Z. Angew. Math. Phys. (2019) 70:86

0044-2275/19/030001-18
published online May 20, 2019
https://doi.org/10.1007/s00033-019-1130-2

Zeitschrift für angewandte Mathematik und Physik ZAMP

Properties of solutions to porous medium problems with different sources and boundary conditions

Tongxing Li, Nicola Pintus and Giuseppe Viglialoro

Abstract. In this paper, we study nonnegative and classical solutions $u = u(\mathbf{x}, t)$ to porous medium problems of the type

$$\begin{cases} u_t = \Delta u^m + g(u, |\nabla u|) & \mathbf{x} \in \Omega, t \in I, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega, \end{cases} \tag{$\Diamond$}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^N$, with $N \geq 1$, $I = (0, t^*)$ is the maximal interval of existence of u, $m > 1$ and $u_0(\mathbf{x})$ is a nonnegative and sufficiently regular function. The problem is equipped with different boundary conditions and depending on such boundary conditions as well as on the expression of the source g, global existence and blow-up criteria for solutions to ($\Diamond$) are established. Additionally, in the three-dimensional setting and when blow-up occurs, lower bounds for the blow-up time t^* are also derived.

Mathematics Subject Classification. 35K55, 35K57, 35A01, 74H35.

Keywords. Nonlinear parabolic problems, Porous medium equations, Global existence, Blow-up, Gradient nonlinearities.

1. Introduction and motivations

It is well–known that several natural phenomena appearing in various physical, chemical and biological applications, are modeled through reaction diffusion equations. Their description, generally given in a cylinder $\Omega \times I$, where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 1)$ with regular boundary $\partial\Omega$, and $I = (0, t^*)$, is formulated by an initial boundary value problem in the unknown $u = u(\mathbf{x}, t)$ reading as

$$\begin{cases} u_t = \nabla \cdot \mathcal{A}(u, \nabla u, \mathbf{x}, t) + \mathcal{B}(u, \nabla u, \mathbf{x}, t) & \mathbf{x} \in \Omega, t \in I, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \text{Boundary conditions on } u & \mathbf{x} \in \partial\Omega, t \in I. \end{cases} \tag{1}$$

As to the question tied to the existence of local (i.e., t^* finite) or global (i.e., $t^* = \infty$) solutions to classes of nonlinear problems of this type, sufficient conditions on $\mathcal{A}$ (as for instance, standard ellipticity behavior) as well as growth and regularity assumptions on both $\mathcal{A}$ and $\mathcal{B}$ guaranteeing this existence are known and have been widely studied in the literature (we refer, for instance, to [6,20,22,23]).

In this paper, we dedicate our attention to problem (1) in the case $\mathcal{A}(u, \nabla u, \mathbf{x}, t) = \nabla u^m$ and $\mathcal{B}(u, \nabla u, \mathbf{x}, t) = g(u, |\nabla u|)$ and endowed with some boundary conditions, i.e.,

$$\begin{cases} u_t = \Delta u^m + g(u, |\nabla u|) & \mathbf{x} \in \Omega, t \in I, \\ k u_{\boldsymbol{\nu}} + h u = 0 & \mathbf{x} \in \partial\Omega, t \in I, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega, \end{cases} \tag{2}$$

where Ω and I were already introduced in the description of (1). Further, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ stands for the outward normal unit vector to the boundary $\partial\Omega$, $\frac{\partial u}{\partial \boldsymbol{\nu}} := u_{\boldsymbol{\nu}}$ is the normal derivative of u, $m > 1$,

Birkhäuser

$k \geq 0$ and $h > 0$. Additionally, $u_0 := u_0(\mathbf{x}) \not\equiv 0$ is a nonnegative sufficiently smooth function (possibly also verifying compatible conditions on $\partial\Omega$), and $g(u, |\nabla u|)$ is a regular function of its arguments and is such that $\underline{u} \equiv 0$ represents a subsolution of the first equation in (2); henceforth, through the maximum principle, the nonnegativity on $\Omega \times I$ of solutions u to (2) remains essentially justified (see [23,38]).

Beyond problems arising in the mathematical models for gas or fluid flow in porous media (see [5] and [43]), the formulation in (2) also describes the evolution of some biological population u occupying a certain domain whose growth is governed by the law of g (see [19]); precisely, the term Δu^m idealizes the spread of the population, the parameter m indicating the speed of propagation: $m > 1$ corresponds to slow, $0 < m < 1$ fast and the limit case $m = 1$ infinity propagation. Moreover, when the coefficient k is zero (the well–known Dirichlet boundary conditions), then the distribution of u on the boundary of the domain maintains constant through the time, while for $k, h > 0$, the Robin boundary conditions are recovered: they model a negative flux on the boundary, virtually meaning that the population u gets out of the domain with rate $-h/k$.

There are several investigations concerning different variants of the initial boundary value problem (2), all devoted to existence and properties of solutions: global and/or local existence, lower and upper bound of blow-up time, blow-up rates and/or asymptotic behavior. In our opinion, the following papers deserve to be referred also because they inspire this present work.

- *Linear diffusion case* $(m = 1)$ *and* $g(u, |\nabla u|) = u^p$, with $p > 1$. For $\Omega = \mathbb{R}^N$, $N \geq 1$, in [4], [12] and [21] it is shown that for $1 < p \leq 1 + (2/N)$ the problem has no global positive solution, while for $p > 1 + (2/N)$ it is possible to fix appropriate initial data u_0 emanating global solutions. When Ω is a bounded and smooth domain of $\mathbb{R}^3$ and Dirichlet boundary conditions are assigned, in [31] a lower bound for the blow-up time of solutions, if blow-up occurs, is derived, and [32] essentially deals with blow-up and global existence questions for the same problem in the N-dimensional setting, with $N \geq 2$, and endowed with Robin boundary conditions.
- *Linear diffusion case* $(m = 1)$ *and* $g(u, |\nabla u|) = k_1 u^p - k_2 |\nabla u|^q$, $k_1, k_2 > 0$ *and* $p, q \geq 1$. In [42] it is proved that for $q = 2p/(p + 1)$ and small $k_2 > 0$ blow-up can occur for any $N \geq 1$, $p > 1$, $(N - 2)p < N + 2$ and without any restriction on the initial data, while lower bounds of the blow-up time, if blow-up occurs, are derived in [27] when k_1 and k_2 are time-dependent functions and under different boundary conditions.
- *Linear diffusion case* $(m = 1)$ *and* $g(u, |\nabla u|) = |\nabla u|^q$, *with* $q > 2$. (The Hamilton–Jacobi equation.) In [36,37], for certain bounded domains Ω of $\mathbb{R}^N$, $N \geq 1$, Dirichlet boundary conditions and regular data u_0, the authors discuss properties of solutions, known in the literature as *gradient blow-up phenomena*, *loss of (classical) boundary conditions* and *recovery of boundary conditions*. In particular, in the two-dimensional setting and for $2 < q \leq 3$, a sharp description of the final blow-up profile of ∇u near an isolated boundary singularity (in both normal and tangential directions) is given.
- *Nonlinear diffusion case* $(m > 1)$ *and* $g(u, |\nabla u|) = u^p$, *with* $p > 1$. For $\Omega = \mathbb{R}^N$, $N \geq 1$, in [14], [15] and [25] it is shown that for $1 < p \leq m + (2/N)$ the problem has no global positive solution, while for $p > m + (2/N)$ there exist initial data u_0 emanating global solutions. When Ω is a bounded and smooth domain of $\mathbb{R}^N$, $N \geq 1$, and under Dirichlet boundary conditions, in [13] is proved that for $1 < p < m$ the problem admits global solutions for all u_0 such that $u_0^{m-1} \in H_0^1(\Omega)$, while for $m < p < m(1 + (2/N)) + (2/N)$ specific initial data produce unbounded solutions (see also [39]).
- *Nonlinear diffusion case* $(m > 1)$ *and* $g(u, |\nabla u|) = -u^p$, *with* $p > 0$. The papers [16], [17] and [34] focus on results dealing with regularity and asymptotic behavior of solutions defined in the whole space $\mathbb{R}^N$, with $N \geq 1$. Additionally, for similar analysis in the case of bounded domains Ω of $\mathbb{R}^N$, $N \geq 1$, we also refer to [26].
- *Nonlinear diffusion case* $(m > 1)$ *and* $g(u, |\nabla u|) = u^p - u^\mu |\nabla u^\alpha|^q$, *with* $p, q, \alpha \geq 1$ *and* $\mu \geq 0$. With Ω bounded and smooth in $\mathbb{R}^N$, $N \geq 1$, and under Dirichlet boundary conditions, in [3] the authors treat the existence of the so-called *admissible solutions* and show that they are globally bounded if $p < \mu + mq$ or $m < p = \mu + mq$, as well as the existence of blowing-up admissible solutions, under

the complementary condition $1 \leq \mu + mq < p$. Similarly, for $\alpha = m + \mu/q$, $m \geq 1, m/2 + \mu/q > 0, 1 \leq q < 2$, existence of global *weak solutions* is addressed in [2].

