frac{r+b}{p}\right)p^*}\,\mathrm{d}x\right)^{\frac{(1-d)\left(r+\frac{p^*}{p}b\right)}{\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]}} \\ &\le A_p \left(\int_\Omega v^{r+d}\,\mathrm{d}x\right)^{\frac{r+1+\frac{p^*}{p}(b-1)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}} \left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{(1-d)\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]}} \\ &\quad \cdot \left(\int_\Omega \Big|\nabla v^{\frac{r+b}{p}}\Big|^p\,\mathrm{d}x\right)^{\frac{p^*}{p}\frac{(1-d)\left(r+\frac{p^*}{p}b\right)}{\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]}} \\ &= A_p \left(\int_\Omega v^{r+d}\,\mathrm{d}x\right)^{\frac{r+1+\frac{p^*}{p}(b-1)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}} \left[\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\right. \\ &\quad \left. \cdot \left(\int_\Omega \Big|\nabla v^{\frac{r+b}{p}}\Big|^p\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\right]^{\frac{\left(\frac{p^*}{p}-1\right)(1-d)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}},\end{aligned} \tag{2.17}$$

where constants p^* and A_p are given by

(i) For $p < N < \frac{\sqrt{5}+1}{2}p$, it follows from the remark in [5, p.158] that

$$p^* = \frac{Np}{N-p}, \quad A_p = \Lambda^{\frac{p^*(1-d)\left(r+\frac{p^*}{p}b\right)}{\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]}}$$

with

$$\Lambda = \frac{1}{N\sqrt{\pi}}\left(\frac{N!\Gamma(N/2)}{2\Gamma(N/p)\Gamma(N+1-N/p)}\right)^{1/N}\left(\frac{N(p-1)}{N-p}\right)^{1-1/p};$$

(ii) For $p = N$, by Lemma 7.13 and inequality (7.37) in [5], we have

$$p^* = 3p, \quad A_p = \left(\frac{3^{3N-2}|\Omega|}{\left(\Upsilon(N)\right)^3 N^2}\right)^{\frac{(1-d)(r+3b)}{(2r+3b)(r+3b-2d)}};$$

(iii) For $p > N$,

$$p^* = 3p, \quad A_p = \left(\tau^{\frac{3p\tau}{(\tau-1)^2}} N^{-\frac{3p}{2}} |\Omega|^{\frac{3p}{N}-2}\right)^{\frac{(1-d)(r+3b)}{(2r+3b)(r+3b-2d)}}$$

with $\tau = \frac{N(p-1)}{p(N-1)}$, which are derived from the proof of [5, Theorem 7.10].

Using the elementary inequality

$$a_1^{p_1}a_2^{p_2} \le p_1a_1 + p_2a_2 \quad \text{for} \quad p_1+p_2 = 1, a_1, a_2, p_1, p_2 > 0, \tag{2.18}$$

we obtain

$$\begin{aligned}\int_\Omega v^{r+1}\,\mathrm{d}x \le A_p\Bigg\{&\frac{r+1+\frac{p^*}{p}(b-1)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}\theta^{-\frac{\left(\frac{p^*}{p}-1\right)(1-d)}{r+1+\frac{p^*}{p}(b-1)}}\int_\Omega v^{r+d}\,\mathrm{d}x\\ &+\frac{\left(\frac{p^*}{p}-1\right)(1-d)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}\theta\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\\ &\cdot\left(\int_\Omega\left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\Bigg\}\end{aligned} \tag{2.19}$$

with the positive constant θ to be determined. Applying the elementary inequality (2.18) again, we have

$$\begin{aligned}
&\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\left(\int_\Omega \left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\\
&=\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}\frac{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\left(\int_\Omega \left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}}\\
&\le\frac{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}\zeta^{-\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}}\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}}\\
&\quad+\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left(\frac{p^*}{p}-1\right)\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}\zeta\int_\Omega \left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x
\end{aligned}\tag{2.20}$$

with the positive constant ζ to be determined. Substituting it into (2.19), we obtain

$$\int_\Omega v^{r+1}\,\mathrm{d}x\le H_1\int_\Omega v^{r+d}\,\mathrm{d}x+H_2\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}}+H_3\int_\Omega \left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x,\tag{2.21}$$

