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Global regularity and decay estimates for 2D magneto-micropolar equations with partial dissipation

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Abstract. This paper examines the global regularity problem and decay estimates for two classes of two-dimensional (2D) magneto-micropolar equations with partial dissipation. By fully exploiting the special structure of the system and using the maximal regularity property of the 1D heat operator, we establish the global existence of classical solution for 2D magneto-micropolar equations with only velocity dissipation and partial magnetic diffusion. In addition, we obtain the global classical solution for small initial data and decay estimates of solution to 2D magneto-micropolar equations with only microrotational dissipation and magnetic diffusion.

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1. Introduction

The magneto-micropolar equations were introduced in [1] to describe the motion of an incompressible, electrically conducting micropolar fluids in the presence of an arbitrary magnetic field. It belongs to a class of fluids with nonsymmetric stress tensor and includes, as special cases, the classical fluids modeled by the Navier–Stokes equations (see, e.g., [7,24,33,34]), magnetohydrodynamic equations (see, e.g., [26]) and micropolar equations (see, e.g., [12,13]). The 3D incompressible magneto-micropolar fluid equations can be written as:

$$\begin{cases} \partial_t u + u\cdot\nabla u = (\mu+\chi)\Delta u - \nabla\pi + b\cdot\nabla b + 2\chi\nabla\times w, \\ \partial_t w + u\cdot\nabla w - \alpha\nabla\nabla\cdot w + 4\chi w = \kappa\Delta w + 2\chi\nabla\times u, \\ \partial_t b + u\cdot\nabla b = \nu\Delta b + b\cdot\nabla u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \\ u(x,y,z,0) = u_0(x,y,z), w(x,y,z,0) = w_0(x,y,z), b(x,y,z,0) = b_0(x,y,z), \end{cases} \tag{1.1}$$

where $(x,y,z)\in\mathbb{R}^3$ and $t\geq 0$, $u(x,y,z,t), w(x,y,z,t), b(x,y,z,t)$ and $\pi(x,y,z,t)$ denote the velocity of the fluid, microrotational velocity, the magnetic field and the hydrostatic pressure, respectively. μ,χ and $\frac{1}{\nu}$ are, respectively, kinematic viscosity, vortex viscosity and magnetic Reynolds number. κ and α are angular viscosities. By setting

$$u = (u_1(x,y,t), u_2(x,y,t), 0),\ w = (0,0,w_3(x,y,t)),\ b = (b_1(x,y,t), b_2(x,y,t), 0),$$

the 3D magneto-micropolar equations reduce to the 2D magneto-micropolar equations

$$\begin{cases} \partial_t u + u\cdot\nabla u = (\mu+\chi)\Delta u - \nabla\pi + b\cdot\nabla b + 2\chi\nabla\times w, \\ \partial_t w + u\cdot\nabla w + 4\chi w = \kappa\Delta w + 2\chi\nabla\times u, \\ \partial_t b + u\cdot\nabla b = \nu\Delta b + b\cdot\nabla u, \\ \nabla\cdot u = 0,\ \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y), w(x,y,0) = w_0(x,y), b(x,y,0) = b_0(x,y), \end{cases} \tag{1.2}$$

where $\nabla \times w = (\partial_y w, -\partial_x w)$ with w for w_3 for notational brevity and $\nabla \times u = \partial_x u_2 - \partial_y u_1$.

The magneto-micropolar equations play an important role in engineering and physics and has attracted considerable attention from the community of mathematical fluids (see, e.g., [20,25,28,29]). When (1.2) has full dissipation (namely, $\mu, \chi, \kappa, \nu > 0$), the global existence and uniqueness of solutions could be obtained easily (see, e.g., [20,28]). However, for the inviscid case (namely, (1.2) with $\mu > 0, \chi > 0, \kappa = \nu = 0$ and Δu replaced by u), the global regularity problem is still a challenging open problem. Therefore, it is natural to study the intermediate cases, namely (1.2) with partial dissipation.

Due to the complex structure of (1.2), when there is only partial dissipation, the global regularity problem can be quite difficult. However, many important progresses have recently been made on this direction (see, e.g., [2–6,8–11,14,21,27,31,32,35,36]). In [14,21,27], the global regularity of the 2D magneto-micropolar equations with various partial dissipation cases was obtained. All these cases contain the full or partial velocity dissipation, microrotational dissipation and magnetic diffusion. Recently, by fully exploiting the structure of the system and the techniques of Littlewood–Paley decomposition, Yamazaki [36] successfully established the global regularity of (1.2) with only velocity dissipation and magnetic diffusion, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times w, \\ \partial_t w + u \cdot \nabla w + 4\chi w = 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), w(x,y,0) = w_0(x,y), b(x,y,0) = b_0(x,y). \end{cases} \tag{1.3}$$

Here our first result is to further improve the results in [36]. More precisely, we consider the global regularity problem of (1.3) with full magnetic Laplacian dissipation replaced by partial magnetic diffusion, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u = (\mu + \chi)\Delta u - \nabla \pi + b \cdot \nabla b + 2\chi \nabla \times w, \\ \partial_t w + u \cdot \nabla w + 4\chi w = 2\chi \nabla \times u, \\ \partial_t b_1 + u \cdot \nabla b_1 = \nu \partial_{yy} b_1 + b \cdot \nabla u_1, \\ \partial_t b_2 + u \cdot \nabla b_2 = \nu \partial_{xx} b_2 + b \cdot \nabla u_2, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), w(x,y,0) = w_0(x,y), b(x,y,0) = b_0(x,y). \end{cases} \tag{1.4}$$

By fully exploiting the special structure of system (1.4) and using the maximal regularity property of the 1D heat operator, we can establish the following result.

Theorem 1.1. *Assume $(u_0, w_0, b_0) \in H^s(\mathbb{R}^2)$ with $s \geq 3$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then, the 2D magneto-micropolar equations (1.4) has a unique global classical solution (u, w, b) satisfying, for any $T > 0$,*

$$\begin{aligned} &(u, w, b) \in C([0,T]; H^s(\mathbb{R}^2)), \\ &u \in L^2(0,T; H^{s+1}(\mathbb{R}^2)), \ \partial_y b_1, \partial_x b_2 \in L^2(0,T; H^s(\mathbb{R}^2)). \end{aligned}$$

Next we consider the global existence and decay estimates to the solution of the 2D magneto-micropolar equations (1.2) with only microrotational dissipation and magnetic diffusion, namely

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\mu + \chi) u = -\nabla \pi + b \cdot \nabla b + 2\chi \nabla \times w, \\ \partial_t w + u \cdot \nabla w + 4\chi w = \kappa \Delta w + 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), w(x,y,0) = w_0(x,y), b(x,y,0) = b_0(x,y). \end{cases} \tag{1.5}$$

We remark that the global regularity problem of (1.5) is still an important open problem. Therefore, it is natural to first consider whether system (1.5) with small initial data has a global classical solution. In fact, we can establish the following theorem.

Theorem 1.2. *Let $s > 2$, and $(u_0, w_0, b_0) \in H^s(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a positive constant $\epsilon = \epsilon(\mu, \kappa, \nu)$ such that if*

$$\|u_0\|_{H^s(\mathbb{R}^2)} + \|w_0\|_{H^s(\mathbb{R}^2)} + \|b_0\|_{H^s(\mathbb{R}^2)} < \epsilon, \tag{1.6}$$

then the 2D magneto-micropolar equations (1.5) has a unique global solution (u, w, b) satisfying, for any $T > 0$,

$$(u, w, b) \in L^\infty(0, T; H^s(\mathbb{R}^2)), (w, b) \in L^2(0, T; H^{s+1}(\mathbb{R}^2)),$$

and

$$\begin{aligned}
&\|u(t)\|_{H^s(\mathbb{R}^2)} + \|w(t)\|_{H^s(\mathbb{R}^2)} + \|b(t)\|_{H^s(\mathbb{R}^2)} \\
&\quad + \int_0^T (\|u(t)\|_{H^s(\mathbb{R}^3)} + \|\nabla w(t)\|_{H^s(\mathbb{R}^3)} + \|\nabla b(t)\|_{H^s(\mathbb{R}^2)})\mathrm{d}t \\
&\le \|u_0\|_{H^s(\mathbb{R}^2)} + \|w_0\|_{H^s(\mathbb{R}^2)} + \|b_0\|_{H^s(\mathbb{R}^2)}.
\end{aligned} \tag{1.7}$$

Remark 1.3. The proof of Theorem 1.2 here is similar to the proof of Theorem 1.3 in [19], and we omit the details.