In the context of this premise, we remark that our investigation is not focused on the question concerning the existence of solutions to system (2), but rather on their maximal interval of existence I. In particular, in the framework of nonnegative *classical solutions*, we follow the same approach used in largely cited papers (see, for instance, [30,31,33,35,40,41] and references cited therein, for linear or nonlinear diffusion equations, even including our same case, i.e., systems like (2) with $m > 1$) where such an existence is a priori assumed. Additionally, as to the lifespan I of these solutions, only two scenarios can appear and they provide the following *extensibility criterion* ([6,7,20]):

$$\begin{aligned} &\rhd)\ I = (0, \infty),\ \text{so that } u \text{ remains bounded for all } \mathbf{x} \in \Omega \text{ and time } t > 0, \\ &\rhd)\ I = (0, t^*), t^* \text{ finite (the blow-up time), so that } \lim_{t \to t^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \end{aligned} \tag{3}$$

By analyzing the expressions of g presented in the previous items, it is reasonable to expect that the contribution of the positive power addendum, representing a source which essentially increases the energy of the system, stimulates the occurrence of the blow-up; conversely, the negative terms have a damping effect, absorb the energy, and so, contrast the power source term.

Exactly in line with the state of the art above reviewed, with this paper, we aim at expanding the underpinning theory of the mathematical analysis for problem (2) when different choices of g, h and k are considered. Indeed, to the best of our knowledge, the interplay between both positive and negative powers of u, or $|\nabla u|$, in the source g and the Robin/Dirichlet boundary conditions has not yet been extensively studied. To be precise, our contribution includes blow-up and global existence criteria for *nonnegative and classical solutions* to (2) and estimates of the blow-up time when it occurs. We proof three theorems, summarized as follows:

- *Criterion for blow-up in* $\mathbb{R}^N$, $N \geq 1$: *Theorem* 3.1. If $g(u, |\nabla u|) = k_1 u^p - k_2 u^q$, $k_1, k_2, h > 0$, $k = 1$ and $p \geq \max\{m, q\}$ with $m, q > 1$, then the lifespan I of the nonnegative classical solution u to problem (2) emanating from any compatible initial data $u_0(\mathbf{x})$ complying with a certain largeness assumption, is finite and u blows up at some finite time t^*.
- *Criterion for global existence in* $\mathbb{R}^N$, $N \geq 1$: *Theorem* 3.2. If $g(u, |\nabla u|) = k_1 u^p - k_2 u^q$, $k_1, k_2, h > 0$, $k = 1$, $p < m$ with $m, q > 1$, then the lifespan I of the nonnegative classical solution u to problem (2) emanating from any initial data $u_0(\mathbf{x})$ is infinite and u is bounded for all time $t > 0$.
- *Lower bound of the blow-up time in* $\mathbb{R}^3$: *Theorem* 3.3. If $g(u, |\nabla u|) = k_1 u^p - k_2 |\nabla u|^q$, $k_1, k_2, h > 0$, $k = 0$, for $2 - 1/p < m < p$ with $p > q \geq 2$, and u is a nonnegative classical solution to problem (2) emanating from any compatible initial data $u_0(\mathbf{x})$ and becoming unbounded in a certain measure at some finite time t^*, then, if k_2 is sufficiently large, there exists T such that $t^* \geq T$.

Remark 1. Even if the main motivation of this paper lies in enhancing the mathematical theory tied to nonlinear partial differential equations, we want to underline that the expressions of the function g given above are justified also by applicative reasons. Indeed, according to [42], a single (biological) species density u occupying a bounded portion of the space evolves in time by displacement, birth/reproduction and death. In particular, the births are described by a superlinear power of such a distribution, the natural deaths by a linear one and the accidental deaths by a function of its gradient; it leads to $u_t = \Delta u + C_1 u^p - C_2 u - C_3 |\nabla u|^q$, with $p, q > 1$ and $C_1, C_2, C_3 > 0$. Adding to this equation homogeneous Dirichlet conditions corresponds to a non-viable environment on the boundary; homogeneous Neumann conditions stand for a totally insulated domain and Robin ones to a domain which allows the species to cross the boundary. Furthermore, other models originally introduced for only a single species describe the population growth through the preceding equation in which the source $C_1 u^p - C_2 u - C_3 |\nabla u|^q$ is replaced by the so-called logistic function $u(a - bu)$, with $a, b > 0$, or more generally by functions independent of the accidental deaths and whose qualitative behavior is $u^l (1 - u)$, with $l \geq 1$. All the mentioned sources have been also employed in chemotaxis models, precisely to describe the self-organizing of living organisms ([1,8,44–48]).

2. Main assumptions and preparatory lemmas

In this section, we give some crucial hypotheses, statements and lemmas which will be considered through the paper in the proofs of Theorems 3.2 and 3.3.

First we give these

Assumptions. *For any integer $N \geq 1$ and real numbers $h > 0, m > 1$, we establish that:*

($\mathcal{H}_1$) Ω is a bounded domain of $\mathbb{R}^N$, star-shaped and convex in two orthogonal directions, whose geometry for some origin $\mathbf{x}_0$ inside Ω is defined throughout

$$m_1 := \frac{1}{\min_{\partial\Omega}((\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nu})} > 0, \quad m_2 := \frac{\max_{\overline{\Omega}}|\mathbf{x}-\mathbf{x}_0|}{\min_{\partial\Omega}((\mathbf{x}-\mathbf{x}_0)\cdot\boldsymbol{\nu})} > 0. \tag{4}$$

($\mathcal{H}_2$) Ω is a bounded smooth domain of $\mathbb{R}^N$ such that

$$\frac{\xi_1(hm)}{hm} > m_1 N + m_2, \tag{5}$$

being m_1 and m_2 as in (4) and $\xi_1(h)$ the first positive eigenvalue associated with the supported membrane problem

$$\begin{cases} \Delta w + \xi(h) w = 0 & \mathbf{x} \in \Omega, \\ w_{\boldsymbol{\nu}} + hw = 0 & \mathbf{x} \in \partial\Omega. \end{cases} \tag{6}$$

Example 1. In order to provide triples (h, m, Ω) for which assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) are satisfied, let us fix N positive numbers $L_i > 0$, $i = 1, 2, \dots, N$ and the N-dimensional rectangle-like domain $R^N_{\{L_1,L_2,\dots,L_N\}}(\mathbf{0})$, with center the origin and sizes $2L_i$. We have that

$$m_1 = \frac{1}{\min\limits_{\{i=1,2,\dots,N\}} L_i}, \quad m_2 = m_1 \sqrt{\sum_{i=1}^N L_i^2}$$

and, moreover, problem (6) can be explicitly solved by the separation of variables technique. Indeed, if we set $w(\mathbf{x}) = w(x_1, x_2, \dots, x_N) = X_1(x_1)X_2(x_2)\dots X_N(x_N)$, it is reduced to

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{X_i^{''}(x_i)}{X_i(x_i)} = \xi(h), \\ \pm X_i^{'}(\pm L_i) + hX_i(\pm L_i) = 0, \quad \text{for all } i = 1, 2, \dots, N. \end{cases} \tag{7}$$

Evidently, this system is composed of N independent second-order ordinary differential problems, reading for each $i = 1, 2, \dots, N$ as

$$\begin{cases} -\frac{X_i^{''}(x_i)}{X_i(x_i)} = \Lambda_i, \\ \pm X_i^{'}(\pm L_i) + hX_i(\pm L_i) = 0, \end{cases} \tag{8}$$

where $\Lambda_i = \Lambda_i(h) > 0$ is precisely the corresponding eigenvalue. ($\Lambda_i \leq 0$ is not compatible with $h > 0$ and $X_i \not\equiv 0$.) Subsequently, for some constants C_1 and C_2, we have that the general integral

$$X_i(x_i) = C_1 \cos(\sqrt{\Lambda_i} x_i) + C_2 \sin(\sqrt{\Lambda_i} x_i),$$

must be such that

$$(hC_1 \pm C_2\sqrt{\Lambda_i})\cos(\sqrt{\Lambda_i} L_i) + (hC_2 \mp C_1\sqrt{\Lambda_i})\sin(\sqrt{\Lambda_i} L_i) = 0.$$