where

$$\begin{aligned}
H_1&=A_p\frac{r+1+\frac{p^*}{p}(b-1)}{r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d}\theta^{-\frac{\left(\frac{p^*}{p}-1\right)(1-d)}{r+1+\frac{p^*}{p}(b-1)}},\\
H_2&=A_p\theta(1-d)\frac{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}{\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}\zeta^{-\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}},\\
H_3&=A_p\theta\zeta\frac{\frac{p^*}{p}\left(r+\frac{p^*}{p}b\right)(1-d)}{\left[r+\frac{p^*}{p}b-\left(\frac{p^*}{p}-1\right)d\right]\left[\left(\frac{p^*}{p}-1\right)r+\frac{p^*}{p}b\right]}.
\end{aligned}\tag{2.22}$$

Combining (2.16) with (2.21), we get

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}t}&\le ra\left[H_1\lambda+C_q\mu\left(\frac{q}{ra+q-1}\right)^q\right]\int_\Omega v^{r+d}\,\mathrm{d}x+\lambda raH_2\left(\int_\Omega v^r\,\mathrm{d}x\right)^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}}\\
&\quad+ra\left[H_3\lambda-(ra-1)\left(\frac{p}{ra+p-2}\right)^p\right]\int_\Omega \left|\nabla v^{\frac{r+b}{p}}\right|^p\mathrm{d}x.
\end{aligned}\tag{2.23}$$

Now, we choose θ to make the coefficient of $\int_\Omega v^{r+d}\,\mathrm{d}x$ vanish and then choose suitable ζ to make the coefficient of $\int_\Omega |\nabla v^{\frac{r+b}{p}}|^p\,\mathrm{d}x$ vanish. It follows that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \le \lambda r a H_2\,[\phi(t)]^{\frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b}}. \tag{2.24}$$

Noticing $r > \frac{\frac{p^*}{p}b}{\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1}$, we find

$$\delta := \frac{\frac{p^*}{p}\left(\frac{p^*}{p}-2\right)(r+b)}{\left[\left(\frac{p^*}{p}\right)^2-3\frac{p^*}{p}+1\right]r-\frac{p^*}{p}b} > 1.$$

Thus, for any $t < T$, integrating (2.24) from 0 to t, we obtain

$$\frac{1}{1-\delta}\left[\frac{1}{[\phi(t)]^{\delta-1}}-\frac{1}{[\phi(0)]^{\delta-1}}\right] \le \lambda r a H_2 t. \tag{2.25}$$

Letting $t \to T^-$, we get a lower bound for blowup time

$$T \ge \frac{1}{H[\phi(0)]^{\delta-1}},$$

where $H = \gamma\lambda H_2(\delta-1)$. Hence, we have established the estimate (2.4).

(2) In the case of $\mu \ge 0$. Our basic strategy in establishing the bounds for blowup time parallels that in the case of $\mu < 0$. We prove that the solution blows up in finite time in measure (2.3) at first. We take $z(x,t)$ as (2.6), and let

$$\rho < \frac{\lambda}{k\left(1+Aw^{-1}\right)},\ k = \frac{1}{m-1},\ 0 < l < \frac{m-p+1}{p(m-1)},\ A > \frac{k}{l}.$$

Define

$$\mathcal{L}_p^2 z = z_t - \Delta_p z - \lambda z^m, \tag{2.26}$$

then we can easily verify that $\mathcal{L}_p^2 z \le 0$. Repeating the procedures in the case of $\mu < 0$, we can conclude that z is a sub-solution of (1.1) with $\mu = 0$. On the other hand, we know the solution u of (1.1) with $\mu \ge 0$, represented by $u_{\{\mu\ge 0\}}(x,t)$, is a super-solution of (1.1) with $\mu = 0$. Therefore, by Lemma 1.1(2), we can further show that

$$u_{\{\mu\ge 0\}}(x,t-t_0) \ge u_{\{\mu=0\}}(x,t-t_0) \ge z(x,t),\quad x\in\Omega,\ t_0 \le t < 1/\rho.$$