At last, using the delicate a priori estimates and the properties of heat operator, we can establish the following decay results for the global solution of system (1.5).

Theorem 1.4. *Let $(u_0, w_0) \in H^1(\mathbb{R}^2)$ and $b_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let (u, w, b) be a global solution to system (1.5), and*

$$\kappa > \frac{4\chi^2}{\mu + \chi}. \tag{1.8}$$

Then, the following decay properties hold

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|w(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{4}{3}}, \tag{1.9}$$

$$\|b(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2}}, \ \|\nabla b(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-1}, \tag{1.10}$$

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} + \|\nabla w(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2}}, \tag{1.11}$$

where the constant C depends on μ, χ, κ, ν and the initial data.

Remark 1.5. Theorem 1.4 holds for any global solution to system (1.5). Theorem 1.2 ensures that Theorem 1.4 is meaningful at least for initial data small.

Remark 1.6. As $b = 0$, system (1.5) reduces to the 2D micropolar equations with partial dissipation

$$\begin{cases}
\partial_t u + u \cdot \nabla u + (\mu + \chi)u = -\nabla\pi + 2\chi\nabla \times w, \\
\partial_t w + u \cdot \nabla w + 4\chi w = \kappa\Delta w + 2\chi\nabla \times u, \\
\nabla \cdot u = 0, \ \nabla \cdot b = 0, \\
u(x, y, 0) = u_0(x, y), w(x, y, 0) = w_0(x, y).
\end{cases} \tag{1.12}$$

The global existence and decay estimates of this system were studied by Dong, Li and Wu in [10]. As particular case of (1.5), Theorem 1.4 here improves the decay rates of u and w in [10].

Remark 1.7. The decay rates (1.10) for b and ∇b obtained in Theorem 1.4 are optimal in the sense that they coincide with the ones of the heat equation. Following the proof given in this paper, one can know that $\|u(t)\|_{L^2(\mathbb{R}^2)} + \|w(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{2}{3}}$ and $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} + \|\nabla b(t)\|_{L^2(\mathbb{R}^2)} + \|\nabla w(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2}}$ with $b_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$ replaced by $b_0 \in H^1(\mathbb{R}^2)$.

The rest of this paper is unfolded as follows: In Sect. 2, we give the proof of Theorem 1.1, and the proof of Theorem 1.4 will be completed in Sect. 3. To simplify the notations, we will write $\int f$ for $\int_{\mathbb{R}^2} f\mathrm{d}x$, $\|f\|_{L^q}$ for $\|f\|_{L^q(\mathbb{R}^2)}$, $\|f\|_{H^s}$ for $\|f\|_{H^s(\mathbb{R}^2)}$, and $\|f\|_{L^q(0,t;L^p)}$ for $(\int_0^t(\int_{\mathbb{R}^2}|f|^p dx)^{\frac{q}{p}}\mathrm{d}\tau)^{\frac{1}{q}}$.

2. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Our main efforts are devoted to prove the uniformly global a priori H^s-bound. Once we get this bound, the global existence of classical solution of (1.4) can be obtained following the Friedrichs method. Without loss of generality, we set $\mu=\frac{1}{2},\chi=\frac{1}{2}$ and $\nu=1$.

We first state and prove the global L^2-bound.

Proposition 2.1. *Assume that (u_0,w_0,b_0) satisfies the conditions stated in Theorem 1.1. Let (u,w,b) be the corresponding solution of the 2D magneto-micropolar equations (1.4). Then for all $t\in[0,T]$, (u,w,b) obeys the following global L^2-bound,*

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\int_0^t(\|\nabla u(\tau)\|_{L^2}^2+\|\partial_y b_1(\tau)\|_{L^2}^2+\|\partial_x b_2(\tau)\|_{L^2}^2)\mathrm{d}\tau\\&\quad\le\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2+\|b_0\|_{L^2}^2.\end{aligned}\tag{2.1}$$

Proof. Taking the L^2 inner product of (1.4) with u, w, b_1 and b_2, respectively, and then adding the resulting equations together, we yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+2\|w\|_{L^2}^2+\|\partial_y b_1\|_{L^2}^2+\|\partial_x b_2\|_{L^2}^2\\&=2\int\nabla\times uw,\end{aligned}\tag{2.2}$$

where we have used the facts

$$\int b\cdot\nabla b\cdot u+\int b\cdot\nabla u\cdot b=0,\quad\int\nabla\times wu=\int\nabla\times uw.$$

Applying Young's inequality, we obtain

$$\int\nabla\times uw\le\frac{1}{4}\|\nabla u\|_{L^2}^2+\|w\|_{L^2}^2.$$

Inserting this bound into (2.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+2\|\partial_y b_1\|_{L^2}^2+2\|\partial_x b_2\|_{L^2}^2\le 0.\tag{2.3}$$

Then integrating in time yields the desired bound. □

Next we start to do the H^1 estimates for u and b. Set $\Omega=\nabla\times u=\partial_x u_2-\partial_y u_1$ and $j=\nabla\times b=\partial_x b_2-\partial_y b_1$, then it follows from the first equation, third equation and fourth equation in (1.4) that

$$\partial_t\Omega+u\cdot\nabla\Omega=\Delta\Omega+b\cdot\nabla j-\Delta w,\tag{2.4}$$

$$\partial_t j+u\cdot\nabla j=\partial_{xxx}b_2-\partial_{yyy}b_1+b\cdot\nabla\Omega+2\partial_x b_1(\partial_x u_2+\partial_y u_1)-2\partial_x u_1(\partial_x b_2+\partial_y b_1).\tag{2.5}$$

Since we have no dissipation on microrotation w, we cannot control the term Δw on the right side of (2.4) directly. To overcome this difficulty, we introduce a combined quantity $G=\Omega-w$, then (2.4) and the second equation in (1.4) yielding

$$\partial_t G+u\cdot\nabla G=\Delta G+b\cdot\nabla j-G+w.\tag{2.6}$$

Proposition 2.2. *Assume that* (u_0, w_0, b_0) *satisfies the conditions stated in Theorem 1.1. Let* (u, w, b) *be the corresponding solution of the 2D magneto-micropolar equations (1.4). Then* (G, j, Ω) *satisfies, for any* $0 < t < T$,

$$\|G(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \|\Omega(t)\|_{L^2}^2 + \int_0^t (\|\nabla G(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{L^2}^2)\mathrm{d}\tau \le C. \tag{2.7}$$

Proof. Taking the L^2 inner product of (2.5) and (2.6) with j and G, respectively, and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|G(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + \|\nabla G\|_{L^2}^2 + \|\partial_{xx} b_1\|_{L^2}^2 + \|\partial_{xx} b_2\|_{L^2}^2 \\
&\quad + \|\partial_{yy} b_1\|_{L^2}^2 + \|\partial_{yy} b_2\|_{L^2}^2 \\
&= \int b \cdot \nabla j G - \|G\|_{L^2}^2 + \int wG + \int b \cdot \nabla \Omega j + 2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1) j \\
&\quad - 2\int \partial_x u_1(\partial_x b_2 + \partial_y b_1) j \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6,
\end{aligned}$$

where we have used the divergence-free conditions $\nabla \cdot u = \nabla \cdot b = 0$ and the fact

$$\begin{aligned}
\int (\partial_{xxx} b_2 - \partial_{yyy} b_1) j &= \int (\partial_{xxx} b_2 - \partial_{yyy} b_1)(\partial_x b_2 - \partial_y b_1) \\
&= \int \partial_{xxx} b_2 \partial_x b_2 - \int \partial_{xxx} b_2 \partial_y b_1 - \int \partial_{yyy} b_1 \partial_x b_2 + \int \partial_{yyy} b_1 \partial_y b_1 \\
&= -\int |\partial_{xx} b_2|^2 - \int |\partial_{xx} b_1|^2 - \int |\partial_{yy} b_2|^2 - \int |\partial_{yy} b_1|^2.
\end{aligned}$$