In order to ensure that the above system in the unknown (C_1, C_2) admits a non–trivial solution, we impose its determinant equal to zero. It yields, for $z_i = L_i\sqrt{\Lambda_i}$

$$\left(h\cos(z_i) - \frac{z_i}{L_i}\sin(z_i)\right)\left(\frac{z_i}{L_i}\cos(z_i) + h\sin(z_i)\right) = 0,$$

or equivalently, dividing by $\cos(z_i)$,

$$\tan(z_i) = -\frac{z_i}{L_i h} \quad \text{or} \quad \tan(z_i) = \frac{hL_i}{z_i}. \tag{9}$$

Since we are dealing with the smallest positive eigenvalue Λ_i to (8), and it is seen that the first positive zero $\hat{z}_i$ of the first equation in (9) belongs to $\in (\pi/2, 3\pi/2)$, we focus on the second one, for which $\hat{z}_i \in (0, \pi/2)$. Finally, the same reasons apply for each independent problem given in (8), so that by superposition we conclude that $\xi(h) = \sum_{i=1}^N \sqrt{\Lambda_i} = \sum_{i=1}^N (\hat{z}_i/L_i)^2$ is the first eigenvalue of (7). From this, we are now in the position to give an example complying with our assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$; in fact, one can numerically check that the triple $(2, 3/2, R^2_{\{\frac{2}{5},\frac{1}{7}\}}(\mathbf{0}))$ infers $\xi(hm) \cong 23.581$, and in turn $\xi(hm) - mh(2/L_1 + \sqrt{L_1^2 + L_2^2}/L_1) \cong 5.395$, so that it suitably fulfills (5).

The forthcoming two lemmas will be employed in the proof of Theorem 3.2 exactly in order to estimate a certain nonzero boundary integral when Robin boundary conditions are considered in system (2). In particular, Lemma 2.2 relies on assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, uses the result in Lemma 2.1 and even though it was already derived in [44, Lemma 3.3], for the sake of completeness we include its proof.

Lemma 2.1. *Let Ω be a domain of $\mathbb{R}^N$ verifying assumption $(\mathcal{H}_1)$. For any nonnegative $C^1(\bar{\Omega})$-function V, we have*

$$\int_{\partial\Omega} V^2 \mathrm{d}s \le m_1 N \int_\Omega V^2 \mathrm{d}\mathbf{x} + 2m_2 \int_\Omega V|\nabla V| \mathrm{d}\mathbf{x}. \tag{10}$$

Proof. This is relation [28, (A.1) of Lemma A.1.] with $n = 1$ and written in terms of the coefficients in (4) of $(\mathcal{H}_1)$. □

Lemma 2.2. *Let Ω be a domain of $\mathbb{R}^N$ satisfying assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Then, for any nonnegative $C^1(\bar{\Omega})$-function V verifying $V_{\boldsymbol{\nu}} + hV = 0$ on $\partial\Omega$, we have*

$$\int_\Omega |\nabla V^m|^2 \mathrm{d}\mathbf{x} \ge \sigma \int_\Omega V^{2m} \mathrm{d}\mathbf{x}, \tag{11}$$

with $\sigma := \eta(hm)$, being $\eta(h) = (\xi_1(h) - h(m_1 N + m_2))/(hm_2 + 1) > 0$.

Proof. For any nonnegative $C^1(\bar{\Omega})$-function V such that $V_{\boldsymbol{\nu}} + hV = 0$ on $\partial\Omega$, the general Poincaré inequality returns this relation for the first eigenvalue $\xi_1(h)$ of (6):

$$\xi_1(h) \int_\Omega V^2 \mathrm{d}\mathbf{x} \le \int_\Omega |\nabla V|^2 \mathrm{d}\mathbf{x} + h \int_{\partial\Omega} V^2 \mathrm{d}s.$$

It can be written, through relation (10) and subsequently the Young inequality with exponents 1/2, as

$$\xi_1(h) \int_\Omega V^2 \mathrm{d}\mathbf{x} \le h(m_1 N + m_2) \int_\Omega V^2 \mathrm{d}\mathbf{x} + (hm_2 + 1) \int_\Omega |\nabla V|^2 \mathrm{d}\mathbf{x}.$$

Hence, since $m_2 > 1$, we also have

$$\int_\Omega |\nabla V|^2 \mathrm{d}\mathbf{x} \ge \eta(h) \int_\Omega V^2 \mathrm{d}\mathbf{x}, \tag{12}$$

with $\eta(h) = (\xi_1(h) - h(m_1N + m_2))/(hm_2 + 1)$. The function $\varphi = V^m$ verifies $\varphi_{\boldsymbol{\nu}} + hm\varphi = 0$, so that (12) provides the

$$\int_\Omega |\nabla V^m|^2 \mathrm{d}\mathbf{x} = \int_\Omega |\nabla\varphi|^2 \mathrm{d}\mathbf{x} \geq \eta(hm)\int_\Omega \varphi^2 \mathrm{d}\mathbf{x} = \sigma\int_\Omega V^{2m} d\mathbf{x},$$

with $\sigma := \eta(hm) = (\xi_1(hm) - hm(m_1N + m_2))/(hmm_2 + 1)$, that is positive by (5). □

Similarly, the remaining two lemmas of this section are necessary to arrange terms emerging in the proof of Theorem 3.3, some of them also depending on $|\nabla u|$. The validity of these lemmas is strictly connected to some interplay between the parameters m, h, p, q defining our main problem, exactly as specified as follows:

Fixing parameters. *Let $p > q \geq 2$. For $2 - 1/p < m < p$, we set*

$$\begin{cases} s = p-1, \quad \mu = \frac{q-1}{s} < 1, \quad d = \frac{m-1}{s} < 1, \\ 1 < \delta < \frac{2}{3}(m+d)\frac{2m+3d-3}{2m+3d-1}, \quad \alpha = \frac{2(m+d)-\delta}{2(m+d)-3\delta} > 1, \\ \beta = \frac{2m+3d-1}{2m+3d-3\alpha} > 1, \quad \sigma = \frac{2(m+d)-3\delta}{2(m+d)} > 0, \quad \gamma = d+\delta > 1. \end{cases} \tag{13}$$

Lemma 2.3. *Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz boundary, and let $m, \alpha, \beta, d, \sigma$ and γ the constants defined in (13). Then for any nonnegative $C^1(\bar{\Omega})$-function V*

$$\int_\Omega V^{m+1}\mathrm{d}\mathbf{x} \leq \frac{\gamma-1}{\gamma-\mu}\varepsilon_1\int_\Omega V^{m+\mu}\mathrm{d}\mathbf{x} + \frac{1-\mu}{\gamma-\mu}\varepsilon_1^{-\frac{\gamma-1}{1-\mu}}\int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x}, \tag{14}$$

where ε_1 is an arbitrarily positive constant.

If, additionally, V vanishes on $\partial\Omega$, there exists a positive constant C_S such that for every $\varepsilon_2 > 0$

$$\begin{aligned} \int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x} \leq\ & C_S^{\frac{3\delta}{m+d}+\frac{6\alpha}{2m+3d}} 3\alpha d\frac{\sigma}{2m+3d}\varepsilon_2\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x} \\ & + \Gamma^{\frac{3\delta}{m+d}+\frac{6\alpha}{2m+3d}} d\sigma\frac{2m+3d-3\alpha}{2m+3d}\varepsilon_2\Big(\int_\Omega V^m d\mathbf{x}\Big)^{\alpha\beta} \\ & + (1-d)\varepsilon_2 C_S^{\frac{3\delta}{m+d}}\sigma\Big(\int_\Omega V^m \mathrm{d}\mathbf{x}\Big)^{\alpha} \\ quad & + C_S^{\frac{3\delta}{m+d}}\frac{3\delta}{2(m+d)}\varepsilon_2^{1-\frac{2(m+d)}{3\delta}}\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}. \end{aligned} \tag{15}$$

Proof. Since $\mu < 1$, for some positive constant $\gamma > 1$, the Young inequality and the consideration of $\varepsilon_1 > 0$ yield

$$\begin{aligned} \int_\Omega V^{m+1}\mathrm{d}\mathbf{x} \leq & \Big(\int_\Omega V^{m+\mu}\mathrm{d}\mathbf{x}\Big)^{\frac{\gamma-1}{\gamma-\mu}}\Big(\int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x}\Big)^{\frac{1-\mu}{\gamma-\mu}} \\ & \frac{\gamma-1}{\gamma-\mu}\varepsilon_1\int_\Omega V^{m+\mu}\mathrm{d}\mathbf{x} + \frac{1-\mu}{\gamma-\mu}\varepsilon_1^{-\frac{\gamma-1}{1-\mu}}\int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x}, \end{aligned}$$

so that the first thesis is shown.