By a similar argument, we have

$$\begin{aligned}\phi(t-t_0) &= \int_\Omega u^{\gamma}_{\{\mu\ge 0\}}(x,t-t_0)\,\mathrm{d}x \\ &\ge \int_{B(0,R(1-\rho t)^l)} u^{\gamma}_{\{\mu\ge 0\}}(x,t-t_0)\,\mathrm{d}x \\ &\ge \int_{B(0,R(1-\rho t)^l)} z^{\gamma}(x,t)\,\mathrm{d}x \\ &\ge \int_{B(0,A(1-\rho t)^l)} z^{\gamma}(x,t)\,\mathrm{d}x \\ &\ge \frac{\Upsilon(N)\left(A(1-\rho t)^l\right)^N}{(1-\rho t)^{k\gamma}}.\end{aligned} \tag{2.27}$$

Recall that $k\gamma - lN > 0$, this entails $\phi(t-t_0)\to\infty$ as $t\to 1/\rho$. Hence, $u_{\{\mu\ge 0\}}(x,t)$ must blow up in finite time in measure (2.3) and the blowup time T satisfies

$$T \le 1/\rho - t_0. \tag{2.28}$$

To keep the presentation as simple as possible, throughout the remainder of this section, we still use u to denote the solution of (1.1) with $\mu\ge 0$, rather than $u_{\{\mu\ge 0\}}(x,t)$. Now, we estimate the lower bound for the blowup time T. We seek bounds for $\int_\Omega u^{m+\gamma-1}\,\mathrm{d}x$ and $\int_\Omega u^{\gamma-1}|\nabla u|^q\,\mathrm{d}x$ in terms of $\phi(t)$ and the first term on the right-hand side of (2.13), which are different from the case $\mu<0$. Using Hölder's inequality and the elementary inequality

$$a_1^{p_1}a_2^{p_2}a_3^{p_3} \le p_1a_1+p_2a_2+p_3a_3 \quad \text{for}\ \ p_1+p_2+p_3=1, a_i,p_i>0 (i=1,2,3), \tag{2.29}$$

we have

$$\begin{aligned}\int_\Omega u^{\gamma-1}|\nabla u|^q\,\mathrm{d}x &= \int_\Omega \left(u^{\gamma-2}|\nabla u|^p\right)^{\frac{q}{p}} u^{(\gamma-1)\left(1-\frac{q}{p}\right)+\frac{q}{p}}\,\mathrm{d}x \\ &\le \left(\int_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x\right)^{\frac{q}{p}} \left(\int_\Omega u^{m+\gamma-1}\,\mathrm{d}x\right)^{\frac{(\gamma-1)\left(1-\frac{q}{p}\right)+\frac{q}{p}}{m+\gamma-1}} |\Omega|^{\frac{m\left(1-\frac{q}{p}\right)-\frac{q}{p}}{m+\gamma-1}} \\ &\le \kappa\frac{q}{p}\int_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x + \frac{(\gamma-1)\left(1-\frac{q}{p}\right)+\frac{q}{p}}{m+\gamma-1}\int_\Omega u^{m+\gamma-1}\,\mathrm{d}x \\ &\quad + \kappa^{-\frac{m+\gamma-1}{m\left(\frac{p}{q}-1\right)-1}}\frac{m\left(1-\frac{q}{p}\right)-\frac{q}{p}}{m+\gamma-1}|\Omega|,\end{aligned} \tag{2.30}$$

where the positive constant κ will be chosen later. By Hölder's inequality, Sobolev's inequality and the elementary inequality (2.29), we deduce