By the definition of G and $\nabla \cdot b = 0$, we have

$$\begin{aligned}
|K_1 + K_4| &= \left| \int b \cdot \nabla j \Omega - \int b \cdot \nabla j w + \int b \cdot \nabla \Omega j \right| \\
&= \left| \int b \cdot \nabla j w \right| \\
&\le \|b\|_{L^\infty} \|\nabla j\|_{L^2} \|w\|_{L^2} \\
&\le C\|b\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2}^{\frac{3}{2}} \|w\|_{L^2} \\
&\le \frac{1}{8}\|\nabla j\|_{L^2}^2 + C\|b\|_{L^2}^2 \|w\|_{L^2}^4,
\end{aligned}$$

where we have also used the Hölder inequality, the Young inequality and the following Gagliardo–Nirenberg inequality

$$\|b\|_{L^\infty} \le C\|b\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2}^{\frac{1}{2}}.$$

Applying the Young inequality to K_3, we yield

$$|K_3| \le \frac{1}{2}\|G\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2.$$

Using again the Hölder inequality, the Young inequality and the following Gagliardo–Nirenberg inequality

$$\|j\|_{L^4} \le C\|j\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2}^{\frac{1}{2}},$$

we obtain

$$\begin{aligned}|K_5+K_6| &\le 8\|\nabla b\|_{L^4}\|j\|_{L^4}\|\nabla u\|_{L^2}\\ &\le C\|j\|_{L^4}^2\|\Omega\|_{L^2}\\ &\le C\|j\|_{L^2}\|\nabla j\|_{L^2}\|\Omega\|_{L^2}\\ &\le \frac{1}{8}\|\nabla j\|_{L^2}^2 + C\|j\|_{L^2}^2(\|G\|_{L^2}^2+\|w\|_{L^2}^2)\end{aligned}$$

Collecting the estimates above, and note that

$$\|\nabla j\|_{L^2}^2 \le 2(\|\partial_{xx}b_1\|_{L^2}^2+\|\partial_{xx}b_2\|_{L^2}^2+\|\partial_{yy}b_1\|_{L^2}^2+\|\partial_{yy}b_2\|_{L^2}^2),$$

then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|G(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2)+\|\nabla G(t)\|_{L^2}^2+\|\nabla j(t)\|_{L^2}^2\\ &\quad\le C(1+\|j(t)\|_{L^2}^2)\|G(t)\|_{L^2}^2 + C(1+\|j(t)\|_{L^2}^2)\|w(t)\|_{L^2}^2 + C\|b(t)\|_{L^2}^2\|w(t)\|_{L^2}^4.\end{aligned}$$

Then Gronwall's inequality and (2.1), together with $\|j\|_{L^2}^2 \le 2(\|\partial_y b_1\|_{L^2}^2+\|\partial_x b_2\|_{L^2}^2)$ and $G=\Omega-w$, yield the desired bound. □

Now we turn to give the L^p bounds for Ω, w and Δb. In order to obtain the desired global bounds, we need to use a regularization property involving the heat operator. Let $K_t(x)=(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ with $x\in\mathbb{R}^d$ and $d\ge 1$ be the heat kernel and set

$$e^{t\Delta}f = K_t(x)*f.$$

Then the following lemma holds (see, e.g., [18]).

Lemma 2.3. (Maximal $L_t^qL_x^p$ regularity for the heat kernel) *Define the operator A by*

$$Af=\int_0^t e^{(t-s)\Delta}\Delta f(s)\mathrm{d}s.$$

Let $p,q\in(1,\infty)$. Then A is bounded from $L^q(0,T;L^p(\mathbb{R}^d))$ to $L^q(0,T;L^p(\mathbb{R}^d))$ for every $T\in(0,\infty]$.

Proposition 2.4. *Assume that (u_0,w_0,b_0) satisfies the conditions stated in Theorem 1.1. Let (u,w,b) be the corresponding solution of the 2D magneto-micropolar equations (1.4). Then for any $0<t<T$,*

$$\|G\|_{L^\infty(0,t;L^p)}+\|\Omega\|_{L^\infty(0,t;L^p)}+\|w\|_{L^\infty(0,t;L^p)}+\|\Delta b\|_{L^4(0,t;L^p)}\le C, \tag{2.8}$$

where $2\le p<\infty$.

Proof. Let $K_t(y)=\frac{1}{4\pi t}e^{-\frac{y^2}{4t}}$ with $y\in\mathbb{R}$ be the heat kernel and set

$$e^{t\partial_{yy}}f=K_t(y)*f.$$

Resorting to the heat kernel, we write the third and fourth equations of (1.4) in the integral form

$$b_1(x,y,t)=e^{t\partial_{yy}}b_{01}+\int_0^t e^{(t-s)\partial_{yy}}(b\cdot\nabla u_1-u\cdot\nabla b_1)(s)\mathrm{d}s, \tag{2.9}$$

$$b_2(x,y,t)=e^{t\partial_{xx}}b_{02}+\int_0^t e^{(t-s)\partial_{xx}}(b\cdot\nabla u_2-u\cdot\nabla b_2)(s)\mathrm{d}s. \tag{2.10}$$

We first bound $\|\Delta b\|_{L^4(0,t;L^p)}$. It is easy to verify that

$$\|\Delta b\|_{L^4(0,t;L^p)}^4\le C_0(\|\partial_{xx}b_1\|_{L^4(0,t;L^p)}^4+\|\partial_{yy}b_1\|_{L^4(0,t;L^p)}^4+\|\partial_{xx}b_2\|_{L^4(0,t;L^p)}^4+\|\partial_{yy}b_2\|_{L^4(0,t;L^p)}^4).$$

Applying ∂_{xx} to (2.10), taking the L^p norm with respect to y, then the L^p norm in x and the L^4 norm in t, then Lemma 2.3 yields

$$\begin{aligned}
&\|\partial_{xx}b_2\|^4_{L^4(0,t;L^p)}\\
&\le \int_0^t \|e^{s\partial_{xx}}\partial_{xx}b_{02}\|^4_{L^p}\mathrm{d}s + C\int_0^t \|(b\cdot\nabla u_2 - u\cdot\nabla b_2)(s)\|^4_{L^p}\mathrm{d}s\\
&\le t\|\partial_{xx}b_{02}\|^4_{L^p} + C\int_0^t (\|b(s)\|^4_{L^\infty}\|\nabla u_2(s)\|^4_{L^p} + \|u(s)\|^4_{L^{2p}}\|\nabla b_2(s)\|^4_{L^{2p}})\mathrm{d}s\\
&\le t\|b_0\|^4_{H^3} + C\int_0^t (\|b(s)\|^4_{L^\infty}\|\Omega(s)\|^4_{L^p} + \|u(s)\|^4_{H^1}\|b(s)\|^{\frac{4(p-1)}{3p-2}}_{L^2}\|\Delta b(s)\|^{\frac{4(2p-1)}{3p-2}}_{L^p})\mathrm{d}s\\
&\le C + C_1\int_0^t (\|b(s)\|^4_{L^\infty}(\|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p}) + \frac{1}{8C_0C_1}\|\Delta b(s)\|^4_{L^p} + 1)\mathrm{d}s,
\end{aligned}$$

where we used the fact $\|K_t(x)\|_{L^1(\mathbb{R})} = 1$ and the Gagliardo–Nirenberg inequality

$$\|\nabla b\|_{L^{2p}} \le C\|b\|^{\frac{p-1}{3p-2}}_{L^2}\|\Delta b\|^{\frac{2p-1}{3p-2}}_{L^p}.$$

Similarly,

$$\|\partial_{yy}b_1\|^q_{L^4(0,t;L^p)} \le C + C_2\int_0^t (\|b(s)\|^4_{L^\infty}(\|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p}) + \frac{1}{8C_0C_2}\|\Delta b(s)\|^4_{L^p} + 1)\mathrm{d}s.$$