On the other hand, let V be such that $V = 0$ on $\partial\Omega$: the Sobolev embedding in $\mathbb{R}^3$, $W_0^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, provides a positive constant C_S such that

$$\int_\Omega \Big(V^{\frac{m+d}{2}}\Big)^6 \mathrm{d}\mathbf{x} \leq C_S^6\Big(\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^3. \tag{16}$$

Now, for $\gamma = d + \delta > 1$, the Hölder inequality leads to

$$\begin{aligned}\int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x} &= \int_\Omega V^{(m+d)+\delta}\mathrm{d}\mathbf{x} \\ &\le \Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{2(m+d)-\delta}{2(m+d)}}\Big(\int_\Omega (V^{\frac{m+d}{2}})^6\mathrm{d}\mathbf{x}\Big)^{\frac{\delta}{2(m+d)}},\end{aligned}\tag{17}$$

so that by replacing (16) into (17), we obtain

$$\int_\Omega V^{(m+d)+\delta}\mathrm{d}\mathbf{x} \le C_S^{\frac{3\delta}{m+d}}\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{2(m+d)-\delta}{2(m+d)}}\Big(\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^{\frac{3\delta}{2(m+d)}}.$$

The introduction of an arbitrary and positive constant ε_2, and an application of the Young inequality, allow us to write [recall (13)]

$$\begin{aligned}\int_\Omega V^{(m+d)+\delta}\mathrm{d}\mathbf{x} &\le C_S^{\frac{3\delta}{m+d}}\Big(\varepsilon_2\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{2(m+d)-\delta}{2(m+d)-3\delta}}\Big)^{\frac{2(m+d)-3\delta}{2(m+d)}} \\ &\quad\times\Big(\varepsilon_2^{1-\frac{2(m+d)}{3\delta}}\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^{\frac{3\delta}{2(m+d)}} \\ &\le C_S^{\frac{3\delta}{m+d}}\varepsilon_2\frac{2(m+d)-3\delta}{2(m+d)}\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{2(m+d)-\delta}{2(m+d)-3\delta}} \\ &\quad + C_S^{\frac{3\delta}{m+d}}\varepsilon_2^{1-\frac{2(m+d)}{3\delta}}\frac{3\delta}{2(m+d)}\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}.\end{aligned}\tag{18}$$

To bound the term $\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{(2(m+d)-\delta)/(2(m+d)-3\delta)}$, let us observe that the Hölder and the Schwarz inequalities give, respectively,

$$\int_\Omega V^{m+1}\mathrm{d}\mathbf{x} \le \Big(\int_\Omega V^{2(m+d)}\mathrm{d}\mathbf{x}\Big)^{\frac{1}{m+2d}}\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{\frac{m+2d-1}{m+2d}},\tag{19}$$

and

$$\int_\Omega V^{2(m+d)}\mathrm{d}\mathbf{x} \le \Big[\int_\Omega \Big(V^{\frac{m+d}{2}}\Big)^6\mathrm{d}\mathbf{x}\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big]^{\frac{1}{2}}.\tag{20}$$

Now, using in (20) relation (16), we get

$$\int_\Omega V^{2(m+d)}\mathrm{d}\mathbf{x} \le C_S^3\Big(\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^{\frac{3}{2}}\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}},$$

and hence (19) reads

$$\begin{aligned}\int_\Omega V^{m+1}\mathrm{d}\mathbf{x} &\le C_S^{\frac{3}{m+2d}}\Big(\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^{\frac{3}{2(m+2d)}}\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2(m+2d)}} \\ &\quad\times\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{\frac{m+2d-1}{m+2d}}.\end{aligned}\tag{21}$$

In addition, we first use again the Hölder inequality to lead to

$$\int_\Omega V^{m+d}\mathrm{d}\mathbf{x} \le \Big(\int_\Omega V^{m+1}\mathrm{d}\mathbf{x}\Big)^d \Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{1-d}, \tag{22}$$

and then we insert this estimate in (21); combining terms, applying

$$a^r b^{1-r} \le ra + (1-r)b, \tag{23}$$

valid for $a, b \ge 0$ and $0 < r < 1$, we arrive at (α as in (13))

$$\begin{aligned}\Big(\int_\Omega V^{m+1}\mathrm{d}\mathbf{x}\Big)^\alpha &\le C_S^{\frac{6\alpha}{2m+3d}}\Big(\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\Big)^{\frac{3\alpha}{2m+3d}}\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{\alpha\frac{2m+3d-1}{2m+3d}}\\ &\le C_S^{\frac{6\alpha}{2m+3d}}\frac{3\alpha}{2m+3d}\int_\Omega |\nabla V^{\frac{m+d}{2}}|^2\mathrm{d}\mathbf{x}\\ &\quad + C_S^{\frac{6\alpha}{2m+3d}}\frac{2m+3d-3\alpha}{2m+3d}\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{\alpha\frac{2m+3d-1}{2m+3d-3\alpha}}.\end{aligned} \tag{24}$$

Hence, by rearranging again (22) with (23) we attain

$$\begin{aligned}\Big(\int_\Omega V^{m+d}\mathrm{d}\mathbf{x}\Big)^\alpha &\le \Big[\Big(\int_\Omega V^{m+1}\mathrm{d}\mathbf{x}\Big)^d\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^{1-d}\Big]^\alpha\\ &\le d\Big(\int_\Omega V^{m+1}\mathrm{d}\mathbf{x}\Big)^\alpha + (1-d)\Big(\int_\Omega V^m\mathrm{d}\mathbf{x}\Big)^\alpha,\end{aligned}$$

so that in view of (24) expression (18) (recall $\gamma = d + \delta$) infers our thesis. □

Lemma 2.4. *Let m, d and δ as in (13). If $c_1, c_2, \ldots, c_6$ are positive real numbers satisfying*

$$c_3 \ge c_5\Big(\frac{c_5}{c_6}\Big)^{\frac{-3\delta}{2(m+d)}}\Big(\frac{3\delta}{2m+2d-3\delta}\Big)^{-\frac{3\delta}{2(m+d)}}\frac{2(m+d)}{2(m+d)-3\delta}, \tag{25}$$

then there exits $\xi_m \in (0, \infty)$ such that

$$c_5\xi_m + c_6\xi_m^{1-\frac{2(m+d)}{3\delta}} - c_3 \le 0. \tag{26}$$

Proof. For any $\xi \in (0, \infty)$, the function $\Phi(\xi) := c_5\xi + c_6\xi^{1-2(m+d)/3\delta}$ attains its minimum at the point

$$\xi_m = \Big(\frac{3\delta c_5}{c_6(2m+2d-3\delta)}\Big)^{\frac{-3\delta}{2(m+d)}}. \tag{27}$$

Therefore, since (25) holds we have

$$c_3 \ge c_5\Big(\frac{c_5}{c_6}\Big)^{\frac{-3\delta}{2(m+d)}}\Big(\frac{3\delta}{2m+2d-3\delta}\Big)^{-\frac{3\delta}{2(m+d)}}\frac{2(m+d)}{2(m+d)-3\delta} = \Phi(\xi_m),$$

and relation (26) is proven. □

3. Analysis and proofs of the main results

In this section, we discuss and give the demonstrations of our main theorems, whose general overview was summarized in Sect. 1.

3.1. A criterion for blow-up

The first theorem is dedicated to understand properties of solutions to system (2) when $g(u, |\nabla u|) = k_1u^p - k_2u^q$ and under Robin boundary conditions. Essentially, we observe that if the power q of the absorption term in g, as well as the coefficient m of the diffusion, does not surpass the power p from the growth contribution, then the occurrence of blow-up phenomena at some finite time may appear for some initial data $u_0(\mathbf{x})$, despite the outflow boundary conditions; in particular, no global solution is expected.

Theorem 3.1. *Let Ω be a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, $k_1, k_2, h > 0$, $k = 1$, $p \geq \max\{m, q\}$, with $m, q > 1$, $g(u, |\nabla u|) = k_1u^p - k_2u^q$ and $u_0(\mathbf{x}) \not\equiv 0$ a nonnegative function from $C^1(\bar{\Omega})$, satisfying the compatibility condition $\frac{\partial u_0(\mathbf{x})}{\partial \boldsymbol{\nu}} + hu_0(\mathbf{x}) = 0$ on $\partial\Omega$. Moreover, let $u \in C^{2,1}(\Omega \times (0, t^*)) \cap C^{1,0}(\bar{\Omega} \times [0, t^*))$ be the nonnegative solution of problem (2). If*

$$\psi(t) := -\frac{p+m}{2m}\int_\Omega |\nabla u^m|^2 \mathrm{d}\mathbf{x} + k_1\int_\Omega u^{p+m}\mathrm{d}\mathbf{x} - k_2\int_\Omega u^{q+m}\mathrm{d}\mathbf{x} - \frac{h(p+m)}{2}\int_{\partial\Omega} u^{2m}\mathrm{d}s, \qquad \text{for all } t \in (0, t^*),$$