$$\begin{aligned}
\int\limits_\Omega u^{m+\gamma-1}\,\mathrm{d}x &\le \left(\int\limits_\Omega u^\gamma\,\mathrm{d}x\right)^{1-\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}} \left(\int\limits_\Omega u^{\frac{p^{**}}{p}(\gamma+p-2)}\,\mathrm{d}x\right)^{\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}}\\
&= \left(\int\limits_\Omega u^\gamma\,\mathrm{d}x\right)^{1-\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}} \left\|u^{\frac{\gamma+p-2}{p}}\right\|_{L^{p^{**}}}^{\frac{p^{**}(m-1)}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}}\\
&\le C_p\left(\int\limits_\Omega u^\gamma\,\mathrm{d}x\right)^{1-\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}} \left(\int\limits_\Omega \left|\nabla u^{\frac{\gamma+p-2}{p}}\right|^p\mathrm{d}x\right)^{\frac{p^{**}}{p}\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}}\\
&\le \chi^{-\frac{2\frac{p^{**}}{p}(m-1)}{\frac{p^{**}}{p}(\gamma+p-2m)+(m-\gamma-1)}}\left[\frac{1}{2}+\frac{(m-1)\left(\frac{1}{2}-\frac{p^{**}}{p}\right)}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\right]C_p^{\left(\frac{1}{2}+\frac{(m-1)\left(\frac{1}{2}-\frac{p^{**}}{p}\right)}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\right)^{-1}}\\
&\quad+\left(\frac{1}{2}-\frac{1}{2}\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\right)\left(\int\limits_\Omega u^\gamma\,\mathrm{d}x\right)^2\\
&\quad+\chi\frac{p^{**}}{p}\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\int\limits_\Omega\left|\nabla u^{\frac{\gamma+p-2}{p}}\right|^p\mathrm{d}x,
\end{aligned}\tag{2.31}$$

where the positive constant χ will be determined later. Here, constants p^{**} and C_p are given by

(i) For $p<N$, $p^{**}=\frac{Np}{N-p}$ and $C_p=\Lambda^{\frac{Np(m-1)}{Np+\gamma p-2N}}$;
(ii) For $p=N$, $p^{**}=2p$ and $C_p=\left(\frac{2^{2N-1}|\Omega|}{(\Upsilon(N))^2N}\right)^{\frac{m-1}{\gamma+2p-4}}$;
(iii) For $p>N$, $p^{**}=2p$ and $C_p=\left(\tau^{\frac{2p\tau}{(\tau-1)^2}}N^{-p}|\Omega|^{\frac{2p}{N}-1}\right)^{\frac{m-1}{\gamma+2p-4}}$,

where Λ and τ are the same as those in the case of $\mu<0$. It follows from (2.13), (2.30) and (2.31) that

$$\begin{aligned}
\frac{\mathrm{d}\phi}{\mathrm{d}t} &\le \left[-\gamma(\gamma-1)+\kappa\mu\gamma\frac{q}{p}\right]\int\limits_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x+\kappa^{-\frac{m+\gamma-1}{m\left(\frac{p}{q}-1\right)-1}}\frac{m\left(1-\frac{q}{p}\right)-\frac{q}{p}}{m+\gamma-1}\mu\gamma|\Omega|\\
&\quad+\gamma\left(\mu\frac{(\gamma-1)\left(1-\frac{q}{p}\right)+\frac{q}{p}}{m+\gamma-1}+\lambda\right)\int\limits_\Omega u^{m+\gamma-1}\,\mathrm{d}x\\
&\le \left[-\gamma(\gamma-1)+\kappa\mu\gamma\frac{q}{p}+M_1\right]\int\limits_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x+M_2\left(\int\limits_\Omega u^\gamma\,\mathrm{d}x\right)^2+M_3,
\end{aligned}\tag{2.32}$$

where

$$
\begin{aligned}
M_1 &= \frac{p^{**}}{p}\frac{(m-1)\chi\gamma}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\left(\mu\frac{(\gamma-1)\big(1-\frac{q}{p}\big)+\frac{q}{p}}{m+\gamma-1}+\lambda\right)\left(\frac{\gamma+p-2}{p}\right)^p,\\
M_2 &= \gamma\left(\mu\frac{(\gamma-1)\big(1-\frac{q}{p}\big)+\frac{q}{p}}{m+\gamma-1}+\lambda\right)\left(\frac{1}{2}-\frac{1}{2}\frac{m-1}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}\right),\\
M_3 &= C_p^{1/M_4}M_4\gamma\chi^{-\frac{2\frac{p^{**}}{p}(m-1)}{\frac{p^{**}}{p}(\gamma+p-2m)+(m-\gamma-1)}}\left(\mu\frac{(\gamma-1)\big(1-\frac{q}{p}\big)+\frac{q}{p}}{m+\gamma-1}+\lambda\right)\\
&\quad+\kappa^{-\frac{m+\gamma-1}{m\left(\frac{p}{q}-1\right)-1}}\frac{m\big(1-\frac{q}{p}\big)-\frac{q}{p}}{m+\gamma-1}\mu\gamma|\Omega|
\end{aligned}
\tag{2.33}
$$

with

$$
M_4 = \frac{1}{2}+\frac{(m-1)\big(\frac{1}{2}-\frac{p^{**}}{p}\big)}{\frac{p^{**}}{p}(\gamma+p-2)-\gamma}.
$$