To close the above estimate, we only need to bound $\|\partial_{xx}b_1\|_{L^q(0,t;L^p)}$ and $\|\partial_{yy}b_2\|_{L^q(0,t;L^p)}$, namely $\|\partial_{xy}b_2\|_{L^q(0,t;L^p)}$ and $\|\partial_{xy}b_1\|_{L^q(0,t;L^p)}$. Applying ∂_{xy} to (2.10), and note that

$$\partial_{xy}(b\cdot\nabla u_2 - u\cdot\nabla b_2) = \partial_{xy}(\partial_x(b_1u_2 - u_1b_2)) = \partial_{xx}(\partial_y(b_1u_2) - \partial_y(u_1b_2)),$$

then we have

$$\begin{aligned}
\partial_{xy}b_2(x,y,t) &= e^{t\partial_{xx}}\partial_{xy}b_{02} + \int_0^t e^{(t-s)\partial_{xx}}\partial_{xy}(b\cdot\nabla u_2 - u\cdot\nabla b_2)(s)\mathrm{d}s\\
&= e^{t\partial_{xx}}\partial_{xy}b_{02} + \int_0^t e^{(t-s)\partial_{xx}}\partial_{xx}(\partial_y(b_1u_2) - \partial_y(u_1b_2))(s)\mathrm{d}s.
\end{aligned}$$

Taking the $L^4(0,t;L^p)$-norm to the above equality and again applying Lemma 2.3, we obtain

$$\begin{aligned}
\|\partial_{xy}b_2\|^q_{L^4(0,t;L^p)} &\le \int_0^t \|e^{s\partial_{xx}}\partial_{xy}b_{02}\|^4_{L^p}\mathrm{d}s + C\int_0^t \|(\partial_y(b_1u_2) - \partial_y(u_1b_2))(s)\|^4_{L^p}\mathrm{d}s\\
&\le t\|\partial_{xy}b_{02}\|^4_{L^p} + C\int_0^t (\|b(s)\|^4_{L^\infty}\|\nabla u(s)\|^4_{L^p} + \|u(s)\|^4_{L^{2p}}\|\nabla b(s)\|^4_{L^{2p}})\mathrm{d}s\\
&\le C + C_3\int_0^t (\|b(s)\|^4_{L^\infty}(\|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p}) + \frac{1}{8C_0C_3}\|\Delta b(s)\|^4_{L^p} + 1)\mathrm{d}s.
\end{aligned}$$

Similarly,

$$\|\partial_{xy}b_1\|^4_{L^4(0,t;L^p)} \leq C + C_4 \int_0^t (\|b(s)\|^4_{L^\infty}(\|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p}) + \frac{1}{8C_0C_4}\|\Delta b(s)\|^4_{L^p} + 1)\mathrm{d}s.$$

Combining the estimates above, we have

$$\|\Delta b\|^4_{L^q(0,t;L^p)} \leq C + C\int_0^t \|b(s)\|^4_{L^\infty}(\|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p})\mathrm{d}s. \tag{2.11}$$

Secondly, we bound $\|G\|_{L^p}$. Multiplying (2.6) by $|G|^{p-2}G$, integrating the resulting equations in $\mathbb{R}^2$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|G(t)\|^p_{L^p} + \frac{4(p-1)}{p^2}\int |\nabla |G|^{\frac{p}{2}}|^2 &\leq \int b\cdot\nabla j|G|^{p-2}G - \|G\|^p_{L^p} + \int w|G|^{p-2}G\\ &\leq \|b\|_{L^\infty}\|\nabla j\|_{L^p}\|G\|^{p-1}_{L^p} + \|w\|_{L^p}\|G\|^{p-1}_{L^p},\end{aligned}$$

where we have used the fact that

$$\int \Delta G|G|^{p-2}G = -\frac{4(p-1)}{p^2}\int |\nabla|G|^{\frac{p}{2}}|^2.$$

Therefore, Young's inequality yield

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|G(t)\|^4_{L^p} &\leq \|b\|_{L^\infty}\|\nabla j\|_{L^p}\|G\|^3_{L^p} + \|w\|_{L^p}\|G\|^3_{L^p}\\ &\leq C(\|b\|^4_{L^\infty}\|\Delta b\|^4_{L^p} + \|G\|^4_{L^p} + \|w\|^4_{L^p}).\end{aligned}$$

Integrating it in $(0,t)$, we obtain

$$\begin{aligned}\|G(t)\|^4_{L^p} &\leq \|G_0\|^4_{L^p} + C\int_0^t (\|b(s)\|^4_{L^\infty}\|\Delta b(s)\|^4_{L^p} + \|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p})\mathrm{d}s\\ &\leq C + C\int_0^t (\|b(s)\|^4_{L^\infty}\|\Delta b(s)\|^4_{L^p} + \|G(s)\|^4_{L^p} + \|w(s)\|^4_{L^p})\mathrm{d}s.\end{aligned} \tag{2.12}$$

Lastly, we bound $\|w\|_{L^p}$. Multiplying the second equation of (1.4) by $|w|^{p-2}w$, integrating the resulting equations in space domain, we get

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^p_{L^p} &\leq \int \nabla\times u|w|^{p-2}w\\ &\leq \|\Omega\|_{L^p}\|w\|^{p-1}_{L^p}\\ &\leq 2(\|G\|_{L^p} + \|w\|_{L^p})\|w\|^{p-1}_{L^p}.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^p} \leq 2\|G\|_{L^p} + 2\|w\|_{L^p}. \tag{2.13}$$

Integrating it in $(0,t)$ and using Hölder's inequality, we have

$$\begin{aligned}\|w(t)\|_{L^p}^4 &\le \|w_0\|_{L^p}^4 + C\int_0^t (\|G(s)\|_{L^p}^4 + \|w(s)\|_{L^p}^4)\mathrm{d}s \\ &\le C + C\int_0^t (\|G(s)\|_{L^p}^4 + \|w(s)\|_{L^p}^4)\mathrm{d}s. \end{aligned} \tag{2.14}$$

Note that Propositions 2.1 and 2.2 and the Gagliardo–Nirenberg inequality imply

$$\|b\|_{L^4(0,t;L^\infty)} \le C\left(\int_0^t \|b(\tau)\|_{L^2}^2\|\nabla j(\tau)\|_{L^2}^2\mathrm{d}\tau\right)^{\frac{1}{4}} \le C. \tag{2.15}$$

Therefore, combining (2.12), (2.14) and (2.11) together, we obtain

$$\|G(t)\|_{L^p}^4 + \|w(t)\|_{L^p}^4 \le C + C\int_0^t ((\|b(s)\|_{L^\infty}^4 + 1)(\|G(s)\|_{L^p}^4 + \|w(s)\|_{L^p}^4))\mathrm{d}s.$$

Then Gronwall's inequality and (2.15) yield

$$\|G(t)\|_{L^p} + \|w(t)\|_{L^p} \le C.$$

It follows from this estimate, (2.11) and the definition of $G = \Omega - w$ that the desired results hold. □

In the following, we prove the L^∞ bounds for Ω, w and ∇b.

Proposition 2.5. *Assume that* (u_0, w_0, b_0) *satisfies the conditions stated in Theorem 1.1. Let* (u, w, b) *be the corresponding solution of the 2D magneto-micropolar equations* (1.4). *Then for any* $0 < t < T$,

$$\|\Omega\|_{L^1(0,t;L^\infty)} + \|w\|_{L^\infty(0,t;L^\infty)} + \|\nabla b\|_{L^1(0,t;L^\infty)} \le C. \tag{2.16}$$

Proof. By (2.1), (2.8) and Sobolev's embedding inequality, we obtain $\|\nabla b\|_{L^1(0,t;L^\infty)} \le C$. Now we turn to bound $\|\Omega\|_{L^1(0,t;L^\infty)}$ and $\|w\|_{L^\infty(0,t;L^\infty)}$. Multiplying (2.6) with ΔG, integrating it in space domain, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla G(t)\|_{L^2}^2 + \|\Delta G(t)\|_{L^2}^2 = L_1 + L_2 + L_3 + L_4, \tag{2.17}$$

where

$$L_1 = -\int \nabla G\cdot\nabla u\cdot\nabla G,\quad L_2 = -\int b\cdot\nabla j\Delta G,\quad L_3 = -\|\nabla G\|_{L^2}^2,\quad L_4 = -\int w\Delta G.$$

By the Hölder inequality, Gagliardo–Nirenberg inequality and Young inequality, we obtain

$$\begin{aligned}|L_1| &\le \|\nabla u\|_{L^2}\|\nabla G\|_{L^4}^2 \\ &\le C\|\Omega\|_{L^2}\|\nabla G\|_{L^2}\|\Delta G\|_{L^2} \\ &\le \frac{1}{4}\|\Delta G\|_{L^2}^2 + C\|\Omega\|_{L^2}^2\|\nabla G\|_{L^2}^2.\end{aligned}$$