is such that $\psi(0) > 0$, then $t^* < \infty$, or equivalently $I = (0, t^*)$. In particular, $\|u(\cdot, t)\|_{L^\infty(\Omega)} \nearrow \infty$ as $t \searrow t^*$ at some time t^* satisfying, for $\varphi(t) := \int_\Omega u^{m+1}\mathrm{d}\mathbf{x}$,

$$t^* < T = \frac{1}{p-1}\frac{\varphi(0)}{\psi(0)}.$$

Proof. Let u be the nonnegative classical solution of (2) satisfying $u_{\boldsymbol{\nu}} = -hu$ on $\partial\Omega$. By a differentiation, we can write

$$\begin{aligned}
\psi'(t) &= -\frac{p+m}{m}\int_\Omega \nabla u^m \cdot (\nabla u^m)_t \mathrm{d}\mathbf{x} + k_1(p+m)\int_\Omega u^{p+m-1}u_t\mathrm{d}\mathbf{x} \\
&\quad - k_2(q+m)\int_\Omega u^{q+m-1}u_t\mathrm{d}\mathbf{x} - h(p+m)m\int_{\partial\Omega} u^{2m-1}u_t\mathrm{d}s \\
&= -\frac{p+m}{m}\int_{\partial\Omega}(u^m)_t\nabla u^m \cdot \boldsymbol{\nu}\mathrm{d}s + \frac{p+m}{m}\int_\Omega (u^m)_t\Delta u^m\mathrm{d}\mathbf{x} \\
&\quad + k_1(p+m)\int_\Omega u^{p+m-1}u_t\mathrm{d}\mathbf{x} - k_2(q+m)\int_\Omega u^{q+m-1}u_t\mathrm{d}\mathbf{x} \\
&\quad - h(p+m)m\int_{\partial\Omega} u^{2m-1}u_t\mathrm{d}s \\
&\geq (p+m)\int_\Omega u^{m-1}u_t(\Delta u^m + k_1u^p - k_2u^q)\mathrm{d}\mathbf{x} \\
&= (p+m)\int_\Omega u^{m-1}(u_t)^2\mathrm{d}\mathbf{x} \geq 0 \quad \text{for all } t \in (0, t^*),
\end{aligned} \tag{28}$$

where we have used the integration by parts formula and the assumption $p \geq q$.

Similarly, as to the evolution of $\varphi(t) := \int_\Omega u^{m+1} d\mathbf{x}$, we derive

$$\begin{aligned}\frac{1}{m+1}\varphi'(t) &= \int_\Omega u^m(\Delta u^m + k_1 u^p - k_2 u^q)\mathrm{d}\mathbf{x}\\
&= -\int_\Omega \nabla u^m \cdot \nabla u^m \mathrm{d}\mathbf{x} + k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} - k_2 \int_\Omega u^{q+m}\mathrm{d}\mathbf{x}\\
&\quad + \int_{\partial\Omega} u^m \nabla u^m \cdot \boldsymbol{\nu}\mathrm{d}s\\
&= -\int_\Omega |\nabla u^m|^2\mathrm{d}\mathbf{x} + k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} - k_2 \int_\Omega u^{q+m}\mathrm{d}\mathbf{x}\\
&\quad - mh\int_{\partial\Omega} u^{2m}\mathrm{d}s\\
&\geq -\frac{p+m}{2m}\int_\Omega |\nabla u^m|^2\mathrm{d}\mathbf{x} + k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} - k_2 \int_\Omega u^{q+m}\mathrm{d}\mathbf{x}\\
&\quad - \frac{h(p+m)}{2}\int_{\partial\Omega} u^{2m}\mathrm{d}s = \psi(t) \quad \text{for all } t \in (0, t^*),\end{aligned} \tag{29}$$

where in this case, we relied on the fact that $p \geq m$. Now, the hypotheses $\psi(0) > 0$, $\varphi(0) > 0$, (28) and (29) yield

$$\psi'(t) > 0,\ \psi(t) > 0,\ \varphi'(t) > 0 \text{ and } \varphi(t) > 0 \text{ on } \quad (0, t^*).$$

Since by the Young inequality, we have that for all $t \in (0, t^*)$

$$\frac{1}{m+1}\varphi'(t) = \int_\Omega u^{\frac{m+1}{2}} u^{\frac{m-1}{2}} u_t \mathrm{d}\mathbf{x} \leq \left(\int_\Omega u^{m+1}\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \left(\int_\Omega u^{m-1}(u_t)^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}},$$

this implies by virtue of the definition of φ, in conjunction with (28) and (29),

$$\varphi(t)\psi'(t) \geq \frac{m+p}{(m+1)^2}\varphi'(t)^2 \geq \frac{m+p}{m+1}\psi(t)\varphi'(t) \quad \text{on} \quad (0, t^*),$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi\varphi^{-\frac{m+p}{m+1}}\right) \geq 0 \quad \text{on} \quad (0, t^*).$$

Subsequently, an integration on $(0, t)$ with $t < t^*$ infers

$$\psi(t) \geq \psi(0)\varphi(0)^{-\frac{m+p}{m+1}}\varphi(t)^{\frac{m+p}{m+1}} \quad \text{on} \quad (0, t).$$

Finally, recalling (29), we have

$$\varphi'(t)\varphi(t)^{-\frac{m+p}{m+1}} \geq (m+1)\psi(0)\varphi(0)^{-\frac{m+p}{m+1}} \quad \text{on} \quad (0, t),$$

and with $(m+p)/(m+1) > 1$ a further integration leads to

$$\frac{1}{\varphi(t)^{\frac{p-1}{m+1}}} \leq \frac{1}{\varphi(0)^{\frac{p-1}{m+1}}} - (p-1)\frac{\psi(0)}{\varphi(0)^{\frac{m+p}{m+1}}}t,$$

that, by virtue of the positivity of φ, cannot hold for $t \geq T = \varphi(0)/((p-1)\psi(0))$. In conclusion, the extensibility criterion (3) implies that $I = (0, t^*)$, for some $t^* < T$. □

Remark 2. As to the above technical requirement $\psi(0) > 0$, we desire to emphasize its consistency with the thesis of Theorem 3.1. Indeed, for a given initial data u_0, with the suitable regularity and properties, such strictly positivity of $\psi(0)$ is essentially achieved when the value of the positive term $k_1 \int\limits_\Omega u_0^{p+m}\mathrm{d}\mathbf{x}$ dominates that from the other remaining contributions. Since this is certainly possible for k_1 sufficiently large, and k_1 is precisely the coefficient associated with the growth source in the model, the conclusion on the unboundedness of the solution u can be rather expected. Similar qualitative arguments apply imposing weak dampening and/or rate outflow effects, i.e., small values of k_2 and/or h. Finally, in support of the overall assumptions of Theorem 3.1, we mention [10,11], where certain numerical methods capable to detect unbounded solutions to porous medium equations defined in bounded intervals of $\mathbb{R}$ and equipped with different boundary conditions are employed.

3.2. A criterion for global existence

In the next result, we are interested to examine the opposite situation described in Theorem 3.1. Precisely, by considering in system (2) again $g(u, |\nabla u|) = k_1 u^p - k_2 u^q$ and Robin boundary conditions, we establish that when the effect of the source (exponent p) is weaker than that of the diffusion (exponent m), the negative flux on the boundary prevents blow-up, even for arbitrary large initial data $u_0(\mathbf{x})$ and any small absorption effect (exponent q).

Theorem 3.2. *Let Ω be a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, satisfying assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) . Moreover, let $k_1, k_2, h > 0$, $k = 1$, $q \geq 1$, $p < m$, with $m > 1$, $g(u, |\nabla u|) = k_1 u^p - k_2 u^q$ and $u_0(\mathbf{x}) \not\equiv 0$ a nonnegative function from $C^0(\bar{\Omega})$. Then the nonnegative solution $u \in C^{2,1}(\Omega \times (0, t^*)) \cap C^{1,0}(\bar{\Omega} \times [0, t^*))$ of problem (2) is global, or equivalently $I = (0, \infty)$.*

Proof. Let u be the nonnegative classical solution of (2) satisfying $u_{\boldsymbol{\nu}} = -hu$ on $\partial\Omega$. By differentiating $\varphi(t) := \int\limits_\Omega u^{m+1}\mathrm{d}\mathbf{x}$, we derive

$$
\begin{aligned}
\frac{1}{m+1}\varphi'(t) &= \int_\Omega u^m(\Delta u^m + k_1 u^p - k_2 u^q)\mathrm{d}\mathbf{x} \\
&= -\int_\Omega \nabla u^m \cdot \nabla u^m \mathrm{d}\mathbf{x} + k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} - k_2 \int_\Omega u^{q+m}\mathrm{d}\mathbf{x} \\
&\quad + \int_{\partial\Omega} u^m \nabla u^m \cdot \boldsymbol{\nu}\mathrm{d}s \\
&\leq -\int_\Omega |\nabla u^m|^2 \mathrm{d}\mathbf{x} + k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} \quad \text{on } (0, t^*),
\end{aligned}
$$

where we have neglected the last two nonpositive integrals.