Now, we choose suitable constants κ and χ to make the coefficient of $\int_\Omega u^{\gamma-2}|\nabla u|^p\,\mathrm{d}x$ vanish. It follows that

$$
\begin{aligned}
\frac{\mathrm{d}\left(\phi(t)+1\right)}{\mathrm{d}t} &\le M_2\left(\phi(t)\right)^2+M_3\\
&\le M\left(\phi(t)+1\right)^2,
\end{aligned}
\tag{2.34}
$$

where $M=\max\{M_2,M_3\}$. Integrating (2.34) from 0 to t for any $t<T$, we have

$$
\frac{1}{\phi(0)+1}-\frac{1}{\phi(t)+1}\le Mt.
$$

Letting $t\to T^-$ and using the fact $\lim_{t\to T^-}\phi(t)=\infty$, we get

$$
T\ge\frac{1}{M(\phi(0)+1)}.
$$

Combining this with (2.28), we obtain (2.5). □

3. Upper bounds for blowup time of gradient blowup solutions

In this section, we derive the upper bound for blowup time of gradient blowup solutions to (1.1). To get the result, we start with some known propositions which ensure the gradient blowup of all solutions in finite time if certain assumptions are satisfied.

Proposition 3.1. (See [20, Theorem 3.2]) *Assume that $\lambda>0$, $\mu>0$ and $q>\max\{p,m\}$. Then, there exists a positive real number K_1 depending on p,m,q,λ,μ and Ω such that, if $\|u_0\|_{L^\infty(\Omega)}>K_1$, then gradient blowup will occur.*

Proposition 3.2. (See [19, Theorem 1.4]) *Assume that $\lambda<0$, $\mu>0$ and p,m,q,λ and μ satisfy one of the following conditions:*

(i) *$q>\max\{p,m\}$;*

(ii) *$q=m>p$ and $\mu\gg|\lambda|$.*

Set $\beta=q/(q-p)$. Then, there exists a positive real number K_2 depending on p,m,q,λ, μ and Ω such that, if $\int_\Omega u_0^{\beta+1}\,\mathrm{d}x>K_2$, then gradient blowup occurs.

Remark 3.1. If $\lambda = 0$ and $\mu > 0$, the conclusion still holds as long as the condition $q > \max\{p, m\}$ is replaced by $q > p > 2$. See [7, Proposition 5.3] for further details.

With the help of these known conclusions, we get upper bounds for blowup time of gradient blowup solutions. Different kinds of upper bounds are established in the following theorem.

Theorem 3.1. *Let u be a solution of (1.1). Assume that $\mu > 0$, then the following conclusions hold.*

(1) *Suppose $\lambda > 0$, $q > \max\{m, p\}$ and the initial data satisfy*

$$\int_\Omega u_0^\alpha \,\mathrm{d}x \geq 1, \quad L_1 \left(\int_\Omega u_0^\alpha \,\mathrm{d}x \right)^{\frac{q+\alpha-1}{\alpha}} + L_2 \left(\int_\Omega u_0^\alpha \,\mathrm{d}x \right)^{\frac{m+\alpha-1}{\alpha}} \geq 2L_3 \tag{3.1}$$

with $\alpha = \frac{2q-p}{q-p}$. If the gradient blowup of the positive solution u occurs in finite time, then the blowup time T satisfies

$$T \leq \frac{2\alpha}{(q-1)L_2} \ln \left(1 + \frac{L_2}{L_1} \|u_0\|_{L^\alpha(\Omega)}^{1-q} \right). \tag{3.2}$$

Furthermore, if $m > 1$, then

$$T \leq \frac{2\alpha}{(q-1)L_2} \ln \frac{L_1 + L_2 \|u_0\|_{L^\alpha(\Omega)}^{1-q}}{L_1 + 2^{\frac{1-q}{m-1}} L_2 |\Omega|^{\frac{1-q}{\alpha}} \|u_0\|_{L^\infty(\Omega)}^{1-q}}. \tag{3.3}$$