Again applying the Hölder inequality and Young inequality, we yield

$$\begin{aligned}|L_2| &\le \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\Delta G\|_{L^2} \\ &\le \frac{1}{4}\|\Delta G\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2,\end{aligned}$$

and

$$|L_4| \leq \frac{1}{4}\|\Delta G\|_{L^2}^2 + \|w\|_{L^2}^2.$$

Inserting these estimates into (2.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla G(t)\|_{L^2}^2 + \|\Delta G\|_{L^2}^2 \leq C\|\Omega\|_{L^2}^2\|\nabla G\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2 + C\|w\|_{L^2}^2.$$

Then Gronwall's inequality, (2.1), (2.7) and (2.15) lead to

$$\|\nabla G(t)\|_{L^2}^2 + \int_0^t \|\Delta G(s)\|_{L^2}^2 \mathrm{d}s \leq C.$$

Therefore,

$$\begin{aligned}\int_0^t \|G(s)\|_{L^\infty}\mathrm{d}s &\leq C\int_0^t (\|G(s)\|_{L^2} + \|\Delta G(s)\|_{L^2})\mathrm{d}s \\ &\leq C\int_0^t (1 + \|G(s)\|_{L^2}^2 + \|\Delta G(s)\|_{L^2}^2)\mathrm{d}s \\ &\leq C. \end{aligned} \tag{2.18}$$

Letting $p \to \infty$ in (2.13), together with (2.18), then Gronwall's inequality implies

$$\|w(t)\|_{L^\infty} \leq C.$$

By the definition of G, we obtain

$$\int_0^t \|\Omega(s)\|_{L^\infty}\mathrm{d}s \leq \int_0^t (\|G(s)\|_{L^\infty} + \|w(s)\|_{L^\infty})\mathrm{d}s \leq C.$$

Thus, the proof of Proposition 2.5 is completed. □

With Propositions 2.1–2.2 and 2.4 –2.5 at our disposal, now we start to get the H^s-estimate to (u, w, b). As preparation we first recall the following calculus inequalities (see, e.g., [15,16]) involving fractional differential operators Λ^s with $s > 0$ and

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi), \quad \widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\mathrm{d}x.$$

Lemma 2.6. *Let $s > 0$. Let $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\begin{aligned}\|\Lambda^s(fg)\|_{L^r} &\leq C\left(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right), \\ \|[\Lambda^s, f]g\|_{L^r} &\leq C\left(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right),\end{aligned}$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$ and C is constant depending on the indices s, r, p_1, q_1, p_2 and q_2.

Proposition 2.7. *Assume that (u_0, w_0, b_0) satisfies the conditions stated in Theorem 1.1. Let (u, w, b) be the corresponding solution of the 2D magneto-micropolar equations (1.4). Then for any $0 < t < T$,*

$$\|u(t)\|_{H^s} + \|w(t)\|_{H^s} + \|b(t)\|_{H^s} \leq C. \tag{2.19}$$

Proof. Applying Λ^s to the first four equations of (1.4), taking the L^2-inner product with $\Lambda^s u$, $\Lambda^s w$, $\Lambda^s b_1$ and $\Lambda^s b_2$, respectively, and adding them together, then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^s w(t)\|_{L^2}^2+\|\Lambda^s b(t)\|_{L^2}^2)+\|\Lambda^s\nabla u\|_{L^2}^2+\|\Lambda^s\partial_y b_1\|_{L^2}^2+\|\Lambda^s\partial_x b_2\|_{L^2}^2\\
&\quad\le-\int\Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u+\int\Lambda^s(b\cdot\nabla b)\cdot\Lambda^s u+\int\Lambda^s\nabla\times w\cdot\Lambda^s u-\int\Lambda^s(u\cdot\nabla w)\Lambda^s w\\
&\qquad+\int\Lambda^s\nabla\times u\Lambda^s w-\int\Lambda^s(u\cdot\nabla b)\cdot\Lambda^s b+\int\Lambda^s(b\cdot\nabla u)\cdot\Lambda^s b\\
&\quad=I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}$$

By divergence-free condition $\nabla\cdot u=0$ and Lemma 2.6, we obtain

$$\begin{aligned}
I_1&=\int\sum_{i,j=1}^{2}(\Lambda^s(u_iu_j)\cdot\Lambda^s\partial_i u_j)\\
&\le C\|u\|_{L^\infty}\|\Lambda^s u\|_{L^2}\|\Lambda^s\nabla u\|_{L^2}\\
&\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$I_2\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|b\|_{L^\infty}^2\|\Lambda^s b\|_{L^2}^2.$$

By Young's inequality, we have

$$\begin{aligned}
I_3+I_5&=2\int\Lambda^s\nabla\times u\Lambda^s w\\
&\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C\|\Lambda^s w\|_{L^2}^2.
\end{aligned}$$

Again applying $\nabla\cdot u=0$ and Lemma 2.6, we yield

$$\begin{aligned}
I_4&=-\int[\Lambda^s,u\cdot\nabla]w\Lambda^s w\\
&\le C(\|\Lambda^s\nabla u\|_{L^2}\|w\|_{L^\infty}+\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2})\|\Lambda^s w\|_{L^2}\\
&\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C(\|w\|_{L^\infty}^2+\|\nabla u\|_{L^\infty})\|\Lambda^s w\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$I_6\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C(\|b\|_{L^\infty}^2+\|\nabla u\|_{L^\infty})\|\Lambda^s b\|_{L^2}^2.$$

Finally,

$$\begin{aligned}
I_7&\le C(\|b\|_{L^\infty}\|\Lambda^s\nabla u\|_{L^2}+\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2})\|\Lambda^s b\|_{L^2}\\
&\le\frac{1}{8}\|\Lambda^s\nabla u\|_{L^2}^2+C(\|b\|_{L^\infty}^2+\|\nabla u\|_{L^\infty})\|\Lambda^s b\|_{L^2}^2.
\end{aligned}$$

Combining the above estimates, and note that $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^\infty}^{\frac{1}{2}}$, then (2.1), (2.7) and (2.16) lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s w(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2) + \|\Lambda^s \nabla u\|_{L^2}^2 + \|\Lambda^s \partial_y b_1\|_{L^2}^2 + \|\Lambda^s \partial_x b_2\|_{L^2}^2 \\
&\quad \le C(1 + \|\nabla u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|w\|_{L^\infty}^2 + \|b\|_{L^\infty}^2)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s w\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) \\
&\quad \le C(1 + \|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s w\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) \\
&\quad \le C(1 + \|\Omega\|_{L^\infty} + \|\nabla b\|_{L^\infty})\log(e + \|u\|_{H^s})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s w\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2),
\end{aligned} \tag{2.20}$$

where we have used the logarithmic inequality

$$\|\nabla u\|_{L^\infty} \le C(1 + \|\Omega\|_{L^\infty}\log(e + \|u\|_{H^s})),\ s > 2$$

of [17] in the last inequality.