On the other hand, since $p < m$, we have thanks to the Young inequality and for some $\varepsilon > 0$

$$
k_1 \int_\Omega u^{p+m}\mathrm{d}\mathbf{x} \leq \varepsilon \int_\Omega u^{2m}\mathrm{d}\mathbf{x} + C(\varepsilon)|\Omega| \quad \text{on } (0, t^*),
$$

with $C(\varepsilon) = (2m\varepsilon/(p+m)k_1)^{(m+p)/(p-m)}(m-p)/2m > 0$ (recall $m > p$). Subsequently,

$$\begin{aligned}\frac{1}{m+1}\varphi'(t) &\leq -\int_\Omega |\nabla u^m|^2 \mathrm{d}\mathbf{x} + \varepsilon \int_\Omega u^{2m}\mathrm{d}\mathbf{x} + C(\varepsilon)|\Omega| \\ &\leq -\sigma \int_\Omega u^{2m}\mathrm{d}\mathbf{x} + \varepsilon\int_\Omega u^{2m}\mathrm{d}\mathbf{x} + C(\varepsilon)|\Omega| \\ &= -(\sigma-\varepsilon)\int_\Omega u^{2m}\mathrm{d}\mathbf{x} + C(\varepsilon)|\Omega| \quad \text{on } (0,t^*),\end{aligned}$$

where we have estimated the integral depending on $|\nabla u^m|^2$ by means of (11) of Lemma 2.2 with, of course, $V = u$. By choosing $\varepsilon = \sigma/2 > 0$, and by taking in consideration that an application of the Young inequality infers

$$-\int_\Omega u^{2m}\mathrm{d}\mathbf{x} \leq -|\Omega|^{\frac{1-m}{1+m}}\varphi^{\frac{2m}{m+1}} \quad \text{on } (0,t^*),$$

the previous estimate reads

$$\varphi'(t) \leq -C_0\varphi^{\frac{2m}{m+1}}(t) + C_1 \quad \text{on } (0,t^*),$$

where $C_0 = (m+1)\sigma|\Omega|^{\frac{1-m}{1+m}}/2$ and $C_1 = (m+1)C(\varepsilon)|\Omega|$; consequently, ODE comparison arguments justify that

$$\varphi(t) \leq C := \max\left\{\varphi(0), \left(\frac{C_1}{C_0}\right)^{\frac{m+1}{2m}}\right\} \quad \text{on } (0,t^*).$$

Finally, well–known extension results for ODE's with locally Lipschitz continuous right side (see, for instance, [18]), show that $t^* = \infty$; indeed, if t^* were finite, $\varphi(t) \nearrow +\infty$ as $t \searrow t^*$ and it would contradict $\varphi(t) \leq C$ on $(0,t^*)$. In conclusion, again the extensibility criterion (3) implies $I = (0,\infty)$. □

Remark 3. Conversely to the demonstration of Theorem 3.1, evidently, the proof of this last theorem remains valid also for $k_2 = 0$, that is in the complete absence of absorption terms in g. In any case, we preferred to consider the expression of the function g in Theorem 3.2 as that in Theorem 3.1 exactly to better highlight the different behaviors of the corresponding solutions to problem (2) despite the same source.

In the behalf of scientific completeness, we mention [9,24] where some questions concerning existence of local-in-time classical solutions to a class of systems tied to (2) are discussed under slightly more general boundary conditions but more specific sources than those considered in Theorems 3.1 and 3.2 ; hence, such theorems do not fall within these contributions.

3.3. Lower bounds of the blow-up time

This last theorem is concerned with lower bounds of the blow-up time t^* for unbounded solutions to (2), when gradient nonlinearities with absorption effects appear in g. More precisely, we define $g(u, |\nabla u|) = k_1 u^p - k_2|\nabla u|^q$ and endow the problem with Dirichlet boundary conditions. We are not aware of general results which straightforwardly infer the existence of unbounded solutions to system (2) under these hypotheses; nevertheless, in the spirit of the result derived in Theorem 3.1, for which blow-up occurs for large initial data and despite negative flux on the boundary, we understand that also in these circumstances seems reasonable to assume the existence of such blowing-up solutions.

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz boundary. Moreover, let $k_1, h > 0$, $k = 0$, p, q, α, β, μ and γ as in (13), $g(u, |\nabla u|) = k_1 u^p - k_2 |\nabla u|^q$ and $u_0(\mathbf{x}) \not\equiv 0$ a nonnegative function from $C^0(\bar{\Omega})$, satisfying the compatibility condition $u_0(\mathbf{x}) = 0$ on $\partial\Omega$. Hence, it is possible to find a positive number Σ with the following property: If k_2 is a positive real satisfying*

$$k_2 \geq k_1 (k_1 \Sigma)^{\frac{1-\mu}{\gamma-1}}, \tag{30}$$

and $u \in C^{2,1}(\Omega \times (0, t^*)) \cap C^0(\bar{\Omega} \times [0, t^*))$ is a nonnegative solution of (2) such that $W(t) \nearrow +\infty$ as $t \searrow t^*$, with some finite t^* and

$$W(t) = \int_\Omega u^{m(p-1)} \mathrm{d}\mathbf{x}, \tag{31}$$

then

$$t^* \geq \frac{W(0)^{-\alpha\beta+1}}{(\mathcal{M} W(0)^{(1-\beta)\alpha} + \mathcal{N})(-\alpha\beta + 1)},$$

$\mathcal{M}$ and $\mathcal{N}$ being two positive computable constants.

Proof. Let u be the nonnegative classical solution of (2) satisfying $u = 0$ on $\partial\Omega$ and t^* be the instant of time where the W-measure (31) associated with u becomes unbounded. For $s = p-1$, let us differentiate with respect to the time t such W-measure. Due to the divergence theorem and the boundary conditions, we obtain

$$\begin{aligned}
W'(t) &= ms \int_\Omega u^{ms-1} [\Delta(u^m) + k_1 u^p - k_2 |\nabla u|^q] \mathrm{d}\mathbf{x} \\
&= -ms \int_\Omega \nabla u^{ms-1} \cdot \nabla(u^m) \mathrm{d}x \\
&\quad + msk_1 \int_\Omega u^{s(m+1)} \mathrm{d}\mathbf{x} - msk_2 \int_\Omega u^{ms-1} |\nabla u|^q \mathrm{d}\mathbf{x} \\
&= -m^2 s(ms-1) \int_\Omega u^{ms-3+m} |\nabla u|^2 \mathrm{d}\mathbf{x} \\
&\quad + msk_1 \int_\Omega u^{s(m+1)} \mathrm{d}\mathbf{x} - msk_2 \int_\Omega u^{ms-1} |\nabla u|^q \mathrm{d}\mathbf{x} \quad \text{on} \quad (0, t^*).
\end{aligned} \tag{32}$$

Now, the statements given in (13) imply $ms + q - 1 > 2$, so we can invoke inequality [29, (2.10)] achieving

$$\begin{aligned}
msk_2 \int_\Omega u^{ms-1} |\nabla u|^q \mathrm{d}\mathbf{x} &= msk_2 \Big(\frac{q}{ms+q-1} \Big)^q \int_\Omega |\nabla u^{\frac{ms+q-1}{q}}|^q \mathrm{d}\mathbf{x} \\
&\geq msk_2 \Big(\frac{2\sqrt{\lambda_1}}{ms+q-1} \Big)^q \int_\Omega u^{ms+q-1} \mathrm{d}\mathbf{x} \quad \text{on} \quad (0, t^*),
\end{aligned} \tag{33}$$

where λ_1 is the optimal Poincaré constant.