Here,

$$L_1 = \mu \left(\Upsilon(N)\right)^{\frac{q}{N}} \left(\frac{q}{q+\alpha-1} \right)^q |\Omega|^{\frac{1-q}{\alpha} - \frac{q}{N}}, \; L_2 = \lambda\alpha|\Omega|^{\frac{1-m}{\alpha}}, \; L_3 = \alpha|\Omega| \left(\frac{q\mu}{p\alpha} \right)^{\frac{p}{p-q}}. \tag{3.4}$$

(2) *Suppose $\lambda \leq 0$, $q = m > p$, $\mu \gg |\lambda|$ or $q > \max\{m, p\}$ and the initial data satisfy*

$$\int_\Omega u_0^\alpha \,\mathrm{d}x \geq (2L_5/L_4)^{\frac{\alpha}{q+\alpha-1}} \tag{3.5}$$

with $\alpha = \frac{2q-p}{q-p}$. If the gradient blowup of the solution u occurs in finite time, then the blowup time T satisfies

$$T \leq \frac{2\alpha}{(q-1)L_4} \left(\|u_0\|_{L^\alpha(\Omega)}^{1-q} - \|u_0\|_{L^\infty(\Omega)}^{1-q} |\Omega|^{\frac{1-q}{\alpha}} \right), \tag{3.6}$$

where $L_i = L_i(p, m, q, \lambda, \mu, N, \Omega)(i = 4, 5)$ are given by (3.20),(3.21).

Proof. In order to obtain the upper bound for blowup time of gradient blowup solutions, we introduce the auxiliary function

$$\Phi(t) = \int_\Omega u^\alpha \,\mathrm{d}x. \tag{3.7}$$

A direct calculation shows that

$$\begin{aligned} \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \alpha \int_\Omega u^{\alpha-1} \left(\Delta_p u + \lambda u^m + \mu |\nabla u|^q \right) \mathrm{d}x \\ &= -\alpha(\alpha-1) \int_\Omega u^{\alpha-2} |\nabla u|^p \,\mathrm{d}x + \lambda\alpha \int_\Omega u^{m+\alpha-1} \,\mathrm{d}x + \mu\alpha \int_\Omega u^{\alpha-1} |\nabla u|^q \,\mathrm{d}x. \end{aligned} \tag{3.8}$$

(1) In the case of $\lambda > 0$. Applying Hölder's inequality and the elementary inequality (2.18) to the first term on the right-hand side of (3.8), we obtain

$$\begin{aligned}\int_\Omega u^{\alpha-2}|\nabla u|^p\,\mathrm{d}x &= \int_\Omega \left(u^{\alpha-1}|\nabla u|^q\right)^{\frac{p}{q}}\,\mathrm{d}x\\ &\le \left(\int_\Omega u^{\alpha-1}|\nabla u|^q\,\mathrm{d}x\right)^{\frac{p}{q}}|\Omega|^{\frac{q-p}{q}}\\ &\le \epsilon\frac{p}{q}\int_\Omega u^{\alpha-1}|\nabla u|^q\,\mathrm{d}x+\frac{q-p}{q}\epsilon^{\frac{p}{p-q}}|\Omega|\end{aligned}\tag{3.9}$$

with $\epsilon = \frac{q\mu}{p\alpha}$. Hence,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ge \mu\int_\Omega u^{\alpha-1}|\nabla u|^q\,\mathrm{d}x + \lambda\alpha\int_\Omega u^{m+\alpha-1}\,\mathrm{d}x - \alpha|\Omega|\epsilon^{\frac{p}{p-q}}.\tag{3.10}$$

We notice that

$$\begin{aligned}\int_\Omega u^{\alpha-1}|\nabla u|^q\,\mathrm{d}x &= \Big(\frac{q}{q+\alpha-1}\Big)^q\int_\Omega\left|\nabla u^{\frac{q+\alpha-1}{q}}\right|^q\mathrm{d}x\\ &\ge C_q\Big(\frac{q}{q+\alpha-1}\Big)^q\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x\end{aligned}\tag{3.11}$$

with $C_q = (\Upsilon(N))^{\frac{q}{N}}|\Omega|^{-\frac{q}{N}}$, where we have used the inequality (7.44) in [5]. On the other hand, Hölder's inequality implies that

$$\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x \ge \left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{q+\alpha-1}{\alpha}}|\Omega|^{\frac{1-q}{\alpha}},\tag{3.12}$$

and

$$\int_\Omega u^{m+\alpha-1}\,\mathrm{d}x \ge \left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{m+\alpha-1}{\alpha}}|\Omega|^{\frac{1-m}{\alpha}}.\tag{3.13}$$