Adding (2.20) and (2.3) together, then Gronwall's inequality, (2.1) and (2.16) yield

$$\|u(t)\|_{H^s}^2 + \|w(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 + \int_0^t (\|u(\tau)\|_{H^{s+1}}^2 + \|\partial_y b_1(\tau)\|_{H^s}^2 + \|\partial_x b_2(\tau)\|_{H^s}^2)\mathrm{d}\tau \le C.$$

□

Finally, with the global bounds in the previous propositions at our disposal, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 We will use Friedrichs method. Let $N > 0$ be an integer and

$$\widehat{J_N f}(\xi) = \chi_{B(0,N)}(\xi)\hat{f}(\xi),$$

where $\chi_{B(0,N)}(\xi) = \begin{cases} 1, & if\ \xi \in B(0,N), \\ 0, & if\ \xi \notin B(0,N), \end{cases}$ $B(0,N) = \{\xi \in \mathbb{R}^2, |\xi| \le N\}$ and $\hat{f}$ is the Fourier transform of f. Set

$$L_N^2 = \{f \in L^2(\mathbb{R}^2)| \quad \mathrm{supp}\hat{f} \subset B(0,N)\}.$$

Let $\mathbb{P}$ denote the Leray projection onto divergence-free vector fields. We consider the following approximate system in the space L_N^2:

$$\begin{cases}
\partial_t u^N + \mathbb{P}J_N(\mathbb{P}J_N u^N \cdot \nabla \mathbb{P}J_N u^N) = \Delta \mathbb{P}J_N^2 u^N + \mathbb{P}J_N(J_N b^N \cdot \nabla J_N b^N) + \mathbb{P}J_N(\nabla \times J_N w^N), \\
\partial_t w^N + J_N(\mathbb{P}J_N u^N \cdot \nabla J_N w^N) + 2J_N^2 w^N = J_N(\nabla \times \mathbb{P}J_N u^N), \\
\partial_t b_1^N + J_N(\mathbb{P}J_N u^N \cdot \nabla J_N b_1^N) = \Delta J_N^2 \partial_{yy} b_1^N + J_N(J_N b^N \cdot \nabla \mathbb{P}J_N u_1^N), \\
\partial_t b_2^N + J_N(\mathbb{P}J_N u^N \cdot \nabla J_N b_2^N) = \Delta J_N^2 \partial_{xx} b_2^N + J_N(J_N b^N \cdot \nabla \mathbb{P}J_N u_2^N), \\
u^N(x,0) = J_N u_0,\ \omega^N = J_N \omega_0,\ b^N(x,0) = J_N b_0.
\end{cases} \tag{2.21}$$

The local existence and uniqueness results to system (2.21) can be obtained by the method similar to Chapter 3 in [22]. Then following the proofs of Propositions 2.1, 2.2, 2.4, 2.5 and 2.7 , we can establish the uniform global bounds, for any $t > 0$,

$$\|u^N(t)\|_{H^s}^2 + \|w^N(t)\|_{H^s}^2 + \|b^N(t)\|_{H^s}^2 \le C. \tag{2.22}$$

Then standard compactness argument allows us to obtain the global existence and uniqueness of the global smooth solution (u, w, b) to system (1.4). Thus, the proof of Theorem 1.1 is completed.

□

3. The proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Since the proof is slightly long, we divide it into five propositions for clarity. The strategy is as follows: as preparations we first establish the H^1 estimates for (u, w, b) in Propositions 3.1 and 3.2; secondly, we prove the decay estimates $\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}$, $\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \leq C(1+t)^{-\frac{2}{3}}$, $\|b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}$ and $\|\nabla b(t)\|_{L^2} \leq C(1+t)^{-1}$ in Propositions 3.3 and 3.5; finally, we give the improved decay estimates $\|u(t)\|_{L^2}+\|w(t)\|_{L^2} \leq C(1+t)^{-\frac{4}{3}}$ in Proposition 3.6, and thus, the proof of Theorem 1.4 is completed.

We first state and prove the global L^2-bound.

Proposition 3.1. *Let the assumptions stated in Theorem 1.4 hold. Then for all $t > 0$, (u, w, b) obey the following global L^2-bound,*

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\epsilon \int_0^t \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau + 8\chi \int_0^t \|w(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\quad + 2\left(\kappa - \frac{4\chi^2}{\mu+\chi-\epsilon}\right) \int_0^t \|\nabla w(\tau)\|_{L^2}^2 \mathrm{d}\tau + 2\nu \int_0^t \|\nabla b(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\leq \|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2, \qquad (3.1)
\end{aligned}$$

where $\epsilon > 0$ is chosen sufficiently small such that $\kappa > \frac{4\chi^2}{\mu+\chi-\epsilon}$.

Proof. Taking the L^2 inner product of (1.5) with u, w and b, respectively, and then adding the resulting equations together, we yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + (\mu+\chi)\|u\|_{L^2}^2 + 4\chi\|w\|_{L^2}^2 + \kappa\|\nabla w\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2 \\
&\quad = 4\chi \int \nabla \times w \cdot u, \qquad (3.2)
\end{aligned}$$

Applying Young's inequality, we obtain

$$\int \nabla \times w \cdot u \leq \frac{\mu+\chi-\epsilon}{4\chi}\|u\|_{L^2}^2 + \frac{\chi}{\mu+\chi-\epsilon}\|\nabla w\|_{L^2}^2.$$

Inserting this bound into (3.2) and then integrating in time, we yield the desired bound. □

Now we turn to do the H^1 estimate of (u, w, b). Set $\Omega = \nabla \times u$ and $j = \nabla \times b$, then it follows from (1.5) that

$$\partial_t \Omega + u \cdot \nabla \Omega + (\mu+\chi)\Omega = b \cdot \nabla j - 2\chi \Delta w, \qquad (3.3)$$

$$\partial_t \nabla w + \nabla(u \cdot \nabla w) + 4\chi \nabla w = \kappa \Delta \nabla w + 2\chi \nabla \Omega, \qquad (3.4)$$

$$\partial_t j + u \cdot \nabla j = \nu \Delta j + b \cdot \nabla \Omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1). \qquad (3.5)$$

Proposition 3.2. *Let the assumptions stated in Theorem 1.4 hold. Then for any $t > 0$,*

$$\begin{aligned}
&\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + 2\epsilon \int_0^t \|\Omega(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\quad + 2\left(\kappa - \frac{4\chi^2}{\mu+\chi-\epsilon} - \epsilon\right) \int_0^t \|\Delta w(\tau)\|_{L^2}^2 \mathrm{d}\tau + \nu \int_0^t \|\nabla j(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\leq (\|\Omega_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|j_0\|_{L^2}^2) e^{C\int_0^t (\|\nabla w(\tau)\|_{L^2}^2 + \|j(\tau)\|_{L^2}^2)\mathrm{d}\tau}, \qquad (3.6)
\end{aligned}$$

where $\epsilon > 0$ *is chosen sufficiently small such that* $\kappa > \frac{4\chi^2}{\mu+\chi-\epsilon} + \epsilon$.

Proof. Taking the L^2 inner product of (3.3), (3.4) and (3.5) with Ω, ∇w and j, respectively, and adding them together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + (\mu+\chi)\|\Omega\|_{L^2}^2 + 4\chi\|\nabla w\|_{L^2}^2 \\
&\quad + \kappa\|\Delta w\|_{L^2}^2 + \nu\|\nabla j\|_{L^2}^2 \\
&\quad = -4\chi\int \Omega\Delta w - \int \nabla w\cdot\nabla u\cdot\nabla w + 2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1)j - 2\int \partial_x u_1(\partial_x b_2 + \partial_y b_1)j \\
&\quad = L_1 + L_2 + L_3 + L_4.
\end{aligned} \tag{3.7}$$

By Young's inequality, we have

$$|L_1| \le (\mu+\chi-\epsilon)\|\Omega\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\epsilon}\|\Delta w\|_{L^2}^2.$$

Using the Hölder inequality, the Young inequality and the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
|L_2| &\le \|\nabla u\|_{L^2}\|\nabla w\|_{L^4}^2 \\
&\le C\|\Omega\|_{L^2}\|\nabla w\|_{L^2}\|\Delta w\|_{L^2} \\
&\le \epsilon\|\Delta w\|_{L^2}^2 + C\|\nabla w\|_{L^2}^2\|\Omega\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$|L_3 + L_4| \le \frac{\nu}{2}\|\nabla j\|_{L^2}^2 + C\|j\|_{L^2}^2\|\Omega\|_{L^2}^2.$$

Inserting the estimates above to (3.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) + \epsilon\|\Omega\|_{L^2}^2 + \left(\kappa - \frac{4\chi^2}{\mu+\chi-\epsilon} - \epsilon\right)\|\Delta w\|_{L^2}^2 \\
&\quad + \frac{\nu}{2}\|\nabla j\|_{L^2}^2 \\
&\quad \le C(\|\nabla w\|_{L^2}^2 + \|j\|_{L^2}^2)\|\Omega\|_{L^2}^2.
\end{aligned}$$

Then Gronwall's inequality yields the desired bound. □

With Propositions 3.1 and 3.2 at our disposal, we now start to prove our decay estimates.