From now on, for simplicity, we indicate $u^s = V$ so to have

$$|\nabla V|^2 = s^2 u^{2(s-1)} |\nabla u|^2. \tag{34}$$

As a consequence, again due to the positions made in (13), it holds that $(m-2)+d>0$, so that using (33) and (34), relation (32) becomes

$$\begin{aligned} W'(t) \leq & -c_1 \int_\Omega V^{(m-2)+d}|\nabla V|^2 d\mathbf{x} + c_2 \int_\Omega V^{m+1} \mathrm{d}\mathbf{x} \\ & - msk_2 \Big(\frac{2\sqrt{\lambda_1}}{ms+q-1}\Big)^q \int_\Omega V^{m+\mu} \mathrm{d}\mathbf{x} \\ = & -c_3 \int_\Omega |\nabla V^{\frac{m+d}{2}}|^2 \mathrm{d}\mathbf{x} + c_2 \int_\Omega V^{m+1}\mathrm{d}\mathbf{x} \\ & - msk_2 \Big(\frac{2\sqrt{\lambda_1}}{ms+q-1}\Big)^q \int_\Omega V^{m+\mu} \mathrm{d}\mathbf{x} \quad \text{on} \quad (0,t^*), \end{aligned} \tag{35}$$

where

$$c_1 = \frac{m^2(ms-1)}{s}, \quad c_2 = msk_1, \quad c_3 = \frac{4}{(m+d)^2} c_1.$$

Now we are in the position to apply Lemma 2.3: by using relation (14) with $\varepsilon_1 = k_2\big(2\sqrt{\lambda_1}/(ms+q-1)\big)^q(\gamma-\mu)/(k_1(\gamma-1))$, (35) is simplified to

$$W'(t) \leq -c_3 \int_\Omega |\nabla V^{\frac{m+d}{2}}|^2 \mathrm{d}\mathbf{x} + c_4 \int_\Omega V^{m+\gamma} \mathrm{d}\mathbf{x} \quad \text{on} \quad (0,t^*),$$

where

$$c_4 = c_2 \frac{1-\mu}{\gamma-\mu} \varepsilon_1^{-\frac{\gamma-1}{1-\mu}},$$

while rearranging the term $\int_\Omega V^{m+\gamma}\mathrm{d}\mathbf{x}$ through (15) we obtain

$$\begin{aligned} W'(t) \leq & \Big(c_5\varepsilon_2 + c_6 \varepsilon_2^{1-\frac{2(m+d)}{3\delta}} - c_3\Big) \int_\Omega |\nabla V^{\frac{m+d}{2}}|^2 \mathrm{d}\mathbf{x} \\ & + \mathcal{M}\Big(\int_\Omega V^m \mathrm{d}\mathbf{x}\Big)^\alpha + \mathcal{N}\Big(\int_\Omega V^m \mathrm{d}\mathbf{x}\Big)^{\alpha\beta} \quad \text{on} \quad (0,t^*), \end{aligned} \tag{36}$$

with

$$\begin{cases} c_5 = C_S^{\frac{3\delta}{m+d}+\frac{6\alpha}{2m+3d}} \frac{3\alpha d c_4 \sigma}{2m+3d}, \quad c_6 = \frac{3\delta c_4 C_S^{\frac{3\delta}{m+d}}}{2(m+d)}, \\ \mathcal{M} = C_S^{\frac{3\delta}{m+d}}(1-d)\varepsilon_2\sigma c_4, \quad \mathcal{N} = C_S^{\frac{3\delta}{m+d}+\frac{6\alpha}{2m+3d}} \frac{2m+3d-3\alpha}{2m+3d}\varepsilon_2 c_4 d\sigma. \end{cases}$$

Hereafter, setting

$$\varSigma = \frac{\frac{C_S^{\frac{3\delta}{m+d}} s^2(m+d)^2}{4m(ms-1)}}{\Big[\frac{\gamma-\mu}{\gamma-1}\Big(\frac{2\sqrt{\lambda_1}}{ms+q-1}\Big)^q\Big]^{\frac{\gamma-1}{1-\mu}}} \frac{1-\mu}{\gamma-\mu} \Big(\frac{6(m+d)C_S^{\frac{6\alpha}{2m+3d}} d\alpha\sigma}{(2m+3d)(2m+2d-3\delta)}\Big)^{1-\frac{3\delta}{2(m+d)}},$$

we observe that using the values of the constants $c_1, c_2, \ldots, c_6$ defined so far, relation (25) is precisely equivalent to (30). Subsequently, Lemma 2.4 warrants that for $\varepsilon_2 = \xi_m$, whose value was computed in

(27), $c_5\varepsilon_2 + c_6\varepsilon_2^{1-2(m+d)/3\delta} - c_3 \le 0$; for such a ε_2, and taking in mind (31), inequality (36) is simplified to

$$W'(t) \le \mathcal{M}\Big(\int_\Omega V^m \mathrm{d}\mathbf{x}\Big)^\alpha + \mathcal{N}\Big(\int_\Omega V^m \mathrm{d}\mathbf{x}\Big)^{\alpha\beta} = \mathcal{M}W^\alpha + \mathcal{N}W^{\alpha\beta} \quad \text{on} \quad (0, t^*). \tag{37}$$

Since we are assuming that $W(t) \nearrow \infty$ as $t \searrow t^*$, $W(t)$ can be nondecreasing, so that $W(t) \ge W(0) > 0$ with $t \in [0, t^*)$, or nonincreasing (possibly presenting oscillations), so that there exists a time t_1 where $W(t_1) = W(0)$. In any case, we can write $W(t) \ge W(0)$ for all $t \in [t_1, t^*)$, where $0 \le t_1 < t^*$. By virtue of (13), $\alpha, \beta > 1$, so that this implies that

$$W(t) \le W(0)^{1-\beta} W(t)^\beta, \quad t \in [t_1, t^*),$$

which, in conjunction with (37), produces

$$W'(t) \le (\mathcal{M}W(0)^{(1-\beta)\alpha} + \mathcal{N})W^{\alpha\beta}, \quad t \in [t_1, t^*). \tag{38}$$

Finally, integrating (38) between t_1 and t^*, we arrive at (recall $W(t_1) = W(0)$) the inequality

$$\begin{aligned}\frac{W(0)^{-\alpha\beta+1}}{-\alpha\beta+1} = \frac{W(\tau)^{-\alpha\beta+1}}{-\alpha\beta+1}\Big|_{t_1}^{t^*} &\le \int_{t_1}^{t^*} (\mathcal{M}W(0)^{(1-\beta)\alpha} + \mathcal{N})\mathrm{d}\tau \\ &\le \int_0^{t^*} (\mathcal{M}W(0)^{(1-\beta)\alpha} + \mathcal{N})\mathrm{d}\tau = (\mathcal{M}W(0)^{(1-\beta)\alpha} + \mathcal{N})t^*,\end{aligned}$$

which concludes the proof. □

The previous theorem is a general extension of the result given in [40, Schaefer, 2008], where the gradient nonlinearity for g does not take part ($k_2 = 0$). Nevertheless, even though in the proof of Theorem 3.3 we used some ideas of [40], the presence of $|\nabla u|^q$ makes the demonstration more complex and requires other necessary derivations, which are somehow tricky and tedious. In particular, these further computations lead inter alia to consider the largeness assumption of k_2 (that is relation (30), not appearing in [40]).

4. Future works

In this concluding section, we leave open some questions, naturally arising from a proper examination of the paper.

- Validity of Theorem 3.3 when the Dirichlet boundary conditions are replaced by the Robin ones. Indeed, the proof of Theorem 3.3 counts on (15), which has been derived by virtue of the fact that $V = 0$ on $\partial\Omega$. If, conversely, $V_\nu + hV = 0$ on $\partial\Omega$ relation (16) is no longer valid and has to be modified adding to its right side a term involving $\int_\Omega V^{m+d} d\mathbf{x}$.
- As specified, in this research, the question of the existence of classical solutions to problem (2) has been circumvented, being these a priori given. Resorting to some weaker solutions concept (as expected for porous media equations), such existence could be possibly accomplished by appropriate regularizing actions on the original problem, the use of mollification techniques and general compactness arguments.
- The gradient nonlinearities $-|\nabla u|^q$ considered in our general nonlinear diffusion problem ($m > 1$) have an absorbing effect on the model. Conversely, in Sect. 1, we presented some papers dealing with the linear diffusion ($m = 1$) Hamilton–Jacobi equation with first-order superquadratic Hamiltonian growth (i.e., $+|\nabla u|^q$, with $q > 2$). Since the treatment of the porous medium equation is rather different and the extension to the range $m > 1$ of general results obtained for $m = 1$ is not an

automatic process, the mentioned contributions could be of inspiration and support to see how far those analyses established in the framework of linear Hamilton–Jacobi equations with superquadratic growth can be generalized to nonlinear diffusion equations with the same growth.

Acknowledgements

The authors express their sincere gratitude to the editors and two anonymous referees for the careful reading of the original manuscript and useful comments that helped to improve the presentation of the results and accentuate important details. GV is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and is partially supported by the research project *Integro-differential Equations and Non-Local Problems*, funded by Fondazione di Sardegna (2017). The research of TL is supported by NNSF of P. R. China (Grant No. 61503171), CPSF (Grant No. 2015M582091), and NSF of Shandong Province (Grant No. ZR2016JL021), DSRF of Linyi University (Grant No. LYDX2015BS001) and the AMEP of Linyi University. The authors would like to thank Professor Cornelis van der Mee for his fruitful remarks during the revision process of this paper.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflicts of interest.