Combining the inequalities (3.10)–(3.13), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ge L_1\left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(t)\right)^{\frac{m+\alpha-1}{\alpha}} - L_3,\tag{3.14}$$

where $L_i (i = 1, 2, 3)$ are given by (3.4). Recalling (3.1), we have

$$L_3 \le \frac{1}{2}\left(L_1\left(\Phi(0)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(0)\right)^{\frac{m+\alpha-1}{\alpha}}\right).$$

It follows that

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} \ge \frac{1}{2}\left(L_1\left(\Phi(0)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(0)\right)^{\frac{m+\alpha-1}{\alpha}}\right) > 0.$$

By the continuity of $\Phi(t)$, we get

$$\Phi(t) > \Phi(0),$$

and

$$L_3 \leq \frac{1}{2}\left(L_1\left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(t)\right)^{\frac{m+\alpha-1}{\alpha}}\right),$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \geq \frac{1}{2}\left(L_1\left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(t)\right)^{\frac{m+\alpha-1}{\alpha}}\right) > 0$$

when $t \in (0, \widetilde{\tau})$ for small $\widetilde{\tau}$. Repeating the procedure, for all $t \in (0, T)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &\geq \frac{1}{2}\left(L_1\left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\left(\Phi(t)\right)^{\frac{m+\alpha-1}{\alpha}}\right)\\ &\geq \frac{1}{2}\left(L_1\left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} + L_2\Phi(t)\right).\end{aligned} \tag{3.15}$$

Integrating (3.15) from 0 to t for any $t < T$, we have

$$\begin{aligned}\exp\left(\frac{L_2(q-1)t}{2\alpha}\right) &\leq \frac{L_1 + L_2\left(\Phi(0)\right)^{\frac{1-q}{\alpha}}}{L_1 + L_2\left(\Phi(t)\right)^{\frac{1-q}{\alpha}}}\\ &\leq 1 + \frac{L_2}{L_1}\left(\Phi(0)\right)^{\frac{1-q}{\alpha}}.\end{aligned} \tag{3.16}$$

Furthermore, if $m > 1$, we can verify that $u \leq 2^{\frac{1}{m-1}}\|u_0\|_{L^\infty(\Omega)}$ by virtue of the proof of [20, Theorem 3.2]. It then follows that

$$\exp\left(\frac{L_2(q-1)t}{2\alpha}\right) \leq \frac{L_1 + L_2\|u_0\|^{1-q}_{L^\alpha(\Omega)}}{L_1 + 2^{\frac{1-q}{m-1}}L_2|\Omega|^{\frac{1-q}{\alpha}}\|u_0\|^{1-q}_{L^\infty(\Omega)}}.$$

Letting $t \to T^-$, we get (3.2) and (3.3).

(2) In the case of $\lambda \leq 0$. The proof of this follows a strategy similar to that in the case of $\lambda > 0$. Estimating the first term on the right-hand side of (3.8) in the same way as the case $\lambda > 0$, we can also get the inequalities (3.9)–(3.12).

Applying Hölder's inequality and the elementary inequality (2.18) to the second term on the right-hand side of (3.10), we obtain

$$\begin{aligned}\int_\Omega u^{m+\alpha-1}\,\mathrm{d}x &\leq \left(\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x\right)^{\frac{m+\alpha-1}{q+\alpha-1}}|\Omega|^{\frac{q-m}{q+\alpha-1}}\\ &\leq \frac{m+\alpha-1}{q+\alpha-1}\varsigma\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x + \frac{q-m}{q+\alpha-1}\varsigma^{\frac{m+\alpha-1}{m-q}}|\Omega|\end{aligned} \tag{3.17}$$

with $\varsigma = -\frac{1}{2\lambda\alpha}C_q\mu\frac{q+\alpha-1}{m+\alpha-1}\left(\frac{q}{q+\alpha-1}\right)^q$ when $\lambda < 0$ and $q > m$. Combining (3.10)–(3.12) with (3.17), we deduce