Proposition 3.3. *Let the assumptions stated in Theorem 1.4 hold. Then*

$$\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}. \tag{3.8}$$

Proof. By (3.1) and (3.6), note that $\|j\|_{L^2} = \|\nabla b\|_{L^2}$ and $\|\Omega\|_{L^2} = \|\nabla u\|_{L^2}$, we have

$$\int_0^\infty \|\nabla w(\tau)\|_{L^2}^2\mathrm{d}\tau + \int_0^\infty \|j(\tau)\|_{L^2}^2\mathrm{d}\tau \le C(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2), \tag{3.9}$$

$$\int_0^\infty \|\nabla u(\tau)\|_{L^2}^2\mathrm{d}\tau = \int_0^\infty \|\Omega(\tau)\|_{L^2}^2\mathrm{d}\tau \le (\|\Omega_0\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2 + \|j_0\|_{L^2}^2)e^{C(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}. \tag{3.10}$$

Using (3.6) and (3.9), for $0 < s < t$, we have

$$\begin{aligned}
&\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \\
&\quad \le (\|\Omega(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2)e^{C\int_s^t(\|\nabla w(\tau)\|_{L^2}^2 + \|j(\tau)\|_{L^2}^2)\mathrm{d}\tau} \\
&\quad \le (\|\Omega(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2)e^{C(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)}.
\end{aligned} \tag{3.11}$$

Integrating (3.11) in $(\frac{t}{2}, t)$ with respect to s, together with (3.9)–(3.10), we obtain

$$\begin{aligned} &t(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2) \\ &\quad \leq 2e^{C(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)} \int_{\frac{t}{2}}^{t} (\|\Omega(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2) \mathrm{d}s \\ &\quad \leq C. \end{aligned}$$

Therefore, for $t \geq 1$, we have

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \leq Ct^{-1} \leq C(1+t)^{-1}. \tag{3.12}$$

For $0 < t < 1$, it follows from (3.6) and (3.9) that

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \leq C \leq C(1+t)^{-1}. \tag{3.13}$$

Then, (3.12) and (3.13) yield (3.8).

□

To give the decay estimates for (u, w, b) and the improved decay estimate for ∇b, as preparation we recall the following estimate for heat operator (see, e.g., [23,30]).

Lemma 3.4. *Let $m \geq 0$, $a > 0$ and $1 \leq p \leq q \leq \infty$. Then for any $t > 0$,*

$$\|\nabla^m e^{a\Delta t} f\|_{L^q(\mathbb{R}^2)} \leq Ct^{-\frac{m}{2} - (\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^2)}, \tag{3.14}$$

where

$$e^{a\Delta t} f(x) = (4\pi a t)^{-1} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4at}} f(y) \mathrm{d}y.$$

Now we can start to establish the desired decay estimates.

Proposition 3.5. *Let the assumptions stated in Theorem 1.4 hold. Then*

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \leq C(1+t)^{-\frac{2}{3}}, \tag{3.15}$$

$$\|b(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}, \ \|\nabla b(t)\|_{L^2} \leq C(1+t)^{-1}. \tag{3.16}$$

Proof. Taking the L^2 inner product to the first and second equations of (1.5) with u and w and then adding the resulting equations together, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2) + (\mu+\chi)\|u\|_{L^2}^2 + 4\chi\|w\|_{L^2}^2 + \kappa\|\nabla w\|_{L^2}^2 \\ &\quad = \int b \cdot \nabla b \cdot u + 4\chi \int \nabla \times w \cdot u \\ &\quad \leq \|b\|_{L^4}\|\nabla b\|_{L^2}\|u\|_{L^4} + (\mu+\chi-\epsilon)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\epsilon}\|\nabla w\|_{L^2}^2 \\ &\quad \leq C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} + (\mu+\chi-\epsilon)\|u\|_{L^2}^2 + \frac{4\chi^2}{\mu+\chi-\epsilon}\|\nabla w\|_{L^2}^2, \end{aligned} \tag{3.17}$$

where $\epsilon > 0$ is chosen sufficiently small such that $\kappa > \frac{4\chi^2}{\mu+\chi-\epsilon}$. Set

$$c = \min\{2\epsilon, 8\chi\}.$$

Then integrating (3.17) in time, we yield

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2\\&\quad\le e^{-ct}(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2)+C\int_0^t e^{-c(t-s)}\|b(s)\|_{L^2}^{\frac12}\|\nabla b(s)\|_{L^2}^{\frac32}\|u(s)\|_{L^2}^{\frac12}\|\nabla u(s)\|_{L^2}^{\frac12}\mathrm{d}s\\&\quad= e^{-ct}(\|u_0\|_{L^2}^2+\|w_0\|_{L^2}^2)+C(A_1+A_2),\end{aligned}\tag{3.18}$$

where

$$A_1=\int_0^{\frac t2} e^{-c(t-s)}\|b(s)\|_{L^2}^{\frac12}\|\nabla b(s)\|_{L^2}^{\frac32}\|u(s)\|_{L^2}^{\frac12}\|\nabla u(s)\|_{L^2}^{\frac12}\mathrm{d}s,$$

$$A_2=\int_{\frac t2}^{t} e^{-c(t-s)}\|b(s)\|_{L^2}^{\frac12}\|\nabla b(s)\|_{L^2}^{\frac32}\|u(s)\|_{L^2}^{\frac12}\|\nabla u(s)\|_{L^2}^{\frac12}\mathrm{d}s.$$

By Hölder's inequality and (3.9)–(3.10), we get

$$\begin{aligned}A_1&\le Ce^{-\frac{ct}{2}}\int_0^{\frac t2}\|\nabla b(s)\|_{L^2}^{\frac32}\|\nabla u(s)\|_{L^2}^{\frac12}\mathrm{d}s\\&\le Ce^{-\frac{ct}{2}}\left(\int_0^{\frac t2}\|\nabla b(s)\|_{L^2}^{2}\mathrm{d}s\right)^{\frac34}\left(\int_0^{\frac t2}\|\nabla u(s)\|_{L^2}^{2}\mathrm{d}s\right)^{\frac14}\\&\le Ce^{-\frac{ct}{2}}.\end{aligned}\tag{3.19}$$

Set

$$\mathcal{M}(t)=\sup_{0\le s\le t}\{(1+s)^{\frac12}(\|\nabla u(s)\|_{L^2}+\|\nabla w(s)\|_{L^2})\}.$$

Then

$$A_2\le C\mathcal{M}^2(t)\int_{\frac t2}^{t}e^{-c(t-s)}(1+s)^{-\frac43}((1+s)^{\frac23}\|u(s)\|_{L^2})^{\frac12}\mathrm{d}s.\tag{3.20}$$

Set

$$\mathcal{N}(t)=\sup_{0\le s\le t}\{(1+s)^{\frac23}(\|u(s)\|_{L^2}+\|w(s)\|_{L^2})\}.$$

Inserting (3.19)–(3.20) into (3.18), we obtain

$$\mathcal{N}^2(t)\le C(1+t)^{\frac43}e^{-\frac{ct}{2}}+C\mathcal{M}^2(t)\mathcal{N}^{\frac12}(t).$$

Then Young's inequality and $\mathcal{M}(t)\le C$ lead to the desired result.

To get the decay estimate of b, we write the third equation of (1.5) into integral form,

$$\begin{aligned} b(t) &= e^{\nu\Delta t}b_0 + \int_0^t e^{\nu\Delta(t-s)}(b\cdot\nabla u - u\cdot\nabla b)(s)\mathrm{d}s \\ &= e^{\nu\Delta t}b_0 + \int_0^{\frac{t}{2}} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s \\ &\quad + \int_{\frac{t}{2}}^t \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s. \end{aligned} \tag{3.21}$$

where $f\otimes g = (f_i g_j)$ defines the tensor product. By Lemma 3.4, for $0 < t < 1$, we have

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le C\|b_0\|_{L^2}.$$

and for $t \ge 1$,

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le Ct^{-\frac{1}{2}}\|b_0\|_{L^1}.$$