References

[1] Aida, M., Tsujikawa, T., Efendiev, M., Yagi, A., Mimura, M.: Lower estimate of the attractor dimension for a chemotaxis growth system. J. Lond. Math. Soc. **74**(2), 453–474 (2006)
[2] Andreu, F., Mazón, J.M., Simondon, F., Toledo, J.: Global existence for a degenerate nonlinear diffusion problem with nonlinear gradient term and source. Math. Ann. **314**(4), 703–728 (1999)
[3] Andreu, F., Mazón, J.M., Simondon, F., Toledo, J.: Blow-up for a class of nonlinear parabolic problems. Asymptot. Anal. **29**(2), 143–155 (2002)
[4] Aronson, D., Weinberger, H.: Multidimensional nonlinear diffusion arising in population genetics. Adv. Math. **30**(1), 33–76 (1978)
[5] Aronson, D.G.: The Porous Medium Equation, pp. 1–46. Springer, Berlin (1986)
[6] Ball, J.M.: Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. Q. J. Math. Oxf. **28**, 473–486 (1977)
[7] Bandle, C., Brunner, H.: Blowup in diffusion equations: a survey. J. Comput. Appl. Math. **97**(1–2), 3–22 (1998)
[8] Cao, X., Zheng, S.: Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source. Math. Methods Appl. Sci. **37**(15), 2326–2330 (2014)
[9] Fellner, K., Latos, E., Pisante, G.: On the finite time blow-up for filtration problems with nonlinear reaction. Appl. Math. Lett. **42**, 47–52 (2015)
[10] Ferreira, R., Groisman, P., Rossi, J.D.: Numerical blow-up for the porous medium equation with a source. Numer. Methods Part. Differ. Equ. **20**(4), 552–575 (2004)
[11] Ferreira, R., Groisman, P., Rossi, J.D.: Numerical blow-up for a nonlinear problem with a nonlinear boundary condition. Math. Models Methods Appl. Sci. **12**(04), 461–483 (2002)
[12] Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo **13**, 109–124 (1966)
[13] Galaktionov, V .A.: A boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$. Differ. Uravn. **17**(5), 836–842 (1981)
[14] Galaktionov, V.A.: Blow-up for quasilinear heat equations with critical Fujita's exponents. Proc. R. Soc. Edinb. Sect. A **124**(3), 517–525 (1994)

[15] Galaktionov, V.A., Kurdyumov, S.P., Mikhailov, A.P., Samarskii, A.A.: Unbounded solutions of the Cauchy problem for the parabolic equation $u_t = \nabla(u^\sigma \nabla u) + u^\beta$. Dokl. Akad. Nauk SSSR **252**(20), 1362–1364 (1980)

[16] Galaktionov, V.A., Shmarev, S.I., Vázquez, J.L.: Behaviour of interfaces in a diffusion–absorption equation with critical exponents. Interfaces Free Bound. **2**(4), 425–448 (2000)

[17] Galaktionov, V.A., Vázquez, J.L.: Extinction for a quasilinear heat equation with absorption I. Technique of intersection comparison. Commun. Partial Differ. Equ. **19**(7–8), 1075–1106 (1994)

[18] Grant, C.: Theory of Ordinary Differential Equations. CreateSpace Independent Publishing Platform (2014)

[19] Gurtin, M.E., MacCamy, R.C.: On the diffusion of biological populations. Math. Biosci. **33**(1–2), 35–49 (1977)

[20] Kielhöfer, H.: Halbgruppen und semilineare Anfangs-Randwertprobleme. Manuscr. Math. **12**(2), 121–152 (1974)

[21] Kobayashi, K., Sirao, T., Tanaka, H.: On the growing up problem for semilinear heat equations. J. Math. Soc. Jpn. **29**(3), 407–424 (1977). 07

[22] Krylov, N.V.: Nonlinear Elliptic and Parabolic Equations of the Second Order. Mathematics and its Applications (Soviet Series), vol. 7. D. Reidel Publishing Co., Dordrecht (1987)

[23] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-Linear Equations of Parabolic Type. In Translations of Mathematical Monographs, volume 23. American Mathematical Society (1988)

[24] Latos, E., Tzanetis, D.: Existence and blow-up of solutions for a semilinear filtration problem. Electron. J. Differ. Equ. **178**, 1–20 (2013)

[25] Levine, H.A.: The role of critical exponents in blowup theorems. SIAM Rev. **32**(2), 262–288 (1990)

[26] Li, Z., Peletier, L.: A comparison principle for the porous media equation with absorption. J. Math. Anal. Appl. **165**(2), 457–471 (1992)

[27] Marras, M., Piro, S., Viglialoro, G.: Lower bounds for blow-up time in a parabolic problem with a gradient term under various boundary conditions. Kodai Math. J. **37**(3), 532–543 (2014)

[28] Payne, L., Philippin, G., Piro, S.V.: Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition. II. Nonlinear Anal. Theory Methods Appl. **73**(4), 971–978 (2010)

[29] Payne, L., Philippin, G., Schaefer, P.: Blow-up phenomena for some nonlinear parabolic problems. Nonlinear Anal. Theory Methods Appl. **69**(10), 3495–3502 (2008)

[30] Payne, L., Philippin, G., Schaefer, P.: Bounds for blow-up time in nonlinear parabolic problems. J. Math. Anal. Appl. **338**(1), 438–447 (2008)

[31] Payne, L., Schaefer, P.: Lower bounds for blow-up time in parabolic problems under Dirichlet conditions. J. Math. Anal. Appl. **328**(2), 1196–1205 (2007)

[32] Payne, L., Schaefer, P.: Blow-up in parabolic problems under Robin boundary conditions. Appl. Anal. **87**(6), 699–707 (2008)

[33] Payne, L.E., Philippin, G.A., Proytcheva, V.: Continuous dependence on the geometry and on the initial time for a class of parabolic problems I. Math. Methods Appl. Sci. **30**(15), 1885–1898 (2007)

[34] Peletier, L., Terman, D.: A very singular solution of the porous media equation with absorption. J. Differ. Equ. **65**(3), 396–410 (1986)

[35] Philippin, G.A., Proytcheva, V.: Some remarks on the asymptotic behaviour of the solutions of a class of parabolic problems. Math. Methods Appl. Sci. **29**(3), 297–307 (2006)

[36] Porretta, A., Souplet, P.: Analysis of the loss of boundary conditions for the diffusive Hamilton–Jacobi equation. Ann. I. H. Poincare An. **34**(7), 1913–1923 (2017)

[37] Porretta, A., Souplet, P.: The profile of boundary gradient blowup for the diffusive Hamilton–Jacobi equation. Int. Math. Res. Not. **2017**(17), 5260–5301 (2017)

[38] Protter, M.H., Weinberger, H.F.: Maximum Principles in Differential Equations. Springer, New York (1984)

[39] Sacks, P.E.: Global behavior for a class of nonlinear evolution equations. SIAM J. Math. Anal. **16**(2), 233–250 (1985)

[40] Schaefer, P.: Lower bounds for blow-up time in some porous medium problems. Proc. Dyn. Syst. Appl. **5**, 442–445 (2008)

[41] Schaefer, P.: Blow-up phenomena in some porous medium problems. Dyn. Syst. Appl. **18**, 103–110 (2009)

[42] Souplet, P.: Finite time blow-up for a non-linear parabolic equation with a gradient term and applications. Math. Methods Appl. Sci. **19**(16), 1317–1333 (1996)

[43] Vázquez, J.: The Porous Medium Equation: Mathematical Theory. Clarendon Press, Oxford Mathematical Monographs (2007)

[44] Viglialoro, G.: Blow-up time of a Keller–Segel-type system with Neumann and Robin boundary conditions. Differ. Integral Equ. **29**(3–4), 359–376 (2016)

[45] Viglialoro, G.: Boundedness properties of very weak solutions to a fully parabolic chemotaxis-system with logistic source. Nonlinear Anal. Real World Appl. **34**, 520–535 (2017)

[46] Viglialoro, G., Woolley, T.: Eventual smoothness and asymptotic behaviour of solutions to a chemotaxis system perturbed by a logistic growth. Discrete Contin. Dyn. Syst. Ser. B. **23**(8), 3023–3045 (2018)

[47] Winkler, M.: Chemotaxis with logistic source: very weak global solutions and their boundedness properties. J. Math. Anal. Appl. **348**(2), 708–729 (2008)

[48] Winkler, M.: Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction. J. Math. Anal. Appl. **384**(2), 261–272 (2011)

Tongxing Li
School of Control Science and Engineering
Shandong University
Jinan 250061 Shandong
People's Republic of China

Nicola Pintus and Giuseppe Viglialoro
Dipartimento di Matematica e Informatica
Università di Cagliari
V. le Merello 92
09123 Cagliari
Italy
e-mail: giuseppe.viglialoro@unica.it

(Received: September 7, 2018; revised: February 1, 2019)