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}t} &\geq \mu C_q\Big(\frac{q}{q+\alpha-1}\Big)^q\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x + \lambda\alpha\int_\Omega u^{m+\alpha-1}\,\mathrm{d}x - \alpha|\Omega|\epsilon^{\frac{p}{p-q}}\\ &\geq \frac{1}{2}\mu C_q\Big(\frac{q}{q+\alpha-1}\Big)^q\int_\Omega u^{q+\alpha-1}\,\mathrm{d}x + \frac{\lambda\alpha(q-m)}{q+\alpha-1}\varsigma^{\frac{m+\alpha-1}{m-q}}|\Omega| - \alpha|\Omega|\epsilon^{\frac{p}{p-q}}\\ &\geq \frac{1}{2}\mu C_q\Big(\frac{q}{q+\alpha-1}\Big)^q|\Omega|^{\frac{1-q}{\alpha}}\left(\int_\Omega u^\alpha\,\mathrm{d}x\right)^{\frac{q+\alpha-1}{\alpha}} + \frac{\lambda\alpha(q-m)}{q+\alpha-1}\varsigma^{\frac{m+\alpha-1}{m-q}}|\Omega| - \alpha|\Omega|\epsilon^{\frac{p}{p-q}}.\end{aligned} \tag{3.18}$$

It follows that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \geq L_4 \left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} - L_5, \tag{3.19}$$

where

$$\begin{aligned} L_4 &= \frac{1}{2}\mu C_q \Big(\frac{q}{q+\alpha-1}\Big)^q |\Omega|^{\frac{1-q}{\alpha}}, \\ L_5 &= -\frac{\lambda\alpha(q-m)}{q+\alpha-1} \varsigma^{\frac{m+\alpha-1}{m-q}} |\Omega| + \alpha|\Omega|\epsilon^{\frac{p}{p-q}} \end{aligned} \tag{3.20}$$

when $\lambda < 0$ and $q > m$. We notice that the estimate (3.19) is still valid, provided that

$$L_4 = \bigg(\mu C_q \Big(\frac{q}{q+\alpha-1}\Big)^q + \lambda\alpha\bigg)|\Omega|^{\frac{1-q}{\alpha}}, \quad L_5 = \alpha|\Omega|\epsilon^{\frac{p}{p-q}} \tag{3.21}$$

when $q = m$, $\mu \gg |\lambda|$ or $\lambda = 0$. From (3.5), we have

$$L_5 \leq \frac{L_4}{2} \left(\Phi(0)\right)^{\frac{q+\alpha-1}{\alpha}}.$$

It follows that

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|_{t=0} \geq \frac{L_4}{2} \left(\Phi(0)\right)^{\frac{q+\alpha-1}{\alpha}} > 0.$$

Using the continuity of $\Phi(t)$, we obtain

$$\Phi(t) > \Phi(0),$$

and

$$L_5 \leq \frac{L_4}{2} \left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}},$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \geq \frac{L_4}{2} \left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}} > 0$$

when $t \in (0, \overline{\tau})$ for small $\overline{\tau} > 0$. Repeating the process, for all $t \in (0, T)$, we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \geq \frac{L_4}{2} \left(\Phi(t)\right)^{\frac{q+\alpha-1}{\alpha}}. \tag{3.22}$$

For any $t < T$, integrating (3.22) from 0 to t, we have

$$\frac{L_4}{2} t \leq \frac{\alpha}{q-1} \left(\left(\Phi(0)\right)^{\frac{1-q}{\alpha}} - \left(\Phi(t)\right)^{\frac{1-q}{\alpha}} \right). \tag{3.23}$$

Hence, the upper bound (3.6) is easily obtained by the fact that $\widetilde{u} = \|u_0\|_{L^\infty(\Omega)}$ is a super-solution of (1.1) when $\lambda \leq 0$ and $\mu > 0$. □

Remark 3.2. As far as we know, this paper is the first one to study blowup time of gradient blowup solutions. It seems natural to ask whether one can derive the lower bound for blowup time when gradient blowup occurs. Unfortunately, we have not found any effective method to obtain related results. We leave it to the interested readers as an open problem.

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