Therefore, for any $t > 0$,

$$\|e^{\nu\Delta t}b_0\|_{L^2} \le C(1+t)^{-\frac{1}{2}}. \tag{3.22}$$

Again applying Lemma 3.4, together with (3.15), we obtain for $t \ge 1$,

$$\begin{aligned} &\left\| \int_0^{\frac{t}{2}} \nabla e^{\nu\Delta(t-s)}(b\otimes u - u\otimes b)(s)\mathrm{d}s \right\|_{L^2} \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-1}\|(b\otimes u - u\otimes b)(s)\|_{L^1}\mathrm{d}s \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-1}\|u(s)\|_{L^2}\|b(s)\|_{L^2}\mathrm{d}s \\ &\le C\int_0^{\frac{t}{2}} (t-s)^{-1}(1+s)^{-\frac{2}{3}}\mathrm{d}s \\ &\le C(1+t)^{-\frac{2}{3}}. \end{aligned} \tag{3.23}$$

Using Lemma 3.4 and the Gagliardo–Nirenberg inequality, together with (3.8) and (3.15), for any $t>0$, we yield

$$\begin{aligned}
&\left\|\int_{\frac{t}{2}}^{t} \nabla e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s\right\|_{L^2}\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{p}}\|(b\otimes u-u\otimes b)(s)\|_{L^p}\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{p}}\|u(s)\|_{L^{2p}}\|b(s)\|_{L^{2p}}\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{p}}\|u(s)\|_{L^2}^{\frac{1}{p}}\|\nabla u(s)\|_{L^2}^{1-\frac{1}{p}}\|b(s)\|_{L^2}^{\frac{1}{p}}\|\nabla b(s)\|_{L^2}^{1-\frac{1}{p}}\mathrm{d}s\\
&\le C\mathcal{M}^{2-\frac{2}{p}}(t)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{p}}(1+s)^{-\frac{2}{3p}-1+\frac{1}{p}}\mathrm{d}s\\
&\le C(1+t)^{-\frac{2}{3p}}
\end{aligned}\tag{3.24}$$

with $1<p\le\frac{4}{3}$. Taking the L^2-norm for space to (3.21), together with (3.22)–(3.24), we obtain for any $t\ge 1$,

$$\|b(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}}+C(1+t)^{-\frac{2}{3}}+C(1+t)^{-\frac{2}{3p}}\le C(1+t)^{-\frac{1}{2}}.\tag{3.25}$$

Note that for $0<t<1$, (3.1) implies

$$\|b(t)\|_{L^2}\le C,$$

then we immediately obtain the first decay estimate in (3.16).

Now we turn to the decay estimate of ∇b. Applying ∇ to (3.21), we yield

$$\begin{aligned}
\nabla b(t)=\nabla e^{\nu\Delta t}b_0&+\int_0^{\frac{t}{2}}\nabla^2 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s\\
&+\int_{\frac{t}{2}}^{t}\nabla^2 e^{\nu\Delta(t-s)}(b\otimes u-u\otimes b)(s)\mathrm{d}s.
\end{aligned}\tag{3.26}$$

By Lemma 3.4, for $0<t<1$, we have

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2}\le C\|\nabla b_0\|_{L^2}.$$

and for $t\ge 1$,

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2}\le Ct^{-1}\|b_0\|_{L^1}.$$

Therefore, for any $t>0$,

$$\|\nabla e^{\nu\Delta t}b_0\|_{L^2}\le C(1+t)^{-1}.\tag{3.27}$$

Using Lemma 3.4, together with (3.15) and (3.25), we have for $t \geq 1$,

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \nabla^2 e^{\nu\Delta(t-s)} (b\otimes u - u\otimes b)(s) \mathrm{d}s \right\|_{L^2} \\
&\quad \leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \|(b\otimes u - u\otimes b)(s)\|_{L^1} \mathrm{d}s \\
&\quad \leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \|u(s)\|_{L^2} \|b(s)\|_{L^2} \mathrm{d}s \\
&\quad \leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} (1+s)^{-\frac{7}{6}} \mathrm{d}s \\
&\quad \leq C(1+t)^{-\frac{3}{2}}.
\end{aligned} \tag{3.28}$$

Applying Lemma 3.4 and Lemma 2.6, together with (3.8), (3.15) and (3.25), for any $t > 0$, we yield

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \nabla^2 e^{\nu\Delta(t-s)} (b\otimes u - u\otimes b)(s) \mathrm{d}s \right\|_{L^2} \\
&\quad = \left\| \int_{\frac{t}{2}}^{t} \Lambda^{-\alpha} \nabla^2 e^{\nu\Delta(t-s)} \Lambda^{\alpha} (b\otimes u - u\otimes b)(s) \mathrm{d}s \right\|_{L^2} \\
&\quad \leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}} \|\Lambda^{\alpha}(b\otimes u - u\otimes b)(s)\|_{L^2} \mathrm{d}s \\
&\quad \leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}} (\|\Lambda^{\alpha} b(s)\|_{L^4} \|u(s)\|_{L^4} + \|\Lambda^{\alpha} u(s)\|_{L^4} \|b(s)\|_{L^4}) \mathrm{d}s \\
&\quad \leq C \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}} (\|b(s)\|_{L^2}^{\frac{1}{2}-\alpha} \|\nabla b(s)\|_{L^2}^{\frac{1}{2}+\alpha} \|u(s)\|_{L^2}^{\frac{1}{2}} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}} \\
&\qquad + \|u(s)\|_{L^2}^{\frac{1}{2}-\alpha} \|\nabla u(s)\|_{L^2}^{\frac{1}{2}+\alpha} \|b(s)\|_{L^2}^{\frac{1}{2}} \|\nabla b(s)\|_{L^2}^{\frac{1}{2}}) \mathrm{d}s \\
&\quad \leq C\mathcal{M}^{1+\alpha}(t) \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{2-\alpha}{2}} \left((1+s)^{-\frac{13}{12}} + (1+s)^{-\frac{13}{12}-\frac{\alpha}{6}} \right) \mathrm{d}s \\
&\quad \leq C(1+t)^{-\frac{13}{12}+\frac{\alpha}{2}},
\end{aligned} \tag{3.29}$$

where $0 < \alpha \leq \frac{1}{6}$. Taking the L^2-norm for space to (3.26), together with (3.27)–(3.29), for $t \geq 1$, we obtain

$$\|\nabla b\|_{L^2} \leq C(1+t)^{-1} + C(1+t)^{-\frac{3}{2}} + C(1+t)^{-\frac{13}{12}+\frac{\alpha}{2}} \leq C(1+t)^{-1}. \tag{3.30}$$

Note that for $0 < t < 1$, (3.6) implies

$$\|\nabla b(t)\|_{L^2} \le C,$$

we thus obtain the second decay estimate in (3.16).

□

Finally, with Propositions 3.3 and 3.5 at our disposal, we can obtain the improved L^2 decay for (u, w).

Proposition 3.6. *Let the assumptions stated in Theorem 1.4 hold. Then*

$$\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \le C(1+t)^{-\frac{4}{3}}. \tag{3.31}$$

Proof. By Young's inequality, we have

$$C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \le \frac{\epsilon}{2}\|u\|_{L^2}^2 + C\|b\|_{L^2}^{\frac{2}{3}}\|\nabla b\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac{2}{3}}.$$

Inserting it into (3.17) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2) + \frac{\epsilon}{2}\|u\|_{L^2}^2 + 4\chi\|w\|_{L^2}^2 \\
&\quad \le C\|b\|_{L^2}^{\frac{2}{3}}\|\nabla b\|_{L^2}^2\|\nabla u\|_{L^2}^{\frac{2}{3}} \\
&\quad \le C(1+t)^{-\frac{8}{3}}.
\end{aligned} \tag{3.32}$$

Set $c_1 = \min\{\epsilon, 8\chi\}$. Integrating (3.32) in time, we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \\
&\le e^{-c_1 t}(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) + C\int_0^t e^{-c_1(t-s)}(1+s)^{-\frac{8}{3}}\mathrm{d}s \\
&= e^{-c_1 t}(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) + C\int_0^{\frac{t}{2}} e^{-c_1(t-s)}(1+s)^{-\frac{8}{3}}\mathrm{d}s + C\int_{\frac{t}{2}}^t e^{-c_1(t-s)}(1+s)^{-\frac{8}{3}}\mathrm{d}s \\
&\le Ce^{-c_1 t} + Ce^{-\frac{c_1 t}{2}} + C(1+t)^{-\frac{8}{3}} \\
&\le C(1+t)^{-\frac{8}{3}},
\end{aligned} \tag{3.33}$$

which immediately implies the desired bound. Thus, the proof of Proposition 3.6 is completed.

□

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